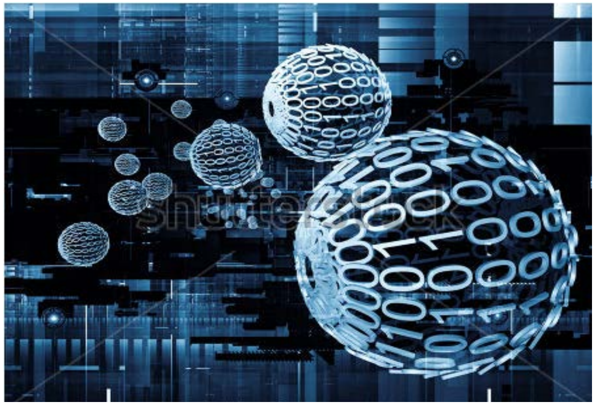
BIRZEIT UNIVERSITY
Faculty of Engineering and Technology
Department of Electrical and Computer Engineering
Modern Communication Systems, ENEE3306



Dr. Mohammad Jubran

Lecture 1

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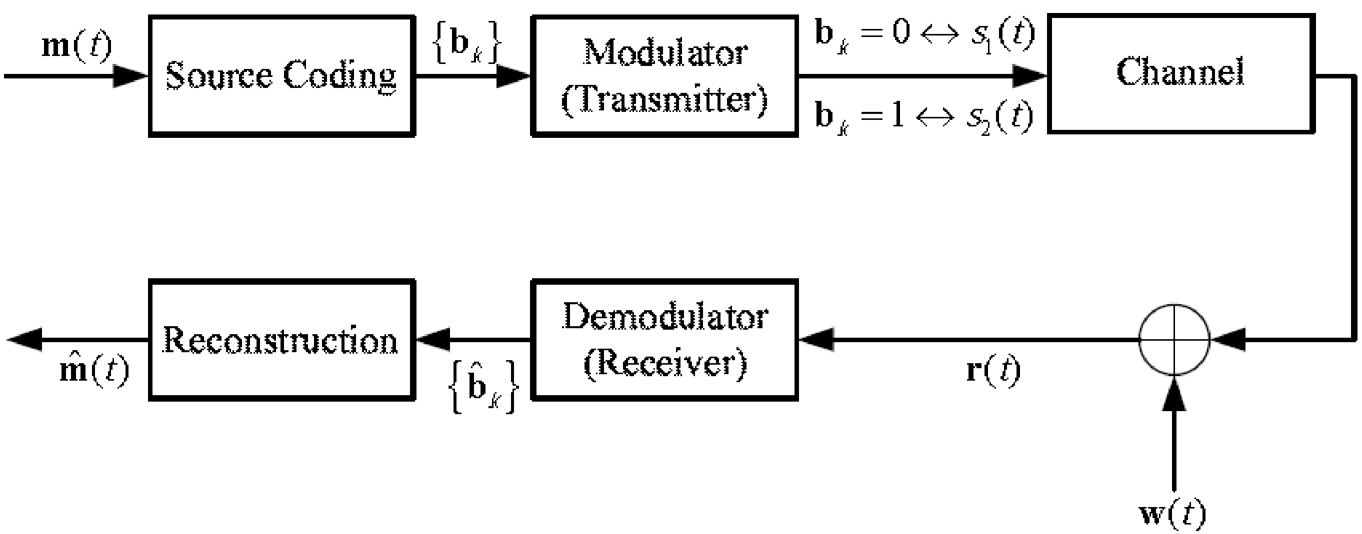
- Bits b_k (0 or 1) are represented by one of two electric waveforms $s_1(t)$ or $s_2(t)$, representation is the function of modulator
- These waveforms are then transmitted through the channel and perturbed by noise
- At the receiver, a decision must be made on the transmitted bit $\hat{b}_k$ based on the received signal $r(t)$
- we will study the optimum receiver for binary data transmission
- The performance of the receiver will be measured in terms of Bit Error Rate (BER)

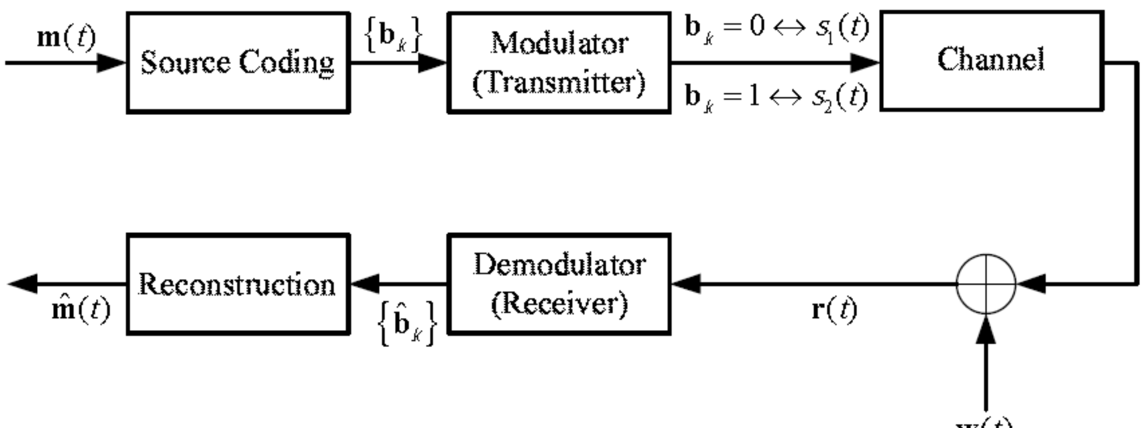


Few assumptions to design the optimal receiver

- Bit duration of b_k is T_b seconds, or the bit rate is $r_b = 1/T_b$ (bits/second)
- Bits in two different time slots are statistically independent.
- A priori probabilities: $P[b_k = 0] = P1$, $P[b_k = 1] = P2$.
- Signals $s_1(t)$ and $s_2(t)$ have a duration of T_b seconds and finite energies:

$$E_1 = \int_0^{T_b} s_1^2(t)dt < \infty, \quad E_2 = \int_0^{T_b} s_2^2(t)dt < \infty$$

- The channel is sufficiently wideband ➔ $s_1(t)$ and $s_2(t)$ are transmitted without distortion, no Intersymbol interference (ISI)



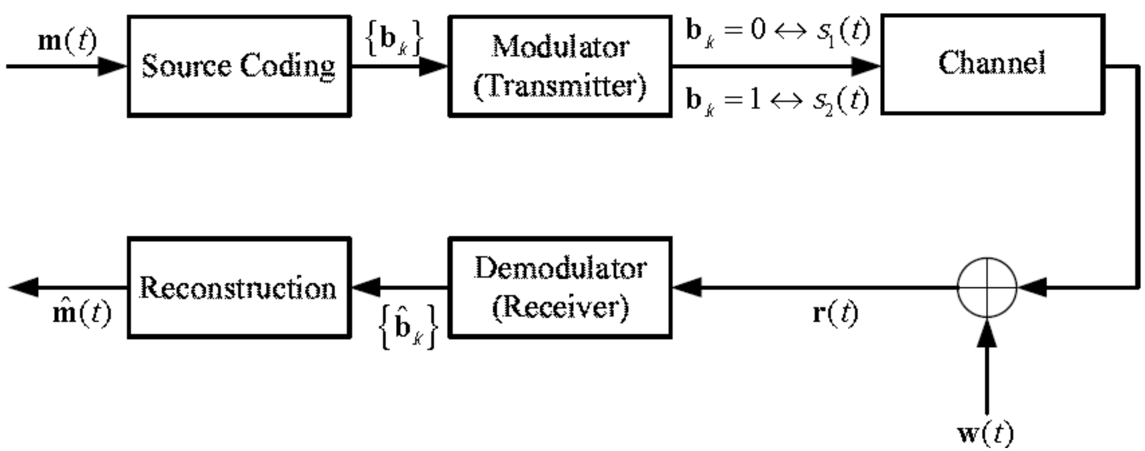
Few assumptions to design the optimal receiver

- Noise w(t) is stationary Gaussian, zero-mean white noise with two-sided power spectral density of N0/2 (watts/Hz):

$$E\{w(t)\}=0, E\{w(t)w(t+\tau)\}=\frac{N_o}{2}\delta(\tau)$$

- Received signal over $[(k-1)T_b, kT_b]$:

$$r(t)=s(t-(k-1)T_b)+w(t), \qquad (k-1)T_b \le t \le kT_b$$



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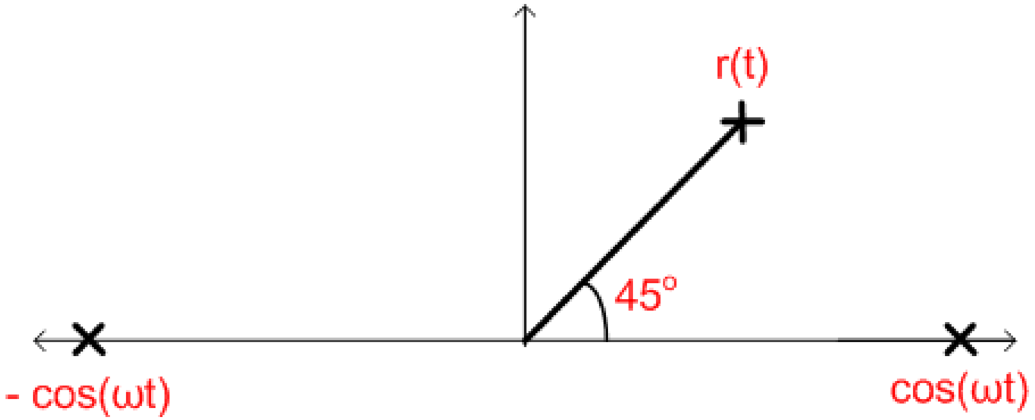
- Objective is to design a receiver (or demodulator) such that the probability of making an error is minimized.
- Shall reduce the problem from the observation of a time waveform to that of observing a set of numbers (which are random variables).
- **Example**: Let $s_1(t)=\text{cost}(wt)$ be the waveform (symbol) represent the binary bit $b_k=0$, and $s_2(t)= -\text{cost}(wt)$ be the waveform (symbol) represent the binary bit $b_k=1$. Now assume one of the waveforms is transmitted and the waveform $r(t)=0.5e^{(j\omega t+0.25\pi)}$ is received. Could you determine what was the transmitted bit $\hat{b}_k$?



- Wish to represent two arbitrary signals $s_1(t)$ and $s_2(t)$ as linear combinations of two orthonormal basis functions $\Phi_1(t)$ and $\Phi_2(t)$.
- $\Phi_1(t)$ and $\Phi_2(t)$ are orthonormal if:

$$\int_0^{T_b} \Phi_1(t)\Phi_2(t)dt = 0$$ ➔ (orthogonality)

$$\int_0^{T_b} \Phi_1^2(t)dt = \int_0^{T_b} \Phi_2^2(t)dt = 1$$ ➔ (normalized to have unity energy)

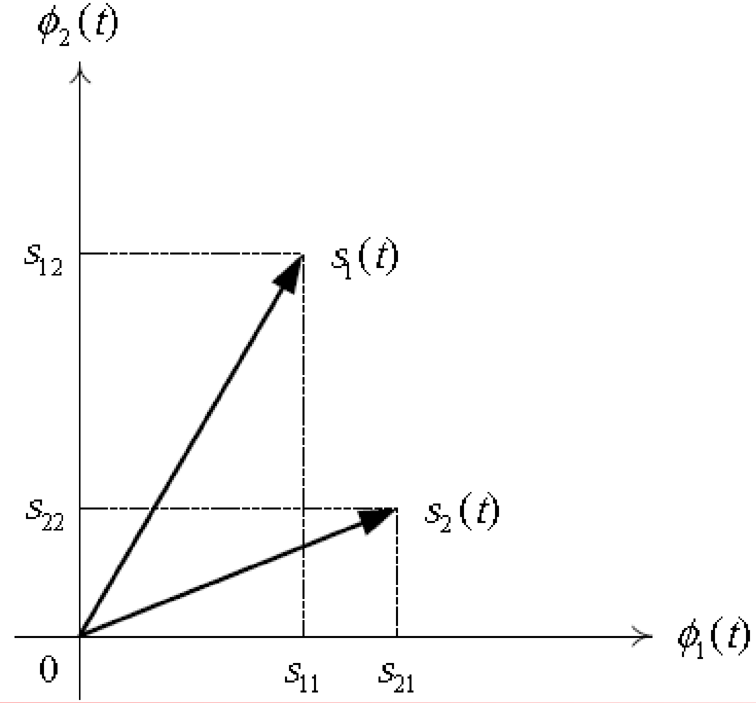
- The representations are

$$s_1(t) = s_{11}\Phi_1(t) + s_{12}\Phi_2(t)$$

$$s_2(t) = s_{21}\Phi_1(t) + s_{22}\Phi_2(t)$$

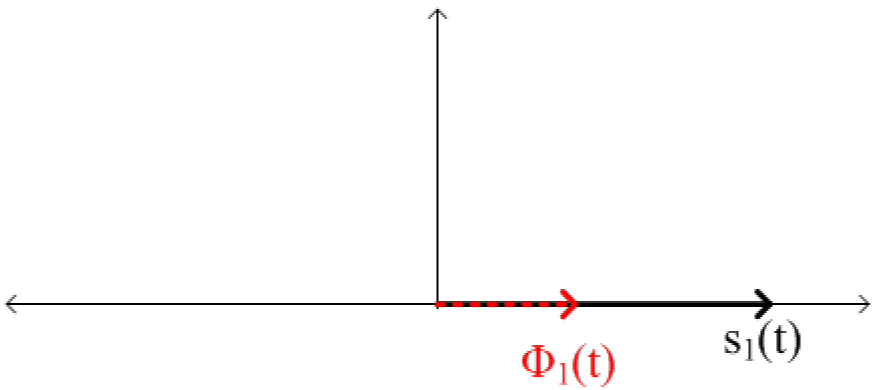
where

$$s_{ij} = \int_0^{T_b} s_i(t)\Phi_j(t)dt, \quad i, j \in \{1,2\}$$



1) Let $\Phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}}$

$$\rightarrow s_{11} = \int_0^{T_b} s_1(t)\Phi_1(t)dt = \int_0^{T_b} s_1(t)\frac{s_1(t)}{\sqrt{E_1}}dt = \frac{E_1}{\sqrt{E_1}} = \sqrt{E_1}$$



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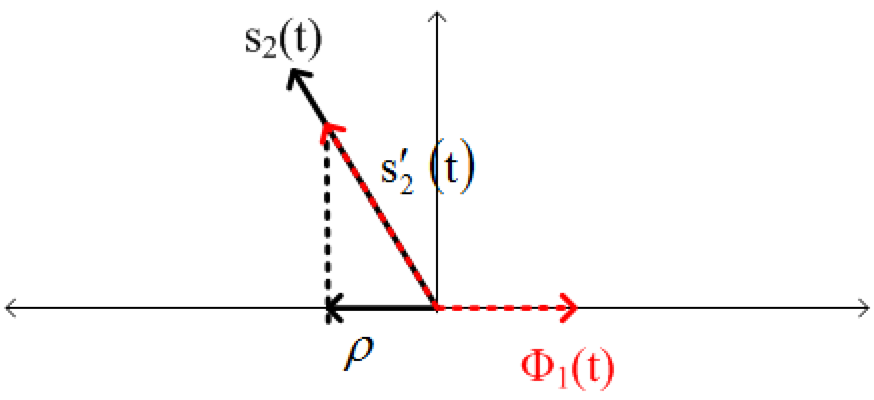
1) Let $\Phi_1(t) = \dfrac{s_1(t)}{\sqrt{E_1}}$

$$\rightarrow \quad s_{11} = \int_0^{T_b} s_1(t)\Phi_1(t)dt = \int_0^{T_b} s_1(t)\frac{s_1(t)}{\sqrt{E_1}}dt = \frac{E_1}{\sqrt{E_1}} = \sqrt{E_1}$$

2) Now, project $s_2'(t) = \dfrac{s_2(t)}{\sqrt{E_2}}$ onto $\Phi_1(t)$ to obtain the correlation coefficient (ρ)

$$\rightarrow \quad \rho = \int_0^{T_b} \frac{s_2(t)}{\sqrt{E_2}}\Phi_1(t)dt = \frac{1}{\sqrt{E_1E_2}}\int_0^{T_b} s_1(t)s_2(t)dt$$



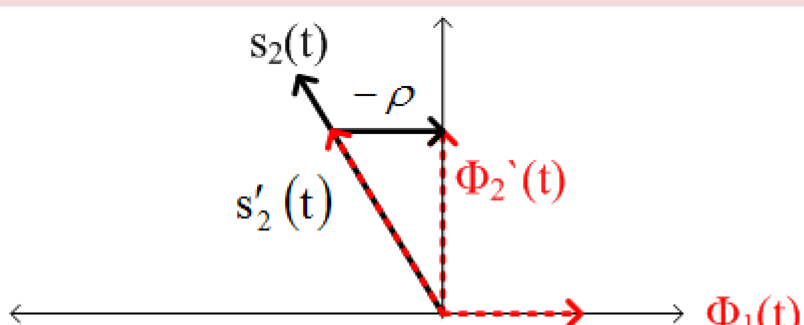
1) Let $\Phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}}$

$$\rightarrow \quad s_{11} = \int_0^{T_b} s_1(t)\Phi_1(t)dt = \int_0^{T_b} s_1(t)\frac{s_1(t)}{\sqrt{E_1}}dt = \frac{E_1}{\sqrt{E_1}} = \sqrt{E_1}$$

2) Now, project $s_2'(t) = \frac{s_2(t)}{\sqrt{E_2}}$ onto $\Phi_1(t)$ to obtain the correlation coefficient (ρ)

$$\rightarrow \quad \rho = \int_0^{T_b} \frac{s_2(t)}{\sqrt{E_2}}\Phi_1(t)dt = \frac{1}{\sqrt{E_1 E_2}}\int_0^{T_b} s_1(t)s_2(t)dt$$

3) Next, Subtract $\rho\,\Phi_1(t)$ from $s_2'(t)$ to obtain $\Phi_2'(t) = \frac{s_2(t)}{\sqrt{E_2}} - \rho\Phi_1(t)$



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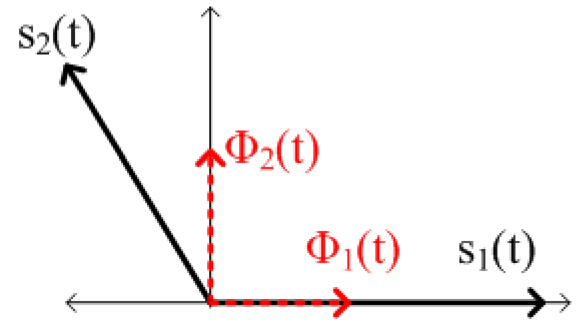
1) Let $\Phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}}$



$$\longrightarrow\ s_{11} = \int_0^{T_b} s_1(t)\Phi_1(t)dt = \int_0^{T_b} s_1(t)\frac{s_1(t)}{\sqrt{E_1}}dt = \frac{E_1}{\sqrt{E_1}} = \sqrt{E_1}$$

2) Now, project $s_2'(t) = \frac{s_2(t)}{\sqrt{E_2}}$ onto $\Phi_1(t)$ to obtain the correlation coefficient (ρ)

$$\longrightarrow\ \rho = \int_0^{T_b} \frac{s_2(t)}{\sqrt{E_2}}\Phi_1(t)dt = \frac{1}{\sqrt{E_1E_2}}\int_0^{T_b} s_1(t)s_2(t)dt$$

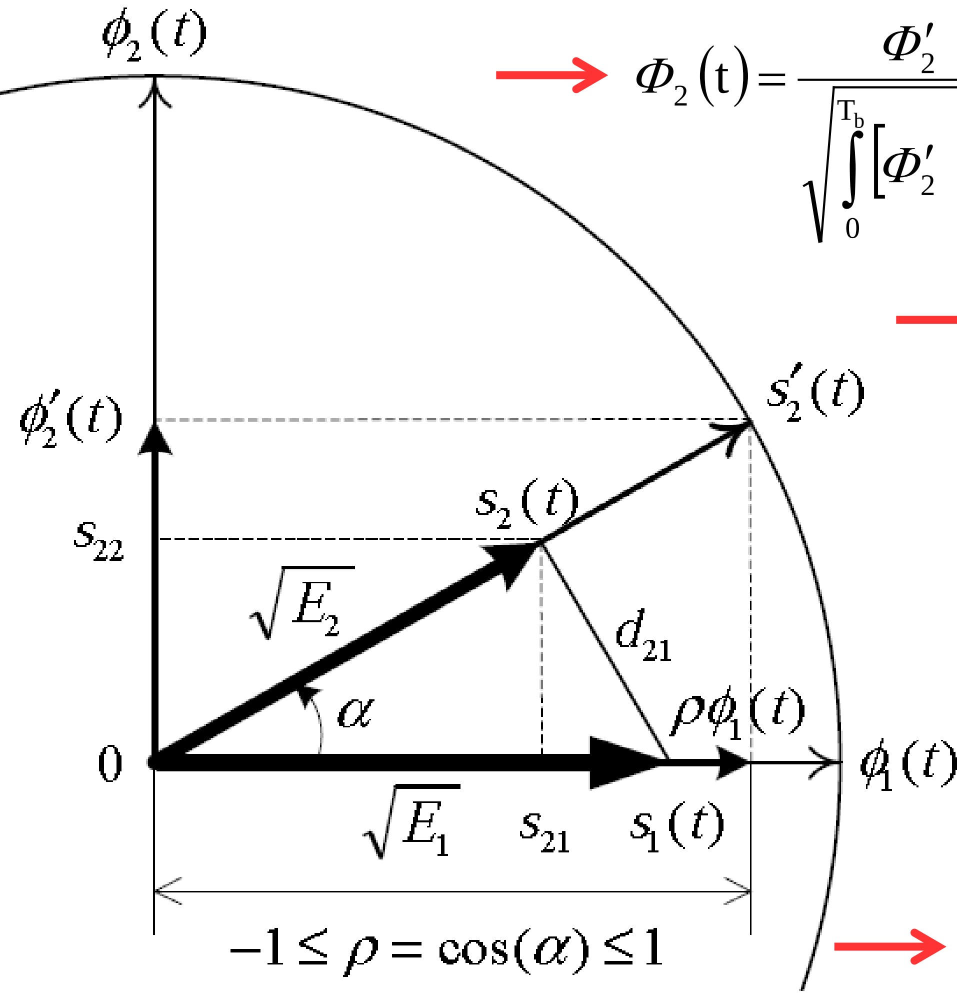
3) Next, Subtract $\rho\,\Phi_1(t)$ from $s_2'(t)$ to obtain $\Phi_2'(t) = \frac{s_2(t)}{\sqrt{E_2}} - \rho\Phi_1(t)$

4) Finally, normalize $\Phi_2'(t)$ to obtain $\Phi_2(t)$

$$\longrightarrow\ \Phi_2(t) = \frac{\Phi_2'(t)}{\sqrt{\int_0^{T_b}[\Phi_2'(t)]^2dt}} = \frac{\Phi_2'(t)}{\sqrt{1-\rho^2}} = \frac{1}{\sqrt{1-\rho^2}}\left[\frac{s_2(t)}{\sqrt{E_2}} - \frac{\rho s_1(t)}{\sqrt{E_1}}\right]$$

$$\Phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}}$$

$$\Phi_2(t) = \frac{\Phi_2'(t)}{\sqrt{\int_0^{T_b} [\Phi_2'(t)]^2 dt}} = \frac{\Phi_2'(t)}{\sqrt{1-\rho^2}} = \frac{1}{\sqrt{1-\rho^2}} \left[\frac{s_2(t)}{\sqrt{E_2}} - \frac{\rho s_1(t)}{\sqrt{E_1}} \right]$$



$$s_{11} = \int_0^{T_b} s_1(t)\Phi_1(t)dt = \sqrt{E_1}$$

$$s_{12} = \int_0^{T_b} s_1(t)\Phi_2(t)dt = ?$$

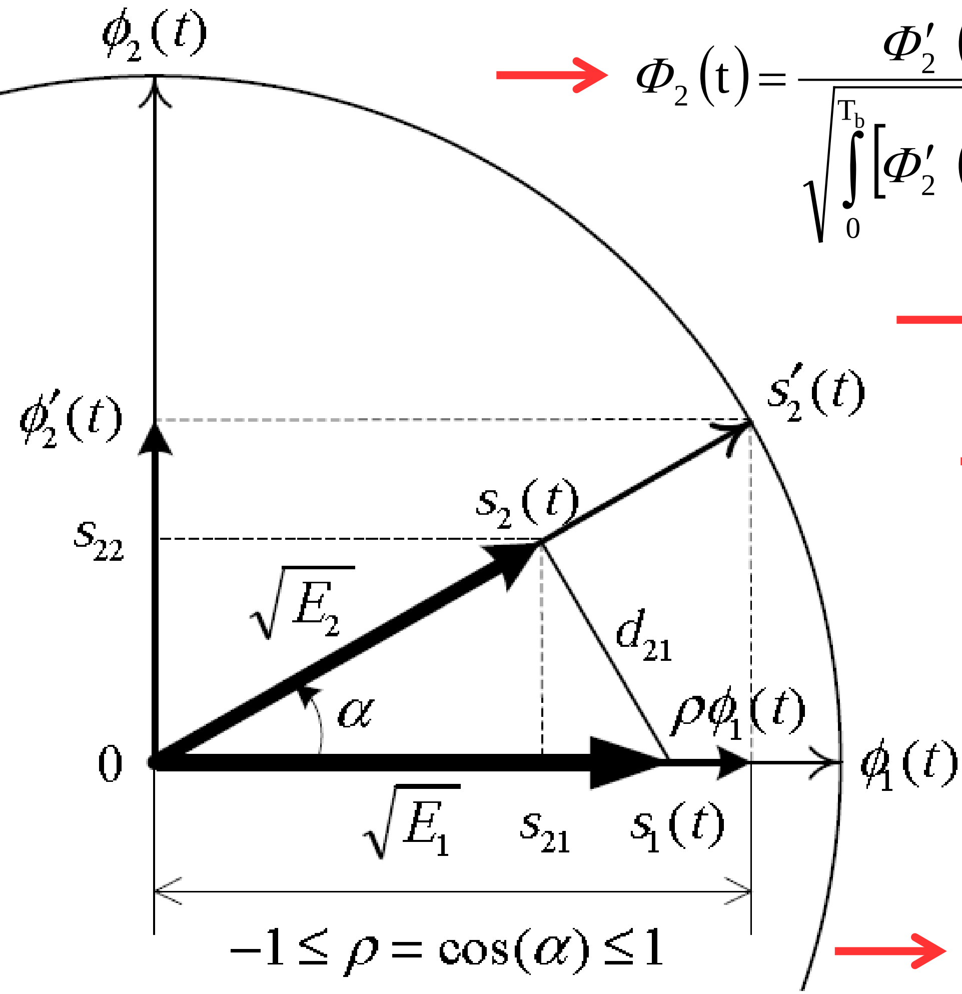
$$s_{21} = \int_0^{T_b} s_2(t)\Phi_1(t)dt = ?$$

$$s_{22} = \int_0^{T_b} s_2(t)\Phi_2(t)dt = ?$$

$$d_{21} = \sqrt{\int_0^{T_b} [s_2(t) - s_1(t)]^2 dt} = ?$$

Do it now

$$\Phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}}$$



$$\Phi_2(t) = \frac{\Phi_2'(t)}{\sqrt{\int_0^{T_b} [\Phi_2'(t)]^2 dt}} = \frac{\Phi_2'(t)}{\sqrt{1-\rho^2}} = \frac{1}{\sqrt{1-\rho^2}}\left[\frac{s_2(t)}{\sqrt{E_2}} - \frac{\rho s_1(t)}{\sqrt{E_1}}\right]$$

$$s_{11} = \int_0^{T_b} s_1(t)\Phi_1(t)dt = \sqrt{E_1}$$

$$s_{12} = \int_0^{T_b} s_1(t)\Phi_2(t)dt = 0$$

$$s_{21} = \int_0^{T_b} s_2(t)\Phi_1(t)dt = \rho\sqrt{E_2}$$

$$s_{22} = \int_0^{T_b} s_2(t)\Phi_2(t)dt = \sqrt{(1-\rho^2)}\sqrt{E_2}$$

$$d_{21} = \sqrt{\int_0^{T_b} [s_2(t) - s_1(t)]^2 dt = E_1 - 2\rho\sqrt{E_1 E_2} + E_2}$$

- Gram-schmidt procedure for more than two signals

1) $\Phi_1(t) = \dfrac{s_1(t)}{\sqrt{\int_0^{T_b} s_1^2(t)\,dt}}$

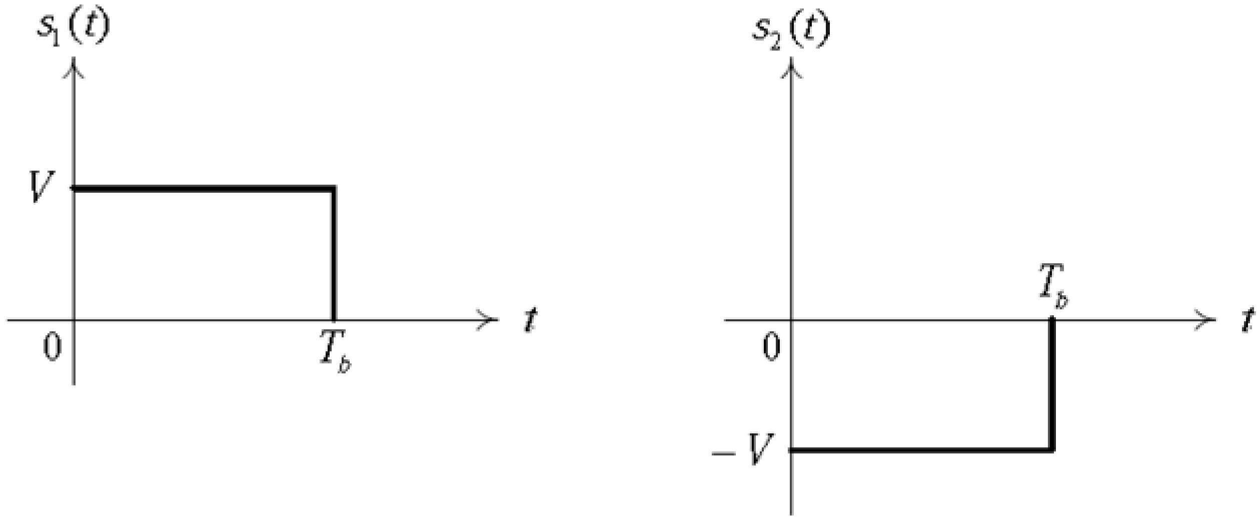
2) $\rho_{ij} = \int_0^{T_b} \dfrac{s_i(t)}{\sqrt{E_i}} \Phi_j(t)\,dt, \qquad j = 1,2,\ldots,i-1$

3) $\Phi_i'(t) = \dfrac{s_i(t)}{\sqrt{E_i}} - \sum_{j=1}^{i-1} \rho_{ij}\Phi_j(t)$

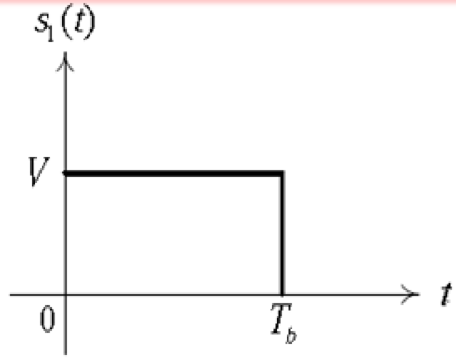
4) $\Phi_i(t) = \dfrac{\Phi_i'(t)}{\sqrt{\int_0^{T_b} \left[\Phi_i'(t)\right]^2 dt}}, \qquad i = 2,3,\ldots, N$

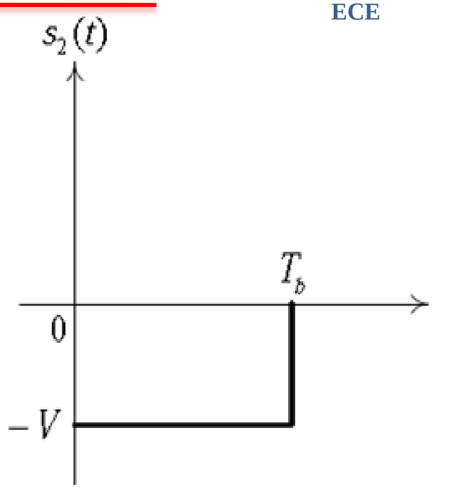
- If the waveforms {si(t)}, i=1,2,…,M form a linearly independent set, then N = M. Otherwise N < M.

Example 5.1: Consider the signal set shown below, determine the orthonormal base functions needed to represent these signals?



Example 5.1 (continue):





$$E_1 = \int_0^{T_b} s_1^2(t)dt = V^2 T_b = E_2 = E$$

$$\Phi_1(t) = \frac{s_1(t)}{\sqrt{E}} = \frac{s_1(t)}{\sqrt{V^2 T_b}}$$

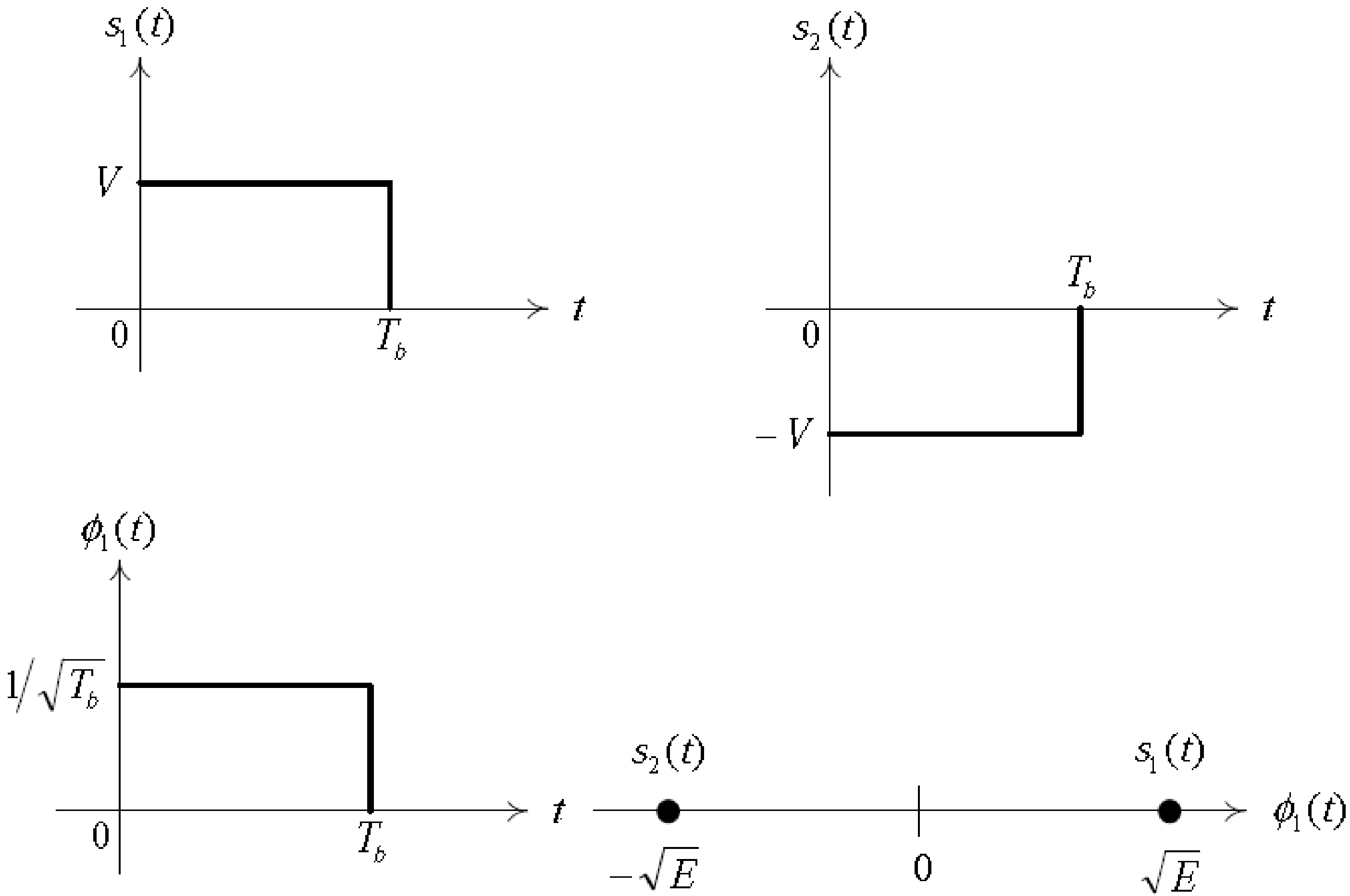
$$\rho_{21} = \int_0^{T_b} \frac{s_2(t)}{\sqrt{E}} \Phi_1(t)dt = \int_0^{T_b} \frac{s_2(t)}{\sqrt{E}} \frac{s_1(t)}{\sqrt{E}} dt = \frac{1}{E}\int_0^{T_b} s_1(t)s_2(t)dt = -1$$

$$\Phi_2'(t) = \frac{s_2(t)}{\sqrt{E}} - \rho_{21}\Phi_1(t) = \frac{s_2(t)}{\sqrt{E}} - (-1)\frac{s_1(t)}{\sqrt{E}} = 0$$

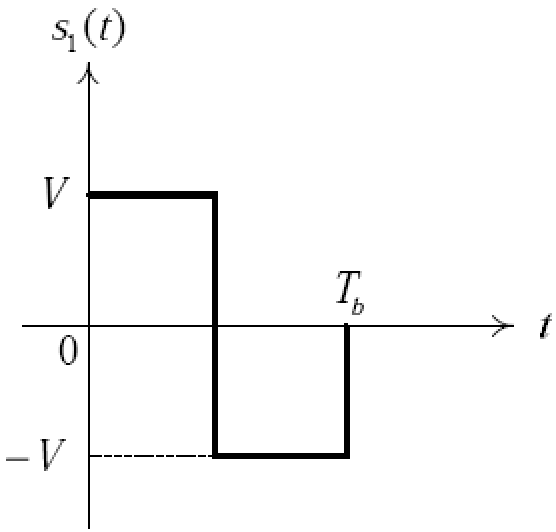
$$s_1(t) = \sqrt{E}\Phi_1(t)$$

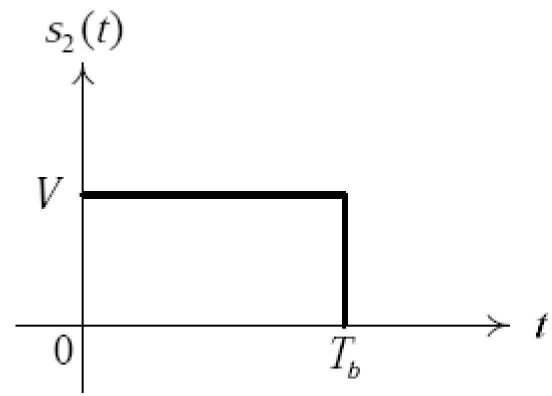
$$s_2(t) = -\sqrt{E}\Phi_1(t)$$

Example 5.1 (continue):

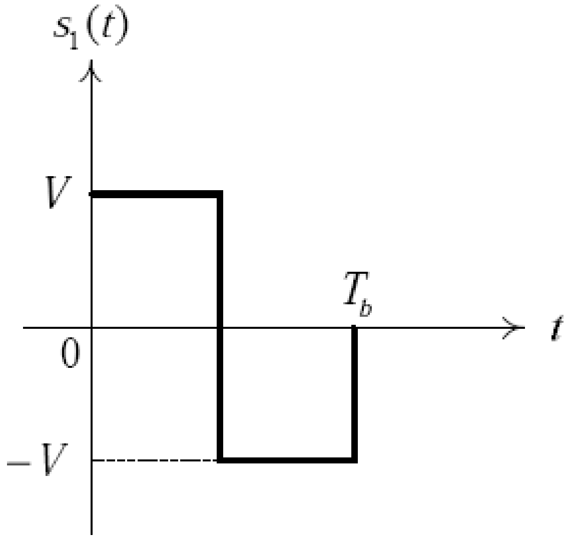


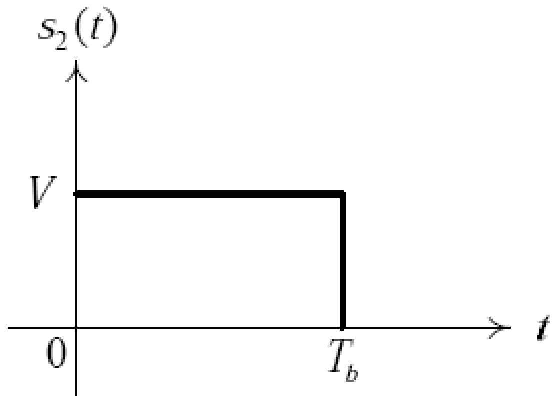
Example 5.2: Consider the signal set shown below, determine the orthonormal base functions needed to represent these signals?





Example 5.2 (continue):





$$\rightarrow E_1 = \int_0^{T_b} s_1^2(t)dt = V^2T_b = E_2 = E \quad \rightarrow \Phi_1(t) = \frac{s_1(t)}{\sqrt{E}} = \frac{s_1(t)}{\sqrt{V^2T_b}}$$

$$\rightarrow \rho_{21} = \int_0^{T_b} \frac{s_2(t)}{\sqrt{E}} \Phi_1(t)dt = \int_0^{T_b} \frac{s_2(t)}{\sqrt{E}} \frac{s_1(t)}{\sqrt{E}} dt = 0$$

$$\rightarrow \Phi_2'(t) = \frac{s_2(t)}{\sqrt{E}} - 0 = \frac{s_2(t)}{\sqrt{E}} = \Phi_2(t)$$

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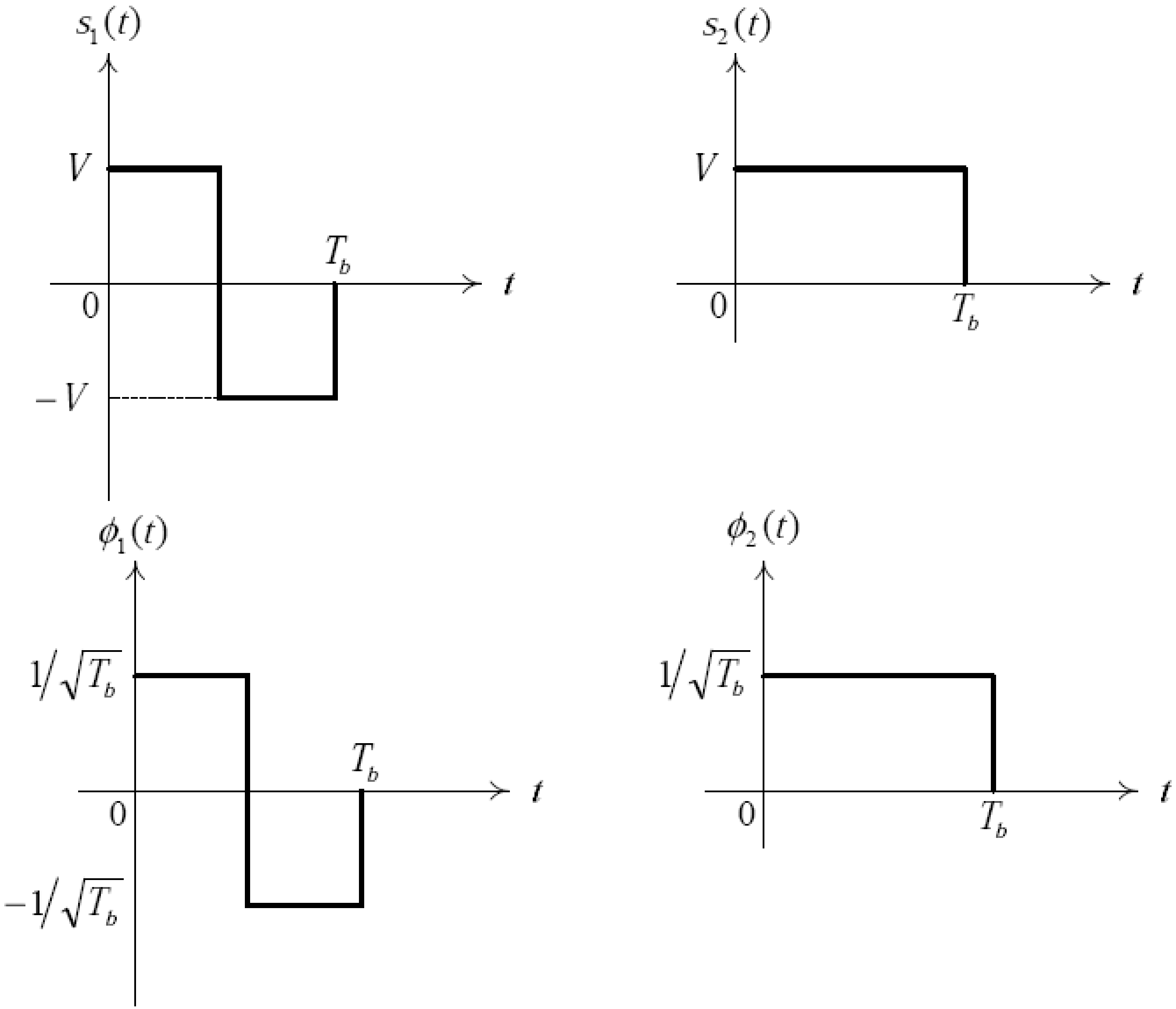
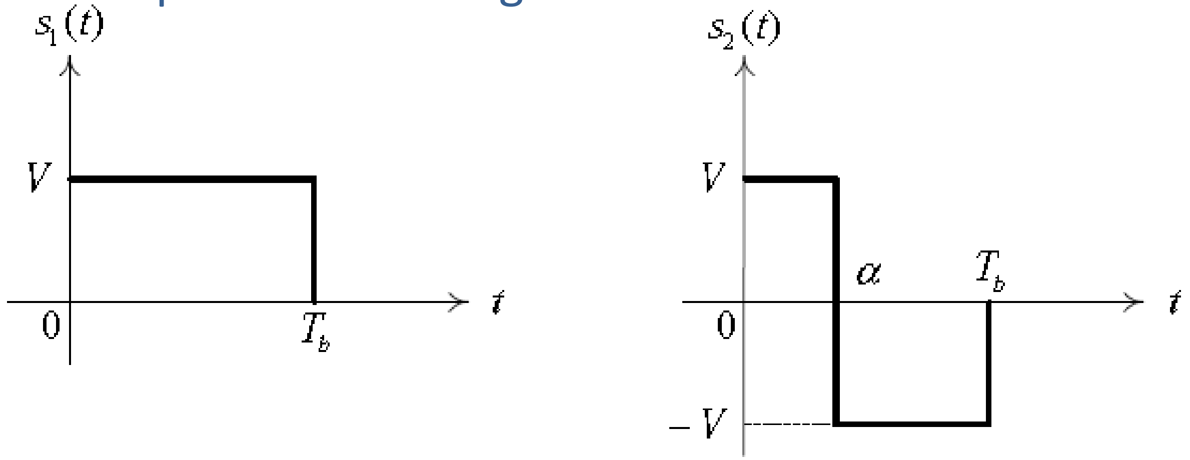
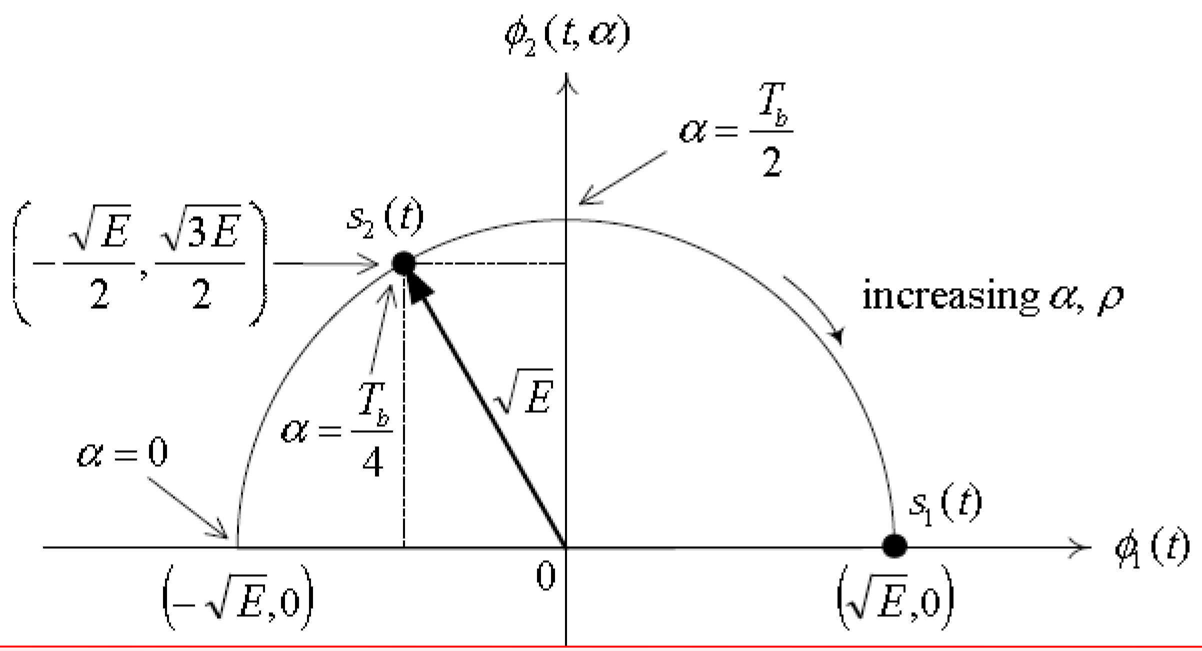
Example 5.2 (continue):

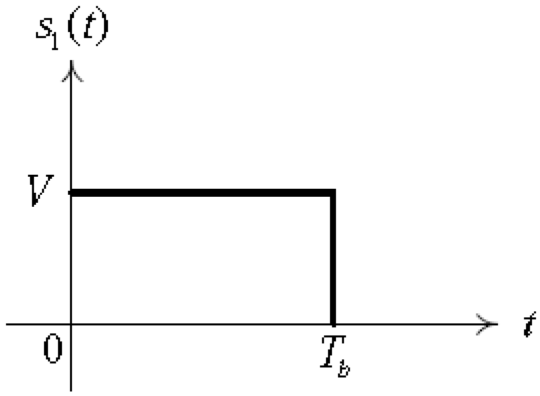


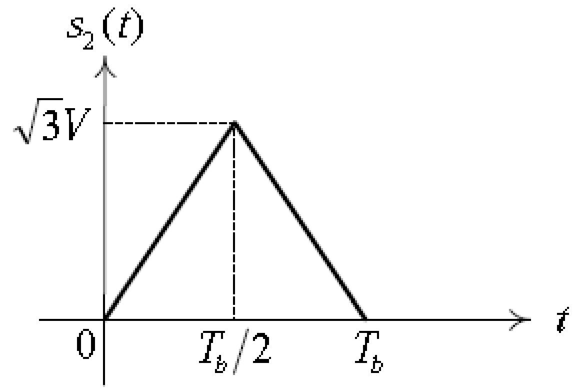
Example 5.3: Consider the signal set shown below, determine the orthonormal base functions needed to represent these signals?



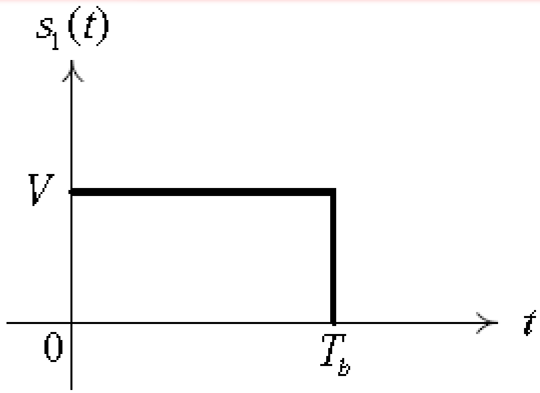


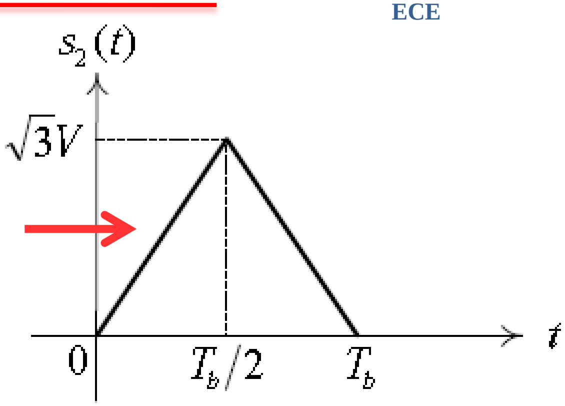
Example 5.4: Consider the signal set shown below, determine the orthonormal base functions needed to represent these signals?





Example 5.4 (continue):





$$s_2(t)=\begin{cases}\dfrac{2\sqrt{3}V}{T_b}t & 0\le t\le \dfrac{T_b}{2}\\ 2\sqrt{3}V\left(1-\dfrac{1}{T_b}t\right) & \dfrac{T_b}{2}\le t\le T_b\\ 0 & \text{o.w}\end{cases}$$

$$E_1=\int_0^{T_b} s_1^2(t)dt=V^2T_b$$

$$E_2=\int_0^{T_b} s_2^2(t)dt$$

$$=\int_0^{T_b/2}\left(\frac{2\sqrt{3}V}{T_b}t\right)^2 dt+\int_{T_b/2}^{T_b}\left(2\sqrt{3}V\left(1-\frac{1}{T_b}t\right)\right)^2 dt$$

$$=\frac{12V^2}{3T_b^{\ 2}}\left(\frac{T_b}{2}\right)^3+12V^2\int_{T_b/2}^{T_b}\left(1-2\frac{1}{T_b}t+\frac{1}{T_b^2}t^2\right)dt$$

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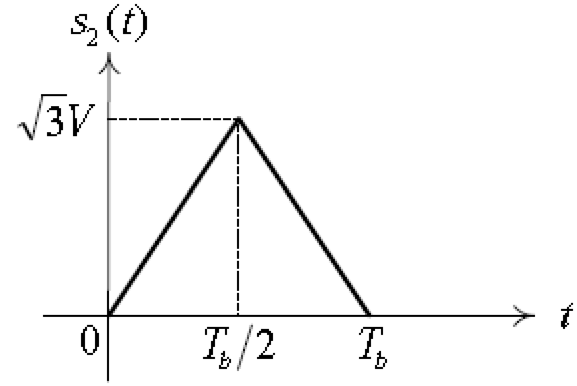
Example 5.4 (continue):

$$s_2(t)=\begin{cases}\dfrac{2\sqrt{3}V}{T_b}t & 0\le t\le \dfrac{T_b}{2}\\ 2\sqrt{3}V\left(1-\dfrac{1}{T_b}t\right) & \dfrac{T_b}{2}\le t\le T_b\\ 0 & \text{o.w}\end{cases}$$



$$E_2=\int_0^{T_b} s_2^2(t)dt$$

$$=\int_0^{T_b/2}\left(\frac{2\sqrt{3}V}{T_b}t\right)^2 dt+\int_{T_b/2}^{T_b}\left(2\sqrt{3}V\left(1-\frac{1}{T_b}t\right)\right)^2 dt$$

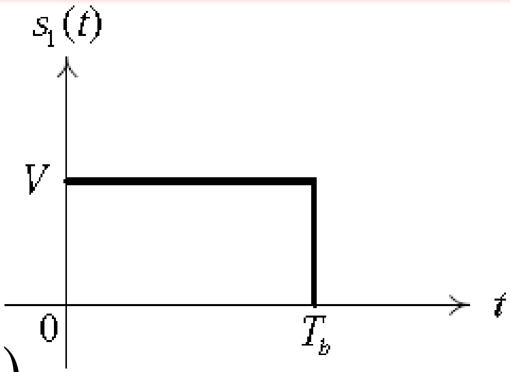
$$=\frac{12V^2}{3T_b^2}\left(\frac{T_b}{2}\right)^3+12V^2\int_{T_b/2}^{T_b}\left(1-2\frac{1}{T_b}t+\frac{1}{T_b^2}t^2\right)dt$$

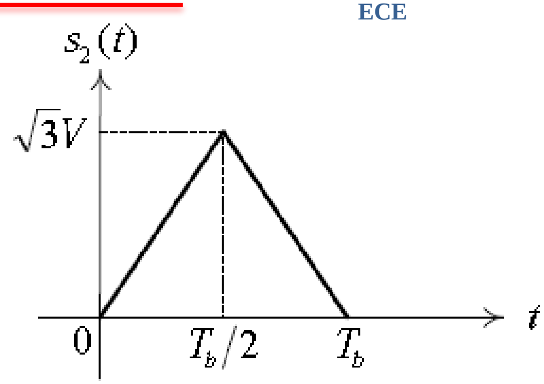
$$=\frac{V^2T_b}{2}+12V^2\left(t-\frac{1}{T_b}t^2+\frac{1}{3T_b^2}t^3\right)\Bigg|_{T_b/2}^{T_b}$$

$$=\frac{V^2T_b}{2}+12V^2\left[\left(T_b-\frac{T_b}{2}\right)-\frac{1}{T_b}\left(T_b^2-\frac{T_b^2}{4}\right)+\frac{1}{3T_b^2}\left(T_b^3-\frac{T_b^3}{8}\right)\right]$$

$$=\frac{V^2T_b}{2}+12V^2\left[\frac{T_b}{2}-\frac{3T_b}{4}+\frac{7T_b}{24}\right]=\frac{V^2T_b}{2}+\frac{V^2T_b}{2}=V^2T_b=E$$

Example 5.4 (continue):





$$\Phi_1(t)=\frac{s_1(t)}{\sqrt{E_1}}=\frac{s_1(t)}{\sqrt{V^2T_b}}$$

$$\rho_{21}=\int_0^{T_b}\frac{s_2(t)}{\sqrt{E_2}}\Phi_1(t)dt=\int_0^{T_b}\frac{s_2(t)}{\sqrt{E_2}}\frac{s_1(t)}{\sqrt{E_1}}dt=\frac{1}{E}\int_0^{T_b}Vs_2(t)dt$$

$$=\frac{V}{V^2T_b}\int_0^{T_b}s_2(t)dt=\frac{1}{VT_b}\times\frac{1}{2}\times T_b\times\sqrt{3}V=\frac{\sqrt{3}}{2}$$

$$\Phi_2'(t)=\frac{s_2(t)}{\sqrt{E}}-\rho_{21}\Phi_1(t)=\frac{s_2(t)}{\sqrt{E}}-\frac{\sqrt{3}}{2}\frac{s_1(t)}{\sqrt{E}}$$

$$\Phi_2(t)=\frac{2}{\sqrt{E}}\left(s_2(t)-\frac{\sqrt{3}}{2}s_1(t)\right)$$

Example 5.4 (continue):

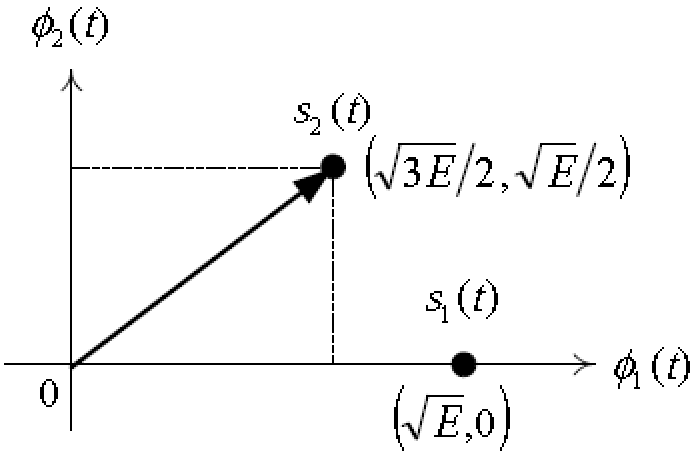
$$\rightarrow s_{11} = \int_0^{T_b} s_1(t)\Phi_1(t)dt = \sqrt{E_1}$$

$$\rightarrow s_{12} = \int_0^{T_b} s_1(t)\Phi_2(t)dt = 0$$

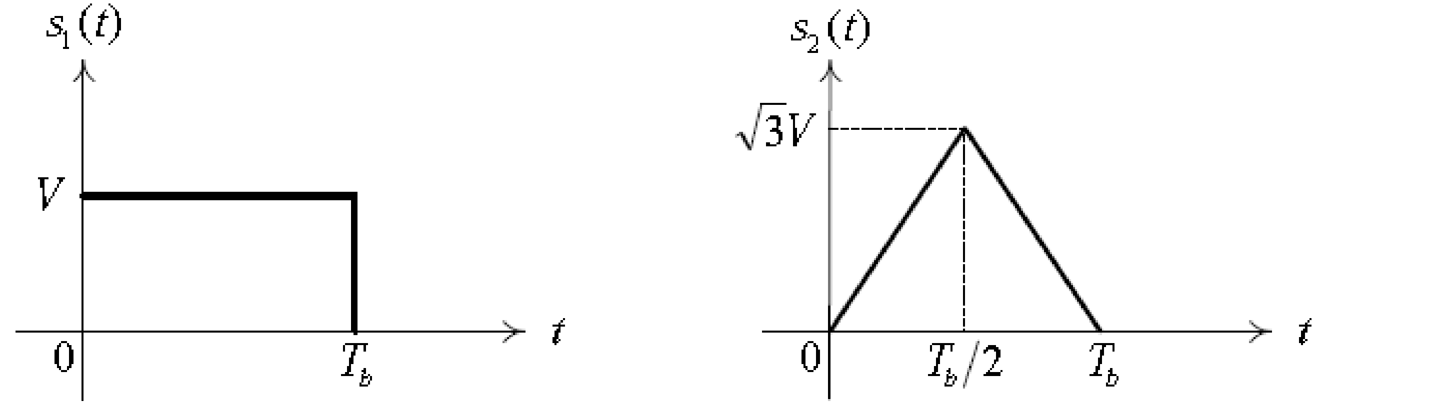
$$\rightarrow s_{21} = \int_0^{T_b} s_2(t)\Phi_1(t)dt = \frac{\sqrt{3}}{2}\sqrt{E_1}$$

$$\rightarrow s_{22} = \int_0^{T_b} s_2(t)\Phi_2(t)dt = \frac{1}{2}\sqrt{E_1}$$

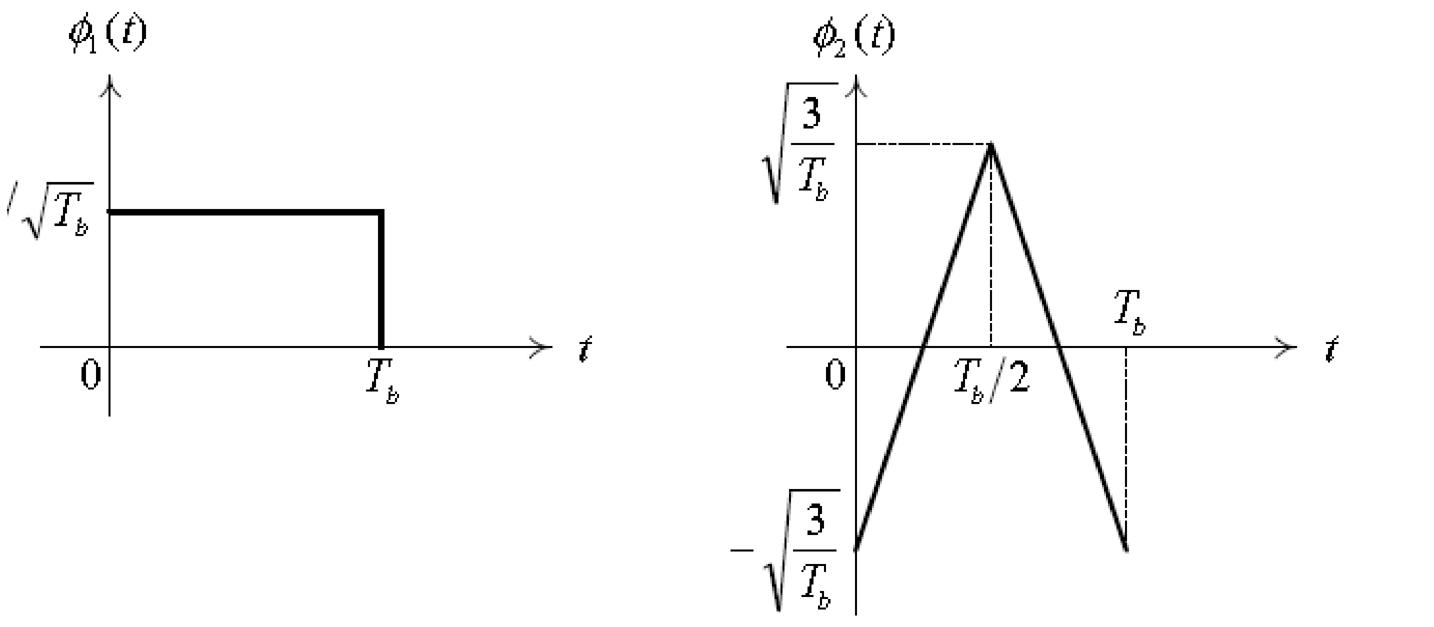
$$\rightarrow d_{21} = \left[\int_0^{T_b} [s_2(t) - s_1(t)]^2 dt\right]^{\frac{1}{2}} = \sqrt{\left(2-\sqrt{3}\right)E}$$



Example 5.4 (continue):



(a)



Example 5.5: Consider the signal $s_1(t)$ and $s_2(t)$ given below, determine the orthonormal base functions needed to represent these signals?

$$s_1 = \sqrt{E}\sqrt{\frac{2}{T_b}}\cos(2\pi f_c t), \quad s_2 = \sqrt{E}\sqrt{\frac{2}{T_b}}\cos(2\pi f_c t + \theta)$$

$$\text{where } f_c = k/2T_b, k: \text{integer}$$

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Example 5.5 (continue):

$$\rightarrow s_1 = \sqrt{E}\sqrt{\frac{2}{T_b}}\cos(2\pi f_c t), \quad s_2 = \sqrt{E}\sqrt{\frac{2}{T_b}}\cos(2\pi f_c t + \theta), f_c = k/2T_b$$

$$\rightarrow E_1 = \int_0^{T_b}\left[\sqrt{E}\sqrt{\frac{2}{T_b}}\cos(2\pi f_c t)\right]^2 dt = \frac{2E}{T_b}\int_0^{T_b}\left[\frac{1+\cos(4\pi f_c t)}{2}\right]dt = E$$

$$\rightarrow E_2 = \int_0^{T_b}\left[\sqrt{E}\sqrt{\frac{2}{T_b}}\cos(2\pi f_c t + \theta)\right]^2 dt = \frac{2E}{T_b}\int_0^{T_b}\left[\frac{1+\cos(4\pi f_c t + 2\theta)}{2}\right]dt = E$$

$$\rightarrow \Phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t)$$

$$\rightarrow \rho_{21} = \int_0^{T_b}\frac{s_2(t)}{\sqrt{E_2}}\Phi_1(t)dt = \frac{2}{T_b}\int_0^{T_b}\cos(2\pi f_c t + \theta)\cos(2\pi f_c t)dt$$

$$= \frac{2}{T_b}\int_0^{T_b}\frac{[\cos(4\pi f_c t + \theta) + \cos(\theta)]}{2}dt = \cos(\theta)$$

Example 5.5 (continue):

$$\rightarrow \Phi_2'(t) = \frac{s_2(t)}{\sqrt{E}} - \rho_{21}\Phi_1(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t + \theta) - \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t)\cos(\theta)$$

$$= \sqrt{\frac{2}{T_b}}\left[\cos(2\pi f_c t)\cos(\theta) - \sin(2\pi f_c t)\sin(\theta)\right] - \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t)\cos(\theta)$$

$$= -\sqrt{\frac{2}{T_b}}\sin(2\pi f_c t)\sin(\theta)$$

$$\rightarrow E_{\Phi_2'} = \int_0^{T_b}\left[\Phi_2'(t)\right]^2 dt = \int_0^{T_b}\left[-\sqrt{\frac{2}{T_b}}\sin(2\pi f_c t)\sin(\theta)\right]^2 dt$$

$$= \frac{2\sin^2(\theta)}{T_b}\int_0^{T_b}\sin^2(2\pi f_c t)dt = \frac{2\sin^2(\theta)}{T_b}\int_0^{T_b}\frac{1-\cos(4\pi f_c t)dt}{2} = \sin^2(\theta)$$

$$\rightarrow \Phi_2(t) = \frac{\Phi_2'(t)}{\sqrt{\int_0^{T_b}\left[\Phi_2'(t)\right]^2 dt}} = -\sqrt{\frac{2}{T_b}}\sin(2\pi f_c t) \quad \text{or} \quad \Phi_2(t) = \sqrt{\frac{2}{T_b}}\sin(2\pi f_c t)$$

Example 5.5 (continue):

$$s_{11} = \int_0^{T_b} s_1(t)\Phi_1(t)dt = \sqrt{E} \qquad s_{12} = \int_0^{T_b} s_1(t)\Phi_2(t)dt = 0$$

$$s_{21} = \int_0^{T_b} s_2(t)\Phi_1(t)dt = \int_0^{T_b}\left[\sqrt{E}\sqrt{\frac{2}{T_b}}\cos(2\pi f_c t+\theta)\right]\left[\sqrt{\frac{2}{T_b}}\cos(2\pi f_c t)\right]dt$$

$$= \frac{2\sqrt{E}}{T_b}\int_0^{T_b}\cos(2\pi f_c t+\theta)\cos(2\pi f_c t)dt = \frac{2\sqrt{E}}{T_b}\int_0^{T_b}\frac{[\cos(4\pi f_c t+\theta)+\cos(\theta)]}{2}dt$$

$$= \sqrt{E}\cos(\theta)$$

$$s_{22} = \int_0^{T_b} s_2(t)\Phi_2(t)dt = \int_0^{T_b}\left[\sqrt{E}\sqrt{\frac{2}{T_b}}\cos(2\pi f_c t+\theta)\right]\left[-\sqrt{\frac{2}{T_b}}\sin(2\pi f_c t)\right]dt$$

$$= -\frac{2\sqrt{E}}{T_b}\int_0^{T_b}\cos(2\pi f_c t+\theta)\sin(2\pi f_c t)dt$$

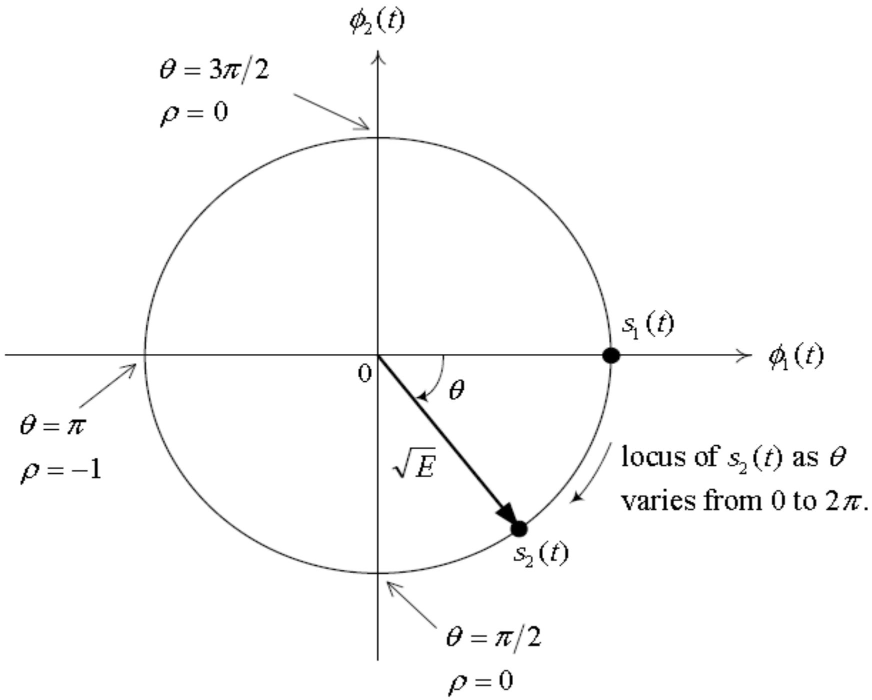
$$= -\frac{2\sqrt{E}}{T_b}\int_0^{T_b}\frac{[\sin(4\pi f_c t+\theta)-\sin(-\theta)]}{2}dt = -\sqrt{E}\sin(\theta)$$

Example 5.5 (continue):

$$s_1(t)=\sqrt{E}\sqrt{\frac{2}{T_b}}\cos(2\pi f_c t)=s_{11}\Phi_1(t)+s_{12}\Phi_2(t)=\sqrt{E}\Phi_1(t)$$

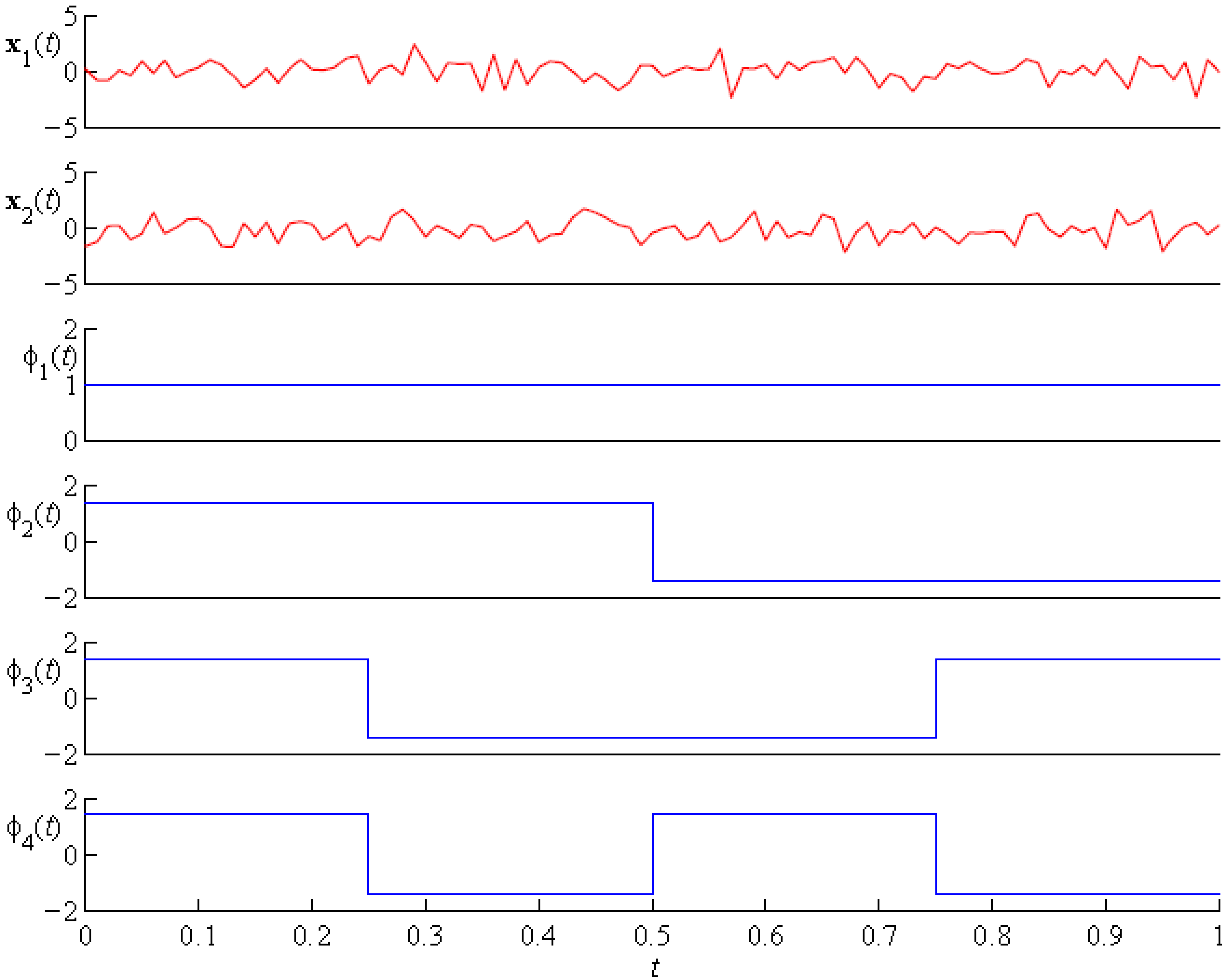
$$s_2(t)=\sqrt{E}\sqrt{\frac{2}{T_b}}\cos(2\pi f_c t+\theta)=s_{21}\Phi_1(t)+s_{22}\Phi_2(t)$$

$$=\sqrt{E}\cos(\theta)\Phi_1(t)-\sqrt{E}\sin(\theta)\Phi_2(t)$$

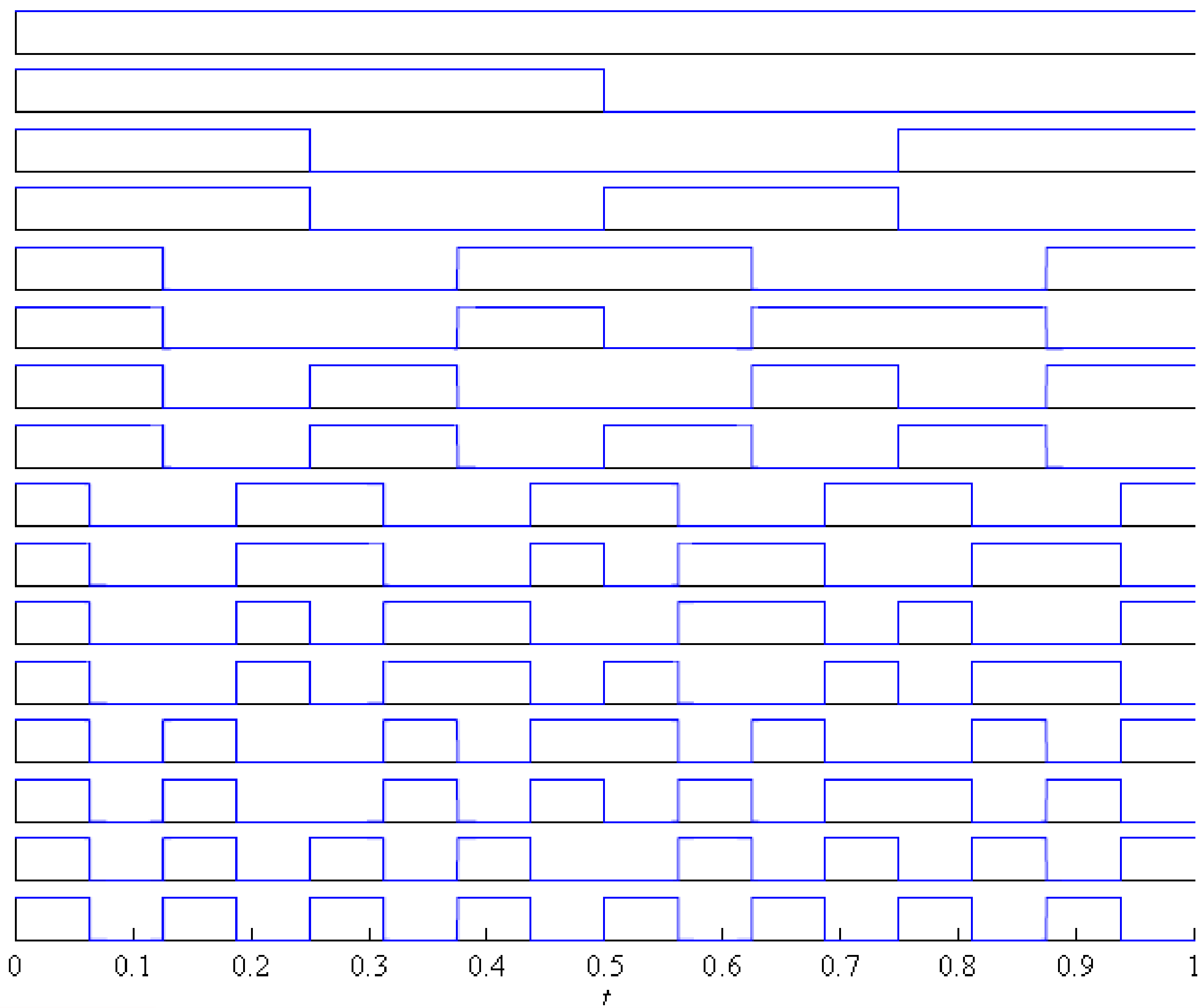


Optimum Receiver – Representation of Noise

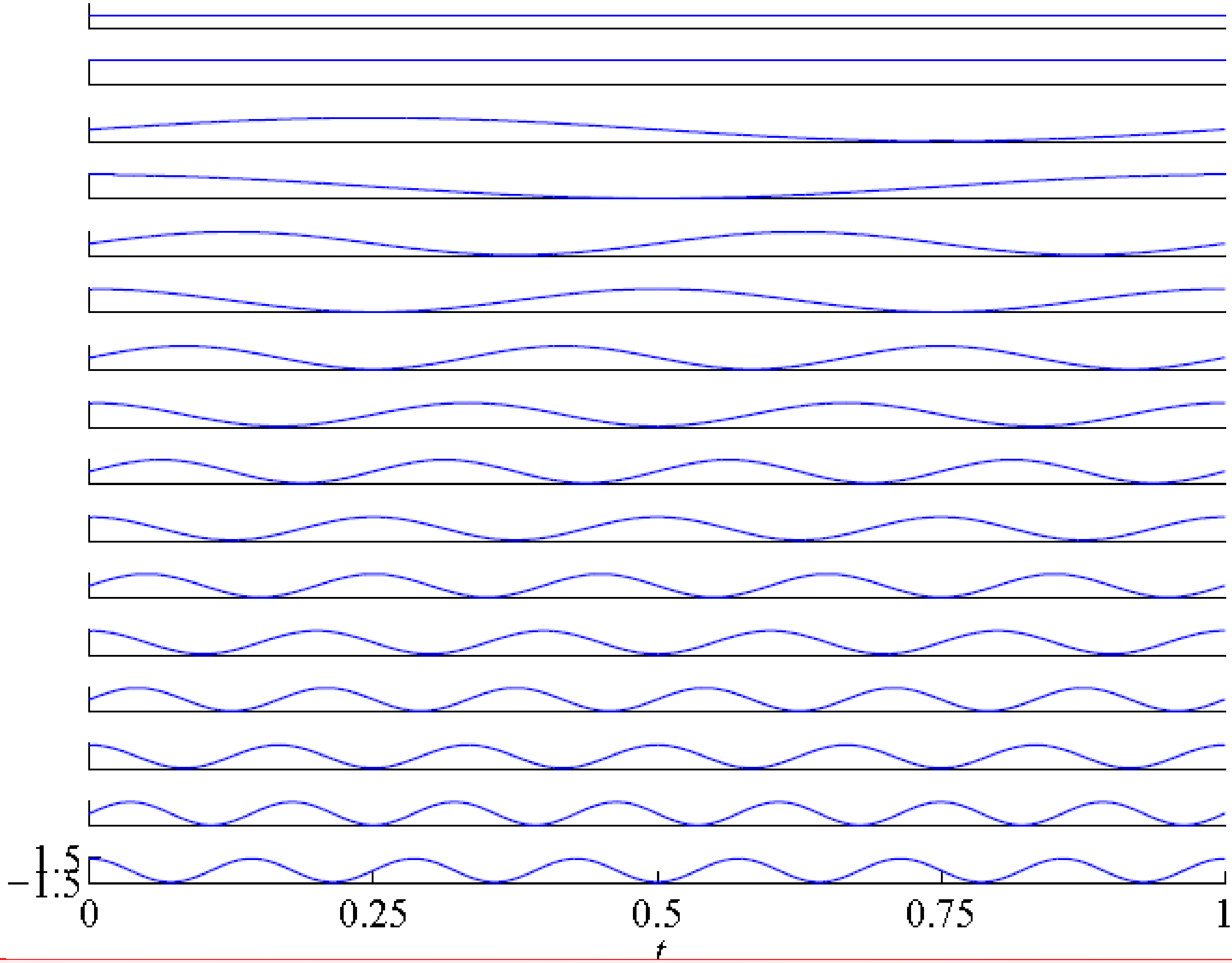
- Representation of Noise with Walsh Functions
- Exact representation of noise with 4 Walsh functions is not possible.



- Representation of Noise with 16 Walsh Functions
- Exact representations might be possible with many more Walsh functions



- The First 16 Sine and Cosine Functions
- Can also use sine and cosine functions (Fourier representation).



- To represent the random noise signal, $w(t)$, in the time interval $[(k-1)T_b, kT_b]$, need to use a complete orthonormal set of known deterministic functions:

$$\longrightarrow \quad w(t) = \sum_{i=1}^{\infty} w_i \Phi_i(t) \quad \text{where} \quad w_i = \int_0^{T_b} w(t)\Phi_i(t)\,dt$$

- The coefficients w_i's are random variables and understanding their statistical properties is imperative in developing the optimum receiver.

- When w(t) is zero-mean and white, then

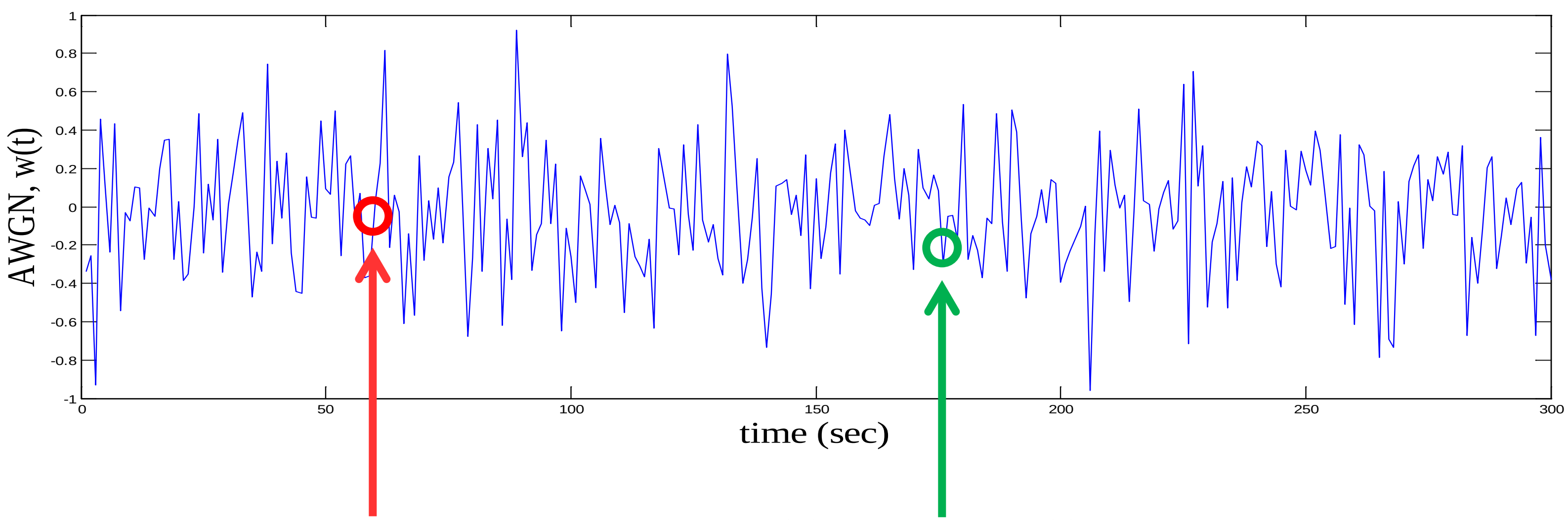
$$E\{w_i\} = E\left\{\int_0^{T_b} w(t)\Phi_i(t)dt\right\} = \int_0^{T_b} E\{w(t)\}\Phi_i(t)dt = 0$$

Noise is zero-mean

$$\rho_{ij} = E\{(w_i - \mu_w)(w_j - \mu_w)\} = E\{w_i w_j\}$$

$$= E\left\{\int_0^{T_b} w(\lambda)\Phi_i(\lambda)d\lambda \int_0^{T_b} w(\tau)\Phi_j(\tau)d\tau\right\}$$

Correlation between the coefficients of the orthonormal base functions



$w(\lambda) = ..+w_i\,\Phi_i(t)+..$ $\qquad$ $w(\tau) = ..+w_j\,\Phi_j(t)+..$

Optimum Receiver – Representation of Noise

- When w(t) is zero-mean and white, then

$$E\{w_i\} = E\left\{\int_0^{T_b} w(t)\Phi_i(t)dt\right\} = \int_0^{T_b} E\{w(t)\}\Phi_i(t)dt = 0$$

Noise is zero-mean

$$E\{w_i w_j\} = E\left\{\int_0^{T_b} w(\lambda)\Phi_i(\lambda)d\lambda \int_0^{T_b} w(\tau)\Phi_j(\tau)d\tau\right\}$$

$$= \int_0^{T_b} \Phi_i(\lambda)\int_0^{T_b} E\{w(\lambda)w(\tau)\}\Phi_j(\tau)d\tau d\lambda$$

Whit Noise $E\{w(\lambda)w(\tau)\} = \frac{N_o}{2}\delta(\lambda-\tau)$

$$= \int_0^{T_b} \Phi_i(\lambda)\int_0^{T_b} \frac{N_o}{2}\delta(\lambda-\tau)\Phi_j(\tau)d\tau d\lambda = \frac{N_o}{2}\int_0^{T_b} \Phi_i(\lambda)\Phi_j(\lambda)d\lambda$$

$$= \begin{cases} \frac{N_o}{2} & i = j \\ 0 & i \neq j \end{cases}$$

- this means that $\{w_1, w_2, w_3, \ldots\}$ are zero-mean and uncorrelated random variables.

- If $w(t)$ is not only zero-mean and white, but also Gaussian ➔ than $\{w_1, w_2, w_3, \ldots\}$ are Gaussian and statistically independent!!!
- The above properties do not depend on how the set that $\{\Phi_1, \Phi_2, \Phi_3, \Phi_4 \ldots\}$ is chosen.
- Shall choose the first two functions $\Phi_1(t)$ and $\Phi_2(t)$, they are used to represent the two signals $s_1(t)$ and $s_2(t)$ exactly. The remaining functions, i.e., $\Phi_3(t)$, $\Phi_4(t)$, . . . , are simply chosen to complete the set.

- Without any loss of generality, concentrate on the first bit interval.
- The received signal is

$$r(t) = s_i(t) + w(t), \quad 0 \le t \le T_b$$

$$= \begin{Bmatrix} s_1(t) + w(t), & \text{if a "0" is transmitted} \\ s_2(t) + w(t), & \text{if a "1" is transmitted} \end{Bmatrix} = s_i(t) + w(t)$$

$$= s_{i1}\Phi_1(t) + s_{i2}\Phi_2(t) + w_1\Phi_1(t) + w_2\Phi_2(t) + w_3\Phi_3(t) + w_4\Phi_4(t) + \cdots$$

$$= (s_{i1} + w_1)\Phi_1(t) + (s_{i2} + w_2)\Phi_2(t) + w_3\Phi_3(t) + w_4\Phi_4(t) + \cdots$$

$$= r_1\Phi_1(t) + r_2\Phi_2(t) + r_3\Phi_3(t) + r_4\Phi_4(t) + \cdots$$

where

$$r_j = \int_0^{T_b} r(t)\Phi_j(t)\,dt \longrightarrow \begin{cases} r_1 = s_{i1} + w_1, \quad r_2 = s_{i2} + w_2 \\ r_3 = w_3, \quad r_4 = w_4, \quad r_5 = w_5, \cdots \end{cases}$$

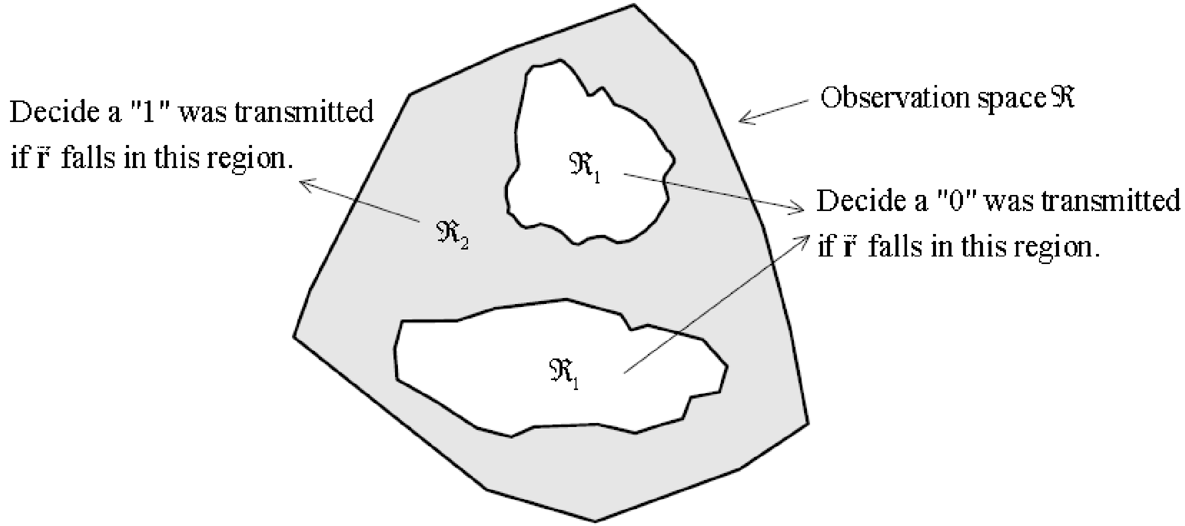
$$r_1 = s_{i1} + w_1, \quad r_2 = s_{i2} + w_2$$

r_j, for $j = 1,2$ can be used to estimate/detect which signal ($s_1(t)$ or $s_2(t)$) was transmitted.

$$r_3 = w_3, r_4 = w_4, r_5 = w_5, \cdots$$

r_j, for $j = 3, 4, 5, \ldots$, does not depend on which signal ($s_1(t)$ or $s_2(t)$) was transmitted.

- The decision can now be based on the observations $r_1, r_2, r_3, r_4, r_5, \ldots$
- The criterion is to minimize the bit error probability.
- In general: consider only the first n terms (n can be very very large), $r = \{r_1, r_2, \ldots, r_n\}$ ⇒ Need to partition the n-dimensional observation space into decision regions.



$$P[\text{error}] = P\left[\begin{array}{l}(\text{"0" decided and "1" transmitted}) \text{ or} \\ (\text{"1" decided and "0" transmitted})\end{array}\right]$$

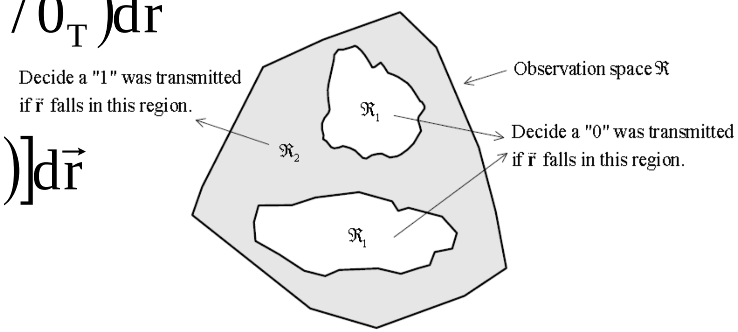
$$= P[0_D, 1_T] + P[1_D, 0_T] = P[0_D / 1_T]P[1_T] + P[1_D / 0_T]P[0_T]$$

$$= P_2 \int_{R_1} f(\vec{r}/1_T)d\vec{r} + P_1 \int_{R_2} f(\vec{r}/0_T)d\vec{r} = P_2 \int_{R-R_2} f(\vec{r}/1_T)d\vec{r} + P_1 \int_{R_2} f(\vec{r}/0_T)d\vec{r}$$

$$= P_2 \int_{R} f(\vec{r}/1_T)d\vec{r} - P_2 \int_{R_2} f(\vec{r}/1_T)d\vec{r} + P_1 \int_{R_2} f(\vec{r}/0_T)d\vec{r}$$

$$= P_2 \int_{R} f(\vec{r}/1_T)d\vec{r} + \int_{R_2} [P_1 f(\vec{r}/0_T) - P_2 f(\vec{r}/1_T)]d\vec{r}$$

$$= P_2 + \int_{R_2} [P_1 f(\vec{r}/0_T) - P_2 f(\vec{r}/1_T)]d\vec{r}$$



To minimize the probability of error we need to make this term negative

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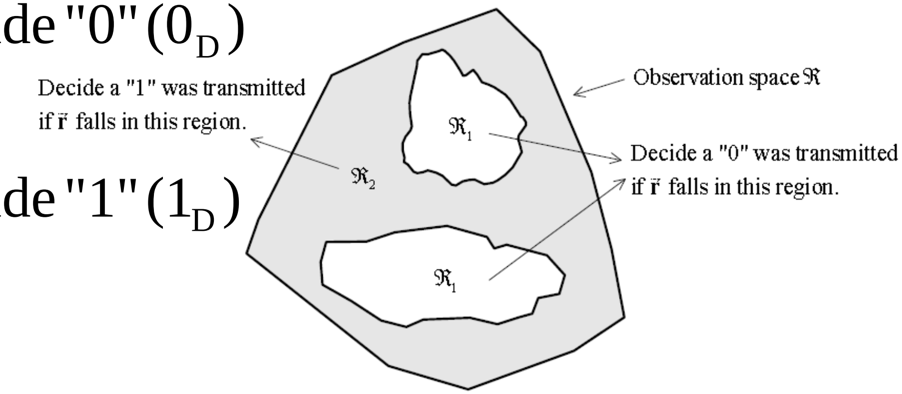
- The minimum error probability decision rule is

$$\longrightarrow P_1 f(\vec{r}/0_T) - P_2 f(\vec{r}/1_T) \geq 0 \Rightarrow \text{ decide "0" } (0_D)$$

$$\longrightarrow P_1 f(\vec{r}/0_T) - P_2 f(\vec{r}/1_T) < 0 \Rightarrow \text{ decide "1" } (1_D)$$



- or it could be written as

$$\longrightarrow P_1 f(\vec{r}/0_T) \geq P_2 f(\vec{r}/1_T) \Rightarrow \text{ decide "0" } (0_D)$$

$$\longrightarrow P_1 f(\vec{r}/0_T) < P_2 f(\vec{r}/1_T) \Rightarrow \text{ decide "1" } (1_D)$$

$$\frac{1}{P_1 f(\vec{r}/0_T)} \underset{0_D}{\overset{1_D}{\gtrless}} \frac{1}{P_2 f(\vec{r}/1_T)}$$

- and so

$$\longrightarrow P_1 f(\vec{r}/0_T) \underset{1_D}{\overset{0_D}{\gtrless}} P_2 f(\vec{r}/1_T) \longrightarrow \frac{f(\vec{r}/0_T)}{f(\vec{r}/1_T)} \underset{1_D}{\overset{0_D}{\gtrless}} \frac{P_2}{P_1} \quad \text{Or} \quad \frac{f(\vec{r}/1_T)}{f(\vec{r}/0_T)} \underset{0_D}{\overset{1_D}{\gtrless}} \frac{P_1}{P_2}$$

- The expression $\frac{f(\vec{r}/1_T)}{f(\vec{r}/0_T)}$ is called the likelihood ratio.

- The decision rule was derived without specifying any statistical properties of the noise process w(t).

$$\frac{f(\vec{r}/0_T)}{f(\vec{r}/1_T)} \underset{1_D}{\overset{0_D}{\gtrless}} \frac{P_2}{P_1} \quad \text{Or} \quad \frac{f(\vec{r}/1_T)}{f(\vec{r}/0_T)} \underset{0_D}{\overset{1_D}{\gtrless}} \frac{P_1}{P_2}$$

- recall that using the chain rule, the conditional pdf could be written as

$$f(\vec{r}/0_T) = f(r_1, r_2, r_3, \cdots/0_T) = f(r_1/0_T)\, f(r_2/r_1, 0_T)\, f(r_3/r_1, r_2, 0_T)\cdots$$

- recall also that for an independent random variables r_1, r_2, r_3, r_4, r_5, …, the conditional *pdf*s could be rewritten as

$$f(r_k/r_1, r_2, \cdots, 0_T) = f(r_k/0_T)$$

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- If the noise is zero-mean, white, and Gaussian ($r_1, r_2, r_3, r_4, r_5, \ldots$ are independent) ➔ the conditional *pdf*s could be rewritten as

$$\longrightarrow f\left(\vec{r}/0_T\right) = f\left(r_1, r_2, r_3, \cdots / 0_T\right) = \prod_{i=1}^{\infty} f\left(r_i / 0_T\right)$$

$$\longrightarrow f\left(\vec{r}/1_T\right) = f\left(r_1, r_2, r_3, \cdots / 1_T\right) = \prod_{i=1}^{\infty} f\left(r_i / 1_T\right)$$

- for $r_1 = s_{i1} + w_1$ ➔ the mean of r_1 is s_{i1} and the variance is $N_o/2$
  - if 0_T $(i=1)$ then $r_1 = s_{11} + w_1$ ➔ the mean of r_1 is s_{11} and the variance is $N_o/2$
  - if 1_T $(i=2)$ then $r_1 = s_{21} + w_1$ ➔ the mean of r_1 is s_{21} and the variance is $N_o/2$

$$\longrightarrow f\left(r_1 / 0_T\right) = \frac{1}{\sqrt{2\pi \times N_o / 2}} \exp\left[\frac{-\left(r_1 - s_{11}\right)^2}{N_o}\right]$$

$$\longrightarrow f\left(r_1 / 1_T\right) = \frac{1}{\sqrt{2\pi \times N_o / 2}} \exp\left[\frac{-\left(r_1 - s_{21}\right)^2}{N_o}\right]$$

- for $r_2 = s_{i2}+w_2$ ➔ the mean of r_2 is s_{i2} and the variance is $N_o/2$
  - if 0_T (i=1) then $r_2 = s_{12}+w_2$ ➔ the mean of r_2 is s_{12} and the variance is $N_o/2$
  - if 1_T (i=2) then $r_2 = s_{22}+w_2$ ➔ the mean of r_2 is s_{22} and the variance is $N_o/2$

$$\rightarrow f(r_2/0_T) = \frac{1}{\sqrt{2\pi \times N_o/2}} \exp\left[\frac{-(r_2 - s_{12})^2}{N_o}\right]$$

$$\rightarrow f(r_2/1_T) = \frac{1}{\sqrt{2\pi \times N_o/2}} \exp\left[\frac{-(r_2 - s_{22})^2}{N_o}\right]$$

- for $r_3 = s_{i3}+w_3$ ➔ the mean of r_3 is $s_{i3}=0$ (only $s_1(t)$, $s_2(t)$ exist and so the projections on Φ_3 is zero) and the variance is $N_o/2$
  - if 0_T (i=1) then $r_2 = s_{13}+w_3$ ➔ the mean of r_3 is Zero and the variance is $N_o/2$
  - if 1_T (i=2) then $r_2 = s_{23}+w_3$ ➔ the mean of r_3 is Zero and the variance is $N_o/2$

$$\rightarrow f(r_3/0_T) = f(r_3/1_T) = \frac{1}{\sqrt{2\pi \times N_o/2}} \exp\left[\frac{-(r_3)^2}{N_o}\right]$$

- Now the conditional *pdf*s could be written as

$$f\left(\vec{r}/0_T\right)=\prod_{i=1}^{\infty} f\left(r_i/0_T\right)=\frac{1}{\sqrt{\pi N_o}}\exp\left[\frac{-\left(r_1-s_{11}\right)^2}{N_o}\right]\times\frac{1}{\sqrt{\pi N_o}}\exp\left[\frac{-\left(r_2-s_{12}\right)^2}{N_o}\right]$$
$$\times\frac{1}{\sqrt{\pi N_o}}\exp\left[\frac{-\left(r_3\right)^2}{N_o}\right]\times\frac{1}{\sqrt{\pi N_o}}\exp\left[\frac{-\left(r_4\right)^2}{N_o}\right]\times\cdots$$

$$f\left(\vec{r}/1_T\right)=\prod_{i=1}^{\infty} f\left(r_i/1_T\right)=\frac{1}{\sqrt{\pi N_o}}\exp\left[\frac{-\left(r_1-s_{21}\right)^2}{N_o}\right]\times\frac{1}{\sqrt{\pi N_o}}\exp\left[\frac{-\left(r_2-s_{22}\right)^2}{N_o}\right]$$
$$\times\frac{1}{\sqrt{\pi N_o}}\exp\left[\frac{-\left(r_3\right)^2}{N_o}\right]\times\frac{1}{\sqrt{\pi N_o}}\exp\left[\frac{-\left(r_4\right)^2}{N_o}\right]\times\cdots$$

- Now the decision rule could be written as

$$\frac{f(\vec{r}/1_T)}{f(\vec{r}/0_T)} = \frac{\frac{1}{\sqrt{\pi N_o}}\exp\left[\frac{-(r_1-s_{21})^2}{N_o}\right]\times\frac{1}{\sqrt{\pi N_o}}\exp\left[\frac{-(r_2-s_{22})^2}{N_o}\right]}{\frac{1}{\sqrt{\pi N_o}}\exp\left[\frac{-(r_1-s_{11})^2}{N_o}\right]\times\frac{1}{\sqrt{\pi N_o}}\exp\left[\frac{-(r_2-s_{12})^2}{N_o}\right]}$$

$$= \frac{\exp\left[\frac{-(r_1-s_{21})^2-(r_2-s_{22})^2}{N_o}\right]}{\exp\left[\frac{-(r_1-s_{11})^2-(r_2-s_{12})^2}{N_o}\right]}$$

$$= \exp\left[\frac{(r_1-s_{11})^2+(r_2-s_{12})^2-(r_1-s_{21})^2-(r_2-s_{22})^2}{N_o}\right] \underset{0_D}{\overset{1_D}{\gtrless}} \frac{P_1}{P_2}$$

- Apply the natural logarithm for both parts (monotonically increasing function)

$$\rightarrow \exp\left[\frac{(r_1-s_{11})^2+(r_2-s_{12})^2-(r_1-s_{21})^2-(r_2-s_{22})^2}{N_o}\right] \underset{0_D}{\overset{1_D}{\gtrless}} \frac{P_1}{P_2}$$

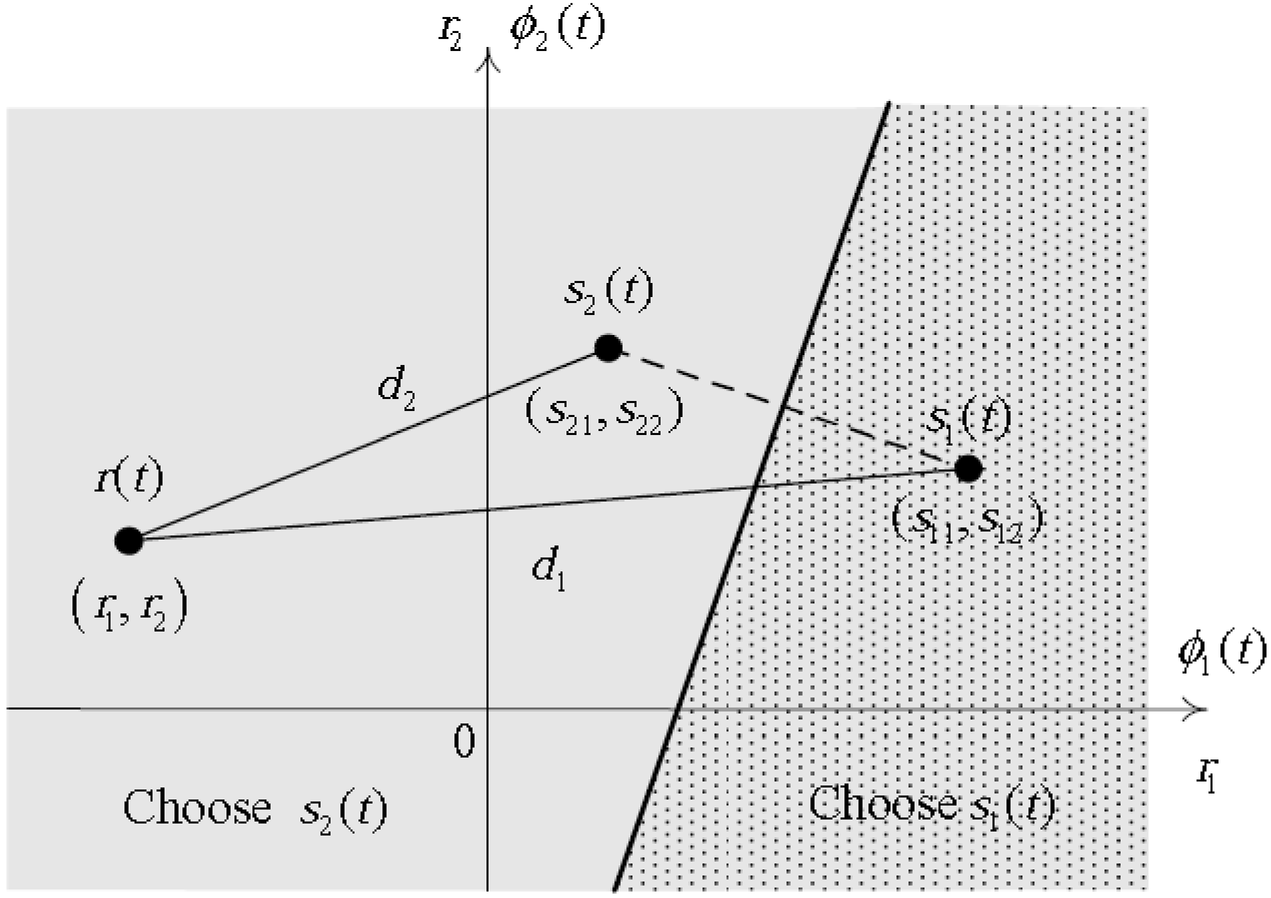
$$\rightarrow (r_1-s_{11})^2+(r_2-s_{12})^2-(r_1-s_{21})^2-(r_2-s_{22})^2 \underset{0_D}{\overset{1_D}{\gtrless}} N_o \ln\left[\frac{P_1}{P_2}\right]$$

- For a special case that $P_1=P_2 \Rightarrow \ln(1)=0$

$$\rightarrow (r_1-s_{11})^2+(r_2-s_{12})^2 \underset{0_D}{\overset{1_D}{\gtrless}} (r_1-s_{21})^2+(r_2-s_{22})^2$$

- For a special case that $P_1=P_2 \Rightarrow \ln(1)=0$

$$\rightarrow \left(r_1 - s_{11}\right)^2 + \left(r_2 - s_{12}\right)^2 \underset{0_D}{\overset{1_D}{\gtrless}} \left(r_1 - s_{21}\right)^2 + \left(r_2 - s_{22}\right)^2$$



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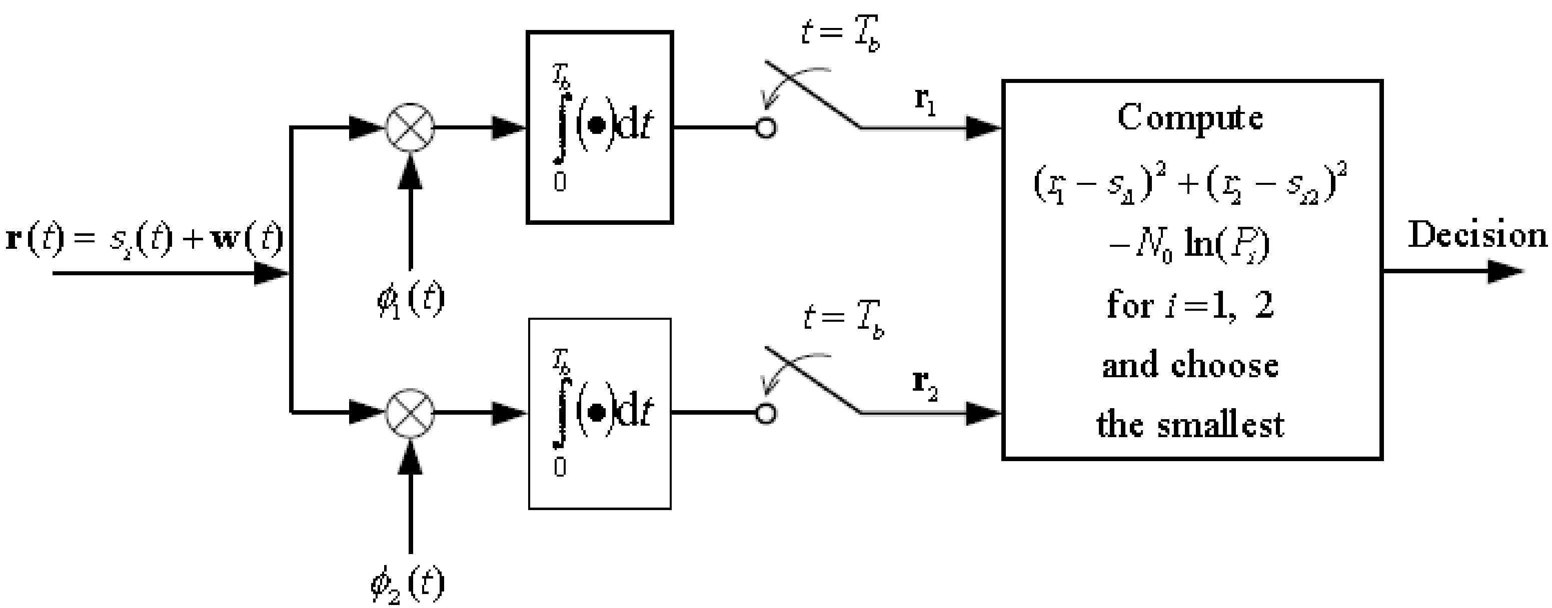
- Correlation receiver or Integrate-and-dump receiver

$$(r_1 - s_{11})^2 + (r_2 - s_{12})^2 - (r_1 - s_{21})^2 - (r_2 - s_{22})^2 \underset{0_D}{\overset{1_D}{\gtrless}} N_o \ln\left[\frac{P_1}{P_2}\right]$$

$$(r_1 - s_{11})^2 + (r_2 - s_{12})^2 - N_o \ln[P_1] \underset{0_D}{\overset{1_D}{\gtrless}} (r_1 - s_{21})^2 - (r_2 - s_{22})^2 - N_o \ln[P_2]$$



$$\rightarrow (r_1 - s_{11})^2 + (r_2 - s_{12})^2 - N_o \ln[P_1] \underset{0_D}{\overset{1_D}{\gtrless}} (r_1 - s_{21})^2 - (r_2 - s_{22})^2 - N_o \ln[P_2]$$

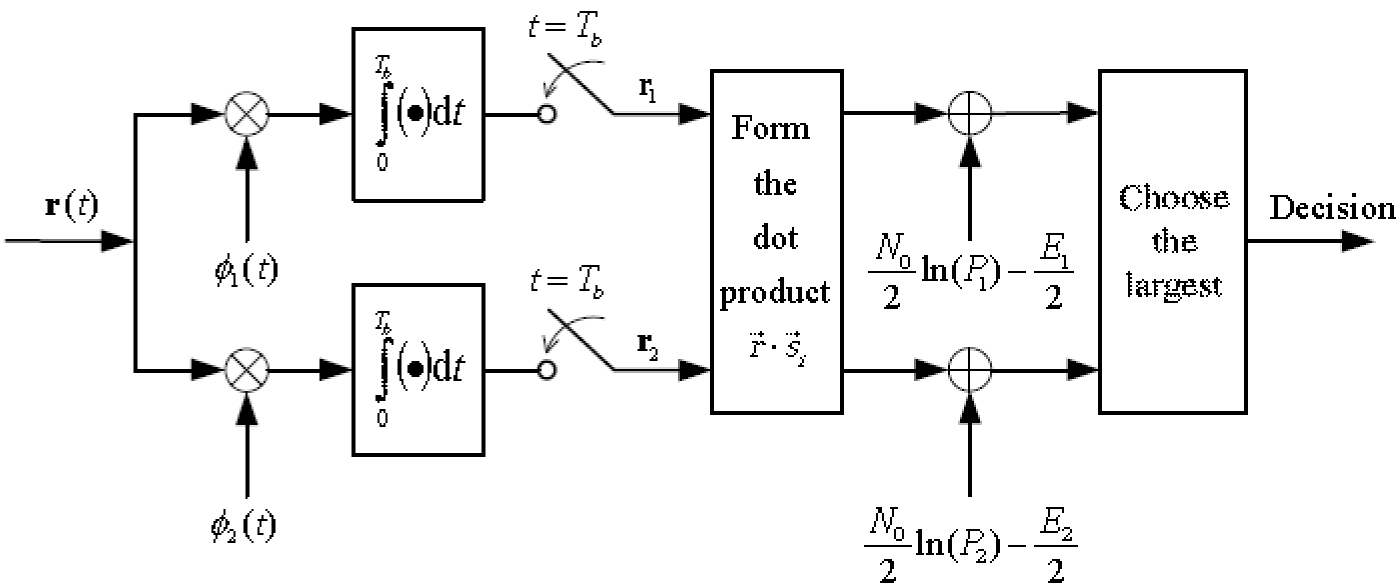
$$\rightarrow \left(r_1^2 - 2r_1 s_{11} + s_{11}^2\right) + \left(r_2^2 - 2r_2 s_{12} + s_{12}^2\right) - N_o \ln[P_1] \underset{0_D}{\overset{1_D}{\gtrless}} \left(r_1^2 - 2r_1 s_{21} + s_{21}^2\right) + \left(r_2^2 - 2r_2 s_{22} + s_{22}^2\right) - N_o \ln[P_2]$$

$$\rightarrow \left(-2r_1 s_{11} - 2r_2 s_{12} + s_{11}^2 + s_{12}^2\right) - N_o \ln[P_1] \underset{0_D}{\overset{1_D}{\gtrless}} \left(-2r_1 s_{21} - 2r_2 s_{22} + s_{21}^2 + s_{22}^2\right) - N_o \ln[P_2]$$

$$\rightarrow \left(-[r_1 \quad r_2].[s_{11} \quad s_{12}] + \frac{E_1}{2}\right) - \frac{N_o}{2} \ln[P_1] \underset{0_D}{\overset{1_D}{\gtrless}} \left(-[r_1 \quad r_2].[s_{21} \quad s_{22}] + \frac{E_2}{2}\right) - \frac{N_o}{2} \ln[P_2]$$

$$\rightarrow [r_1 \quad r_2].[s_{11} \quad s_{12}] - \frac{E_1}{2} + \frac{N_o}{2} \ln[P_1] \underset{1_D}{\overset{0_D}{\gtrless}} [r_1 \quad r_2].[s_{21} \quad s_{22}] - \frac{E_2}{2} + \frac{N_o}{2} \ln[P_2]$$

$$[r_1 \quad r_2].[s_{11} \quad s_{12}] - \frac{E_1}{2} + \frac{N_o}{2}\ln[\mathrm{P}_1] \underset{1_D}{\overset{0_D}{\gtrless}} [r_1 \quad r_2].[s_{21} \quad s_{22}] - \frac{E_2}{2} + \frac{N_o}{2}\ln[\mathrm{P}_2]$$



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- Recall: Convolution formula

$$y(t)=r(t)*h(t)=\int_{-\infty}^{\infty} r(\tau)h(t-\tau)d\tau=\int_{0}^{T_b} r(\tau)\Phi(\tau)d\tau$$

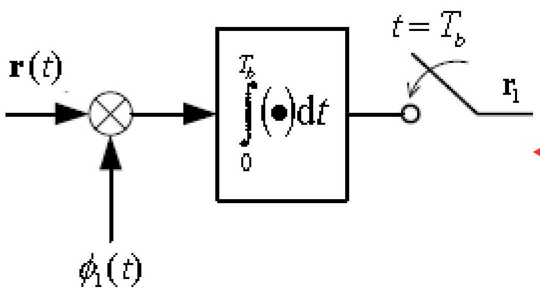
We want to make the convolution to equal the right side, and this is possible if

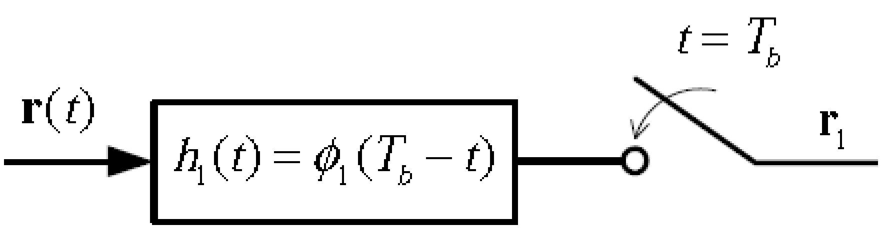
$$h(t-\tau)=\Phi(\tau) \quad \text{at} \quad t=T_b$$
$$h(T_b-\tau)=\Phi(\tau)$$
$$h(-\tau)=\Phi(\tau-T_b)$$
$$h(\tau)=\Phi(T_b-\tau)$$
$$h(t)=\Phi(T_b-t)$$

$$y(t)=r(t)*h(t)=r(t)*\Phi(T_b-t)=\int_{-\infty}^{\infty} r(\tau)\Phi(t-T_b+\tau)d\tau$$

- Evaluate this at $t=T_b$

$$y(T_b)=\int_{-\infty}^{\infty} r(\tau)\Phi(T_b-T_b+\tau)d\tau=\int_{-\infty}^{\infty} r(\tau)\Phi(\tau)d\tau$$





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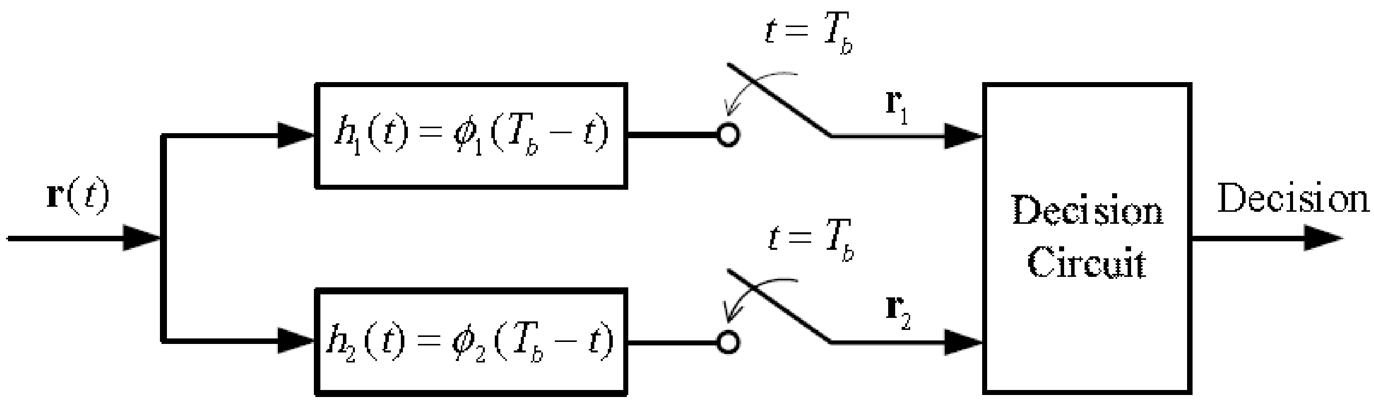
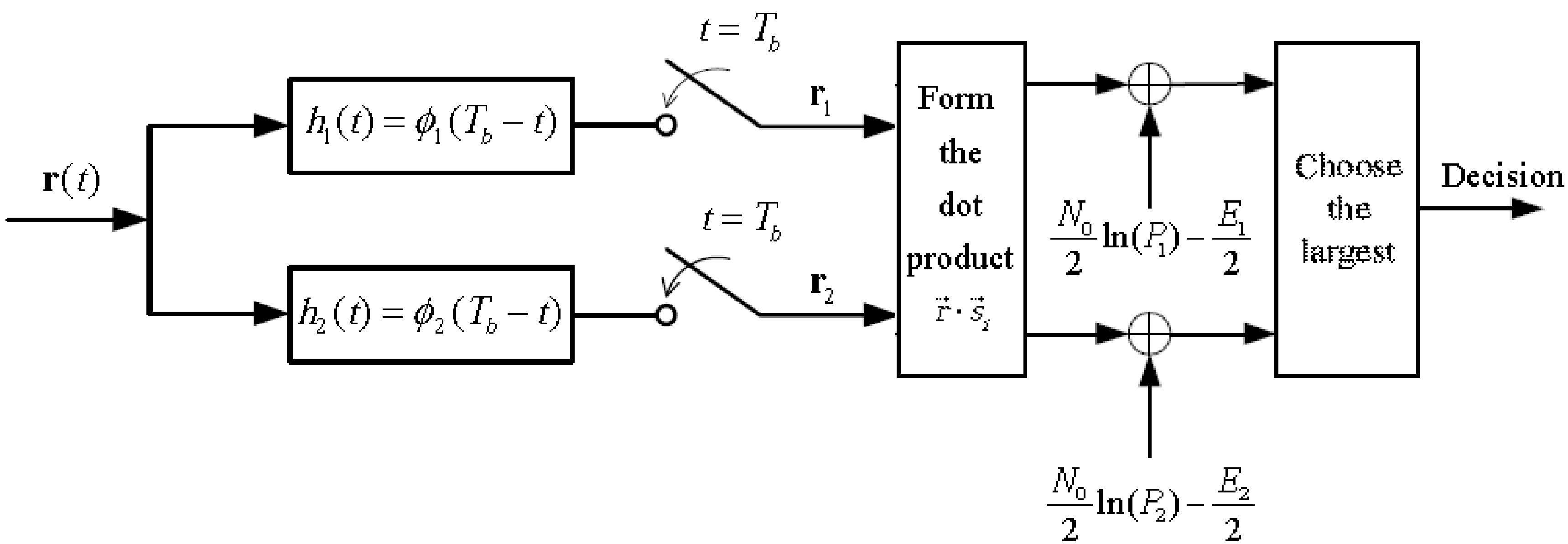
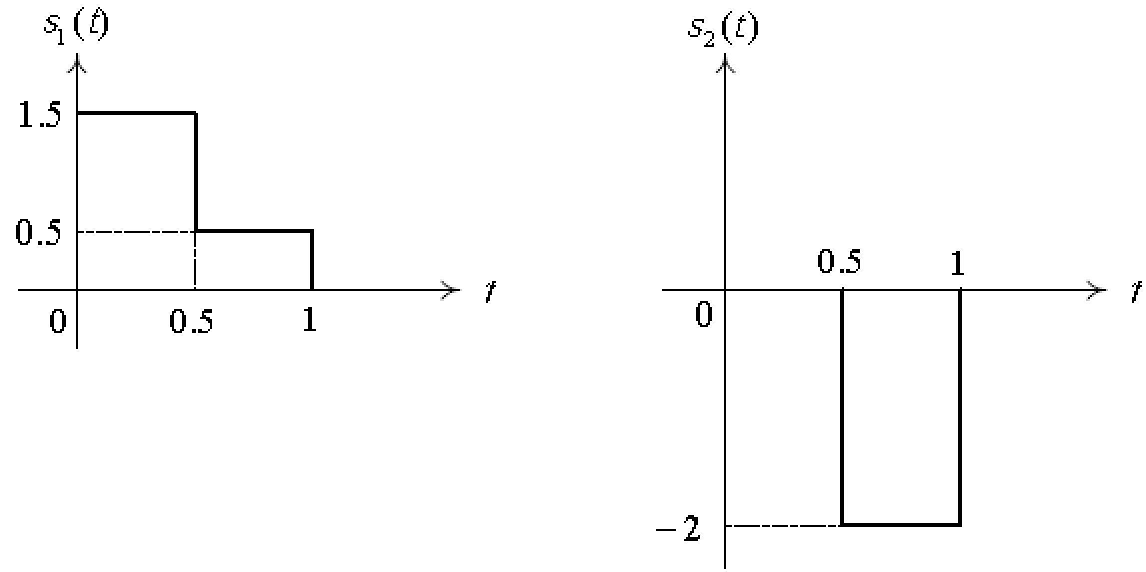




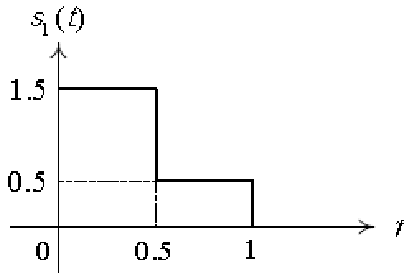
Example 5.6: Consider the signal set shown below,

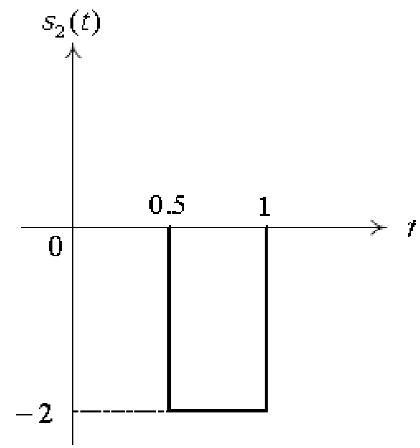
1) determine the orthonormal base functions needed to represent these signals?
2) Draw the signal space diagram?
3) Find and draw the optimum decision region?



Example 5.6: Consider the signal set shown below,

1) determine the orthonormal base functions needed to represent these signals?

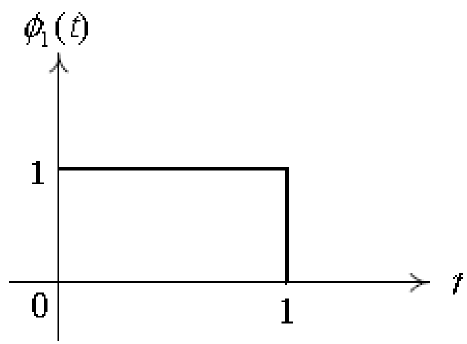


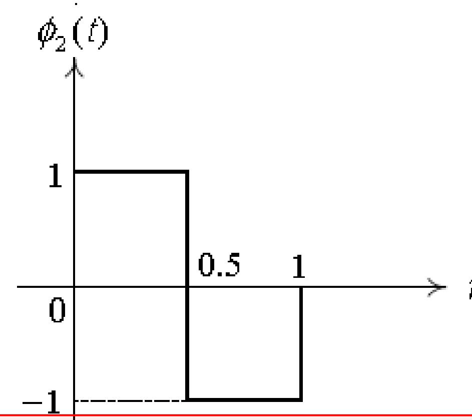


- The orthonormal base functions by inspection

$$\rightarrow s_1 = s_{11}\Phi_1(t) + s_{12}\Phi_2(t) = \Phi_1(t) + 0.5\Phi_2(t)$$

$$\rightarrow s_2 = s_{21}\Phi_1(t) + s_{22}\Phi_2(t) = -\Phi_1(t) + \Phi_2(t)$$



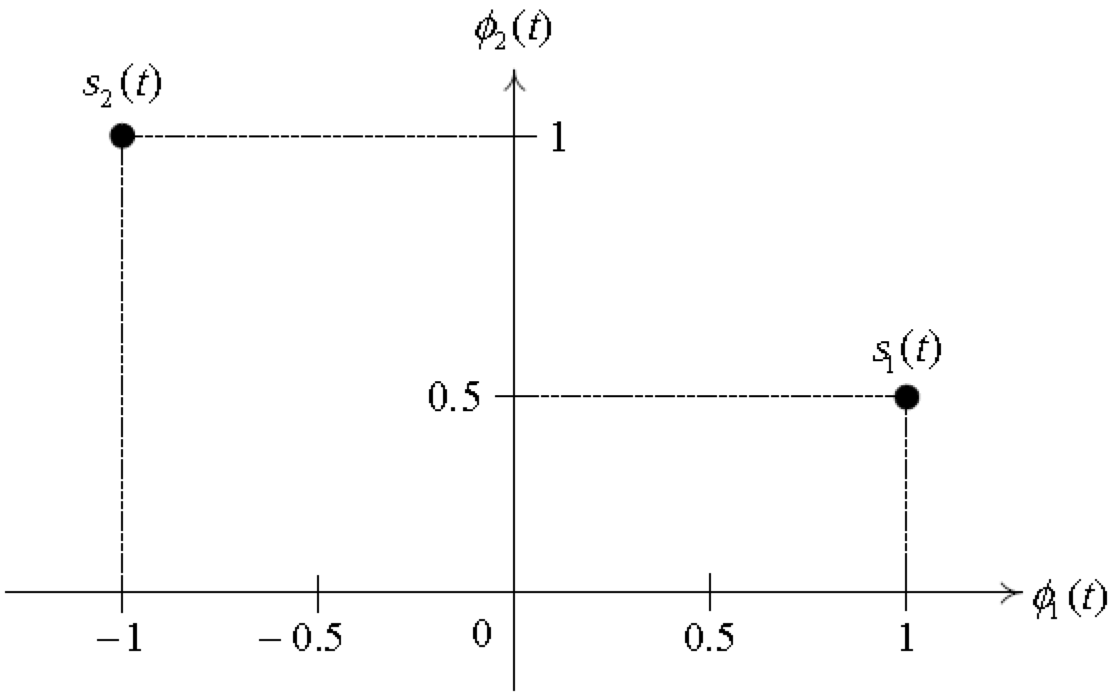


Example 5.6: Consider the signal set shown below,

2) Draw the signal space diagram?

$$s_1 = s_{11}\Phi_1(t) + s_{12}\Phi_2(t) = \Phi_1(t) + 0.5\Phi_2(t)$$

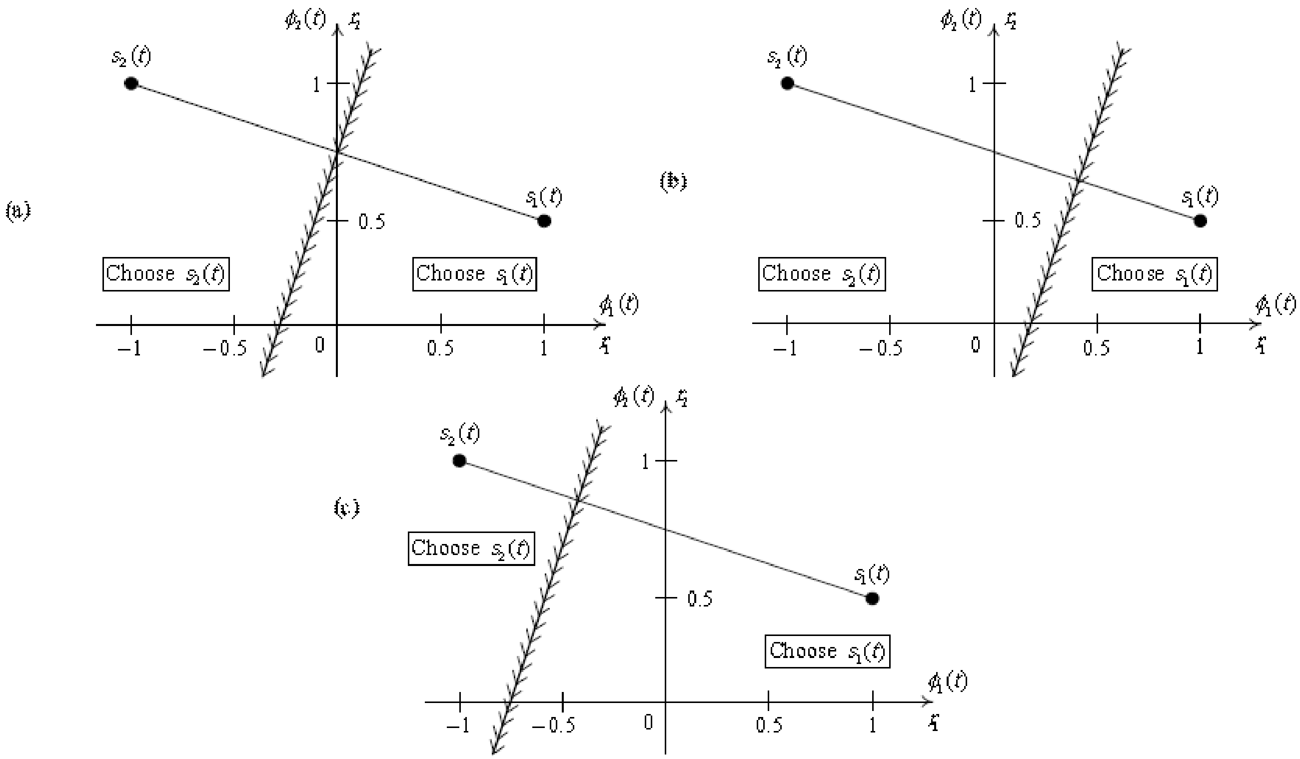
$$s_2 = s_{21}\Phi_1(t) + s_{22}\Phi_2(t) = -\Phi_1(t) + \Phi_2(t)$$



Example 5.6: Consider the signal set shown below,

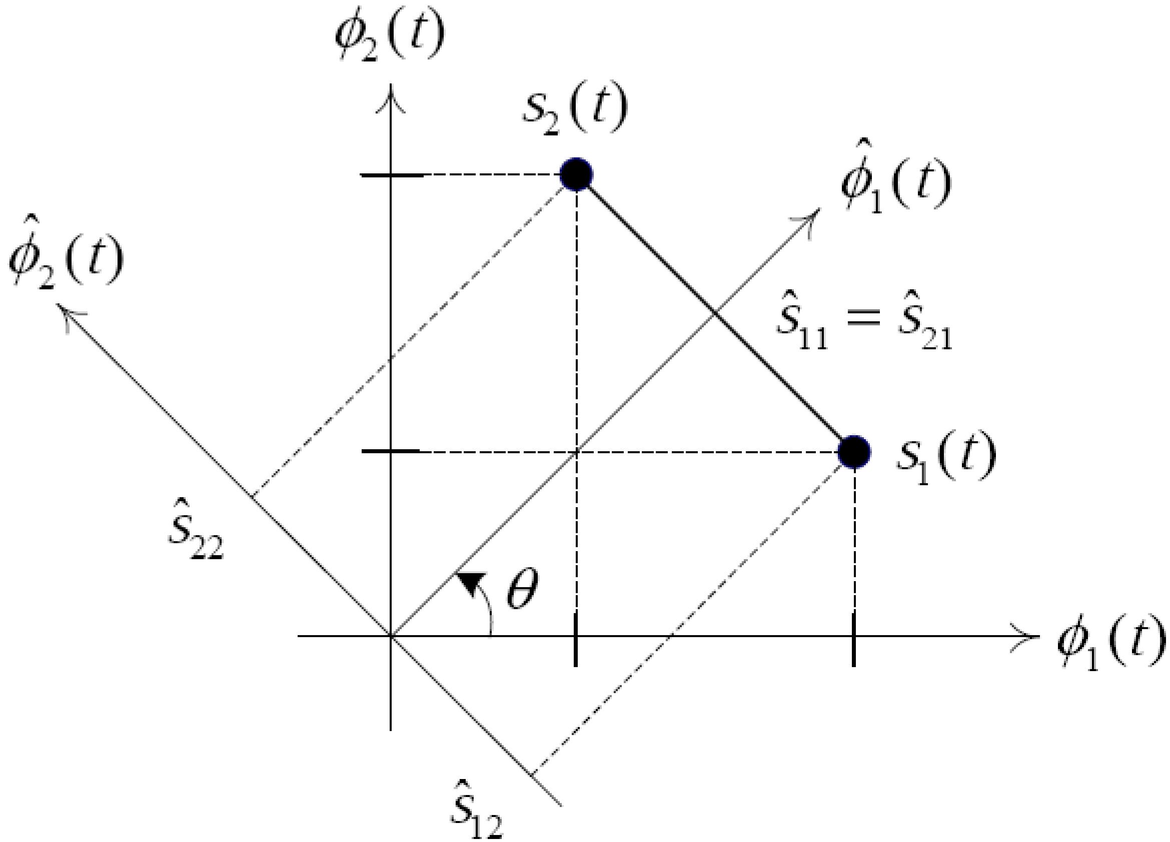
3) Find and draw the optimum decision region?

$$(r_1 - s_{11})^2 + (r_2 - s_{12})^2 \underset{0_D}{\overset{1_D}{\gtrless}} (r_1 - s_{21})^2 + (r_2 - s_{22})^2 + N_o \ln\left[\frac{P_1}{P_2}\right]$$



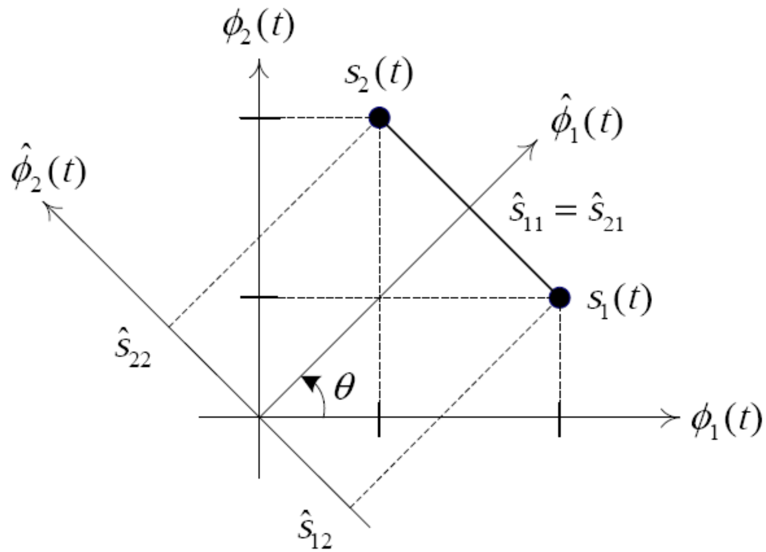
(a) $P_1 = P_2 = 0.5$, (b) $P_1 = 0.25$, $P_2 = 0.75$. (c) $P_1 = 0.75$, $P_2 = 0.25$.

- Can we implement the optimum receiver using one Correlator/match filter?
- Yes, possible by a judicious choice of the Orthonormal basis.



$$\begin{bmatrix} \hat{\Phi}_1(t) \\ \hat{\Phi}_2(t) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \Phi_1(t) \\ \Phi_2(t) \end{bmatrix}$$

- Now, we project the received $r(t)=s_i(t)+w(t)$ onto the new Orthonormal basis functions.



$$\rightarrow \frac{f\left(\vec{\hat{r}}/1_T\right)}{f\left(\vec{\hat{r}}/0_T\right)} \underset{0_D}{\overset{1_D}{\gtrless}} \frac{P_1}{P_2}$$

$$\rightarrow \frac{f\left(\hat{r}_1,\hat{r}_2,\hat{r}_3,\hat{r}_4,\cdots/1_T\right)}{f\left(\hat{r}_1,\hat{r}_2,\hat{r}_3,\hat{r}_4,\cdots/0_T\right)} \underset{0_D}{\overset{1_D}{\gtrless}} \frac{P_1}{P_2}$$

$$\rightarrow \frac{f\left(\hat{r}_1/1_T\right)f\left(\hat{r}_2/\hat{r}_1,1_T\right)f\left(\hat{r}_3/\hat{r}_1,\hat{r}_2,1_T\right)\cdots}{f\left(\hat{r}_1/0_T\right)f\left(\hat{r}_2/\hat{r}_1,0_T\right)f\left(\hat{r}_3/\hat{r}_1,\hat{r}_2,0_T\right)\cdots} \underset{0_D}{\overset{1_D}{\gtrless}} \frac{P_1}{P_2}$$

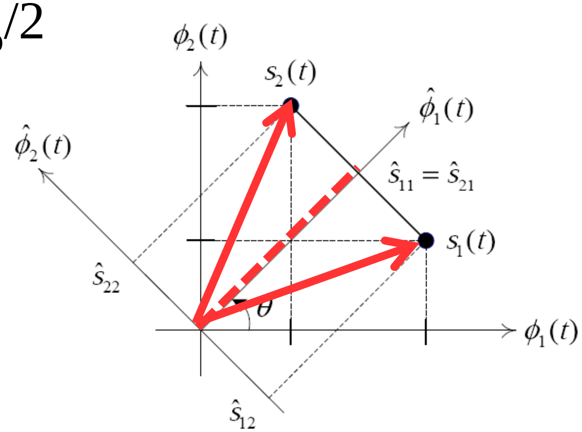
- Due to the independence condition discussed earlier

$$\rightarrow \frac{f\left(\hat{r}_1/1_T\right)f\left(\hat{r}_2/1_T\right)f\left(\hat{r}_3/1_T\right)\cdots}{f\left(\hat{r}_1/0_T\right)f\left(\hat{r}_2/0_T\right)f\left(\hat{r}_3/0_T\right)\cdots} \underset{0_D}{\overset{1_D}{\gtrless}} \frac{P_1}{P_2}$$

$$\rightarrow \frac{f\left(\hat{r}_1/1_T\right)f\left(\hat{r}_2/1_T\right)f\left(\hat{r}_3/1_T\right)\cdots}{f\left(\hat{r}_1/0_T\right)f\left(\hat{r}_2/0_T\right)f\left(\hat{r}_3/0_T\right)\cdots} \underset{0_D}{\overset{1_D}{\gtrless}} \frac{P_1}{P_2}$$

- for $\hat{r}_1 = \hat{s}_{i1} + \hat{w}_1$ → the mean of $\hat{r}_1$ is $\hat{s}_{i1}$ and the variance is $N_o/2$
  - if 0_T (i=1) then $\hat{r}_1 = \hat{s}_{11} + \hat{w}_1$ → the mean of $\hat{r}_1$ is $\hat{s}_{11}$ and the variance is $N_o/2$
  - if 1_T (i=2) then $\hat{r}_1 = \hat{s}_{21} + \hat{w}_1$ → the mean of $\hat{r}_1$ is $\hat{s}_{21}$ and the variance is $N_o/2$
- for $\hat{r}_2 = \hat{s}_{i2} + \hat{w}_2$ → the mean of $\hat{r}_2$ is $\hat{s}_{i2}$ and the variance is $N_o/2$
  - if 0_T (i=1) then $\hat{r}_2 = \hat{s}_{12} + \hat{w}_2$ → the mean of $\hat{r}_2$ is $\hat{s}_{12}$ and the variance is $N_o/2$
  - if 1_T (i=2) then $\hat{r}_2 = \hat{s}_{22} + \hat{w}_2$ → the mean of $\hat{r}_2$ is $\hat{s}_{22}$ and the variance is $N_o/2$
- for $\hat{r}_3 = \hat{w}_3$ → $\hat{r}_3$ is zero mean with variance of $N_o/2$
- Also notice that $\hat{s}_{11} = \hat{s}_{21}$ → $f\left(\hat{r}_1/0_T\right) = f\left(\hat{r}_1/1_T\right)$
- The decision rule will be reduced to

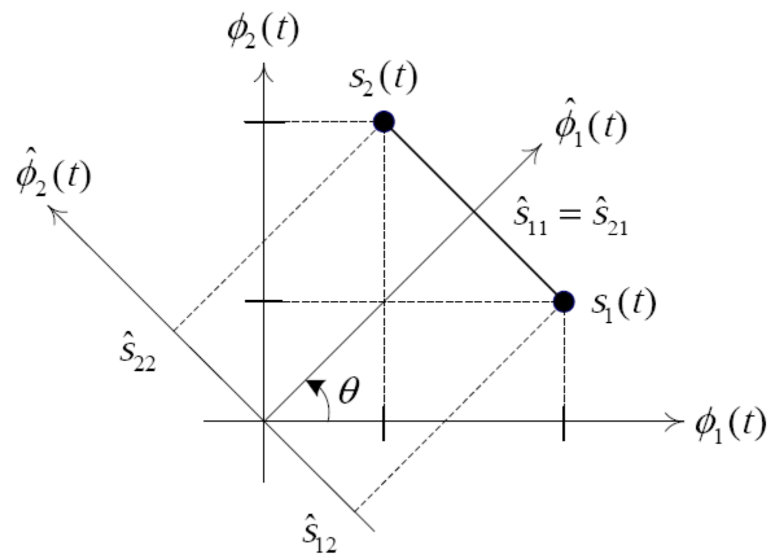
$$\rightarrow \frac{f\left(\hat{r}_2/1_T\right)}{f\left(\hat{r}_2/0_T\right)} \underset{0_D}{\overset{1_D}{\gtrless}} \frac{P_1}{P_2}$$



- From last slide: for $\hat{r}_2 = \hat{s}_{i2} + \hat{w}_2$ ➔ the mean of $\hat{r}_2$ is $\hat{s}_{i2}$ and the variance is $N_o/2$
  - if 0_T (i=1) then $\hat{r}_2 = \hat{s}_{12} + \hat{w}_2$ ➔ the mean of $\hat{r}_2$ is $\hat{s}_{12}$ and the variance is $N_o/2$
  - if 1_T (i=2) then $\hat{r}_2 = \hat{s}_{22} + \hat{w}_2$ ➔ the mean of $\hat{r}_2$ is $\hat{s}_{22}$ and the variance is $N_o/2$

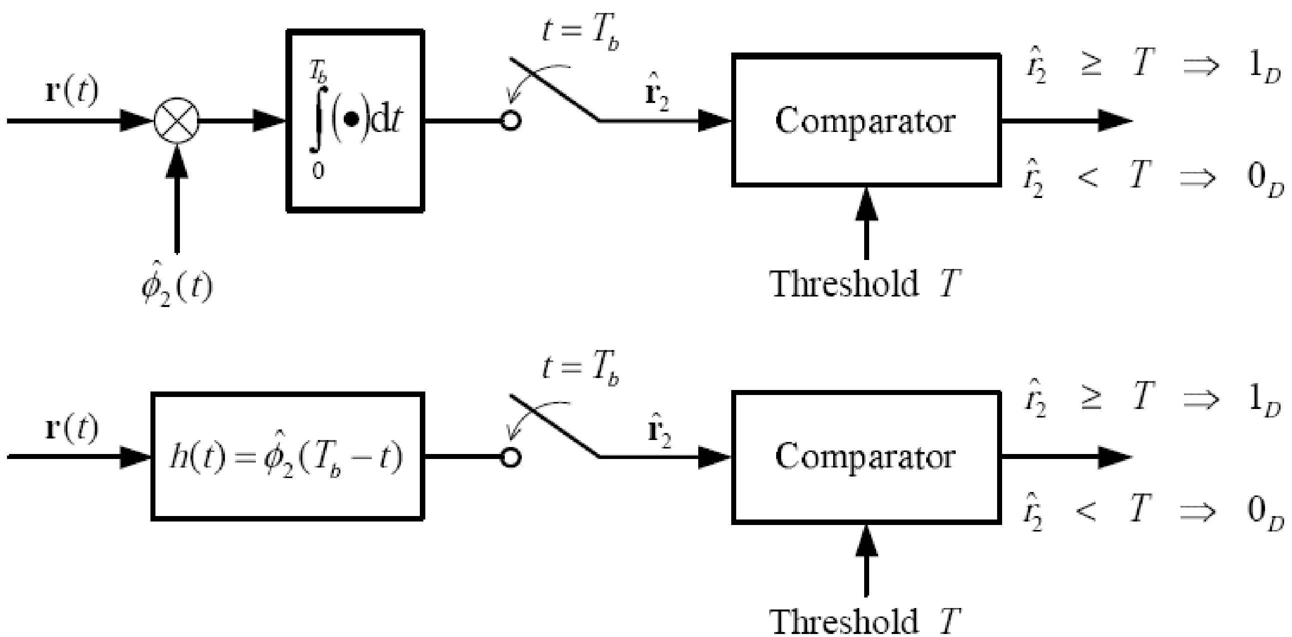
$$\rightarrow \frac{f(\hat{r}_2/1_T)}{f(\hat{r}_2/0_T)} = \frac{f(\hat{s}_{22}+\hat{w}_2)}{f(\hat{s}_{12}+\hat{w}_2)} = \frac{\frac{1}{\sqrt{\pi N_o}}\exp\left[\frac{-(\hat{r}_2-\hat{s}_{22})^2}{N_o}\right]}{\frac{1}{\sqrt{\pi N_o}}\exp\left[\frac{-(\hat{r}_2-\hat{s}_{12})^2}{N_o}\right]} \underset{0_D}{\overset{1_D}{\gtrless}} \frac{P_1}{P_2}$$

$$\rightarrow \hat{r}_2 \underset{0_D}{\overset{1_D}{\gtrless}} T = \frac{\hat{s}_{22}+\hat{s}_{12}}{2} + \left(\frac{N_o/2}{\hat{s}_{22}-\hat{s}_{12}}\right)\ln\left(\frac{P_1}{P_2}\right)$$



- But how to find $\hat{\Phi}(t)$

$$\hat{\Phi}_2(t) = \frac{s_2(t) - s_1(t)}{\sqrt{\int_0^{T_b} [s_2(t) - s_1(t)]^2 dt}} = \frac{s_2(t) - s_1(t)}{\left(E_2 - 2\rho\sqrt{E_1 E_2} + E_1\right)^{\frac{1}{2}}}$$

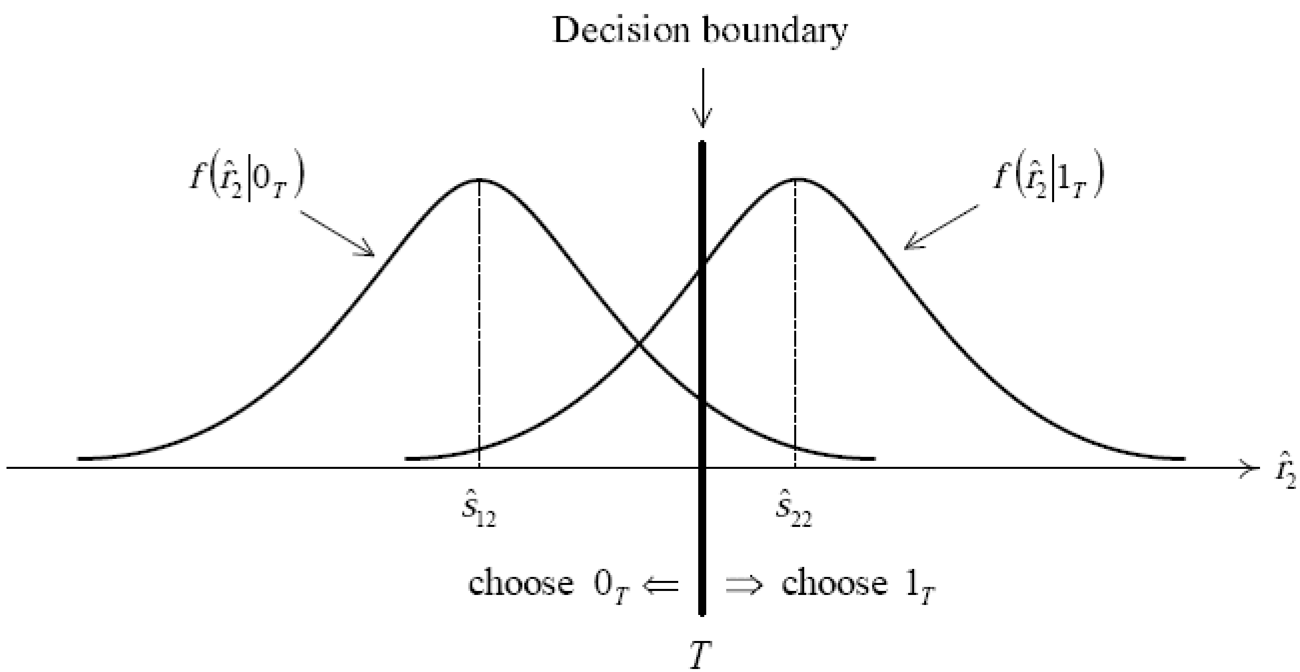


- To detect b_k, compare $\hat{r}_2$ to the threshold value T

$$\hat{r}_2 = \int_0^{T_b} r(t)\hat{\Phi}_2(t)\,dt$$

$$T = \frac{\hat{s}_{22}+\hat{s}_{12}}{2} + \left(\frac{N_o/2}{\hat{s}_{22}-\hat{s}_{12}}\right)\ln\left(\frac{P_1}{P_2}\right)$$

$$P[error] = P\left[\begin{matrix}(\text{"0"}\ decided\ and\ \text{"1"}\ transmitted)\ \text{or} \\ (\text{"1"}\ decided\ and\ \text{"0"}\ transmitted)\end{matrix}\right]$$

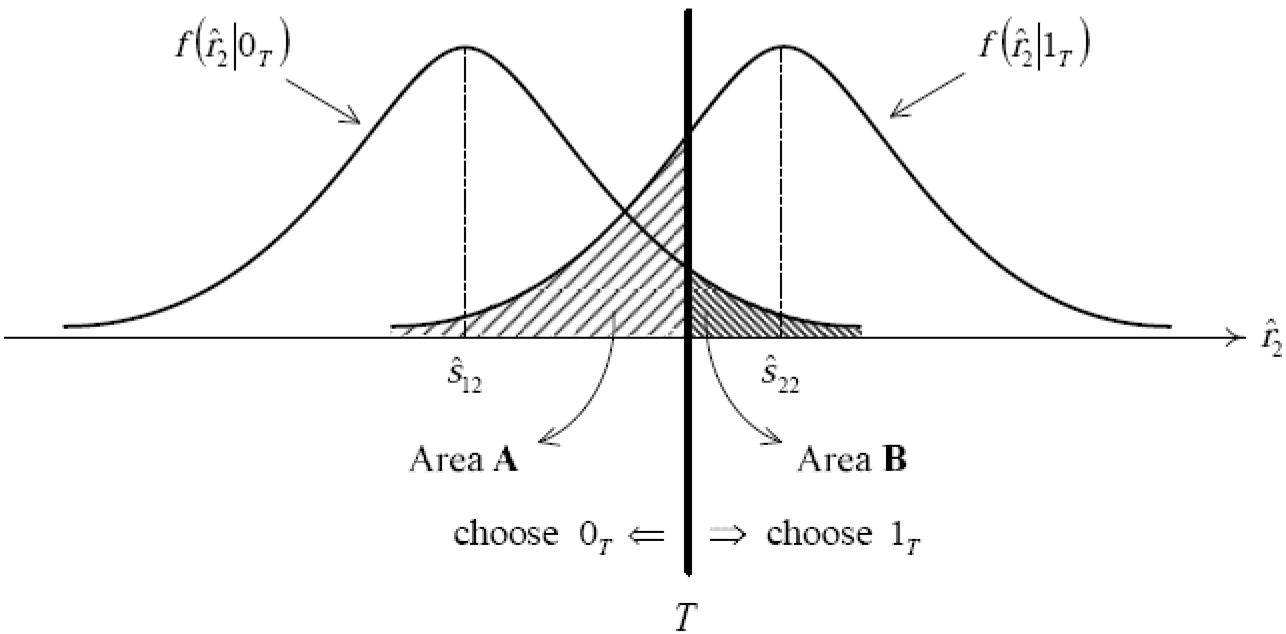


- To detect b_k, compare $\hat{r}_2$ to the threshold value T

$$\hat{r}_2 = \int_0^{T_b} r(t)\hat{\Phi}_2(t)dt$$

$$T = \frac{\hat{s}_{22}+\hat{s}_{12}}{2} + \left(\frac{N_o/2}{\hat{s}_{22}-\hat{s}_{12}}\right)\ln\left(\frac{P_1}{P_2}\right)$$

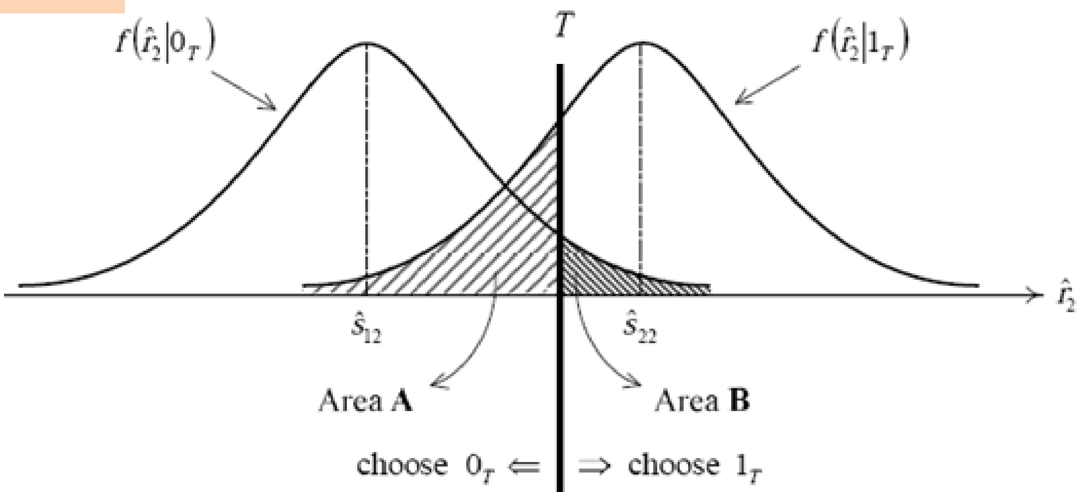
$$P[error] = P\begin{bmatrix}(\text{"0" decided and "1" transmitted})\text{ or} \\ (\text{"1" decided and "0" transmitted})\end{bmatrix}$$



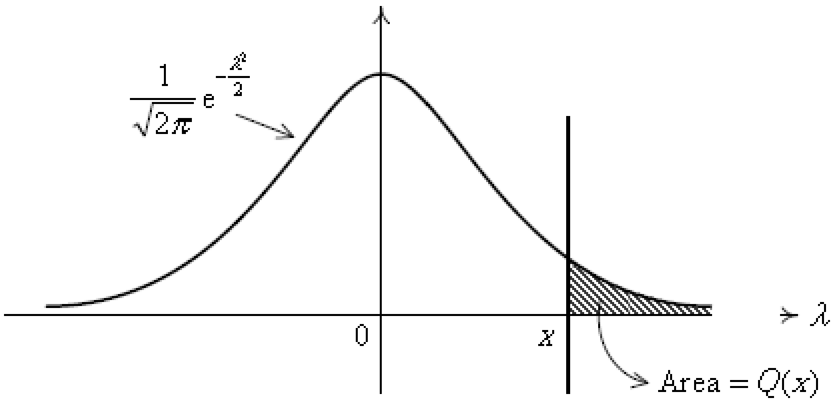
$$\rightarrow P[\text{error}] = P\left[\begin{array}{l}(\text{"0" decided and "1" transmitted})\text{ or}\\(\text{"1" decided and "0" transmitted})\end{array}\right]$$

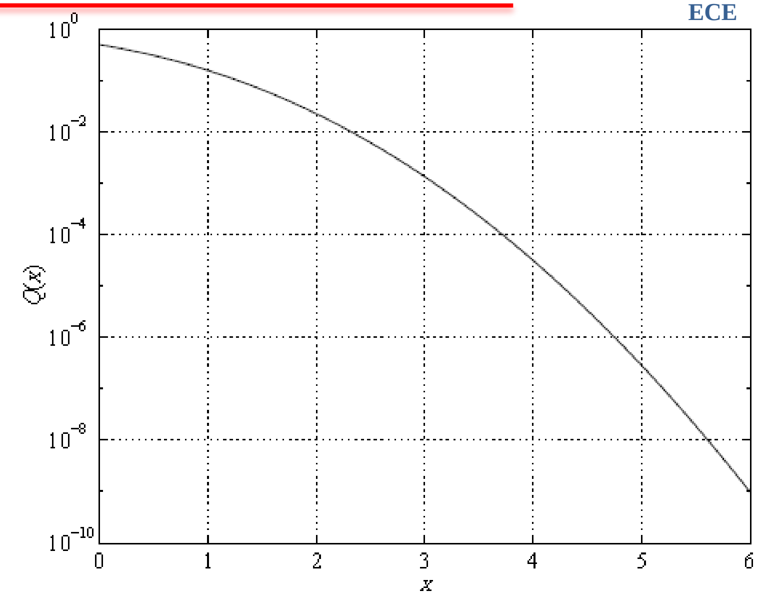
$$\rightarrow = P[0_T, 1_D] + P[1_T, 0_D] = P[0_T]P[1_D / 0_T] + P[1_T]P[0_D / 1_T]$$

$$= \underbrace{P_1 \int_T^{\infty} f(\hat{r}_2 / 0_T) d\hat{r}_2}_{\text{Area B}} + \underbrace{P_2 \int_{-\infty}^{T} f(\hat{r}_2 / 1_T) d\hat{r}_2}_{\text{Area A}} = P_1 Q\left(\frac{T - \hat{s}_{12}}{\sqrt{N_o/2}}\right) + P_2\left[1 - Q\left(\frac{T - \hat{s}_{22}}{\sqrt{N_o/2}}\right)\right]$$



$$Q(x)=\frac{1}{\sqrt{2\pi}}\int_{x}^{\infty} e^{-\frac{\lambda^2}{2}}\,d\lambda$$





- If $P_1=P_2$, receiver performance could be caulculated as

$$T=\frac{\hat{s}_{22}+\hat{s}_{12}}{2}+\left(\frac{N_o/2}{\hat{s}_{22}-\hat{s}_{12}}\right)\ln\left(\frac{P_1}{P_2}\right)=\frac{\hat{s}_{22}+\hat{s}_{12}}{2}$$

$$P[\text{error}]=\frac{1}{2}Q\left(\frac{T-\hat{s}_{12}}{\sqrt{N_o/2}}\right)+\frac{1}{2}\left[1-Q\left(\frac{T-\hat{s}_{22}}{\sqrt{N_o/2}}\right)\right]$$

$$\rightarrow P[\text{error}] = \frac{1}{2} Q\left(\frac{T - \hat{s}_{12}}{\sqrt{N_o/2}}\right) + \frac{1}{2}\left[1 - Q\left(\frac{T - \hat{s}_{22}}{\sqrt{N_o/2}}\right)\right]$$

$$= \frac{1}{2} Q\left(\frac{\frac{\hat{s}_{22} + \hat{s}_{12}}{2} - \hat{s}_{12}}{\sqrt{N_o/2}}\right) + \frac{1}{2}\left[1 - Q\left(\frac{\frac{\hat{s}_{22} + \hat{s}_{12}}{2} - \hat{s}_{22}}{\sqrt{N_o/2}}\right)\right]$$

$$= \frac{1}{2} Q\left(\frac{\hat{s}_{22} - \hat{s}_{12}}{2\sqrt{N_o/2}}\right) + \frac{1}{2}\left[1 - Q\left(\frac{\hat{s}_{12} - \hat{s}_{22}}{2\sqrt{N_o/2}}\right)\right]$$

$$= \frac{1}{2} Q\left(\frac{\hat{s}_{22} - \hat{s}_{12}}{2\sqrt{N_o/2}}\right) + \frac{1}{2} Q\left(-\frac{\hat{s}_{12} - \hat{s}_{22}}{2\sqrt{N_o/2}}\right) = Q\left(\frac{\hat{s}_{22} - \hat{s}_{12}}{2\sqrt{N_o/2}}\right)$$

$$= Q\left(\frac{\text{distance between the signals}}{2 \times \text{noise RMS value}}\right)$$

$$\rightarrow P[\text{error}] = Q\left(\frac{\hat{s}_{22} - \hat{s}_{12}}{2\sqrt{N_o/2}}\right) = Q\left(\frac{\text{dis tan ce between the signals}}{2 \times \text{noise RMS value}}\right)$$

- Probability of error decreases as either the two signals become more dissimilar (increasing the distances between them) or the noise power becomes less.

- To maximize the distance between the two signals one chooses them so that they are placed 180◦ from each other → $s_1(t) = -s_1(t)$, i.e., antipodal signaling.

- The error probability does not depend on the signal shapes but only on the distance between them.

- The complementary error function is defined as:

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-\lambda^2} d\lambda = 1 - \operatorname{erf}(x)$$

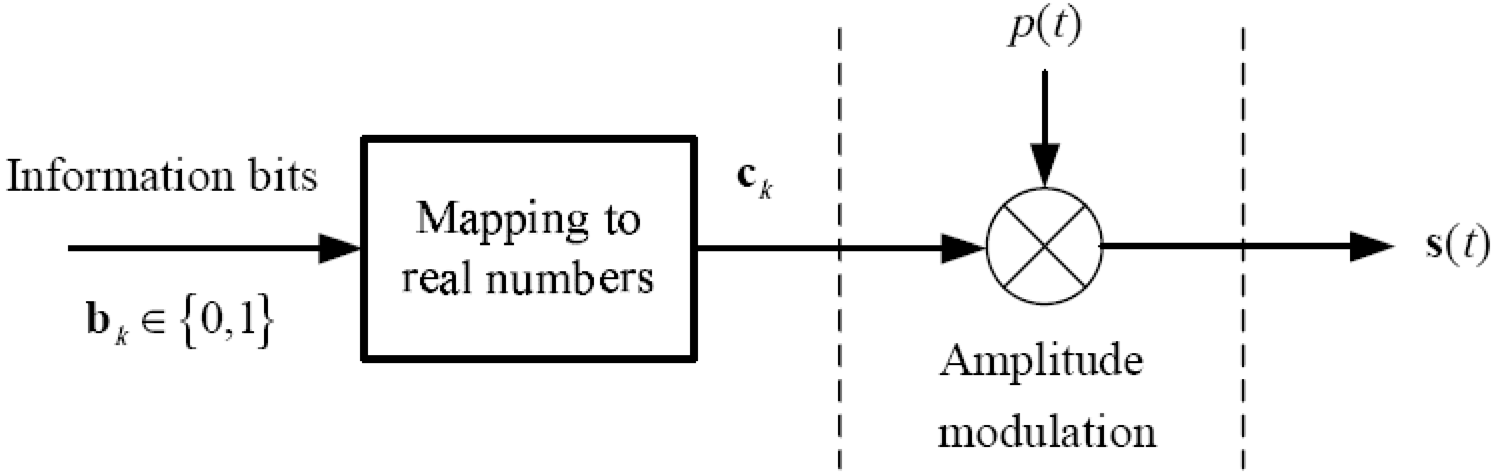
- erfc-function and the Q-function are related by:

$$Q(x) = \frac{1}{2} \operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right)$$

$$\operatorname{erfc}(x) = 2Q\left(\sqrt{2}x\right)$$

$$Q^{-1}(x) = \sqrt{2}\operatorname{erfc}^{-1}(2x)$$

- C_k is drawn from a finite set of real numbers with a probability that is known.
- Example of C_k :
  - C_k : {-1,1} antipodal signaling
  - C_k : {0,1} on-off keying
  - C_k : {-1,0,1} pseudoternary line coding
  - C_k : {±1, ±3, ±5,…} M-ary amplitude-shift keying
- p(t) is a pulse wave of duration T_b



- You must study section 3.2.3 in text book
- The transmitted signal is

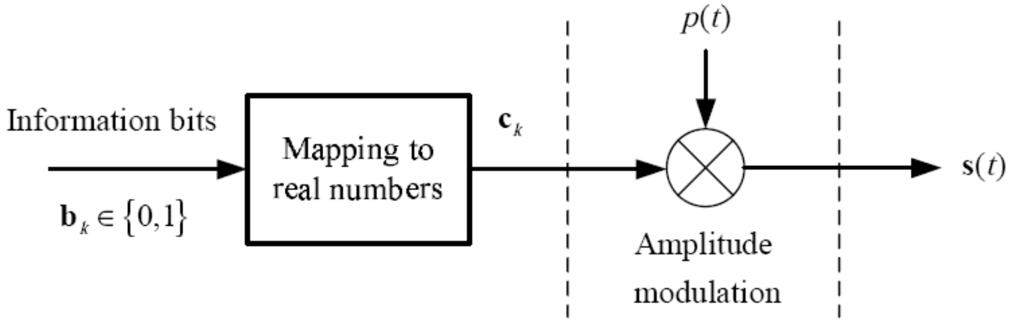
$$s(t) = \sum_{k=-\infty}^{\infty} c_k p(t - kT_b)$$

- To find PSD, truncate the random process to a time interval of $-T = -NT_b$ to $T = NT_b$:

$$s_T(t) = \sum_{k=-N}^{N} c_k p(t - kT_b)$$

- Take the Fourier transform of the truncated process:

$$S_T(f) = \sum_{k=-N}^{N} c_k F(p(t - kT_b)) = P(f) \sum_{k=-N}^{N} c_k e^{-j2\pi f k T_b}$$



- Apply the basic definition of PSD (refer to section 3.2.3 in text book):

$$S(f) = \lim_{T\to\infty} \frac{E\left\{\left|S_T(f)\right|^2\right\}}{2T}, \qquad S_T(f) = P(f)\sum_{k=-N}^{N} c_k e^{-j2\pi f k T_b}$$

$$= \lim_{N\to\infty} \frac{\left|P(f)\right|^2}{(2N+1)T_b} E\left\{\left|\sum_{k=-N}^{N} c_k e^{-j2\pi f k T_b}\right|^2\right\}$$

$$= \lim_{N\to\infty} \frac{\left|P(f)\right|^2}{(2N+1)T_b} E\left\{\left[\sum_{k=-N}^{N} c_k e^{-j2\pi f k T_b}\right]\left[\sum_{l=-N}^{N} c_l e^{-j2\pi f l T_b}\right]^*\right\}$$

$$= \lim_{N\to\infty} \frac{\left|P(f)\right|^2}{(2N+1)T_b} E\left\{\sum_{k=-N}^{N}\sum_{l=-N}^{N} c_k c_l^* e^{-j2\pi f k T_b} e^{j2\pi f l T_b}\right\}$$

$$= \lim_{N\to\infty} \frac{\left|P(f)\right|^2}{(2N+1)T_b} E\left\{\sum_{k=-N}^{N}\sum_{l=-N}^{N} c_k c_l^* e^{-j2\pi f (k-l) T_b}\right\}$$

$$= \lim_{N\to\infty} \frac{\left|P(f)\right|^2}{(2N+1)T_b} E\left\{\sum_{k=-N}^{N}\sum_{m=k+N}^{m=k-N} c_k c_{k-m}^* e^{-j2\pi f (m) T_b}\right\}$$

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$$S(f) = \lim_{T\to\infty} \frac{E\left\{\left|S_T(f)\right|^2\right\}}{2T}$$

$$= \lim_{N\to\infty} \frac{\left|P(f)\right|^2}{(2N+1)T_b} E\left\{ \sum_{k=-N}^{N} \sum_{m=k+N}^{m=k-N} c_k c_{k-m}^* e^{-j2\pi f m T_b} \right\}$$

$$= \lim_{N\to\infty} \frac{\left|P(f)\right|^2}{(2N+1)T_b} \left[\sum_{k=-N}^{N} \sum_{m=k+N}^{m=k-N} E\left\{c_k c_{k-m}^*\right\} e^{-j2\pi f m T_b} \right]$$

$$= \lim_{N\to\infty} \frac{\left|P(f)\right|^2}{(2N+1)T_b} \left[\sum_{k=-N}^{N} \sum_{m=k+N}^{m=k-N} R_c(m) e^{-j2\pi f m T_b} \right]$$

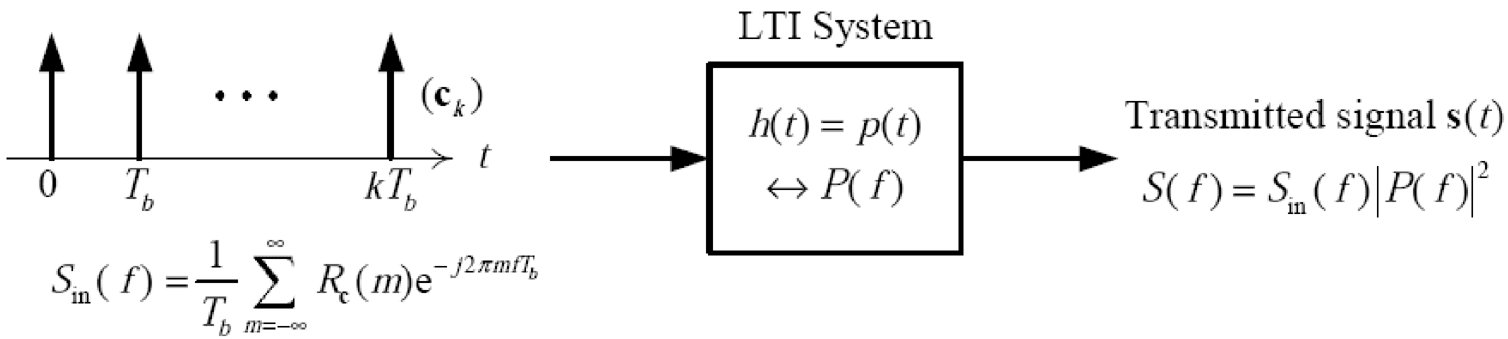
Discrete Autocorrelation

$$= \lim_{N\to\infty} \frac{\left|P(f)\right|^2}{(2N+1)T_b} \left[(2N+1) \sum_{m=-\infty}^{m=\infty} R_c(m) e^{-j2\pi f m T_b} \right]$$

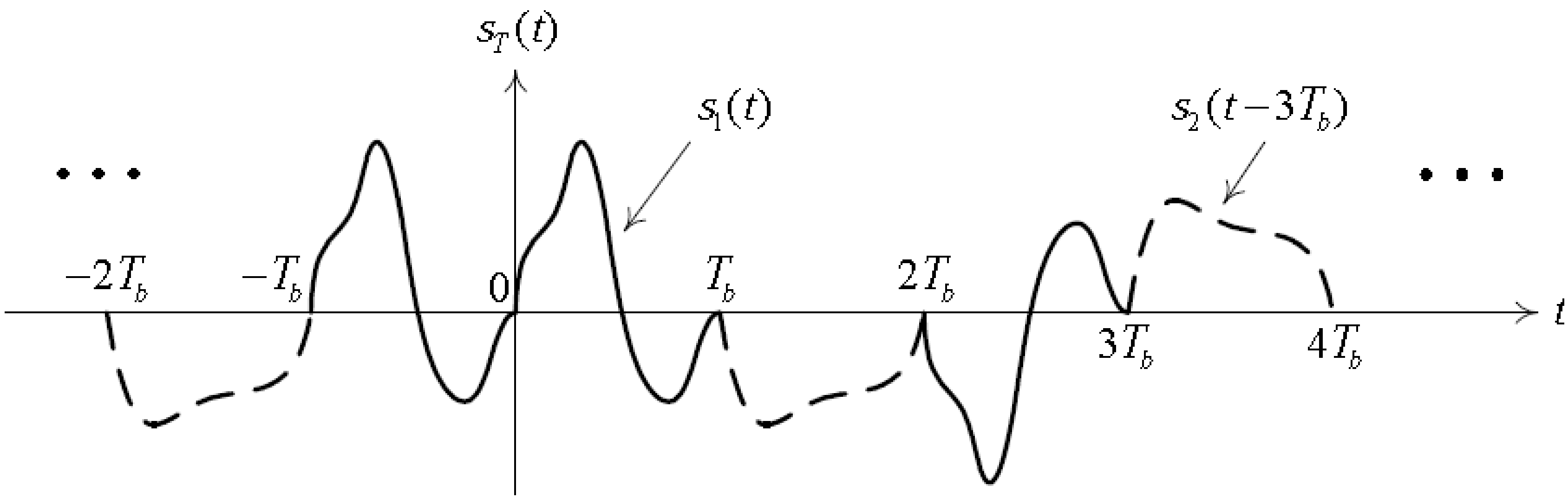
$$= \frac{\left|P(f)\right|^2}{T_b} \sum_{m=-\infty}^{m=\infty} R_c(m) e^{-j2\pi f m T_b}$$

- The output PSD is the input PSD multiplied by $|P(f)|^2$, a transfer function.

$$S(f) = \lim_{T\to\infty} \frac{E\left\{|S_T(f)|^2\right\}}{2T} = \frac{|P(f)|^2}{T_b} \sum_{m=-\infty}^{m=\infty} R_c(m) e^{-j2\pi f m T_b}$$



➢ Applicable to any binary modulation with arbitrary a priori probabilities, but restricted to statistically independent bits..



$$s_T(t)=\sum_{k=-\infty}^{\infty} g_k(t), \quad g_k(t)=\begin{cases} s_1(t-kT_b), & \text{with probability } p_1 \\ s_2(t-kT_b), & \text{with probability } p_2 \end{cases}$$

- Decompose $s_T(t)$ into a sum of a DC and an AC components:

$$s_T(t) = \underbrace{E\{s_T(t)\}}_{\text{DC}} + \underbrace{s_T(t) - E\{s_T(t)\}}_{\text{AC}} = v(t) + q(t)$$

$$v(t) = E\{s_T(t)\} = \sum_{k=-\infty}^{\infty}\left[p_1 s_1(t-kT_b) + p_2 s_2(t-kT_b)\right]$$

$$S_v(f) = \sum_{n=-\infty}^{\infty} |D_n|^2 \delta\left(f - \frac{n}{T_b}\right), \qquad D_n = \frac{1}{T_b}\left[p_1 S_1\left(\frac{n}{T_b}\right) + p_2 S_2\left(\frac{n}{T_b}\right)\right]$$

- where $S_1(f)$ and $S_2(f)$ are the FTs of $s_1(t)$ and $s_2(t)$.

$$S_v(f) = \sum_{n=-\infty}^{\infty} \left|\frac{p_1 S_1\left(\frac{n}{T_b}\right) + p_2 S_2\left(\frac{n}{T_b}\right)}{T_b}\right|^2 \delta\left(f - \frac{n}{T_b}\right)$$

- To calculate $S_q(f)$, apply the basic definition of PSD:

$$S_q(f) = \lim_{T\to\infty} \frac{E\left\{\left|G_T(f)\right|^2\right\}}{T} = \frac{p_1 p_2}{T_b}\left|S_1(f) - S_2(f)\right|^2$$

- Finally,

$$S_{s_T}(f) = S_q(f) + S_v(f)$$

$$= \frac{p_1 p_2}{T_b}\left|S_1(f) - S_2(f)\right|^2 + \sum_{n=-\infty}^{\infty}\left|\frac{p_1 S_1\left(\frac{n}{T_b}\right) + p_2 S_2\left(\frac{n}{T_b}\right)}{T_b}\right|^2 \delta\left(f - \frac{n}{T_b}\right)$$

- Notice: The output power spectral density depends on the Fourier Transform of the signal used to represent “0” and “1”, and the a priori probabilities of the data from the source.