A First Course in Digital Communications

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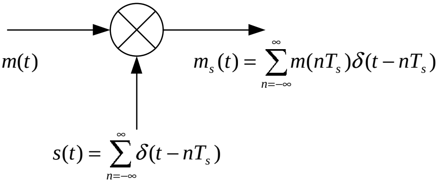
Introduction

- Though many message sources are inherently digital in nature, two of the most common message sources, audio and video, are analog, i.e., they produce continuous time signals.
- To make analog messages amenable for digital transmission sampling, quantization and encoding are required.
  - *Sampling*: How many samples per second are needed to exactly represent the signal and how to reconstruct the analog message from the samples?
  - *Quantization*: To represent the sample value by a digital symbol chosen from a finite set. What is the choice of a discrete set of amplitudes to represent the continuous range of possible amplitudes and how to measure the distortion due to quantization?
  - *Encoding*: Map the quantized signal sample into a string of digital, typically binary, symbols.

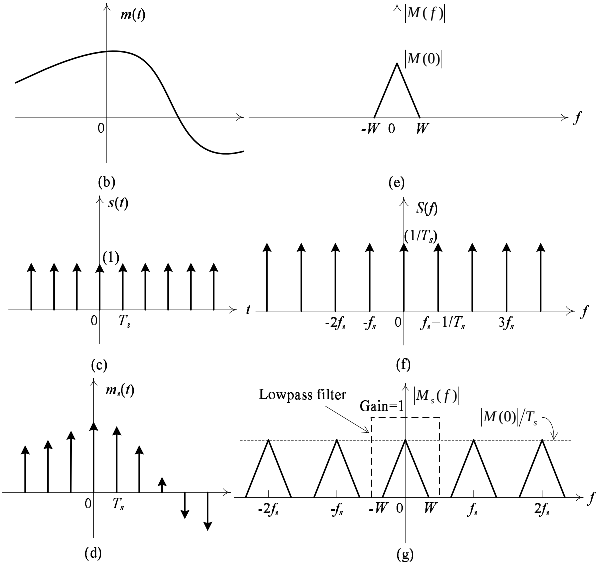
Ideal (or Impulse) Sampling



(a)

- T_s is the period of the impulse train, also referred to as the *sampling period*.
- The inverse of the sampling period, $f_s = 1/T_s$, is called the sampling frequency or *sampling rate*.
- It is intuitive that the higher the sampling rate is, the more accurate the representation of $m(t)$ by $m_s(t)$ is.
- What is the minimum sampling rate for the sampled version $m_s(t)$ to exactly represent the original analog signal $m(t)$?

Illustration of Ideal Sampling



Spectrum of the Sampled Waveform

$$m_s(t) = m(t)s(t) \leftrightarrow M_s(f) = M(f) * S(f)$$

$$M_s(f) = M(f) * \underbrace{\left[\frac{1}{T_s}\sum_{n=-\infty}^{\infty}\delta(f - nf_s)\right]}_{S(f)} = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} M(f - nf_s).$$

- If the bandwidth of $m(t)$ is limited to W Hertz, $m(t)$ can be completely recovered from $m_s(t)$ by an ideal lowpass filter of bandwidth W if $f_s \geq 2W$.
- When $f_s < 2W$ (under-sampling), the copies of $M(f)$ overlap and it is not possible to recover $m(t)$ by filtering $\Rightarrow$ *aliasing*.

Reconstruction of $m(t)$

$$M_s(f) = \mathcal{F}\{m_s(t)\} = \sum_{n=-\infty}^{\infty} m(nT_s)\mathcal{F}\{\delta(t - nT_s)\} = \sum_{n=-\infty}^{\infty} m(nT_s)\exp(-j2\pi n f T_s)$$

$$M(f) = \frac{M_s(f)}{f_s} = \frac{1}{f_s}\sum_{n=-\infty}^{\infty} m(nT_s)\exp(-j2\pi n f T_s), \quad -W \le f \le W.$$

$$\begin{aligned}
m(t) &= \mathcal{F}^{-1}\{M(f)\} = \int_{-\infty}^{\infty} M(f)\exp(j2\pi f t)\mathrm{d}f \\
&= \int_{-W}^{W} \frac{1}{f_s}\sum_{n=-\infty}^{\infty} m(nT_s)\exp(-j2\pi n f T_s)\exp(j2\pi f t)\mathrm{d}f \\
&= \frac{1}{f_s}\sum_{n=-\infty}^{\infty} m(nT_s)\int_{-W}^{W} \exp[j2\pi f(t - nT_s)]\mathrm{d}f \\
&= \sum_{n=-\infty}^{\infty} m(nT_s)\frac{\sin[2\pi W(t - nT_s)]}{\pi f_s(t - nT_s)} = \sum_{n=-\infty}^{\infty} m\left(\frac{n}{2W}\right)\mathrm{sinc}(2Wt - n)
\end{aligned}$$

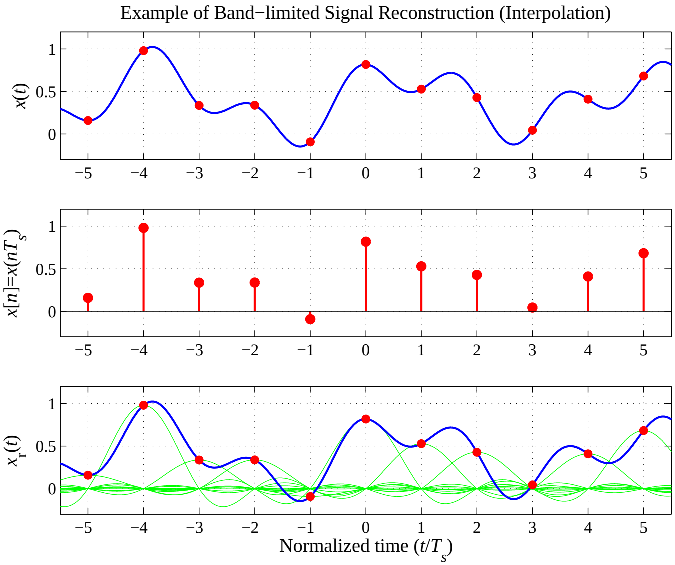
Sampling Theorem

Theorem

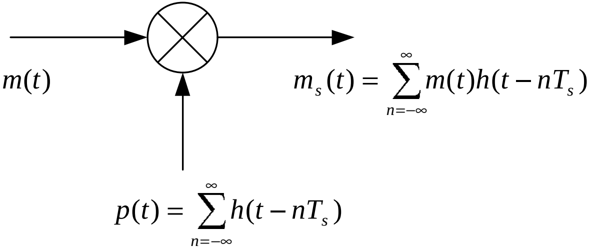
A signal having no frequency components above W Hertz is completely described by specifying the values of the signal at periodic time instants that are separated by at most $1/2W$ seconds.

- $f_s \geq 2W$ is known as the *Nyquist criterion*, the sampling rate $f_s = 2W$ is called the *Nyquist rate* and its reciprocal called the *Nyquist interval*.
- Ideal sampling is not practical $\Rightarrow$ Need practical sampling methods.

Bandlimited Interpolation



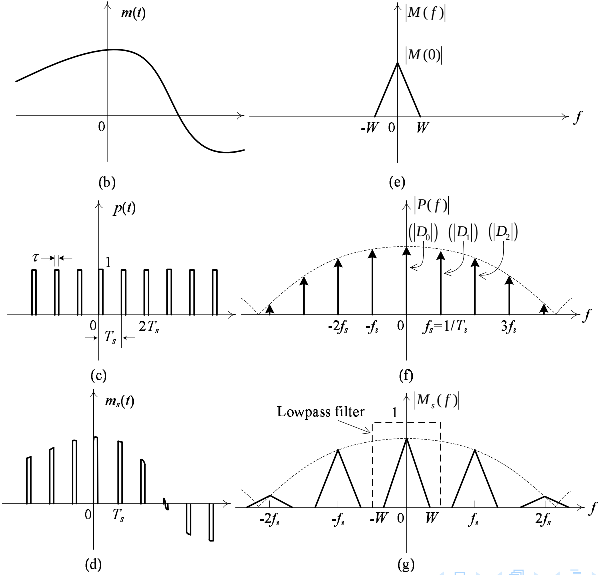
Natural Sampling



(a)

- In the above, $h(t) = 1$ for $0 \leq t \leq \tau$ and $h(t) = 0$ otherwise.
- The pulse train $p(t)$ is also known as the gating waveform.
- Natural sampling requires only an on/off gate.

Illustration of Natural Sampling



Signal Reconstruction in Natural Sampling

Write the periodic pulse train $p(t)$ in a Fourier series as:

$$p(t) = \sum_{n=-\infty}^{\infty} D_n \exp(j2\pi n f_s t), \quad D_n = \frac{\tau}{T_s} \mathrm{sinc}\left(\frac{n\tau}{T_s}\right) \mathrm{e}^{-j\pi n\tau/T_s}.$$

The sampled waveform and its Fourier transform are

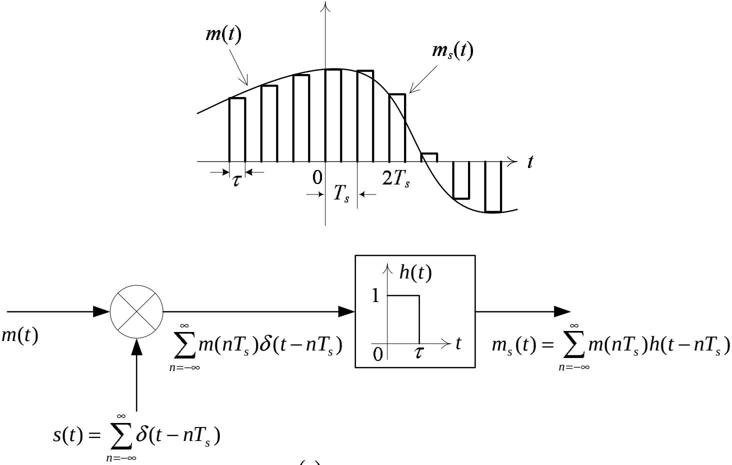
$$m_s(t) = m(t) \sum_{n=-\infty}^{\infty} D_n \exp(j2\pi n f_s t).$$

$$M_s(f) = \sum_{n=-\infty}^{\infty} D_n \mathcal{F}\{m(t) \exp(j2\pi n f_s t)\} = \sum_{n=-\infty}^{\infty} D_n M(f - n f_s).$$

The original signal $m(t)$ can still be reconstructed using a lowpass filter as long as the Nyquist criterion is satisfied.

Flat-Top Sampling

Flat-top sampling is the most popular sampling method and involves two simple operations: *sample* and *hold*.



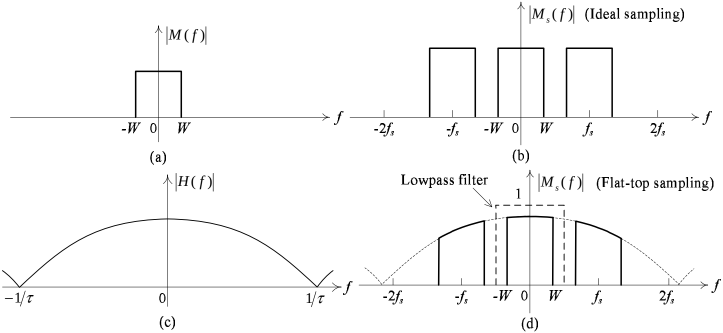
(a)

Spectrum of $m_s(t)$ in Flat-Top Sampling

$$m_s(t) = \left[m(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \right] * h(t).$$

$$M_s(f) = \mathcal{F}\left\{ m(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \right\} \mathcal{F}\{h(t)\} = \frac{1}{T_s} H(f) \sum_{n=-\infty}^{\infty} M(f - nf_s),$$

where $H(f) = \mathcal{F}\{h(t)\} = \tau \mathrm{sinc}(f\tau)\exp(-j\pi f\tau)$.

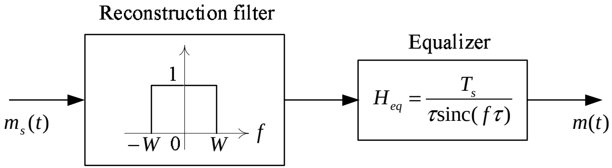


(a) (b) (c) (d)

Equalization

- Not possible to reconstruct $m(t)$ using an lowpass filter, even when the Nyquist criterion is satisfied.
- The distortion due to $H(f)$ can be corrected by connecting an *equalizer* in cascade with the lowpass reconstruction filter.
- Ideally, the amplitude response of the equalizer is

$$|H_{\text{eq}}| = \frac{T_s}{|H(f)|} = \frac{T_s}{\tau \text{sinc}(f\tau)}$$

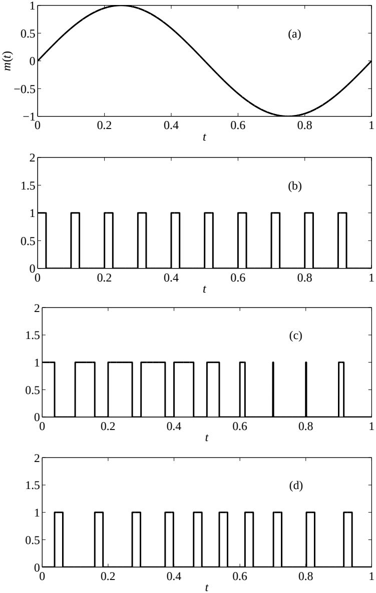


(b)

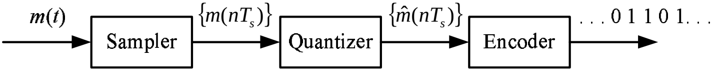
Pulse Modulation

- In pulse modulation, some parameter of a *pulse train* is varied in accordance with the sample values of a message signal.
- Pulse-amplitude modulation (PAM): *amplitudes* of regularly spaced pulses are varied.
  - PAM transmission does not improve the noise performance over baseband modulation, but allows multiplexing, i.e., sharing the same transmission media by different sources.
  - The multiplexing advantage offered by PAM comes at the expense of a larger transmission bandwidth.
- Pulse-width modulation (PWM): *widths* of the individual pulses are varied.
- Pulse-position modulation (PPM): *position* of a pulse relative to its original time of occurrence is varied.
- Pulse modulation techniques are still analog modulation. For digital communications of an analog source, quantization of sampled values is needed.

PWM & PPM Waveforms with a Sinusoidal Message

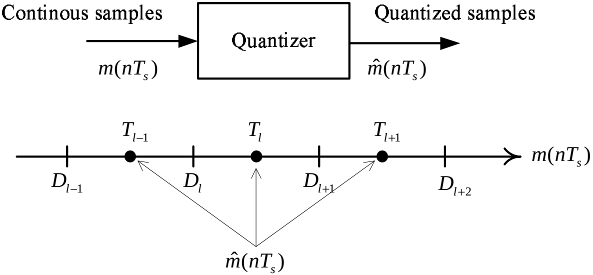


Quantization



- Quantization is to transform $m(nT_s)$ into a discrete amplitude $\hat{m}(nT_s)$ taken from a *finite* set.
- If the spacing between two adjacent amplitude levels is sufficiently small, then $\hat{m}(nT_s)$ can be made practically indistinguishable from $m(nT_s)$.
- There is always a loss of information associated with the quantization process, no matter how fine one may choose the finite set of the amplitudes $\Rightarrow$ Not possible to *completely* recover the sampled signal from the quantized signal.

Memoryless Quantization



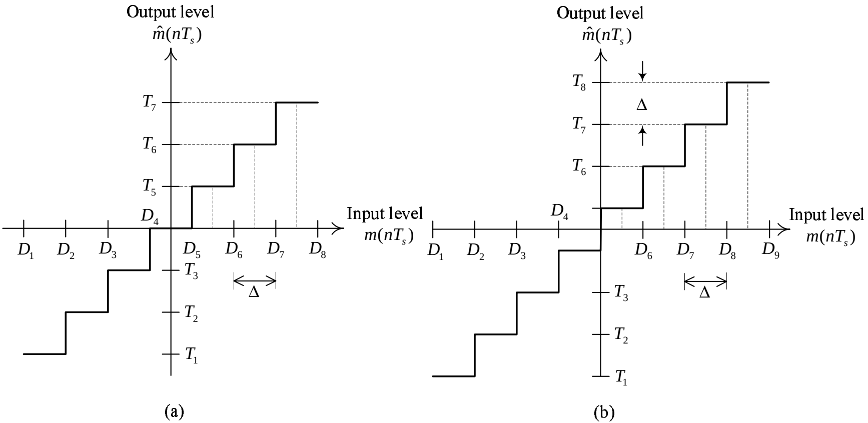
- Quantization of current sample value is independent of earlier/later samples.
- The lth interval is determined by the *decision levels* (also called the *threshold levels*) D_l and D_{l+1}:

$$\mathcal{I}_l : \{D_l < m \leq D_{l+1}\}, \quad l = 1, \ldots, L.$$

- Signal amplitudes in $\mathcal{I}_l$ are all represented by one amplitude $T_l \in \mathcal{I}_l$ (*target level* or *reconstruction level*).

Uniform Quantizer

- *Step-size* is the same and the target level is in the middle of the interval: $T_l = \frac{D_l + D_{l+1}}{2}$.
- *Midtread* and *midrise* input/output characteristics:



Input and Output of A Midrise Uniform Quantizer

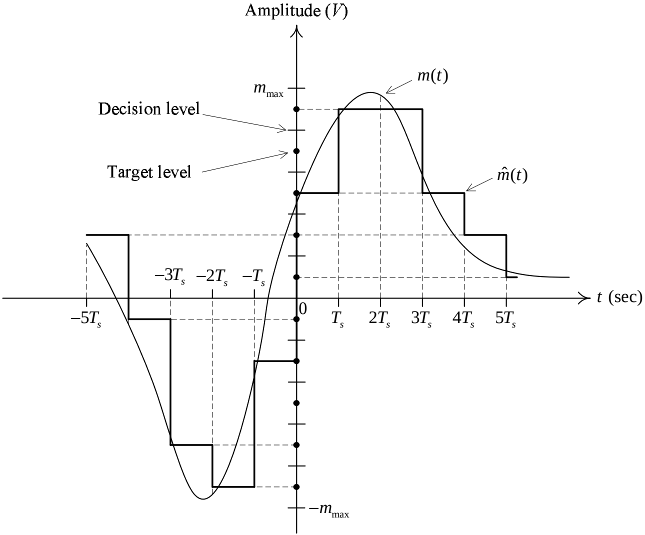
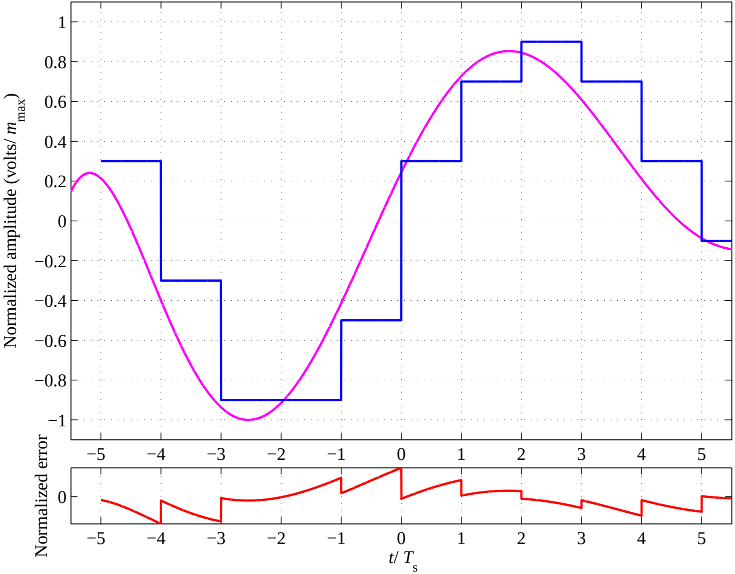


Illustration of Quantization Error



Signal-to-Quantization Noise Ratio (SNR_q)

- Model the input as a zero-mean random variable $\mathbf{m}$ with some pdf $f_{\mathbf{m}}(m)$.
- Assume the amplitude range of $\mathbf{m}$ is $-m_{\max} \leq \mathbf{m} \leq m_{\max} \Rightarrow$ the quantization step-size is $\Delta = \frac{2m_{\max}}{L}$.
- Let $\mathbf{q} = \mathbf{m} - \hat{\mathbf{m}}$ be the quantization error, then $-\Delta/2 \leq \mathbf{q} \leq \Delta/2$.
- If Δ is sufficiently small (L is sufficiently large), $\mathbf{q}$ is approximately *uniform* over $[-\Delta/2, \Delta/2]$:

$$f_{\mathbf{q}}(q) = \begin{cases} \frac{1}{\Delta}, & -\frac{\Delta}{2} < q \leq \frac{\Delta}{2} \\ 0, & \text{otherwise} \end{cases}.$$

- The mean of $\mathbf{q}$ is zero, while its variance is:

$$\sigma_{\mathbf{q}}^2 = \int_{-\Delta/2}^{\Delta/2} q^2 f_{\mathbf{q}}(q)\mathrm{d}q = \int_{-\Delta/2}^{\Delta/2} q^2 \left(\frac{1}{\Delta}\right) \mathrm{d}q = \frac{\Delta^2}{12} = \frac{m_{\max}^2}{3L^2}.$$

- With $L = 2^R$, where R is the number of bits needed to represent each target level, then $\sigma_{\mathbf{q}}^2 = \frac{m_{\max}^2}{3\times 2^{2R}}$
- The average message power is $\sigma_{\mathbf{m}}^2 = \int_{-m_{\max}}^{m_{\max}} m^2 f_{\mathbf{m}}(m)\mathrm{d}m$.
- The signal-to-quantization noise ratio is

$$\mathrm{SNR_q} = \left(\frac{3\sigma_{\mathbf{m}}^2}{m_{\max}^2}\right) 2^{2R} = \frac{3 \times 2^{2R}}{F^2}.$$

- F is called the *crest factor* of the message, defined as,

$$F = \frac{\text{Peak value of the signal}}{\text{RMS value of the signal}} = \frac{m_{\max}}{\sigma_{\mathbf{m}}}.$$

- $\mathrm{SNR_q}$ increases *exponentially* with the number of bits per sample R and decreases with the square of the message's crest factor.

- Expressed in decibels, $\mathrm{SNR_q}$ is

$$\begin{aligned} 10\log_{10}\mathrm{SNR_q} &= 6.02R + 10\log_{10}\left(\frac{\sigma_{\mathbf{m}}^2}{m_{\max}^2}\right) + 4.77 \\ &= 6.02R - 20\log_{10}F + 4.77 \end{aligned}$$

An additional 6-dB improvement in $\mathrm{SNR_q}$ *is obtained for each bit added to represent the continuous signal sample* (6-dB rule).

Optimal Quantizer

- Uniform quantizer is not optimal in terms of minimizing the signal-to-quantization noise ratio.
- In general, the decision levels are constrained to satisfy:

$$\begin{aligned} D_1 &= -m_{\max}, \\ D_{L+1} &= m_{\max}, \\ D_l &\leq D_{l+1}, \quad \text{for } l = 1, 2, \ldots L. \end{aligned}$$

- The average quantization noise power is

$$N_{\mathrm{q}} = \sum_{l=1}^{L} \int_{D_l}^{D_{l+1}} (m - T_l)^2 f_{\mathbf{m}}(m) \mathrm{d}m.$$

- To obtain the optimal quantizer that maximizes the $\mathrm{SNR}_{\mathrm{q}}$, one needs to find the set of $2L-1$ variables $\{D_2, D_3, \ldots, D_L, T_1, T_2, \ldots, T_L\}$ to minimize N_{q}.

- Differentiate N_{q} with respect to D_j and set the result to 0:

$$\frac{\partial N_{\mathrm{q}}}{\partial D_j} = f_{\mathbf{m}}(D_j)\left[(D_j - T_{j-1})^2 - (D_j - T_j)^2\right] = 0,\ j = 2, 3, \ldots L.$$

$$D_l^{\mathrm{opt}} = \frac{T_{l-1} + T_l}{2}, \quad l = 2, 3, \ldots L.$$

$\Rightarrow$ The decision levels are the midpoints of the target values!

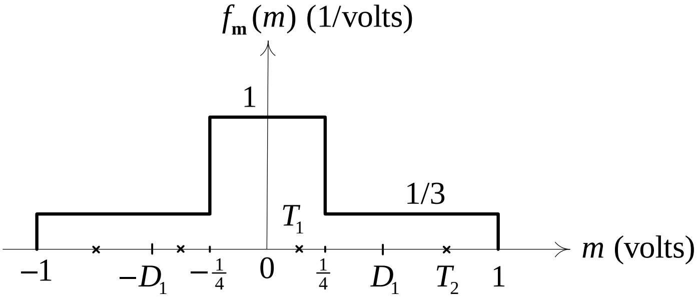
- Differentiate N_{q} with respect to T_j and set the result to 0:

$$\frac{\partial N_{\mathrm{q}}}{\partial T_j} = -2\int_{D_j}^{D_{j+1}} (m - T_j) f_{\mathbf{m}}(m)\mathrm{d}m = 0,\ j = 1, 2, \ldots L.$$

$$T_l^{\mathrm{opt}} = \frac{\int_{D_l}^{D_{l+1}} m f_{\mathbf{m}}(m)\mathrm{d}m}{\int_{D_l}^{D_{l+1}} f_{\mathbf{m}}(m)\mathrm{d}m}, \quad l = 1, 2, \ldots, L.$$

$\Rightarrow$ The target value for a quantization region should be chosen to be the *centroid* of that region.

Example of Optimal Quantizer Design (Problem 4.6)



$$T_1 = \frac{\int_0^{D_1} m f_{\mathbf{m}}(m)\mathrm{d}m}{\int_0^{D_1} f_{\mathbf{m}}(m)\mathrm{d}m} = \frac{\int_0^{1/4} m\mathrm{d}m + \frac{1}{3}\int_{1/4}^{D_1} m\mathrm{d}m}{\frac{1}{4} + \left(D_1 - \frac{1}{4}\right)\frac{1}{3}} = \frac{1+8D_1^2}{8+16D_1} \quad (1)$$

$$T_2 = \frac{\int_{D_1}^{1} m f_{\mathbf{m}}(m)\mathrm{d}m}{\int_{D_1}^{1} f_{\mathbf{m}}(m)\mathrm{d}m} = \frac{1-D_1^2}{2(1-D_1)} = \frac{1+D_1}{2}, \quad D_1 = \frac{T_1+T_2}{2} \quad (2)$$

$$\therefore 2D_1 = \frac{1+8D_1^2}{8+16D_1} + \frac{1+D_1}{2} \Rightarrow 4D_1^2 + D_1 - \frac{5}{4} = 0 \Rightarrow D_1 = 0.4478 \quad (3)$$

$$T_1 = 0.1717; \quad T_2 = 0.7239. \quad (4)$$

Lloyd-Max Conditions and Iterative Algorithm

$$D_l^{\text{opt}} = \frac{T_{l-1} + T_l}{2}, \quad (5) \qquad T_l^{\text{opt}} = \frac{\int_{D_l}^{D_{l+1}} m f_{\mathbf{m}}(m)\mathrm{d}m}{\int_{D_l}^{D_{l+1}} f_{\mathbf{m}}(m)\mathrm{d}m}. \quad (6)$$

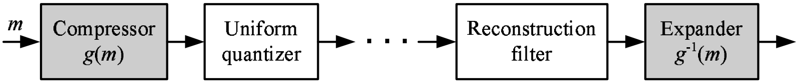
$$l = 2, 3, \ldots L$$

1. Start by specifying an arbitrary set of decision levels (for example the set that results in equal-length regions) and then find the target values using (6).
2. Determine the new decision levels using (5).
3. The two steps are iterated until the parameters do not change significantly from one step to the next.

The optimal quantizer needs to know pdf $f_{\mathbf{m}}(m)$ and is designed for a specific $m_{\max}$ $\Rightarrow$ Prefer quantization methods that are robust to source statistics and changes in the signal's power level.

Robust Quantizers

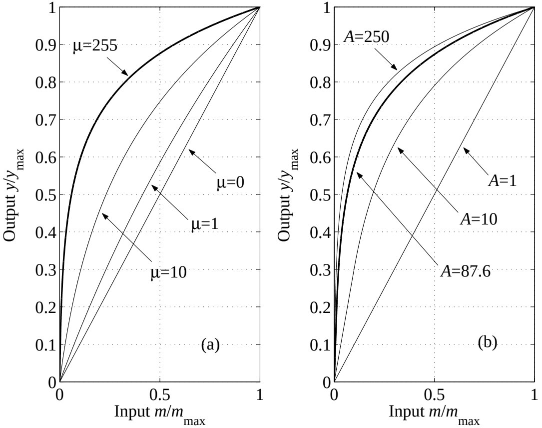
- When the message signal is uniformly distributed, the optimal quantizer is a uniform quantizer ⇒ As long as the distribution of the message signal is close to uniform, the uniform quantizer works fine.
- For a voice signal, there exists a higher probability for smaller amplitudes and a lower probability for larger amplitudes ⇒ it is more efficient to design a quantizer with more quantization regions at lower amplitudes and less quantization regions at larger amplitudes (i.e., nonuniform quantization).
- Robust method for performing nonuniform quantization is to use *compander=compressor+ expander*.



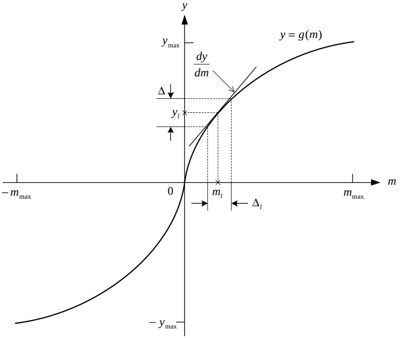
μ-law and A-law Companders

$$y = y_{\max} \frac{\ln\left[1 + \mu\left(|m|/m_{\max}\right)\right]}{\ln(1+\mu)} \operatorname{sgn}(m), \quad (\mu\text{-law})$$

$$y = \begin{cases} y_{\max} \frac{A(|m|/m_{\max})}{1+\ln A} \operatorname{sgn}(m), & 0 < \frac{|m|}{m_{\max}} \le \frac{1}{A} \\ y_{\max} \frac{1+\ln[A(|m|/m_{\max})]}{1+\ln A} \operatorname{sgn}(m), & \frac{1}{A} < \frac{|m|}{m_{\max}} < 1 \end{cases}, \quad (A\text{-law})$$

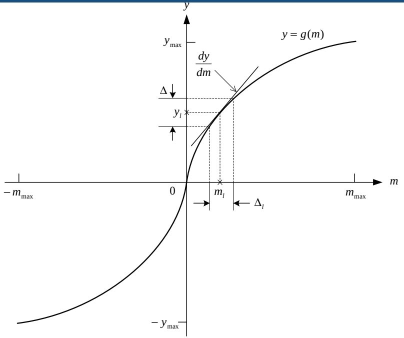


$\mathrm{SNR_q}$ of Non-Uniform Quantizers



When $L \gg 1$, Δ and Δ_l are small $\Rightarrow f_{\mathbf{m}}(m)$ is a constant $f_{\mathbf{m}}(m_l)$ over Δ_l and m_l is at the midpoint of the lth quantization region.

$$\begin{aligned} N_{\mathrm{q}} &= \sum_{l=1}^{L} \int_{m_l - \frac{\Delta_l}{2}}^{m_l + \frac{\Delta_l}{2}} (m - m_l)^2 f_{\mathbf{m}}(m) \mathrm{d}m \\ &\cong \sum_{l=1}^{L} f_{\mathbf{m}}(m_l) \int_{m_l - \frac{\Delta_l}{2}}^{m_l + \frac{\Delta_l}{2}} (m - m_l)^2 \mathrm{d}m = \sum_{l=1}^{L} \frac{\Delta_l^3}{12} f_{\mathbf{m}}(m_l). \end{aligned}$$



$$\frac{\Delta}{\Delta_l} = \left.\frac{\mathrm{d}g(m)}{\mathrm{d}m}\right|_{m=m_l} \Rightarrow N_\mathrm{q} = \frac{\Delta^2}{12}\sum_{l=1}^{L}\frac{f_\mathbf{m}(m_l)}{\left(\left.\frac{\mathrm{d}g(m)}{\mathrm{d}m}\right|_{m=m_l}\right)^2}\Delta_l.$$

Since $L \gg 1$, approximate the summation by an integral to obtain

$$N_\mathrm{q} = \frac{\Delta^2}{12}\int_{-m_\mathrm{max}}^{m_\mathrm{max}}\frac{f_\mathbf{m}(m)}{\left(\frac{\mathrm{d}g(m)}{\mathrm{d}m}\right)^2}\mathrm{d}m = \frac{y_\mathrm{max}^2}{3L^2}\int_{-m_\mathrm{max}}^{m_\mathrm{max}}\frac{f_\mathbf{m}(m)}{\left(\frac{\mathrm{d}g(m)}{\mathrm{d}m}\right)^2}\mathrm{d}m.$$

$\mathrm{SNR_q}$ of μ-law Compander

$$\frac{\mathrm{d}g(m)}{\mathrm{d}m} = \frac{y_{\max}}{\ln(1+\mu)} \frac{\mu(1/m_{\max})}{1+\mu(|m|/m_{\max})}.$$

$$N_{\mathrm{q}} = \frac{y_{\max}^2}{3L^2} \frac{\ln^2(1+\mu)}{y_{\max}^2\left(\frac{\mu}{m_{\max}}\right)^2} \int_{-m_{\max}}^{m_{\max}} \left[1+\mu\left(\frac{|m|}{m_{\max}}\right)\right]^2 f_{\mathbf{m}}(m)\mathrm{d}m$$

$$= \frac{m_{\max}^2}{3L^2} \frac{\ln^2(1+\mu)}{\mu^2} \int_{-m_{\max}}^{m_{\max}} \left[1+2\mu\left(\frac{|m|}{m_{\max}}\right)+\mu^2\left(\frac{|m|}{m_{\max}}\right)^2\right] f_{\mathbf{m}}(m)\mathrm{d}m.$$

Since $\int_{-m_{\max}}^{m_{\max}} f_{\mathbf{m}}(m)\mathrm{d}m = 1$, $\int_{-m_{\max}}^{m_{\max}} m^2 f_{\mathbf{m}}(m)\mathrm{d}m = \sigma_{\mathbf{m}}^2$ and $\int_{-m_{\max}}^{m_{\max}} |m| f_{\mathbf{m}}(m)\mathrm{d}m = E\{|\mathbf{m}|\}$, then

$$N_{\mathrm{q}} = \frac{m_{\max}^2}{3L^2} \frac{\ln^2(1+\mu)}{\mu^2} \left[1+2\mu\frac{E\{|\mathbf{m}|\}}{m_{\max}}+\mu^2\frac{\sigma_{\mathbf{m}}^2}{m_{\max}^2}\right].$$

$$\mathrm{SNR_q} = \frac{\sigma_{\mathbf{m}}^2}{N_{\mathrm{q}}} = \frac{3L^2\mu^2}{\ln^2(1+\mu)} \frac{(\sigma_{\mathbf{m}}^2/m_{\max}^2)}{1 + 2\mu(E\{|\mathbf{m}|\}/m_{\max}) + \mu^2(\sigma_{\mathbf{m}}^2/m_{\max}^2)}.$$

Define $\sigma_n^2 = \frac{\sigma_{\mathbf{m}}^2}{m_{\max}^2}$, then $\frac{E\{|\mathbf{m}|\}}{\sigma_{\mathbf{m}}} \frac{\sigma_{\mathbf{m}}}{m_{\max}} = \frac{E\{|\mathbf{m}|\}}{\sigma_{\mathbf{m}}}\sigma_n$. Therefore,

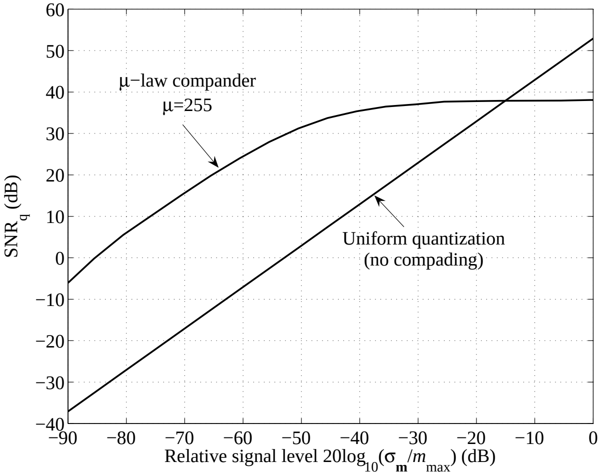
$$\mathrm{SNR_q}(\sigma_n^2) = \frac{3L^2\mu^2}{\ln^2(1+\mu)} \frac{\sigma_n^2}{1 + 2\mu\sigma_n \frac{E\{|\mathbf{m}|\}}{\sigma_{\mathbf{m}}} + \mu^2\sigma_n^2}.$$

If $\mu \gg 1$ then the dependence of $\mathrm{SNR_q}$ on the message's characteristics is very small and $\mathrm{SNR_q}$ can be approximated as

$$\mathrm{SNR_q} = \frac{3L^2}{\ln^2(1+\mu)}.$$

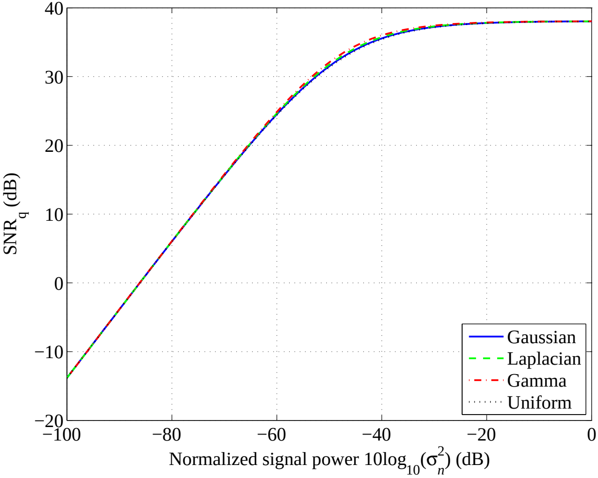
For practical values of $\mu = 255$ and $L = 256$, $\mathrm{SNR_q} = 38.1\mathrm{dB}$.

8-bit Quantizer for the Gaussian-Distributed Message



One sacrifices performance for larger input power levels to obtain a performance that remains robust over a wide range of input levels.

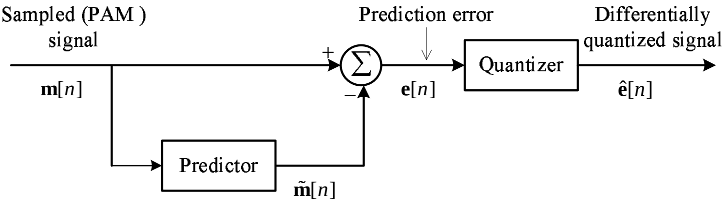
$\mathrm{SNR_q}$ with 8-bit μ-law quantizer ($L = 256$, $\mu = 255$)



Insensitive to variations in input signal power and also insensitive to the actual pdf model – Both desirable properties.

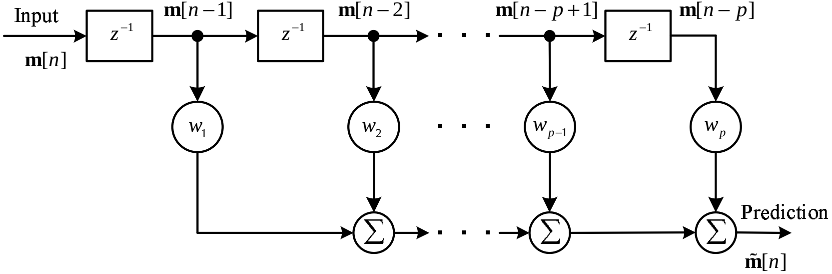
Differential Quantizers

- Most message signals (e.g., voice or video) exhibit a high degree of correlation between successive samples.
- Redundancy can be exploited to obtain a better $\mathrm{SNR_q}$ for a given L, or conversely for a specified $\mathrm{SNR_q}$ the number of levels L can be reduced:
  1. Use the previous sample values to predict the next sample value and then transmit the difference.
  2. Quantize and transmit the prediction error, $\mathbf{e}[n] = \mathbf{m}[n] - \tilde{\mathbf{m}}[n]$.



If $|e_{\max}| = |m_{\max} - km_{\max}| = |1-k|m_{\max}$ is less than $m_{\max}$ then the quantization noise power is reduced!

Linear Predictor



Select $\{w_i\}$ to *minimize* the variance of prediction error:

$$\sigma_{\mathbf{e}}^2 = E\left\{\left(\mathbf{m}[n] - \sum_{i=1}^{p} w_i \mathbf{m}[n-i]\right)^2\right\} = E\{\mathbf{m}^2[n]\} - 2\sum_{i=1}^{p} w_i E\{\mathbf{m}[n]\mathbf{m}[n-i]\} + \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j E\{\mathbf{m}[n-i]\mathbf{m}[n-j]\}.$$

Normal Equations (or the *Yule-Walker Equations*)

- With $R_{\mathbf{m}}(k) = E\{\mathbf{m}[n]\mathbf{m}[n+k]\}$ the *autocorrelation* of $\{\mathbf{m}[n]\}$,

$$\sigma_{\mathbf{e}}^2 = R_{\mathbf{m}}(0) - 2\sum_{i=1}^{p} w_i R_{\mathbf{m}}(i) + \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j R_{\mathbf{m}}(i-j).$$

- Take the partial derivative of $\sigma_{\mathbf{e}}^2$ with respect to each coefficient w_i and set the results to zero to yield:

$$\begin{bmatrix} R_{\mathbf{m}}(0) & R_{\mathbf{m}}(1) & R_{\mathbf{m}}(2) & \cdots & R_{\mathbf{m}}(p-1) \\ R_{\mathbf{m}}(-1) & R_{\mathbf{m}}(0) & R_{\mathbf{m}}(1) & \cdots & R_{\mathbf{m}}(p-2) \\ R_{\mathbf{m}}(-2) & R_{\mathbf{m}}(-1) & R_{\mathbf{m}}(0) & \cdots & R_{\mathbf{m}}(p-3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_{\mathbf{m}}(-p+1) & R_{\mathbf{m}}(-p+2) & R_{\mathbf{m}}(-p+3) & \cdots & R_{\mathbf{m}}(0) \end{bmatrix}.$$

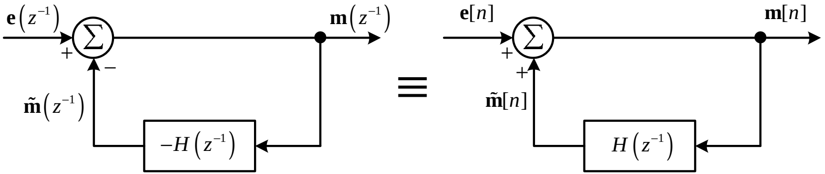
$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_p \end{bmatrix} = \begin{bmatrix} R_{\mathbf{m}}(1) \\ R_{\mathbf{m}}(2) \\ R_{\mathbf{m}}(3) \\ \vdots \\ R_{\mathbf{m}}(p) \end{bmatrix}.$$

Reconstruction of $\mathbf{m}[n]$ from the Differential Samples

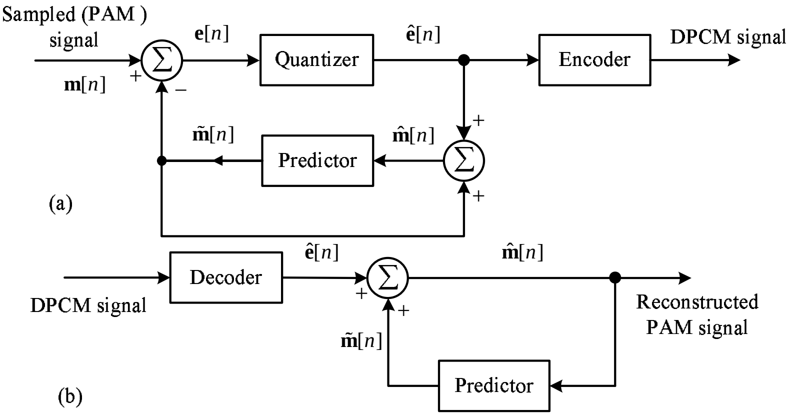
Ignore the quantization error and look at the reconstruction of $\mathbf{m}[n]$ from the differential samples $\mathbf{e}[n]$.

$$\mathbf{e}[n] = \mathbf{m}[n] - \sum_{i=1}^{p} w_i \mathbf{m}[n-i].$$

$$\begin{aligned}
\mathbf{e}(z^{-1}) &= \mathbf{m}(z^{-1}) - \sum_{i=1}^{p} w_i z^{-i} \mathbf{m}(z^{-1}) = \mathbf{m}(z^{-1}) - \mathbf{m}(z^{-1}) \sum_{i=1}^{p} w_i z^{-i} \\
&= \mathbf{m}(z^{-1}) - \mathbf{m}(z^{-1}) H(z^{-1}) \Rightarrow \mathbf{m}(z^{-1}) = \frac{1}{1 - H(z^{-1})} \mathbf{e}(z^{-1}).
\end{aligned}$$



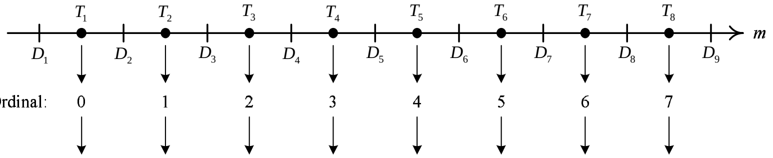
Under quantization noise error, use DPCM to eliminate the effect of previous quantization noise samples.



$$\begin{aligned}\hat{\mathbf{m}}[n] &= \tilde{\mathbf{m}}[n] + \hat{\mathbf{e}}[n] = \tilde{\mathbf{m}}[n] + (\mathbf{e}[n] - \mathbf{q}[n]) \\ &= (\tilde{\mathbf{m}}[n] + \mathbf{e}[n]) - \mathbf{q}[n] = \mathbf{m}[n] - \mathbf{q}[n].\end{aligned}$$

Pulse-Code Modulation (PCM)

- A PCM signal is obtained from the quantized PAM signal by encoding each quantized sample to a *digital codeword*.
- In binary PCM each quantized sample is digitally encoded into an R-bit binary codeword, where $R = \lceil \log_2 L \rceil + 1$.
- Binary digits of a PCM signal can be transmitted using many efficient modulation schemes.
- There are several mappings: Natural binary coding (NBC), *Gray* mapping, foldover binary coding (FBC), etc.



Ordinal:	0	1	2	3	4	5	6	7
NBC:	000	001	010	011	100	101	110	111
Gray:	000	001	011	010	110	111	101	100
FBC:	011	010	001	000	100	101	110	111