Solutions Manual
"A First Course in Digital Communications"

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Preface

This Solutions Manual was last updated in August, 2011. We appreciate receiving comments and corrections you might have on the current version. Please send emails to `ha.nguyen@usask.ca` or `shwedyk@ee.umanitoba.ca`.

Chapter 2

Deterministic Signal Characterization and Analysis

P2.1 First let us establish (or review) the relationships for real signals

(a) Given $\{A_k, B_k\}$, i.e., $s(t) = \sum_{k=0}^{\infty}\left[A_k \cos(2\pi k f_r t) + B_k \sin(2\pi k f_r t)\right]$, A_k, B_k are real. Then $C_k = \sqrt{A_k^2 + B_k^2}$, $\theta_k = \tan^{-1}\frac{B_k}{A_k}$, where we write $s(t)$ as

$$s(t) = \sum_{k=0}^{\infty} C_k \cos(2\pi k f_r t - \theta_k)$$

If we choose to write it as

$$s(t) = \sum_{k=0}^{\infty} C_k \cos(2\pi k f_r t + \theta_k)$$

then the phase becomes $\theta_k = -\tan^{-1}\frac{B_k}{A_k}$. Further

$$\begin{aligned} D_k &= \frac{A_k - jB_k}{2} \\ D_{-k} &= D_k^*, \quad k = 1, 2, 3, \ldots \\ D_0 &= A_0 \ (B_0 = 0 \text{ always}). \end{aligned}$$

(b) Given $\{C_k, \theta_k\}$ then $A_k = C_k \cos\theta_k$, $B_k = C_k \sin\theta_k$. They are obtained from:

$$s(t) = \sum_{k=0}^{\infty} C_k \cos(2\pi k f_r t - \theta_k) = \sum_{k=0}^{\infty}\left[C_k \cos\theta_k \cos(2\pi k f_r t) + C_k \sin\theta_k \sin(2\pi k f_r t)\right]$$

Now $s(t)$ can written as:

$$s(t) = \sum_{k=0}^{\infty} C_k \left\{\frac{e^{j[2\pi k f_r t - \theta_k]} + e^{-j[2\pi k f_r t - \theta_k]}}{2}\right\}$$

$$s(t) = C_0 \underbrace{\left[\frac{e^{-j\theta_0} + e^{j\theta_0}}{2}\right]}_{\cos\theta_0} + \sum_{k=1}^{\infty}\left[\frac{C_k}{2} e^{-j\theta_k} e^{j2\pi k f_r t} + \frac{C_k}{2} e^{j\theta_k} e^{-j2\pi k f_r t}\right]$$

Therefore

$$\begin{aligned} D_0 &= C_0 \cos\theta_0 \text{ where } \theta_0 \text{ is either 0 or } \pi \\ D_k &= \frac{C_k}{2}\mathrm{e}^{-j\theta_k}, \quad k = 1, 2, 3, \ldots \end{aligned}$$

What about negative frequencies? Write the third term as $\frac{C_k}{2}\mathrm{e}^{j\theta_k}\mathrm{e}^{j[2\pi k\cdot(-f_r)\cdot t]}$, where $(-f_r)$ is interpreted as negative frequency. Therefore $D_{-k} = \frac{C_k}{2}\mathrm{e}^{+j\theta_k}$, i.e., $D_{-k} = D_k^*$.

(c) Given $\{D_k\}$, then $A_k = 2\mathcal{R}\{D_k\}$ and $B_k = -2\mathcal{I}\{D_k\}$. Also $C_k = 2|D_k|$, $\theta_k = -\angle D_k$, where D_k is in general complex and written as $D_k = |D_k|\mathrm{e}^{j\angle D_k}$.

Remark: Even though given any set of the coefficients, we can find the other 2 sets, we can only determine $\{A_k, B_k\}$ or $\{D_k\}$ from the signal, $s(t)$, i.e., there is no method to determine $\{C_k, \theta_k\}$ directly.

Consider now that $s(t)$ is a complex, periodic time signal with period $T = \frac{1}{f_r}$, i.e., $s(t) = s_R(t) + js_I(t)$ where the real and imaginary components, $s_R(t)$, $s_I(t)$, are each periodic with period $T = \frac{1}{f_r}$. Again we represent $s(t)$ in terms of the orthogonal basis set $\{\cos(2\pi k f_r t), \sin(2\pi k f_r t)\}_{k=1,2,\ldots}$. That is

$$s(t) = \sum_{k=0}^{\infty}\left[A_k\cos(2\pi k f_r t) + B_k \sin(2\pi k f_r t)\right] \tag{2.1}$$

where $A_k = \frac{2}{T}\int_{t\in T} s(t)\cos(2\pi k f_r t)\mathrm{d}t$; $B_k = \frac{2}{T}\int_{t\in T} s(t)\sin(2\pi k f_r t)\mathrm{d}t$ are now complex numbers.

One approach to finding A_k, B_k is to express $s_R(t), s_I(t)$ in their own individual Fourier series, and then combine to determine A_k, B_k. That is

$$s_R(t) = \sum_{k=0}^{\infty}\left[A_k^{(R)}\cos(2\pi k f_r t) + B_k^{(R)}\sin(2\pi k f_r t)\right]$$

$$s_I(t) = \sum_{k=0}^{\infty}\left[A_k^{(I)}\cos(2\pi k f_r t) + B_k^{(I)}\sin(2\pi k f_r t)\right]$$

$$\Rightarrow s(t) = \sum_{k=0}^{\infty}\left[\underbrace{\left(A_k^{(R)} + jA_k^{(I)}\right)}_{=A_k}\cos(2\pi k f_r t) + \underbrace{\left(B_k^{(R)} + jB_k^{(I)}\right)}_{=B_k}\sin(2\pi k f_r t)\right]$$

(d) So now suppose we are given $\{A_k, B_k\}$. How do we determine $\{C_k, \theta_k\}$ and D_k? Again, (2.1) can be written as

$$s(t) = \sum_{k=0}^{\infty}\left[\left(\frac{A_k - jB_k}{2}\right)\mathrm{e}^{j2\pi k f_r t} + \left(\frac{A_k + jB_k}{2}\right)\mathrm{e}^{-j2\pi k f_r t}\right]$$

As before, define D_k as $D_k \equiv \frac{A_k - jB_k}{2}$ and note that the term $\sum_{k=0}^{\infty}\frac{A_k + jB_k}{2}\mathrm{e}^{-j2\pi k f_r t}$ can be written as

$$\sum_{k=-\infty}^{0}\frac{A_k + jB_{-k}}{2}\mathrm{e}^{j2\pi k f_r t} = \sum_{k=-\infty}^{0}\left(\frac{A_k - jB_k}{2}\right)\mathrm{e}^{j2\pi k f_r t} = \sum_{k=-\infty}^{0} D_k \mathrm{e}^{j2\pi k f_r t}$$

Therefore $s(t) = \sum_{k=-\infty}^{\infty} D_k e^{j2\pi k f_r t}$, where $D_k = \frac{A_k - jB_k}{2}$, $k = \pm 1, \pm 2, \ldots$ and $D_0 = A_0 - jB_0$ (note that B_0 is not necessary equal to zero now).

Equation (2.1) can also be written as:

$$s(t) = \sum_{k=0}^{\infty} \sqrt{A_k^2 + B_k^2} \cos\left(2\pi k f_r t - \tan^{-1}\frac{B_k}{A_k}\right)$$

from which it follows that $C_k \equiv \sqrt{A_k^2 + B_k^2}$ and $\theta_k = -\tan^{-1}\frac{B_k}{A_k}$.

Remarks:

(i) $D_k \neq D_k^*$ if $s(t)$ is complex.

(ii) The amplitude, C_k, and θ_k are in general complex quantities and as such lose their usual physical meanings.

(iii) A complex signal, $s(t)$ would arise not from the physical phenomena but from our analysis procedure. This occurs for instance when we go to an equivalent baseband model.

Consider now that $\left\{D_k \equiv \frac{A_k - jB_k}{2}\right\}$ are given. Then since $D_{-k} = \frac{A_k + jB_k}{2}$, we have, upon adding and subtracting D_k, D_{-k}: $A_k = D_k + D_{-k}$; $B_k = j(D_k - D_{-k})$.

$\{C_k, \theta_k\}$ can be determined from $\{A_k, B_k\}$, which in turn can be determined from D_k as above.

P2.2 (a) $A_k = \frac{2}{T}\int_{t\in T} s(t)\cos(2\pi k f_r t)\mathrm{d}t = \frac{2}{T}\int_{t\in T} s(t)\cos(2\pi(-k) f_r t)\mathrm{d}t = A_{-k}$, i.e., even.

(b) $B_k = \frac{2}{T}\int_{t\in T} s(t)\sin(2\pi k f_r t)\mathrm{d}t = -\frac{2}{T}\int_{t\in T} s(t)\sin(2\pi(-k) f_r t)\mathrm{d}t = -B_{-k}$, i.e., odd.

(c) $C_k = \sqrt{A_k^2 + B_k^2}$; $C_{-k} = \sqrt{(A_{-k})^2 + (B_{-k})^2} = \sqrt{(A_k)^2 + (-B_k)^2} = C_k$, i.e., even (note that squaring an odd real function always gives an even function).

(d) $\theta_k = -\tan^{-1}\left(\frac{B_k}{A_k}\right)$; $\theta_{-k} = -\tan^{-1}\left(\frac{B_{-k}}{A_{-k}}\right) = -\tan^{-1}\left(-\frac{B_k}{A_k}\right) = \tan^{-1}\left(\frac{B_k}{A_k}\right) = -\theta_k$, i.e., odd (note that $\tan^{-1}(\cdot)$ is an odd function).

(e) $D_k = \frac{A_k - jB_k}{2}$; $D_{-k} = \frac{A_{-k} - jB_{-k}}{2} = \frac{A_k + jB_k}{2} = D_k^*$.

Remarks: Properties (a), (b) are true for complex signals as well. But not (c), (d), (e), at least not always.

P2.3 $f_r = \frac{1}{T} = \frac{1}{8}$ (Hz).

$A_1 = 2\mathcal{R}(D_1) = 0$; $B_1 = -2\mathcal{I}(D_1) = -2$.

$A_5 = 2\mathcal{R}(D_5) = 4$; $B_5 = -2\mathcal{I}(D_5) = 0$.

$C_1 = 2|D_1| = 2$; $\theta_1 = \angle D_1 = \frac{\pi}{2}$.

$C_5 = 2|D_5| = 4$; $\theta_5 = 0$.

$\therefore s(t) = -2\sin\left(2\pi\left(\frac{1}{8}t\right)\right) + 4\cos\left(2\pi\left(\frac{5}{8}t\right)\right) = 2\cos\left(2\pi\left(\frac{1}{8}t + \frac{\pi}{2}\right)\right) + 4\cos\left(2\pi\left(\frac{5}{8}t\right)\right)$.

P2.4 The fundamental period, f_r, is $f_r = f_c$ (Hz).

(a) Write $s(t)$ as $s(t) = \underbrace{\frac{V}{2}e^{j\alpha}}_{D_1}\underbrace{e^{j2\pi f_c t}}_{k=1} + \underbrace{\frac{V}{2}e^{-j\alpha}}_{D_{-1}}\underbrace{e^{-j2\pi f_c t}}_{k=-1}$.

(i) $D_1 = \frac{V}{2}\mathrm{e}^{j\alpha}$, $D_{-1} = \frac{V}{2}\mathrm{e}^{-j\alpha} = D_1^*$.

(ii) $\alpha = 0 \Rightarrow D_1 = \frac{V}{2} = D_{-1}$.

(iii) $\alpha = \pi \Rightarrow D_1 = -\frac{V}{2} = D_{-1}$.

(iv) $\alpha = -\pi \Rightarrow$ same as (iii), not much difference between $+\pi$ and $-\pi$.

(v) $\alpha = \frac{\pi}{2} \Rightarrow D_1 = j\frac{V}{2}$, $D_{-1} = -j\frac{V}{2}$.

(vi) $\alpha = -\frac{\pi}{2} \Rightarrow D_1 = -j\frac{V}{2}$, $D_{-1} = j\frac{V}{2}$.

(b) The magnitude spectrum is the same for all the cases, as expected, since only the phase of the sinusoid changes. It looks like:

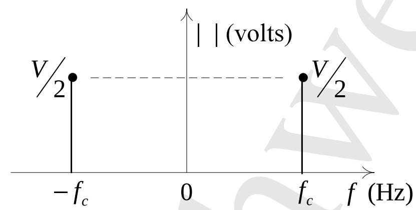


Figure 2.1

The phase spectra of cosines change as shown in Fig. 2.2. For (iii) and (iv), which plot(s) do you prefer and why?

Note: In all cases, the magnitude and phase spectra are even and odd functions, respectively, as they should be.

(c) $s(t) = V\cos\alpha\cos(2\pi f_c t) - V\sin\alpha\sin(2\pi f_c t)$

(i) $A_1 = V\cos\alpha$, $B_1 = -V\sin\alpha$.

(ii) $A_1 = V$, $B_1 = 0$.

(iii) $A_1 = -V$, $B_1 = 0$.

(iv) same as (iii).

(v) $A_1 = 0$, $B_1 = -V$.

(vi) $A_1 = 0$, $B_1 = V$.

For $\{C_k, \theta_k\}$ we have $C_1 = V$ always and $\theta_1 = +\alpha$.

P2.5 $s(t) = 2\sin(2\pi t - 3) + 6\sin(6\pi t)$.

(a) The frequency of the first sinusoid is 1 Hz and that of the second one is $\frac{6\pi}{2\pi} = 3$ Hz. The fundamental frequency is $f_r = \text{GCD}\{1, 3\} = 1$ Hz. Rewrite $\sin(x)$ as $\cos(x - \frac{\pi}{2})$. Then

$$s(t) = \underbrace{2}_{C_1}\cos\left(2\pi t - \underbrace{\left(3 + \frac{\pi}{2}\right)}_{\theta_1}\right) + \underbrace{1}_{C_3}\cos\left(2\pi(3)t - \underbrace{\frac{\pi}{2}}_{\theta_3}\right).$$

Use $|D_k| = \frac{C_k}{2}$, $\angle D_k = \theta_k$, $D_{-k} = D_k^*$ relationships to obtain:

$\therefore D_1 = 1\mathrm{e}^{j(3+\frac{\pi}{2})}$, $D_3 = \frac{1}{2}\mathrm{e}^{j\frac{\pi}{2}}$.

$\therefore D_{-1} = 1\mathrm{e}^{-j(3+\frac{\pi}{2})}$, $D_{-3} = \frac{1}{2}\mathrm{e}^{-j\frac{\pi}{2}}$.

(b) See Fig. 2.3.

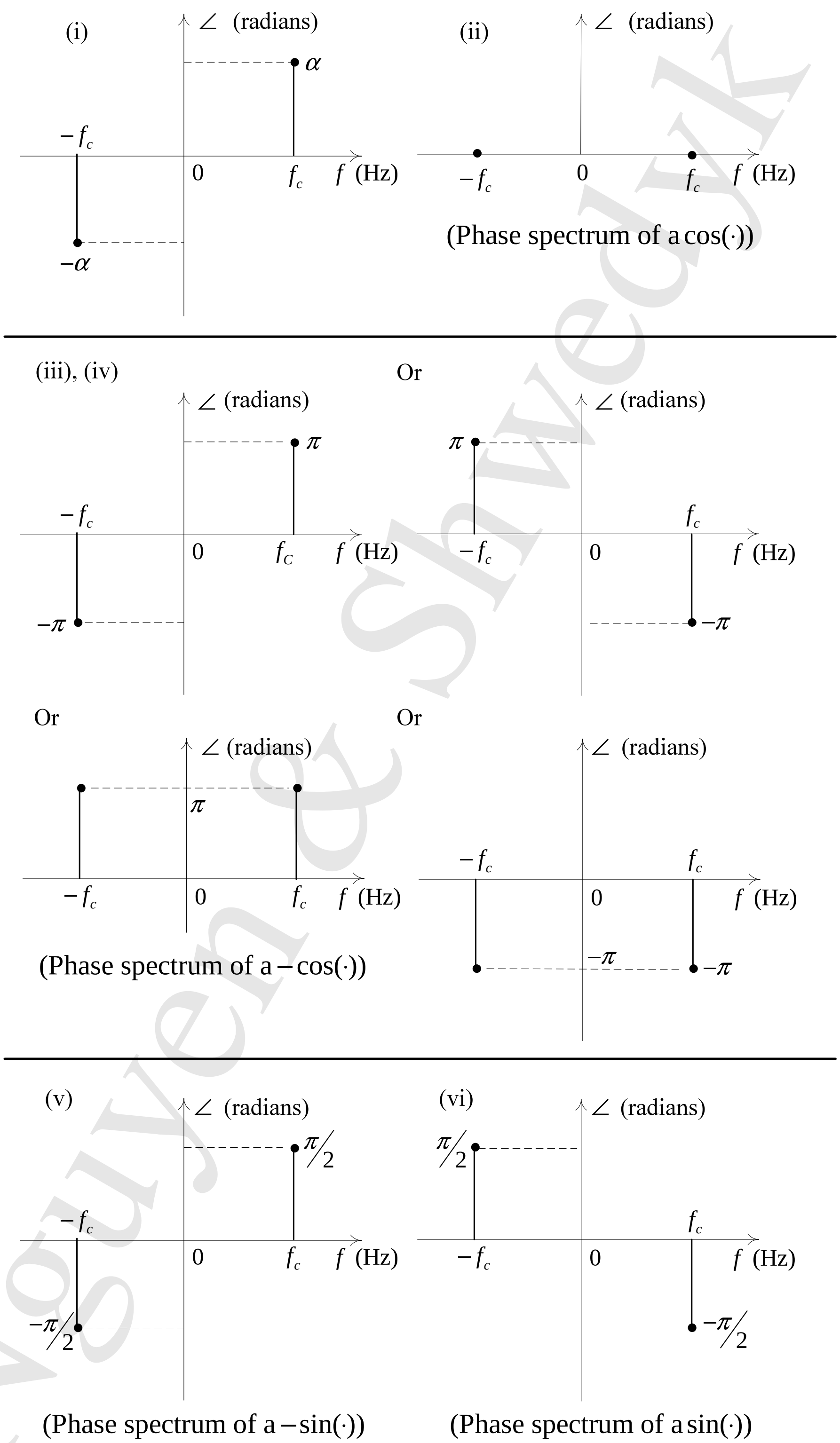


Figure 2.2

P2.6 Classification of signals is as follows:

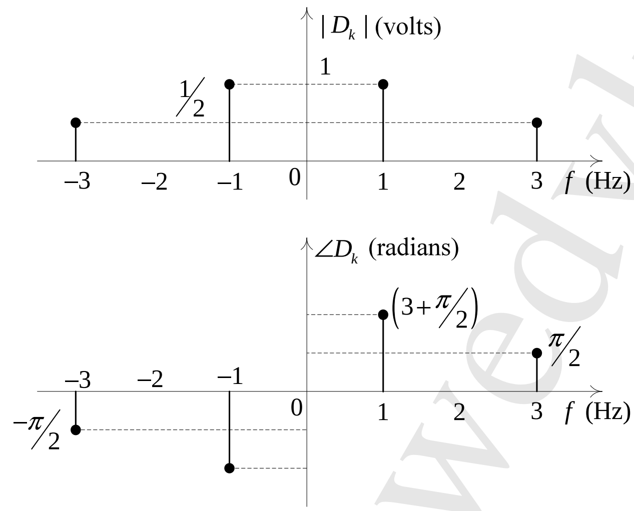


Figure 2.3

(a) It has odd (but not halfwave) symmetry.

(b) Neither even, nor odd but show a halfwave symmetry.

(c) Even symmetry, also halfwave symmetry, therefore even quarterwave symmetry.

(d) (i) odd symmetry, also halfwave $\Rightarrow$ odd quarterwave symmetry.

(ii) neither even, nor odd, nor halfwave.

(iii) even but not halfwave.

(e) Neither even, nor odd, nor halfwave.

(f) Any even signal still remains even since $s'(t) = \text{DC} + s(t)$ and $s'(-t) = \text{DC} + s(-t) = \text{DC} + s(t) = s'(t)$.

Any odd signal is no longer odd. Note that a DC is an even signal.

Any halfwave symmetric signal is no longer halfwave symmetric. Again a DC signal is not halfwave symmetric and this component destroys the halfwave symmetry. Note that any signal with a nonzero DC component cannot be halfwave symmetric.

Regarding the values of the Fourier series coefficients, all that changes is the D_0, C_0 or A_0 value, i.e., the DC component value. All other coefficients are unchanged.

(g) In general the classification changes completely. But for certain specific time shift the classification may remain the same. To see the effect of a time shift on the coefficients consider first

$$D_k^{(\text{shifted})} = \frac{1}{T}\int_{t\in T} s(t-\tau)\mathrm{e}^{-j2\pi k f_r t}\mathrm{d}t \overset{\lambda=t-\tau}{=} \mathrm{e}^{-j2\pi k f_r \tau}\underbrace{\left[\frac{1}{T}\int_{\lambda\in T} s(\lambda)\mathrm{e}^{-j2\pi k f_r \lambda}\mathrm{d}\lambda\right]}_{D_k}$$

$$\therefore D_k^{(\text{shifted})} = \mathrm{e}^{-j2\pi k f_r \tau} D_k$$

$$D_k^{(\text{shifted})} = \frac{A_k^{(\text{shifted})} - jB_k^{(\text{shifted})}}{2} = \left[\cos(2\pi k f_r \tau) - j\sin(2\pi k f_r \tau)\right]\left[\frac{A_k - jB_k}{2}\right]$$

$$\therefore A_k^{(\text{shifted})} = A_k \cos(2\pi k f_r \tau) - B_k \sin(2\pi k f_r \tau).$$
$$\therefore B_k^{(\text{shifted})} = A_k \sin(2\pi k f_r \tau) + B_k \cos(2\pi k f_r \tau).$$

Now
$$D_k^{(\text{shifted})} = \frac{C_k^{(\text{shifted})}}{2} e^{j\theta_k^{(\text{shifted})}} = e^{-j2\pi k f_r \tau} \underbrace{\left\{\frac{C_k}{2} e^{j\theta_k}\right\}}_{D_k}$$

$$\begin{aligned} \therefore C_k^{(\text{shifted})} &= C_k, \\ \therefore \theta_k^{(\text{shifted})} &= \theta_k - j2\pi k f_r \tau. \end{aligned}$$

Basically a time shift leaves the amplitude of the sinusoid unchanged and results in a linear change in the phase.

P2.7 (a) $s(-t) = s_1(-t) + s_2(-t) = s_1(t) + s_2(t) = s(t) \Rightarrow$ even.

(b) $s(-t) = s_1(-t) + s_2(-t) = s_1(t) - s_2(t)$, no symmetry.

(c) $s\left(t \pm \frac{T}{2}\right) = s_1\left(t \pm \frac{T}{2}\right) + s_2\left(t \pm \frac{T}{2}\right) = -s_1(t) - s_2(t) = -s(t)$. Therefore $s(t)$ is halfwave symmetric.

(d) Can't say anything.

(e) Again cannot say anything.

(f) Consider $s\left(t \pm \frac{T}{2}\right) = s_1\left(t \pm \frac{T}{2}\right) + s_2\left(t \pm \frac{T}{2}\right) = -s_1(t) - s_2(t) = -s(t)$. Therefore $s(t)$ is halfwave symmetric.

Now if both $s_1(t)$, $s_2(t)$ are even then $s(t)$ is even and therefore it is even quarterwave symmetric. If both are odd, $s(t)$ is odd and $s(t)$ is odd quarterwave symmetric. However if one is even and the other is odd then $s(t)$ is only halfwave symmetric.

(g) $s(t)$ is even but not halfwave symmetric.

P2.8 $s(t) = s_1(t)s_2(t)$

(a) $s(-t) = s_1(-t)s_2(-t) = s_1(t)s_2(t) = s(t) \Rightarrow$ even.

(b) $s(-t) = s_1(-t)s_2(-t) = -s_1(t)s_2(t) = -s(t) \Rightarrow$ odd.

(c) $s\left(t \pm \frac{T}{2}\right) = s_1\left(t \pm \frac{T}{2}\right) s_2\left(t \pm \frac{T}{2}\right) = [-s_1(t)][-s_2(t)] = s(t)$. Therefore $s(t)$ is not halfwave symmetric. Note that the fundamental period changes to $\frac{T}{2}$.

(d) Neither even nor halfwave symmetric.

(e) Neither odd nor halfwave symmetric.

(f) Have 3 possibilities: (i) $s_1(t)$, $s_2(t)$ are both even quarterwave; (ii) $s_1(t)$, $s_2(t)$ are both odd quarterwave; (iii) one is odd quarterwave, the other is even quarterwave.

(i) From (a) it follows that $s(t)$ is even. From (c) it follows that it is not halfwave symmetric.

(ii) Easy to show that $s(t)$ is even. But $s\left(t \pm \frac{T}{2}\right) = s_1\left(t \pm \frac{T}{2}\right) s_2\left(t \pm \frac{T}{2}\right) = [-s_1(t)][-s_2(t)] = s_1(t)s_2(t) = s(t) \neq -s\left(t \pm \frac{T}{2}\right)$, therefore not halfwave symmetric.

(iii) $s(t)$ is odd (follows from (b)) but again $s\left(t \pm \frac{T}{2}\right) = s(t) \neq -s\left(t \pm \frac{T}{2}\right)$, therefore not halfwave symmetric.

Again note that in each case the fundamental period changes to $\frac{T}{2}$.

(g) $s(t)$ is even. Is it halfwave symmetric?

$$s\left(t \pm \frac{T}{2}\right) = s_1\left(t \pm \frac{T}{2}\right) s_2\left(t \pm \frac{T}{2}\right) = -s_1\left(t \pm \frac{T}{2}\right) s_2(t) \neq -s(t)$$

No.

P2.9 (a) Check for 2 conditions: that the magnitude is even and the phase is odd. If both are satisfied then the signal is real. If one or the other or neither condition is satisfied then the signal is complex.

(i) real, (ii) real (note the phase of π is the same as $-\pi$), (iii) complex (phase spectrum is not odd).

(b) If a harmonic frequency has a phase of π (or 0) then there is only a $\cos(\cdot)$ term at this frequency. Similarly if the phase is $\frac{\pi}{2}$ (or $\frac{3\pi}{2}$) then there is only a $\sin(\cdot)$ term at that frequency. Finally if the signal is even then it only has $\cos(\cdot)$ terms in the Fourier series, if odd then only $\sin(\cdot)$ terms.

Therefore the signal in (ii) is even (and real). No signal is odd.

P2.10 The two spectra are

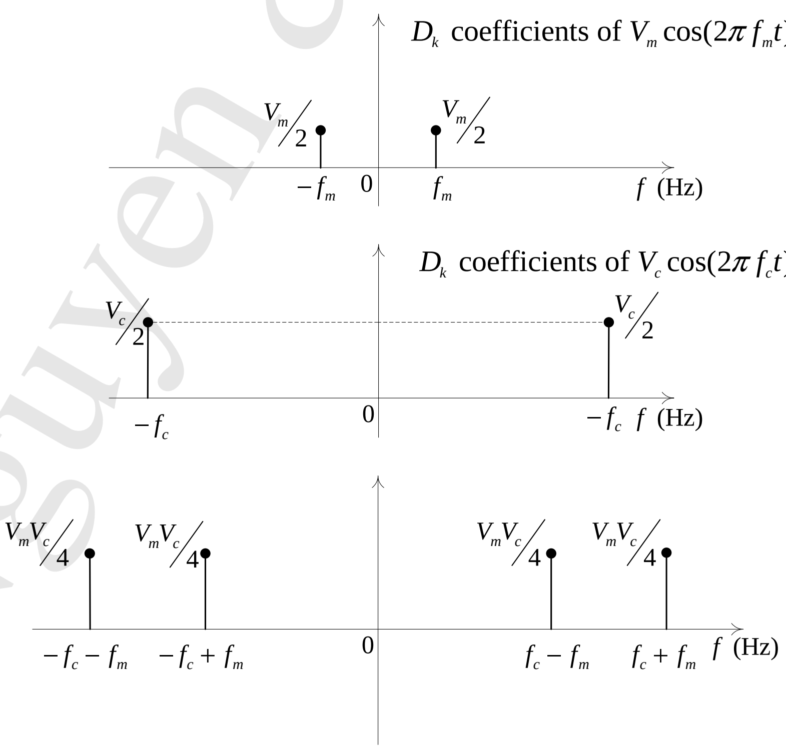


Figure 2.4

P2.11 The difference between two modulations of P2.1 and P2.2 is that the one in P2.2 has a spectral component at f_c or equivalently the modulating signal, $s_1(t) = V_{\text{DC}} + V_m \cos(2\pi f_m t)$, has a spectral component at $f = 0$ (i.e., DC component).

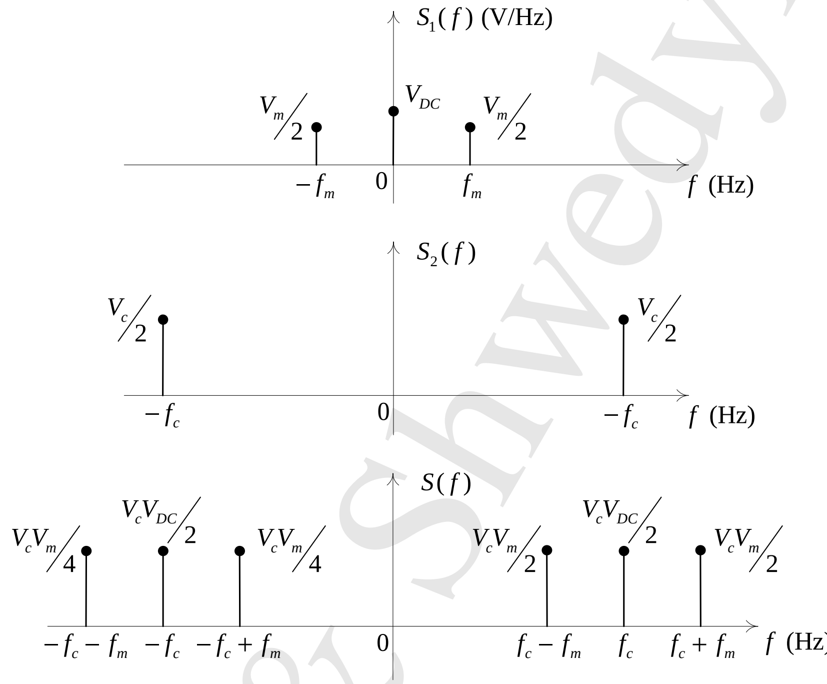


Figure 2.5

Remarks:

(i) One can use the frequency shift property to determine the spectrum of $s(t)$.

(ii) The phases of all the spectra are zero, a consequence of only cosines being involved, both as carrier and message signals. A good exercise is to change some (or all) of the signals to sines and redo the spectra.

(iii) The simplest way to find the spectra is to use some high-school trig, namely $\cos(x)\cos(y) = \frac{\cos(x+y)+\cos(x-y)}{2}$. No need to know convolution, frequency shift property, etc.

P2.12 (a) $f_i(t) = \frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}t}\left[2\pi f_c t - V_m \sin(2\pi f_m t)\right] = f_c - V_m f_m \cos(2\pi f_m t)$ (Hz).

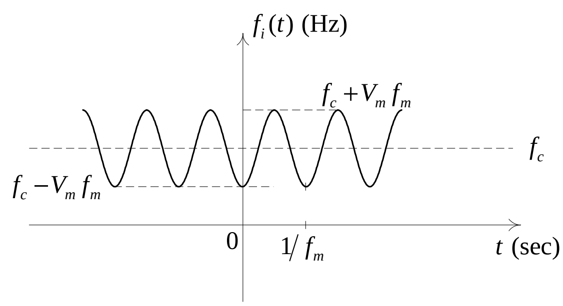


Figure 2.6

(b) Consider

$$s\left(t+\frac{k}{f_m}\right) = V_c\cos\left[2\pi f_c\left(t+\frac{k}{f_m}\right)-V_m\sin\left(2\pi f_m\left(t+\frac{k}{f_m}\right)\right)\right]$$
$$= V_c\cos\left[2\pi f_c t+2\pi k\underbrace{\frac{f_c}{f_m}}_{n}-V_m\sin\left(2\pi f_m t+2\pi k\right)\right] = s(t)$$

Choose $k=1$, i.e., $s\left(t+\frac{1}{f_m}\right)=s(t) \Rightarrow$ periodic with period $T=\frac{1}{f_m}$ (sec).

(c) Consider $s_1(t)=\mathrm{e}^{j[2\pi f_c t-V_m\sin(2\pi f_m t)]}$

$$s_1\left(t+\frac{k}{f_m}\right)=\mathrm{e}^{j[2\pi f_c t+kn2\pi-V_m\sin(2\pi f_m t+k2\pi)]}=\underbrace{\mathrm{e}^{j2\pi kn}}_{1}\underbrace{\mathrm{e}^{j[2\pi f_c t-V_m\sin(2\pi f_m t)]}}_{s_1(t)}$$

Therefore $s_1(t)$ is periodic with period $\frac{1}{f_m}$ (sec). Its Fourier coefficients are determined as follows:

$$D_k=\frac{1}{T}\int_{-\frac{T}{2}=-\frac{1}{2f_m}}^{\frac{T}{2}=\frac{1}{2f_m}}\mathrm{e}^{j[2\pi f_c t-V_m\sin(2\pi f_m t)]}\mathrm{e}^{-j2\pi k f_m t}\mathrm{d}t$$

Change variable to $\lambda=2\pi f_m T$

$$D_k=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{e}^{j[(n-k)\lambda-V_m\sin\lambda]}\mathrm{d}\lambda=J_{n-k}(V_m),$$

where by definition $J_n(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{e}^{-j[n\lambda-x\sin\lambda]}\mathrm{d}\lambda$ is nth order Bessel function.

Remarks: The spectrum depends on V_m, the amplitude of the modulating signals. In fact V_m determines the maximum instantaneous frequency (see plot in (a)) which implies that the larger V_m is, the wider the spectrum bandwidth becomes. Of course in theory there are spectral components out to infinity but in practise a finite bandwidth would capture most of the power of the transmitted signal. The restriction that f_c is an integer multiple of f_m is easily removed but for this the Fourier transform is needed since the signal is no longer periodic in general.

P2.13 (a)

$$s_1(t) = 1+\sqrt{2}\mathrm{e}^{j\frac{\pi}{4}}\mathrm{e}^{j2\pi t}+\sqrt{2}\mathrm{e}^{-j\frac{\pi}{4}}\mathrm{e}^{-j2\pi t}+5\mathrm{e}^{j2.21}\mathrm{e}^{j2\pi(2t)}+5\mathrm{e}^{-j2.21}\mathrm{e}^{-j2\pi(2t)}$$
$$+5\mathrm{e}^{-j2.21}\mathrm{e}^{j2\pi(3t)}+5\mathrm{e}^{j2.21}\mathrm{e}^{-j2\pi(3t)}$$
$$= 1+2\sqrt{2}\cos\left(2\pi t+\frac{\pi}{4}\right)+10\cos\left(4\pi t+2.21\right)+10\cos\left(6\pi t+2.21\right)$$

where f_r is taken to be 1 Hz.

$s_2(t) = 2\cos\left(2\pi t + \frac{\pi}{4}\right)$

$$
\begin{aligned}
s_1(t)s_2(t) &= 2\cos\left(2\pi t + \frac{\pi}{4}\right) + 2\sqrt{2}\cos^2\left(2\pi t + \frac{\pi}{4}\right) \\
&\quad +20\cos(4\pi t + 2.21)\cos\left(2\pi t + \frac{\pi}{4}\right) \\
&\quad +20\cos(6\pi t - 2.21)\cos\left(2\pi t + \frac{\pi}{4}\right) \\
&= 2\cos\left(2\pi t + \frac{\pi}{4}\right) \\
&\quad +\sqrt{2}\left(1 + \cos\left(4\pi t + \frac{\pi}{2}\right)\right) + 10\cos\left(2\pi t + \frac{\pi}{4} - 2.21\right) \\
&\quad +10\cos\left(6\pi t + \frac{\pi}{4} + 2.21\right) + 10\cos\left(4\pi t - 2.21 - \frac{\pi}{4}\right) \\
&\quad +10\cos\left(8\pi t - 2.21 + \frac{\pi}{4}\right) \\
&= \sqrt{2} + 2\cos\left(2\pi t + \frac{\pi}{4}\right) \\
&\quad +10\cos\left(2\pi t + \frac{\pi}{4} - 2.21\right) + \sqrt{2}\cos\left(4\pi t + \frac{\pi}{2}\right) \\
&\quad +10\cos\left(4\pi t - \frac{\pi}{4} - 2.21\right) + 10\cos\left(6\pi t + 2.21 + \frac{\pi}{4}\right) \\
&\quad +10\cos\left(8\pi t - 2.21 + \frac{\pi}{4}\right)
\end{aligned}
$$

$D_0 = \sqrt{2}$; $D_1 = \mathrm{e}^{j\frac{\pi}{4}} + 5\mathrm{e}^{j\left(\frac{\pi}{4} - 2.21\right)}$; $D_2 = \frac{\sqrt{2}}{2}\mathrm{e}^{j\frac{\pi}{2}} + 5\mathrm{e}^{-j\left(\frac{\pi}{4} + 2.21\right)}$;
$D_3 = 5\mathrm{e}^{j\left(\frac{\pi}{4} + 2.21\right)}$; $D_4 = 5\mathrm{e}^{-j\left(\frac{\pi}{4} - 2.21\right)}$; $D_{-k} = D_k^*$

P2.14 The signal $s_1(t)$ is shown below:

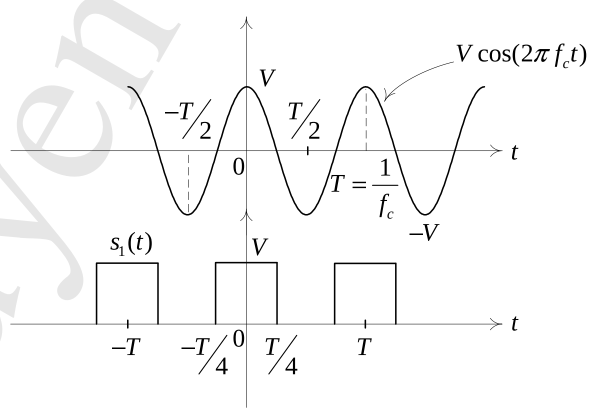


Figure 2.7

The Fourier coefficients of $s_1(t)$ are given by

$$D_k^{[s_1(t)]} = \frac{1}{T}\int_{-\frac{T}{4}}^{\frac{T}{4}} \mathrm{e}^{-j2\pi\frac{k}{T}t}\mathrm{d}t = \frac{\sin\left(\frac{\pi k}{2}\right)}{\pi k}$$

$$D_k^{[V\cos(2\pi f_c t)]} = \begin{cases} \frac{V}{2}, & k = \pm 1 \\ 0, & k \text{ otherwise} \end{cases} = \frac{V}{2}\delta_D(k-1) + \frac{V}{2}\delta_D(k+1).$$

where $\delta_D(\cdot)$ is interpreted as a "discrete" delta function, i.e.,

$$\delta_D(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

$$\begin{aligned}
\therefore D_k^{[s(t)]} &= D_k^{[s_1(t)]} * D_k^{[V\cos(2\pi f_c t)]} \\
&= \sum_{n=-\infty}^{\infty} \frac{\sin\pi(\frac{k-n}{2})}{\pi(k-n)} \left[\frac{V}{2}\delta_D(n-1) + \frac{V}{2}\delta_D(n+1)\right] \\
&= \frac{V}{2}\frac{\sin\left(\pi(\frac{k-1}{2})\right)}{\pi(k-1)} + \frac{V}{2}\frac{\sin\left(\pi(\frac{k+1}{2})\right)}{\pi(k+1)}
\end{aligned}$$

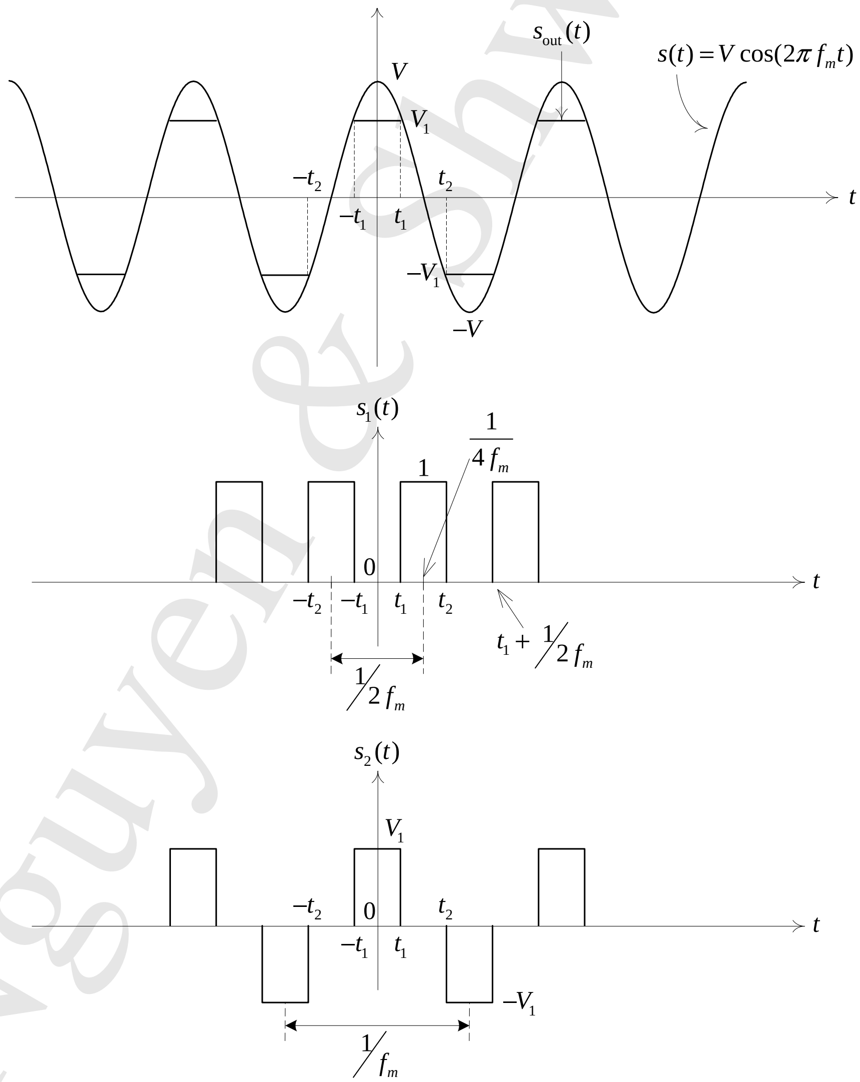


Figure 2.8

P2.15 It is obvious that we are interested in the case $V_1 < V$, otherwise there is no saturation (see Figure 2.8).

The time t_1 is given by: $V\cos(2\pi f_m t_1) = V_1 \Rightarrow t_1 = \frac{1}{2\pi f_m}\cos^{-1}\left(\frac{V_1}{V}\right)$.

The time t_2 is given by $t_2 = \frac{1}{4f_m} + \left(\frac{1}{4f_m} - t_1\right) = \frac{1}{2f_m} - \frac{1}{2\pi f_m}\cos^{-1}\left(\frac{V_1}{V}\right)$.

Lastly $t_2 - t_1 = \frac{1}{2f_m} - \frac{1}{\pi f_m}\cos^{-1}\left(\frac{V_1}{V}\right) \equiv \Delta t$.

We are now in a position to draw the waveforms $s_1(t)$, $s_2(t)$ as in Figure 2.9. Note that $s_1(t)$ is periodic with period $\frac{1}{2f_m}$ and has a DC component equal to $\frac{t_2-t_1}{T}$. On the other hand $s_2(t)$ is periodic with period $\frac{1}{f_m}$.

Now we determine the Fourier coefficients for $s_1(t)$, $s_2(t)$. The basic pulse shapes for $s_1(t)$ (over time interval $-\frac{1}{4f_m}$ to $\frac{1}{4f_m}$) and $s_2(t)$ (over time interval $-\frac{1}{2f_m}$ to $\frac{1}{2f_m}$) are shown in Fig. 2.9.

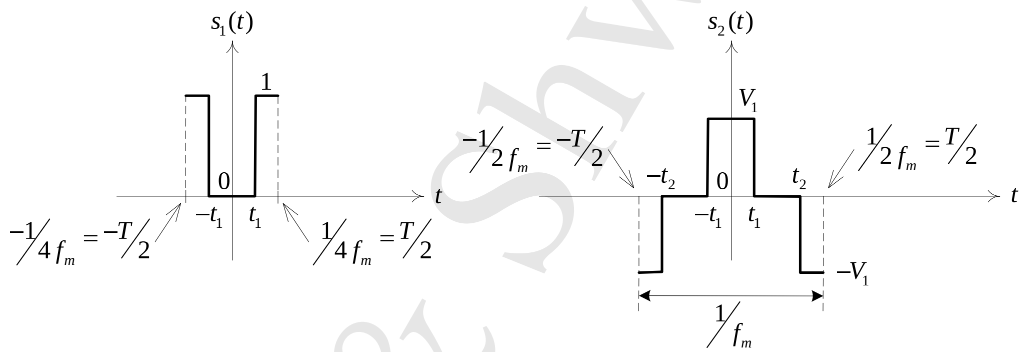


Figure 2.9

Fourier coefficients of $s_1(t)$:

$$\begin{aligned}
D_k^{[s_1(t)]} &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} s_1(t)\mathrm{e}^{-j2\pi\frac{k}{T}t}\mathrm{d}t, \text{ where } T = \frac{1}{2f_m} \\
&= \frac{1}{T}\left[\int_{-\frac{T}{2}}^{t_1} \mathrm{e}^{-j2\pi\frac{k}{T}t}\mathrm{d}t + \int_{t_1}^{\frac{T}{2}} \mathrm{e}^{-j2\pi\frac{k}{T}t}\mathrm{d}t\right] \\
&= \frac{\sin(\pi k) - \sin\left(\frac{\pi k t_1}{T}\right)}{\pi k} = \frac{-\sin\left(k\cos^{-1}\left(\frac{V_1}{V}\right)\right)}{\pi k} \\
&\quad \text{where } \frac{t_1}{T} = 2f_m t_1 = \frac{1}{\pi}\cos^{-1}\left(\frac{V_1}{V}\right),\ k = \pm 1, \pm 2, \ldots, \\
D_0^{[s_1(t)]} &= \frac{t_2 - t_1}{T}.
\end{aligned}$$

Fourier coefficients of $s_2(t)$:

$$\begin{aligned}
D_k^{[s_2(t)]} &= \frac{1}{T}\left[\int_{-\frac{T}{2}}^{-t_2} -V_1\mathrm{e}^{-j2\pi\frac{k}{T}t}\mathrm{d}t + \int_{-t_1}^{t_1} V_1\mathrm{e}^{-j2\pi\frac{k}{T}t}\mathrm{d}t + \int_{t_2}^{\frac{T}{2}} -V_1\mathrm{e}^{-j2\pi\frac{k}{T}t}\mathrm{d}t\right] \\
&= \frac{V_1}{\pi k}\left[\sin\left(\frac{2\pi k t_2}{T}\right) + \sin\left(\frac{2\pi k t_1}{T}\right)\right], \text{ where } T = \frac{1}{f_m}.
\end{aligned}$$

Now $\frac{t_1}{T} = \frac{1}{2\pi}\cos^{-1}\left(\frac{V_1}{V}\right)$ and $\frac{t_2}{T} = \frac{1}{2} - \frac{1}{2\pi}\cos^{-1}\left(\frac{V_1}{V}\right)$. Therefore

$$D_k^{[s_2(t)]} = [1-\cos(\pi k)]\sin\left(k\cos^{-1}\left(\frac{V_1}{V}\right)\right).$$

Consider now the coefficients of $s_1(t)s(t)$ which is a convolution of the Fourier coefficients of $s_1(t)$ and $s(t)$. The Fourier coefficients of $s(t)$ are simply $D_1 = \frac{V}{2}$ ($k=1$ or $f=f_m$) and $D_{-1} = \frac{V}{2}$ ($k=-1$ or $f=-f_m$). All other coefficients are zero.

Graphically:

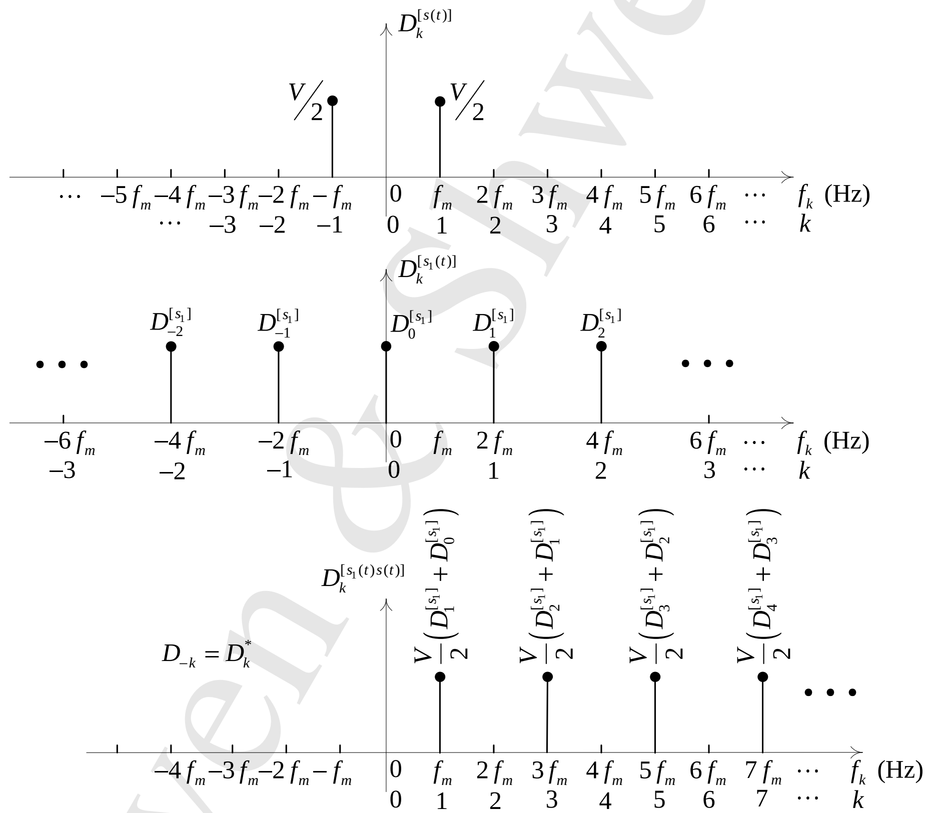


Figure 2.10

Convolve the Fourier series of $s_1(t), s(t)$ graphically, namely flip $D_k^{[s(t)]}$ around the vertical axis, slide it along the horizontal axis, multiply by $D_k^{[s_1(t)]}$ and sum the product. The plot looks as in Fig. 2.10.

$$\begin{aligned} D_k^{[s_1(t)s(t)]} &= \begin{cases} D_{2k-1}^{[s_1(t)]} + D_{2k-2}^{[s_1(t)]}, & k=1,3,5,\ldots \\ 0, & k=0,2,4,\ldots \end{cases}, \\ D_{-k}^{[s_1(t)s(t)]} &= \left\{D_k^{[s_1(t)s(t)]}\right\}^*. \end{aligned}$$

Finally, $D_k^{[s_{\text{out}}(t)]} = D_k^{[s_1(t)s(t)]} + D_k^{[s_2(t)]}$.

P2.16 (a)

$$\begin{aligned} S(f) &= V\int_{-\beta}^{\beta} \mathrm{e}^{-j2\pi f t}\mathrm{d}t = V\left[\frac{\mathrm{e}^{j2\pi f\beta}-\mathrm{e}^{-j2\pi f\beta}}{j2\pi f}\right] \\ &= V\frac{\sin(2\pi f\beta)}{\pi f} = 2V\beta\frac{\sin(2\pi f\beta)}{2\pi f\beta} \end{aligned}$$

(b) $D_k^{[\alpha T]} = \frac{1}{\alpha T}\frac{2V\beta\sin(2\pi f\beta)}{2\pi f\beta}\bigg|_{f=kf_r=k/\alpha T} = \frac{1}{\alpha T}2V\beta\frac{\sin\left(2\pi\frac{k\beta}{\alpha T}\right)}{\left(2\pi\frac{k\beta}{\alpha T}\right)}$

(c) Plots of $D_k^{[\alpha T]} = \frac{1}{2\alpha}\frac{\sin\left(\frac{\pi k}{2\alpha}\right)}{\left(\frac{\pi k}{2\alpha}\right)}$ are shown in Fig. 2.11 for $\alpha = [1, 1.5, 2, 3, 5]$ (we set $V = \beta = 1$ and $T = 4$).

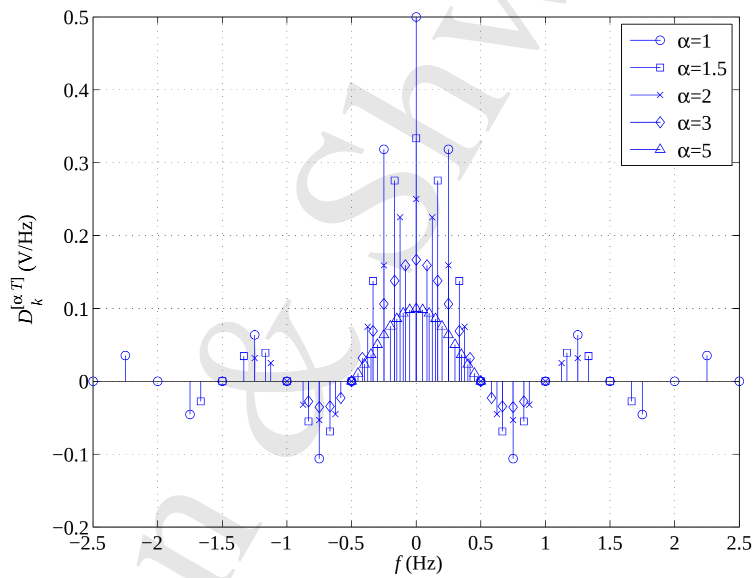


Figure 2.11

(d) As $\alpha \to \infty$, the amplitude (magnitude) of $D_k^{[\alpha T]}$ goes to zero; however the "envelope" is always a sinc(·), i.e., that of $S(f)$. Note also that the frequency spacing between the frequency components, Δf is proportional to $\frac{1}{\alpha}$ and it goes to zero as $\alpha \to \infty$.

P2.17 The basic difference in the magnitude/phase plots of the Fourier series representation and the Fourier transform representation is that the Fourier transform magnitude plot will have impulses at the fundamental (and harmonic) frequency(ies) of strength $= |D_k|$. This reflects the fact that the signal is periodic and that we have finite power concentrated in a "zero" interval. The phase spectrum is the same.

P2.18 (a) (i) $S(f) = \frac{V_1}{2}\left[\delta(f-\sqrt{2})+\delta(f+\sqrt{2})\right] + \frac{V_2}{2}\left[\delta(f-2)+\delta(f+2)\right]$ (V/Hz)

(ii) $S(f) = \frac{V_1}{2}\left[\delta(f-\sqrt{2})+\delta(f+\sqrt{2})\right] + \frac{V_2}{2}\left[\delta(f-2\sqrt{2})+\delta(f+2\sqrt{2})\right]$

(b) (i) Not periodic. The ratio of the two frequencies of the 2 sinusoids, $\sqrt{2}$ and 2 Hz, is not a rational number.

(ii) Periodic, since the ratio of the 2 frequencies is a rational number, $\frac{2\sqrt{2}}{\sqrt{2}} = 2$. The spectra look as follows:

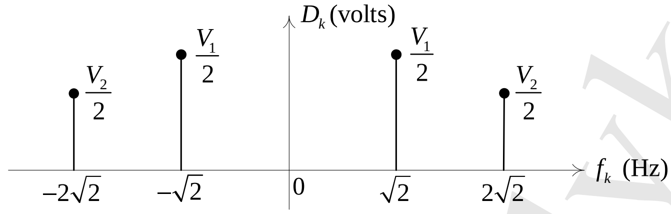


Figure 2.12

P2.19 (a) $S(f) = \int_{-\infty}^{\infty} m(t) \left[\frac{e^{j2\pi f_c t} - e^{-j2\pi f_c t}}{2}\right] e^{-j2\pi f t} dt = \frac{M(f-f_c)+M(f+f_c)}{2}$

(b) See Figure 2.13.

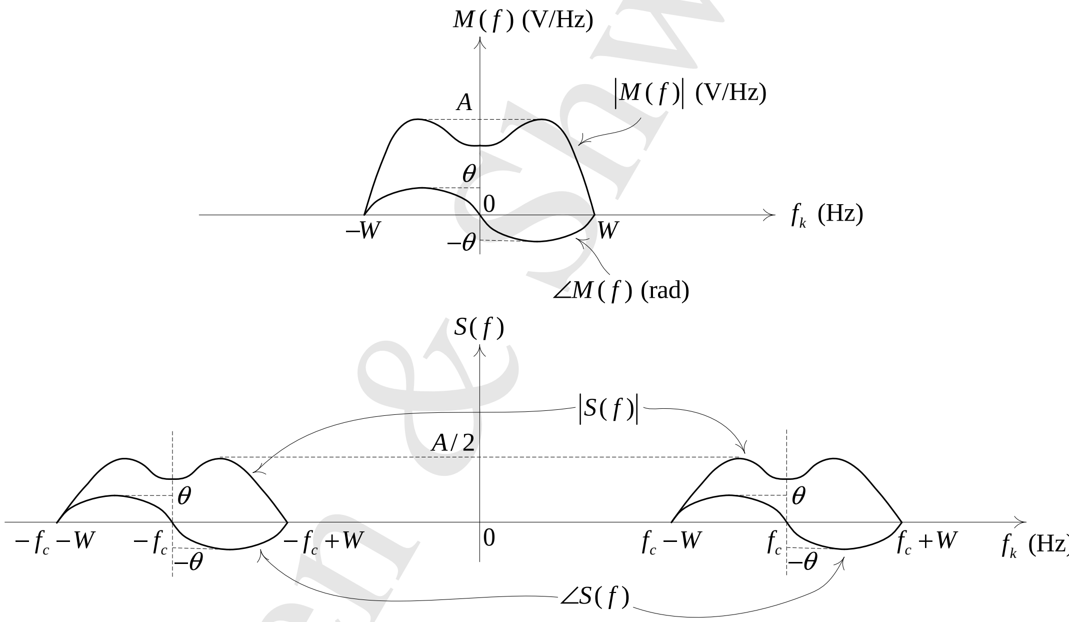


Figure 2.13

P2.20 First find the Fourier series of $e^{-jV_m \sin(2\pi f_m t)}$:

$$
\begin{aligned}
D_k &= \frac{1}{T}\int_0^T e^{-jV_m \sin(2\pi f_m t)} e^{-j2\pi k f_m t} dt \quad \text{where } T = \frac{1}{f_m} \\
&\overset{\lambda = 2\pi f_m t}{=} \frac{1}{T}\int_0^{2\pi f_m T} e^{-jV_m \sin\lambda} e^{-jk\lambda} \frac{d\lambda}{2\pi f_m} \\
&\overset{f_m T = 1}{=} \frac{1}{2\pi}\int_0^{2\pi} e^{-j(k\lambda - V_m \sin\lambda)} d\lambda \\
&= J_k(V_m) \text{ (kth order Bessel function).}
\end{aligned} \tag{2.2}
$$

Then the Fourier transform is: $\sum_{k=-\infty}^{\infty} J_k(V_m)\delta(f - kf_m)$. Multiplying by $e^{j2\pi f_c t}$ shifts the above spectrum by f_c, i.e.,

$$
s(t) \longleftrightarrow S(f) = V_c \sum_{k=-\infty}^{\infty} J_k(V_m)\delta(f - f_c - kf_m).
$$

P2.21 Differentiate the 1st signal twice:

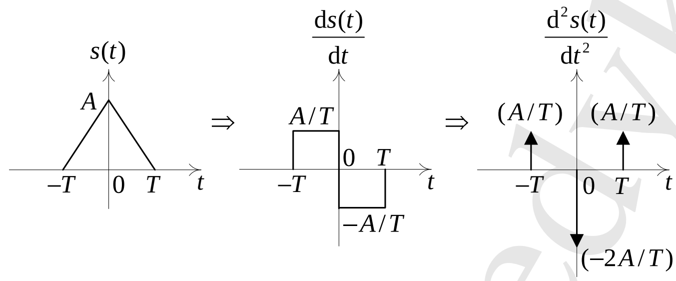


Figure 2.14

Then

$$\mathcal{F}\left\{\frac{\mathrm{d}^2 s(t)}{\mathrm{d}t^2}\right\} = \frac{A}{T}\mathrm{e}^{j2\pi fT} - \frac{2A}{T} + \frac{A}{T}\mathrm{e}^{-j2\pi fT} = \frac{2A}{T}\left[\cos(2\pi fT) - 1\right]$$

$$S(f) = \frac{\mathcal{F}\left\{\frac{\mathrm{d}^2 s(t)}{\mathrm{d}t^2}\right\}}{(j2\pi f)^2} = \frac{A}{T}\frac{1-\cos(2\pi fT)}{2\pi^2 f^2}\ (\mathrm{V/Hz}).$$

Consider now the 2nd signal. The 1st derivation looks like

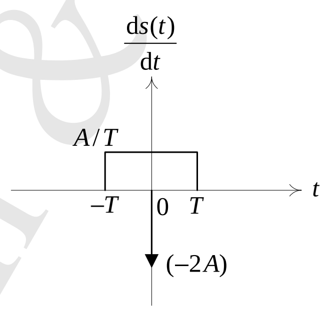


Figure 2.15

We could differentiate the rectangular pulse again but we have done it so many times that its transform is almost engrained in our minds.

$$\begin{aligned}
\mathcal{F}\left\{\frac{\mathrm{d}s(t)}{\mathrm{d}t}\right\} &= -2A + \frac{A\sin(2\pi fT)}{\pi fT} \\
\mathcal{F}\left\{s(t)\right\} &= \frac{A}{j2\pi f}\left[-2 + \frac{\sin(2\pi fT)}{\pi fT}\right].
\end{aligned}$$

P2.22 For reference the Fourier transform of a rectangular pulse, centered at $t = 0$, width $2T$, height A, is $AT\frac{\sin(2\pi fT)}{2\pi fT}$.

Now decompose $s(t)$ into the sum of rectangular pulses, $s_1(t)$ and $s_2(t)$ (note that this decomposition is not unique):

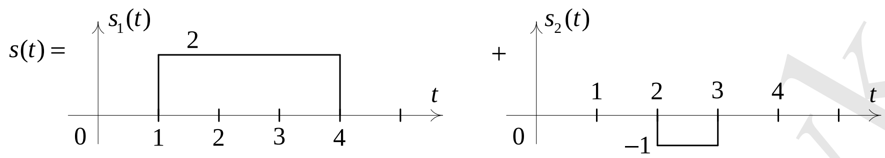


Figure 2.16

$s_1(t)$ is a shifted version of a rectangular pulse of width $2T = 3$ and height $A = 2$; shifted by 2.5 seconds. Therefore,

$$\mathcal{F}\left\{s_1(t)\right\} = \underbrace{\mathrm{e}^{-j2\pi(2.5)f}}_{\text{due to shifting}} 2(3/2)\frac{\sin(2\pi f(3/2))}{2\pi f(3/2)} = 3\mathrm{e}^{-j5\pi f}\frac{\sin(3\pi f)}{3\pi f}.$$

$s_2(t)$ is a rectangular pulse of width $2T = 1$, height $A = -1$; shift of 2.5 seconds. Thus,

$$\mathcal{F}\left\{s_1(t)\right\} = -\frac{1}{2}\mathrm{e}^{-j5\pi f}\frac{\sin(\pi f)}{\pi f}.$$

Finally,

$$\mathcal{F}\left\{s(t)\right\} = \mathcal{F}\left\{s_1(t)\right\} + \mathcal{F}\left\{s_2(t)\right\} = 3\mathrm{e}^{-j5\pi f}\frac{\sin(3\pi f)}{3\pi f} - \frac{1}{2}\mathrm{e}^{-j5\pi f}\frac{\sin(\pi f)}{\pi f}$$

P2.23 Consider the 1st figure.

$$S(f) = 1\mathrm{e}^{j\frac{\pi}{2}}[u(f+2) - u(0)] + 1\mathrm{e}^{-j\frac{\pi}{2}}[u(0) - u(f-2)]$$

$$s(t) = \int_{-2}^{0} 1\mathrm{e}^{j\frac{\pi}{2}}\mathrm{e}^{j2\pi ft}\mathrm{d}f + \int_{0}^{2} 1\mathrm{e}^{-j\frac{\pi}{2}}\mathrm{e}^{j2\pi ft}\mathrm{d}f = \frac{1-\cos(4\pi t)}{\pi t}$$

The spectrum in the 2nd figure is: $S(f) = 1\mathrm{e}^{-j\frac{\pi}{4}f}\,[u(f+2) - u(f-2)]$.

$$\therefore \quad s(t) = \int_{-2}^{2} \mathrm{e}^{-j\frac{\pi}{4}f}\mathrm{e}^{j2\pi ft}\mathrm{d}f = \frac{-2\cos(4\pi t)}{2\pi t - \frac{\pi}{4}}$$

Remark: The problem is a small illustration that the phase is also important in determining the shape of a signal. An interesting question is: Is the ear more sensitive to phase or magnitude distortions in an audio signal? Is the eye more sensitive to phase or magnitude distortions in a image? For this one needs to resort to Matlab and experiment since we are now in the realm of psychophysics.

P2.24 $s_{\cos}(t) = A\cos(2\pi f_m t)$ where $f_m = 1/2T \Rightarrow S_{\cos}(f) = (A/2)\left[\delta(f-f_m) + \delta(f+f_m)\right]$.
$s_{\text{rect}}(t) = u\left(t+\frac{T}{2}\right) - u\left(t-\frac{T}{2}\right) \Rightarrow S_{\text{rect}}(f) = \frac{T\sin(\pi fT)}{\pi fT}$.

$$\begin{aligned} S(f) = S_{\cos}(f) * S_{\text{rect}}(f) &= \frac{AT}{2}\int_{-\infty}^{\infty}\left[\delta(\lambda - f_m) + \delta(\lambda + f_m)\right]\frac{\sin(\pi(f-\lambda)T)}{\pi(f-\lambda)T}\mathrm{d}\lambda \\ &= \frac{2AT}{\pi}\frac{\cos(\pi fT)}{1-4(\pi fT)^2}. \end{aligned}$$

P2.25 A rectangular signal of width W (sec) and amplitude A (volts), i.e., $s_{\text{rect}}(t) = A\left[u\left(t+\frac{W}{2}\right] - u\left(t-\frac{W}{2}\right)\right]$ has Fourier transform $AW\frac{\sin(\pi f W)}{\pi f W}$.

(a) The signal can be decomposed into 2 shifted rectangular signals; each of width T seconds; amplitude $\pm A$ volts; one shifted $T/2$ seconds to the left, the other $T/2$ seconds to the right. Now $\mathcal{F}\{s(t-\tau)\} = \mathrm{e}^{-j2\pi f\tau}s(t)$. Here $\tau = +T/2$ (or $-T/2$). Also $W = T$.

$$\begin{aligned}\therefore S(f) &= AT\mathrm{e}^{-j2\pi f(T/2)}\frac{\sin(\pi f T)}{\pi f T} + AT\mathrm{e}^{-j2\pi f(-T/2)}\frac{\sin(\pi f T)}{\pi f T} \\ &= 2jAT\frac{\sin^2(\pi f T)}{\pi f T} \text{ (V/Hz).}\end{aligned}$$

(b) The derivative of $s(t)$ results in 3 impulses, i.e.,

$$\frac{\mathrm{d}s(t)}{\mathrm{d}t} = A\delta(t+T) - 2A\delta(t) + A\delta(t-T)$$

$$\therefore \quad \mathcal{F}\left\{\frac{\mathrm{d}s(t)}{\mathrm{d}t}\right\} = A\mathrm{e}^{j2\pi f T} - 2A + A\mathrm{e}^{-j2\pi f T}.$$

$$\therefore \quad S(f) = \frac{\mathcal{F}\left\{\frac{\mathrm{d}s(t)}{\mathrm{d}t}\right\}}{j2\pi f} = 2A\frac{\cos(2\pi f T) - 1}{j2\pi f} = 2jAT\frac{\sin^2(\pi f T)}{\pi f T} \text{ (V/Hz).}$$

P2.26 (a)

$$\begin{aligned}X(f) &= \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-j2\pi f t}\mathrm{d}t \\ X^*(f) &= \left[\int_{-\infty}^{\infty} x(t)\mathrm{e}^{-j2\pi f t}\mathrm{d}t\right]^* = \int_{-\infty}^{\infty} x^*(t)\left(\mathrm{e}^{-j2\pi f t}\right)^*\mathrm{d}t^* \\ &= \int_{-\infty}^{\infty} x(t)\mathrm{e}^{j2\pi f t}\mathrm{d}t = \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-j2\pi(-f)t}\mathrm{d}t = X(-f),\end{aligned}$$

where the following were used: $x(t)$ is real as is $\mathrm{d}t$; the complex conjugate of a sum equals the sum of complex conjugates (and integration for engineers is essentially a summation operation); the complex conjugate of a product equals the product of complex conjugates.

(b) Now

$$\begin{aligned}\int_{-\infty}^{\infty} x(t)y(t)\mathrm{d}t &= \int_{t=-\infty}^{\infty}\mathrm{d}t x(t)\int_{f=-\infty}^{\infty} Y(f)\mathrm{e}^{j2\pi f t}\mathrm{d}f \\ &= \int_{f=-\infty}^{\infty}\mathrm{d}f Y(f)\underbrace{\int_{t=-\infty}^{\infty} x(t)\mathrm{e}^{-j2\pi(-f)t}\mathrm{d}t}_{=X(-f)=X^*(f)} = \int_{f=-\infty}^{\infty} X^*(f)Y(f)\mathrm{d}f.\end{aligned}$$

And interchanging the roles of $x(t)$, $y(t)$ we have

$$\int_{f=-\infty}^{\infty} X^*(f)Y(f)\mathrm{d}f = \int_{f=-\infty}^{\infty} X(f)Y^*(f)\mathrm{d}f.$$

If $x(t) = y(t)$, we get

$$\underbrace{\int_{-\infty}^{\infty} x^2(t)\mathrm{d}t}_{\text{This has units of (volts)}^2\cdot\text{sec=joules}} = \int_{-\infty}^{\infty} \underbrace{|X(f)|^2}_{\text{which means this has units of joules/Hz}} \mathrm{d}f$$

or $|X(f)|^2$ is an *energy density spectrum*, i.e., it tells us how the energy of $x(t)$ is distributed in the frequency domain.

P2.27 (a)

$$E_s = \int_0^{\infty} \mathrm{e}^{-2at}\mathrm{d}t = \frac{1}{2a} \text{ (joules).}$$

(b)

$$E_s = \int_{-\infty}^{\infty} |X(f)|^2 \,\mathrm{d}f = \int_{-\infty}^{\infty} \frac{\mathrm{d}f}{a^2 + 4\pi^2 f^2} = \frac{1}{\pi a}\tan^{-1}\left(\frac{f}{a/2\pi}\right)\Bigg|_{f=0}^{\infty} = \frac{1}{2a} \text{ (joules).}$$

(c)

$$\begin{aligned}
& 2\int_0^W |X(f)|^2 \,\mathrm{d}f = 0.95\left(\frac{1}{2a}\right) \\
\Rightarrow\quad & 2\int_0^W \frac{\mathrm{d}f}{2\pi^2\left(f^2 + \frac{a^2}{4\pi^2}\right)} = \frac{1}{\pi a}\tan^{-1}\left(\frac{f}{a/2\pi}\right)\Bigg|_{f=0}^{W} = \frac{1}{\pi a}\tan^{-1}\left(\frac{2\pi W}{a}\right) \\
\Rightarrow\quad & \text{Solve } \tan^{-1}\left(\frac{2\pi W}{a}\right) = \frac{0.95\pi}{2} \\
\Rightarrow\quad & W = \frac{a}{2\pi}\tan\left(\frac{0.95\pi}{2}\right) = 2.02a \text{ (Hz).}
\end{aligned}$$

Note that the required bandwidth is proportional to $\frac{1}{\text{time constant}}$ $\Rightarrow$ the smaller the time constant, the larger the bandwidth and vice versa. Makes intuitive sense, n'est-ce pas?

P2.28 (a) Since $s_T(t) = \int_{-\infty}^{\infty} S_T(f)\mathrm{e}^{j2\pi ft}\mathrm{d}f$. Write $s_T^2(t)$ as:

$$\begin{aligned}
s_T^2(t) &= \int_{f=-\infty}^{\infty} S_T(f)\mathrm{e}^{j2\pi ft}\mathrm{d}f \cdot \int_{\lambda=-\infty}^{\infty} S_T(\lambda)\mathrm{e}^{j2\pi\lambda t}\mathrm{d}\lambda \\
\therefore\quad P &= \lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty} s_T^2(t)\mathrm{d}t \\
&= \lim_{T\to\infty}\frac{1}{T}\int_{t=-\infty}^{\infty}\left[\int_{f=-\infty}^{\infty} S_T(f)\mathrm{e}^{j2\pi ft}\mathrm{d}f\int_{\lambda=-\infty}^{\infty} S_T(\lambda)\mathrm{e}^{j2\pi\lambda t}\mathrm{d}\lambda\right]\mathrm{d}t
\end{aligned}$$

Interchange the order of integration:

$$\begin{aligned}
P &= \lim_{T\to\infty}\frac{1}{T}\int_{f=-\infty}^{\infty}\mathrm{d}f S_T(f)\int_{\lambda=-\infty}^{\infty}\mathrm{d}\lambda S_T(\lambda)\int_{t=-\infty}^{\infty}\mathrm{e}^{j2\pi(\lambda+f)t}\mathrm{d}t \\
&= \lim_{T\to\infty}\frac{1}{T}\int_{f=-\infty}^{\infty}\mathrm{d}f S_T(f)\int_{\lambda=-\infty}^{\infty}\mathrm{d}\lambda\delta(\lambda+f)S_T(\lambda) \\
&= \lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty}\mathrm{d}f S_T(f)S_T^*(f) = \lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty}|S_T(f)|^2\,\mathrm{d}f
\end{aligned}$$

In the above derivation we made use of the facts that $\int_{t=-\infty}^{\infty}\mathrm{e}^{j2\pi(\lambda+f)t}\mathrm{d}t = \delta(\lambda+f)$ and $\int_{\lambda=-\infty}^{\infty}\mathrm{d}\lambda\delta(\lambda+f)S_T(\lambda) = S_T(-f) = S_T^*(f)$ (since $s_T(t)$ is real).

Note that as $T \to \infty$, the integral $\int_{-\infty}^{\infty} |S_T(f)|^2 \mathrm{d}f \to \int_{-\infty}^{\infty} s_T^2(t)\mathrm{d}t \to \infty$. To overcome this difficulty; interchange the integration and limiting operations (mathematicians may worry about this, engineers shall assume the functions exhibit "reasonable" behaviour). Then

$$P = \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{|S_T(f)|^2}{T} \mathrm{d}f.$$

(b) From $P = \lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{\infty} s_T^2(t)\mathrm{d}t$, P has a unit of $\frac{\text{volts}^2\cdot\text{sec}}{\text{sec}} = \frac{\text{joules}}{\text{sec}} = \text{watts} \Rightarrow \frac{|S_T(f)|^2}{T}$ has a unit of $\frac{\text{watts}}{\text{Hz}}$. Call the limit a power spectrum density, i.e., it shows how the power of $s(t)$ is distributed in the frequency domain.

P2.29

$$\begin{aligned}
\mathcal{F}\{R(\tau)\} &= \int_{\tau=-\infty}^{\infty} R(\tau)\mathrm{e}^{-j2\pi f\tau}\mathrm{d}\tau \\
&= \lim_{T\to\infty}\frac{1}{T}\int_{\tau=-\infty}^{\infty}\left[\int_{t=-\infty}^{\infty} s_T(t)s_T(t+\tau)\mathrm{d}t\right]\mathrm{e}^{-j2\pi f\tau}\mathrm{d}\tau \\
&= \lim_{T\to\infty}\frac{1}{T}\int_{t=-\infty}^{\infty}\mathrm{d}t\, s_T(t) \qquad \underbrace{\int_{\tau=-\infty}^{\infty} s_T(t+\tau)\mathrm{e}^{-j2\pi f\tau}\mathrm{d}\tau}_{\overset{\lambda=t+\tau}{=}\mathrm{e}^{j2\pi ft}\int_{\lambda=-\infty}^{\infty} s_T(\lambda)\mathrm{e}^{-j2\pi f\lambda}\mathrm{d}\lambda=\mathrm{e}^{j2\pi ft}S_T(f)} \\
&= \lim_{T\to\infty}\frac{1}{T}\int_{t=-\infty}^{\infty}\mathrm{d}t\, s_T(t)\mathrm{e}^{j2\pi ft}S_T(f) \\
&= \lim_{T\to\infty}\frac{1}{T}S_T(f)\underbrace{\int_{t=-\infty}^{\infty}\mathrm{d}t\, s_T(t)\mathrm{e}^{j2\pi ft}}_{=S_T(-f)=S_T^*(f)} \\
\therefore \mathcal{F}\{R(\tau)\} &= \lim_{T\to\infty}\frac{|S_T(f)|^2}{T}.
\end{aligned}$$

P2.30 (a)

$$R(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty} s_T(t)s_T(t+\tau)\mathrm{d}t = \lim_{T\to\infty}\frac{V^2(T-\tau)}{T} = V^2 \text{ watts}$$

$$\Rightarrow \quad S(f) = V^2\delta(f) \text{ watts/Hz, all the power is concentrated at } f=0 \text{ (DC only)}$$

(b)

$$R(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty} s_T(t)s_T(t+\tau)\mathrm{d}t = \lim_{T\to\infty}\frac{V^2(T/2-\tau)}{T} = \frac{V^2}{2} \text{ watts}$$

$$\Rightarrow \quad S(f) = \frac{V^2}{2}\delta(f) \text{ watts/Hz (half a DC, half the power)}$$

(c)

$$\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} V\cos(2\pi f_c t+\theta)\, V\cos[2\pi f_c(t+\tau)+\theta]\,\mathrm{d}t$$

$$= \lim_{T\to\infty}\frac{1}{T}\underbrace{\int_{-T/2}^{T/2}\frac{V^2}{2}\cos[2\pi f_c(2t+\tau)+2\theta]\,\mathrm{d}t}_{=0} + \lim_{T\to\infty}\frac{1}{T}\frac{V^2}{2}\underbrace{\int_{-T/2}^{T/2}\cos(2\pi f_c\tau)\,\mathrm{d}t}_{T\cos(2\pi f_c\tau)}$$

$$\therefore R(\tau) = \frac{V^2}{2}\cos(2\pi f_c\tau) \text{ watts and } S(f) = \frac{V^2}{4}\left[\delta(f-f_c)+\delta(f+f_c)\right] \text{ watts/Hz.}$$

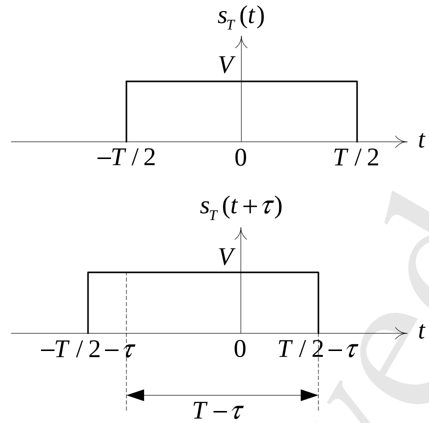


Figure 2.17

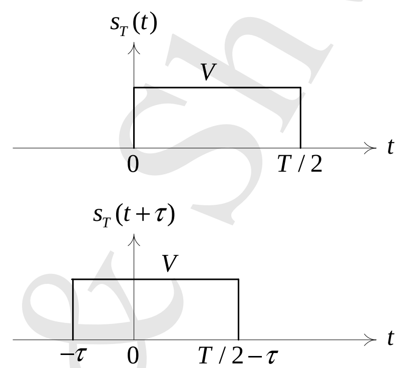


Figure 2.18

The average power of a sinusoid is $\frac{V^2}{2}$ watts. Half of it is concentrated at $f = f_c$ and half at $f = -f_c$.

P2.31

$$
\begin{aligned}
R(\tau) &= \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} s_T(t) s_T(t+\tau) \mathrm{d}t \\
R(\tau + T_s) &= \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} s_T(t) \underbrace{s_T(t+\tau+T_s)}_{s_T(t+\tau)} \mathrm{d}t = R(\tau)
\end{aligned}
$$

where T_s is the period of $s(t)$.

P2.32 (a) Multiplication in the time domain results in a differentiation in the frequency domain.

(b)

$$
\begin{aligned}
S(f) &= \int_{-\infty}^{\infty} s(t) \mathrm{e}^{-j2\pi f t} \mathrm{d}t \\
\frac{\mathrm{d}S(f)}{\mathrm{d}f} &= \int_{-\infty}^{\infty} (-j2\pi t) s(t) \mathrm{e}^{-j2\pi f t} \mathrm{d}t = -j2\pi \int_{-\infty}^{\infty} t s(t) \mathrm{e}^{-j2\pi f t} \mathrm{d}t
\end{aligned}
$$

which shows that $ts(t)$ and $-\frac{1}{2\pi j} \frac{\mathrm{d}S(f)}{\mathrm{d}f}$ are a Fourier transform pair. Continuing, it is easily shown that $t^n s(t) \longleftrightarrow \left(-\frac{1}{2\pi j}\right)^n \frac{\mathrm{d}^n S(f)}{\mathrm{d}f^n}$ are a Fourier transform pair.

P2.33 Duality states that if $s(t) \longleftrightarrow S(f)$ then $s(f) \longleftrightarrow S(-t)$. Therefore, if

$$\begin{aligned} V\mathrm{e}^{-a|t|} &\longleftrightarrow \frac{2Va}{a^2+4\pi^2 f^2} \\ \text{then} \qquad V\mathrm{e}^{-a|f|} &\longleftrightarrow \frac{2Va}{a^2+4\pi^2 f^2} = \frac{V2a}{4\pi^2\left[\frac{a^2}{4\pi^2}+t^2\right]}. \end{aligned}$$

Adjust V, a to get $\frac{1}{1+t^2} \Rightarrow \frac{a^2}{4\pi^2} = 1 \Rightarrow a = 2\pi$ and $\frac{2Va}{a^2} = \frac{2V}{a} = \frac{2V}{2\pi} = 1 \Rightarrow V = \pi$.

$$\therefore \quad \frac{1}{1+t^2} \longleftrightarrow \pi\mathrm{e}^{-2\pi|f|}.$$

It is easy enough to check that

$$\frac{1}{1+t^2} = \mathcal{F}^{-1}\left\{\pi\mathrm{e}^{-2\pi|f|}\right\} = \int_{-\infty}^{\infty} \pi\mathrm{e}^{-2\pi|f|}\mathrm{e}^{2\pi ft}\mathrm{d}f.$$

P2.34 P2.33 shows that $s(t) = \frac{1}{1+t^2} \longleftrightarrow \pi\mathrm{e}^{-2\pi|f|} = S(f)$ are a Fourier transform pair.

From P2.32 we have $ts(t) = \frac{t}{1+t^2} \longleftrightarrow \frac{1}{-j2\pi}\frac{\mathrm{d}S(f)}{\mathrm{d}f}$ are a Fourier transform pair, or

$$\frac{t}{1+t^2} \longleftrightarrow \frac{1}{-2j}\frac{\mathrm{d}(\mathrm{e}^{-2\pi|f|})}{\mathrm{d}f}$$

.

Now with $f < 0$ we have $\frac{\mathrm{d}(\mathrm{e}^{2\pi f})}{\mathrm{d}f} = 2\pi\mathrm{e}^{2\pi f}$, and with $f > 0$ we have $\frac{\mathrm{d}(\mathrm{e}^{-2\pi f})}{\mathrm{d}f} = -2\pi\mathrm{e}^{-2\pi f}$. Thus,

$$\begin{aligned} \frac{\mathrm{d}(\mathrm{e}^{-2\pi|f|})}{\mathrm{d}f} &= -2\pi\mathrm{e}^{-2\pi|f|}\,\mathrm{sgn}(f) \\ \therefore \quad \frac{t}{1+t^2} &\longleftrightarrow -j\pi\mathrm{e}^{-2\pi|f|}\,\mathrm{sgn}(f). \end{aligned}$$

P2.35 Certainly $\int_{-\infty}^{\infty}[s_1(t)+\lambda s_2(t)]^2\,\mathrm{d}t \geq 0$ for all λ or

$$\underbrace{\int_{-\infty}^{\infty} s_1^2(t)\mathrm{d}t}_{c} + \underbrace{\left[2\int_{-\infty}^{\infty} s_1(t)s_2(t)\mathrm{d}t\right]}_{b}\lambda + \underbrace{\left[\int_{-\infty}^{\infty} s_2^2(t)\mathrm{d}t\right]}_{a}\lambda^2 \geq 0.$$

The above is a quadratic in λ with coefficients a, b, c and roots $\lambda_{1,2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$. For the polynomial to be always ≥ 0 requires that the roots to be complex (at very minimum a double root), i.e., the quadratic should never intersect the λ axis, at best it may only touch it. Therefore, $b^2 - 4ac \leq 0$ or

$$\begin{aligned} 4\left[\int_{-\infty}^{\infty} s_1(t)s_2(t)\mathrm{d}t\right]^2 &\leq 4\int_{-\infty}^{\infty} s_1^2(t)\mathrm{d}t\int_{-\infty}^{\infty} s_2^2(t)\mathrm{d}t \\ \text{or} \quad \left|\int_{-\infty}^{\infty} s_1(t)s_2(t)\mathrm{d}t\right| &\leq \sqrt{\int_{-\infty}^{\infty} s_1^2(t)\mathrm{d}t\int_{-\infty}^{\infty} s_2^2(t)\mathrm{d}t} \\ &= \sqrt{\int_{-\infty}^{\infty} s_1^2(t)\mathrm{d}t}\sqrt{\int_{-\infty}^{\infty} s_2^2(t)\mathrm{d}t} = \sqrt{E_1}\sqrt{E_2}. \end{aligned}$$

Equality holds when $s_1(t) = Ks_2(t)$, i.e., one signal is a scaled version of the other.

P2.36

$$\left|\int_{-\infty}^{\infty} S_1(f)S_2^*(f)\mathrm{d}f\right| = \left|\int_{-\infty}^{\infty} S_1^*(f)S_2(f)\mathrm{d}f\right| \le \sqrt{\int_{-\infty}^{\infty}|S_1(f)|^2\mathrm{d}f}\sqrt{\int_{-\infty}^{\infty}|S_2(f)|^2\mathrm{d}f}.$$

P2.37 Let $s_1(t) = s(t)$, $s_2(t) = s(t+\tau)$. Then

$$\Bigg|\underbrace{\int_{-\infty}^{\infty} s(t)s(t+\tau)\mathrm{d}t}_{R_s(\tau)}\Bigg| \le \sqrt{\underbrace{\int_{-\infty}^{\infty} s^2(t)\mathrm{d}t}_{R_s(0)}\underbrace{\int_{-\infty}^{\infty} s^2(t+\tau)\mathrm{d}t}_{R_s(0)}}.$$

Therefore, $|R_s(\tau)| \le R_s(0)$.

Remark: Above was done for energy signals but readily shown to be true for power signals.

P2.38

$$s(0) = \int_{-\infty}^{\infty} S(f)\mathrm{d}f \le \int_{-\infty}^{\infty}|S(f)|\,\mathrm{d}f; \quad S(0) = \int_{-\infty}^{\infty} s(t)\mathrm{d}t \le \int_{-\infty}^{\infty}|s(t)|\,\mathrm{d}t$$

$$WT = \frac{\int_{-\infty}^{\infty}|S(f)|\,\mathrm{d}f}{2S(0)} \cdot \frac{\int_{-\infty}^{\infty}|s(t)|\,\mathrm{d}t}{s(0)} \ge \frac{1}{2}.$$

P2.39 (a)

$$\begin{aligned} s(t) &= V\mathrm{e}^{-at}u(t) \\ E_s &= \int_{-\infty}^{\infty} s^2(t)\mathrm{d}t = V^2\int_{-\infty}^{\infty}\mathrm{e}^{-2at}(t)\mathrm{d}t = \frac{V^2}{2a} \text{ (joules)}. \end{aligned}$$

(b) Let $V = 1$

$$\frac{2}{a^2}\int_0^{f_\kappa}\frac{1}{1+\frac{4\pi^2f^2}{a^2}}\mathrm{d}f = \frac{\kappa}{2a}.$$

Let $f_n = \frac{2\pi f}{a}$, i.e., change variables. Then

$$\begin{aligned} \frac{2}{a^2}\int_0^{\frac{2\pi}{a}f_\kappa}\frac{1}{1+f_n^2}\,a\,\frac{\mathrm{d}f_n}{2\pi} &= \kappa\left(\frac{1}{2a}\right) \\ \text{or} \qquad \tan^{-1}\left(\frac{2\pi}{a}f_\kappa\right) &= \frac{\pi\kappa}{2} \Rightarrow \frac{f_\kappa}{a} = \frac{1}{2\pi}\tan\left(\frac{\pi\kappa}{2}\right). \end{aligned}$$

(c) $s(0) = 1 \Rightarrow T = \int_0^{\infty}\mathrm{e}^{-at}\mathrm{d}t = \frac{1}{a}$. With this T and f_κ found in (b) we have

$$Tf_\kappa = \frac{f_\kappa}{a} = \frac{1}{2\pi}\tan\left(\frac{\pi\kappa}{2}\right).$$

(d) Matlab plot.

P2.40 (a)

$$E_s = \int_{-\infty}^{\infty} s^2(t)\mathrm{d}t = \int_{-\infty}^{\infty}\mathrm{e}^{-2a|t|}\mathrm{d}t = 2\int_0^{\infty}\mathrm{e}^{-2at}\mathrm{d}t = \frac{1}{a} \text{ (joules), where } V = 1.$$

(b) Using Equation (2.62c), f_κ is determined from

$$2\int_0^{f_\kappa} \frac{16\pi^2 f^2}{(a^2+4\pi^2 f^2)^2}\mathrm{d}f = \frac{\kappa}{a} \Leftrightarrow \int_0^{f_\kappa} \frac{f^2}{a^4\left(1+\frac{4\pi^2}{a^2}f^2\right)^2}\mathrm{d}f = \frac{\kappa}{32\pi^2 a}$$

Let integration variable be $\lambda = \frac{2\pi f}{a}$. Then,

$$\int_0^{\frac{2\pi f_\kappa}{a}} \frac{\lambda^2}{(1+\lambda^2)^2}\mathrm{d}\lambda = \frac{\kappa\pi}{4}$$

Change variable again to $x=\lambda^2 \Rightarrow \mathrm{d}\lambda = \frac{\mathrm{d}x}{2\sqrt{x}}$.

$$\int_{\lambda=0}^{\frac{2\pi f_\kappa}{a}} \frac{\lambda^2}{(1+\lambda^2)^2}\mathrm{d}\lambda = \int_{x=0}^{\frac{4\pi^2 f_\kappa^2}{a^2}} \frac{\sqrt{x}}{2\,(1+x)^2}\mathrm{d}x.$$

To integrate, use G&R, p.71, 2.313.5, where $z_1 = a+bx$, $a=b=1$. Let $f_n^2 = \frac{4\pi^2 f_\kappa^2}{a^2}$. The integral becomes

$$-\left.\frac{\sqrt{x}}{2(1+x)}\right|_0^{f_n^2} + \frac{1}{4}\int_0^{f_n^2} \frac{\mathrm{d}x}{(1+x)\sqrt{x}}$$

and using 2.211 (same page in G&R)

$$\int_0^{f_n^2} \frac{\mathrm{d}x}{(1+x)\sqrt{x}} = 2\tan^{-1}\sqrt{x}\Big|_0^{f_n^2}.$$

Putting it all together we have

$$-\frac{f_n}{2(1+f_n^2)} + \frac{1}{2}\tan^{-1}(f_n) = \frac{\kappa\pi}{4} \tag{2.3}$$

as the equation that defines f_n. Need to solve in Matlab for various values of κ.

(c) $s(0)=1 \Rightarrow T = \int_{-\infty}^{\infty}\left|\mathrm{e}^{-a|t|}\,\mathrm{sgn}(t)\right|\mathrm{d}t = 2\int_0^\infty \mathrm{e}^{-at}\mathrm{d}t = \frac{2}{a}$.
In (b), after solving for f_n we get f_κ to be $f_\kappa = cf_n$, $c=\frac{a}{2\pi} \Rightarrow Tf_\kappa = \left(\frac{2}{a}\right)\left(\frac{a}{2\pi}\right)f_n = \frac{1}{\pi}f_n$ (where to repeat f_n is the solution of (2.3) for various values of κ).

(d) Matlab work.

P2.41

$$\begin{aligned} s(t) &\longleftrightarrow S(f) \\ \frac{\mathrm{d}^n s(t)}{\mathrm{d}t^n} = s_1(t) + K\delta(t) &\longleftrightarrow S_1(f)+K \end{aligned}$$

where $S_1(f)\to 0$ as $f\to\infty$. Now

$$\mathcal{F}\left\{\frac{\mathrm{d}^n s(t)}{\mathrm{d}t^n}\right\} = (j2\pi f)^n S(f) \Rightarrow S(f) = \frac{S_1(f)}{(j2\pi f)^n} + \frac{K}{(j2\pi f)^n}.$$

Since $S_1(f)\to 0$ as $f\to\infty$, the above means that $S(f)\to\frac{K}{(j2\pi f)^n}$ for large n, i.e., the asymptotic behaviour of the spectrum depends on how many times the signal can be differentiated before an impulse(s) appears.

How many times can one differentiate a pure sinusoid (i.e., one that lasts from $-\infty$ to $+\infty$) before an impulse appears? How "concentrated" is the amplitude spectrum of the sinusoid?

P2.42 (a)

$$
\begin{aligned}
S_{\text{DSB-SC}}(f) &= \int_{-\infty}^{\infty} m(t)\left[\frac{e^{j2\pi f_c t}+e^{-j2\pi f_c t}}{2}\right]e^{-j2\pi f t}dt \\
&= \frac{1}{2}\int_{-\infty}^{\infty} m(t)e^{-j2\pi(f-f_c)t}dt + \frac{1}{2}\int_{-\infty}^{\infty} m(t)e^{-j2\pi(f+f_c)t}dt \\
&= \frac{M(f-f_c)}{2}+\frac{M(f+f_c)}{2}.
\end{aligned}
$$

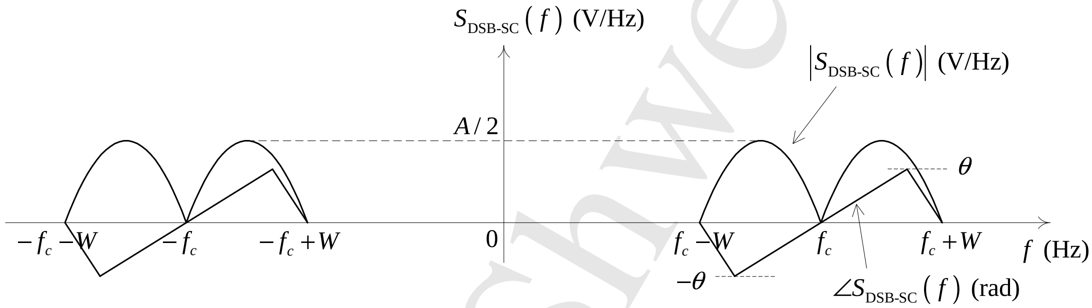


Figure 2.19

(b) *Note:* The magnitude spectrum is a shifted and scaled version of the original spectrum while the phase spectrum is simply a shifted version.

(c) The passband bandwidth is $2W$ Hz.

How would the above change if the carrier is $\sin(2\pi f_c t)$ instead of $\cos(2\pi f_c t)$?

P2.43 The difference between $S_{\text{DSB-SC}}(f)$ of P2.42 and $S_{\text{AM}}(f)$ is a spectral component at $\pm f_c$ due to the carrier (or if you wish a DC component added to $m(t)$). It looks like in Fig. 2.20.

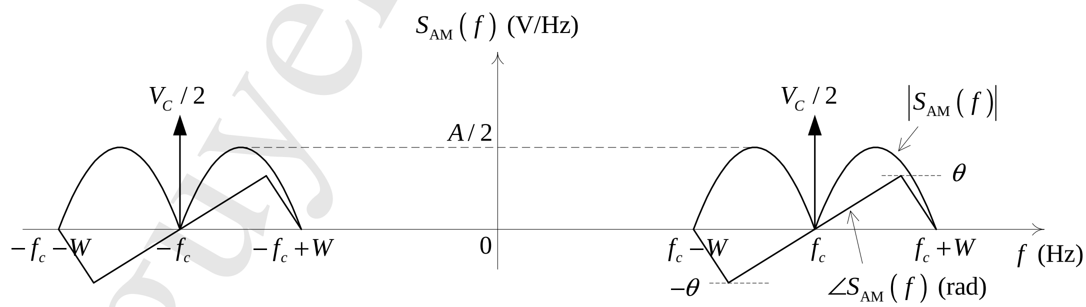


Figure 2.20

Note: We invariably assume that the message, $m(t)$, has zero DC. DC is useful for powering stuff but contains no information of interest to be transmitted. The reason to insert a DC is to simplify the demodulator – as we see in the next 2 problems.

P2.44 By now we are beginning to appreciate that multiplying a signal in the time domain by a pure sinusoid simply shifts the spectrum and scales it. It is the shift that is important, the scaling is somewhat arbitrary and practically would be eventually controlled by a "volume" control.

(a) For DSB-SC; $S_1(f) = \left[S_{\text{DSB-SC}}(f - f_c) + S_{\text{DSB-SC}}(f + f_c)\right]/2$ and looks like in Fig. 2.21 (note that the impulses in the figure only occur for AM, not for DSB-SC).

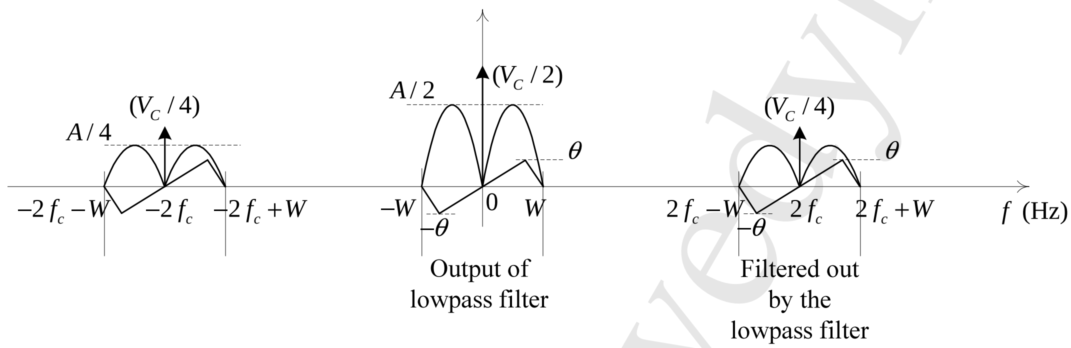


Figure 2.21

(b) The AM spectrum is the same as the DSB-SC except for the DC component that results in the impulses at 0, $\pm 2f_c$. Normally the demodulated signal, i.e., $s_1(t)$, would not only be passed through a lowpass filter but also through a (well designed) high pass filter to eliminate the DC component.

(c) Of course, as usual, multiplying by a sinusoid shifts the spectrum, scales it, etc. But it is scaling that important, perhaps a more appropriate terminology would be scaling and phasing. Let us look at the problem in 2 ways (not completely distinct). First in the time domain:

$$s_1(t) = s(t)\sin(2\pi f_c t) = m(t)\cos(2\pi f_c t)\sin(2\pi f_c t).$$

Now

$$\sin x \cos y = \frac{\sin(x+y) + \sin(x-y)}{2} \Rightarrow s_1(t) = \frac{1}{2}m(t)\sin\left[2\pi(2f_c)t\right]$$

which shows that the spectrum of $s_1(t)$ is that of $M(f)$ shifted by $\pm 2f_c \Rightarrow$ output of the lowpass filter is ZERO.
In the frequency domain:

$$s_1(t) = s(t)\left[\frac{e^{j2\pi f_c t} - e^{-j2\pi f_c t}}{2j}\right] \Rightarrow S_1(f) = \frac{1}{2j}\left[S(f - f_c) - S(f + f_c)\right].$$

Considering $\frac{1}{2j}$ to be just a scaling factor, albeit a complex one, $S_1(f)$ is $S(f)$ shifted up and down (more appropriately to the left and right) by f_c. But note the minus sign in front of $S(f + f_c)$. This results in the spectrum around $f = 0$ canceling out – but we know this already from the time domain analysis.

Finally the $\frac{1}{2j}$ not only scales but adds a phase of $\pm\frac{\pi}{2}$. Putting this all together the spectrum $S_1(f)$ looks like:

To generalize one should consider $s_1(t) = s(t)\cos(2\pi f_c t + \theta)$, where θ would represent the phase difference between the oscillator at the transmitter (modulator) and the one at the receiver (demodulator), and see what happens to the output of the demodulator.

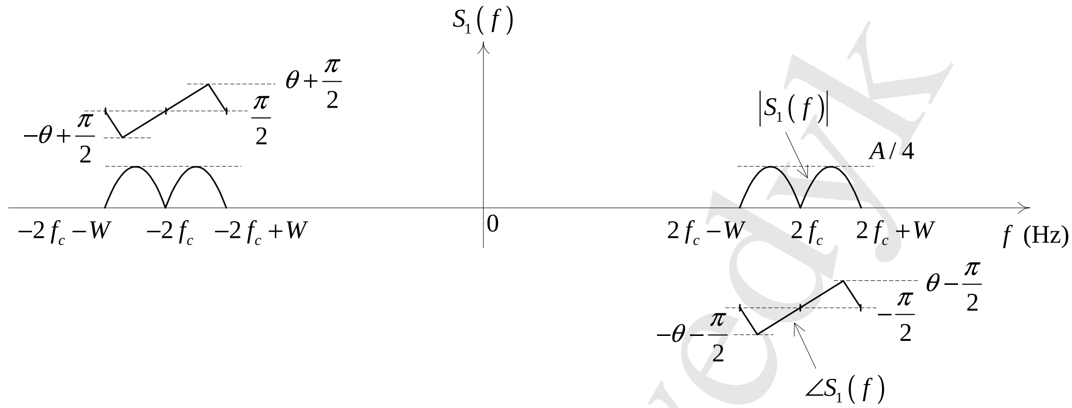


Figure 2.22

The last chapter in the text considers a circuit (phase locked loop) which estimates the incoming phase of the received signal. The next problem looks at a demodulation technique that does not require knowledge of the phase, hence it is called noncoherent. But alas it only works for AM.

P2.45 (a) $s_1(t) = |[V_c + m(t)]\cos 2\pi f_c t| = |V_c + m(t)|\,|\cos 2\pi f_c t|$. But $V_c + m(t) \geq 0$. Therefore, $|V_c + m(t)| = V_c + m(t)$. And $|\cos 2\pi f_c t|$ is a fullwave rectified sinusoid of period $= \frac{1}{2f_c}$ or $f_r = 2f_c$. It therefore has a Fourier series of $\sum_{k=-\infty}^{\infty} D_k e^{j2\pi k f_r t}$, where most importantly D_0 (the DC value) $\neq 0$.

The spectrum of $s_1(t) = [V_c + m(t)]\,|\cos 2\pi f_c t| = \sum_{k=-\infty}^{\infty} D_k\,[V_c + m(t)]\,e^{j2\pi k f_r t}$, then consists of shifted versions of the spectrum of $[V_c + m(t)]$, shifted by $kf_r = k2f_c$ Hz, with the kth shift weighted by D_k.

The output of the lowpass filter is $s_{\text{out}} = D_0\,[V_c + m(t)]$ since the spectrum components at $\pm 2f_c, \pm 4f_c$, etc, are filtered out. The DC term $D_0 V_c$ is easily filtered out by a high pass filter whose output would be $D_0 m(t)$ (assuming $m(t)$ does not have significant energy around $f = 0$ – typically the case).

(b) $s_1(t)$ now is $s_1(t) = |m(t)|\,|\cos 2\pi f_c t| = \sum_{k=-\infty}^{\infty} D_k\,|m(t)|\,e^{j2\pi k f_r t}$. Therefore the spectrum of $|m(t)|$ will lie in a larger bandwidth, theoretically infinite but practically in some bandwidth, say $W_c = \kappa W$ ($\kappa > 1$ but say less than 3). Then by adjusting the bandwidth of the lowpass filter to W_c, the output would be $D_0|m(t)|$, a distorted version of $m(t)$.

Of course, $V_c = 0$ is an extreme condition but distortion shall exist as long as $V_c < \max|m(t)|$ since then $|V_c + m(t)| \neq V_c + m(t)$.

P2.46 (a) The double sideband suppressed carrier spectrum looks like:

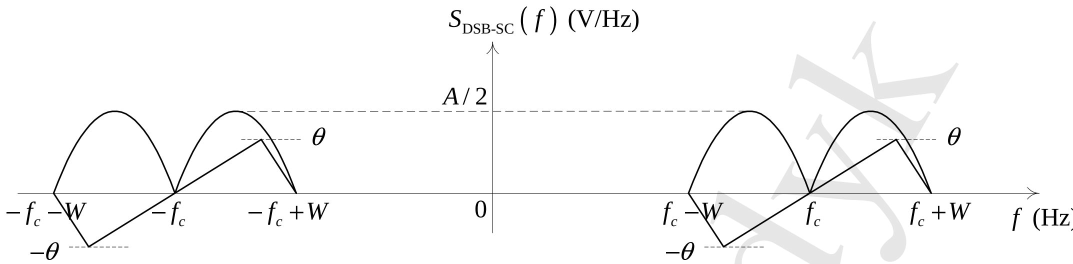


Figure 2.23

And the function $\frac{1+\text{sgn}(f-f_c)}{2}+\frac{1-\text{sgn}(f+f_c)}{2}$ plots as:

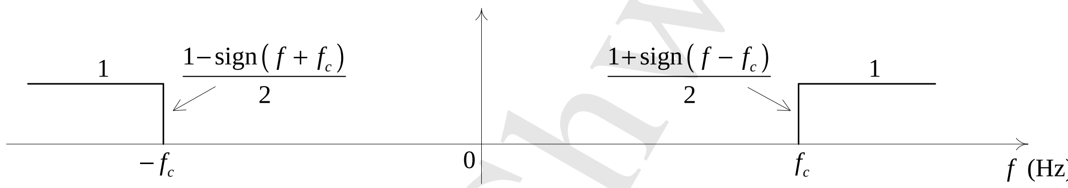


Figure 2.24

Multiplying the 2 frequency functions together results in

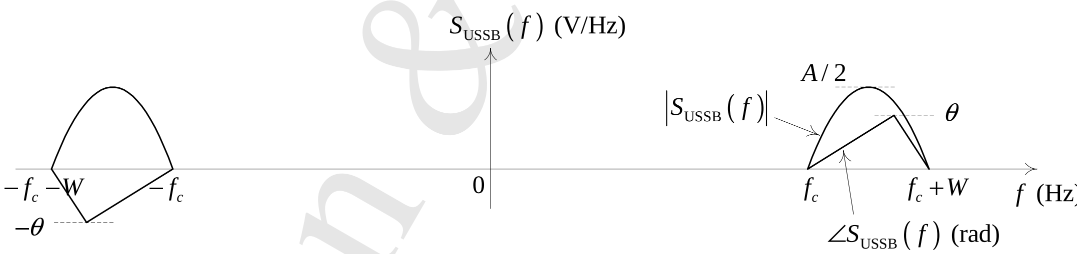


Figure 2.25

Note that the magnitude function is even and the phase function is odd. This tells us that the time domain signal is real.

(b) $s(t) = s_{\text{USSB}}(t)2\cos 2\pi f_c t = s_{\text{USSB}}(t)\left[e^{j2\pi f_c t}+e^{-j2\pi f_c t}\right]$. Therefore, $S(f) = S_{\text{USSB}}(f-f_c)+S_{\text{USSB}}(f+f_c)$, i.e., the upper sideband spectrum shifted by $\pm f_c$ Hz.

(c) Since the spectrum is exactly that of $m(t)$ and the Fourier transform is unique.

P2.47 (a) $s_{\text{DSB-SC}}(f) = \frac{M(f-f_c)+M(f+f_c)}{2}$ and some very simple algebra gives (P2.11).

(b) $m(t)\cos(2\pi f_c t)$.

(c) $M(t)\,\text{sgn}(f)$ is a product in the frequency domain which means that in the time domain we have a convolution, i.e, $m(t)*h(t)$ where $h(t) = \mathcal{F}^{-1}\{\text{sgn}(f)\}$. A shift of the spectrum by f_c is accomplished my multiplying the time function by $e^{j2\pi f_c t}$. Therefore, $[M(f-f_c)\,\text{sgn}(f-f_c)] \longleftrightarrow \left[m(t)*\mathcal{F}^{-1}\{\text{sgn}(f)\}\right]e^{j2\pi f_c t}$.

(d) The shift is now $-f_c$ Hz, hence $[M(f-f_c)\,\text{sgn}(f+f_c)] \longleftrightarrow \left[m(t)*\mathcal{F}^{-1}\{\text{sgn}(f)\}\right]e^{-j2\pi f_c t}$.

(e) Use the fact that the inverse transform is a linear operation, i.e., superposition holds.

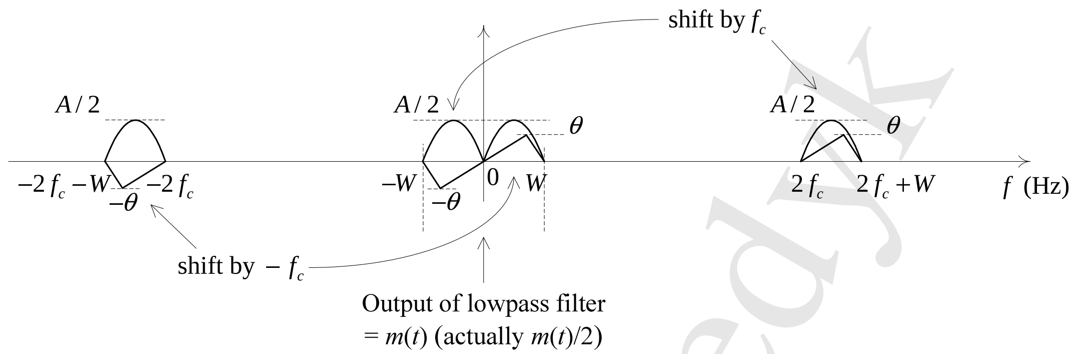


Figure 2.26

(f) No. The frequency function, $\text{sgn}(f)$ has a magnitude spectrum $= |\text{sgn}(f)| = 1$, an even function but its phase spectrum is $\angle\text{sgn}(f) = 0$ for $f \geq 0$ and $= \pi$ for $f < 0$, which is not an odd function.

(g) Yes. The magnitude spectrum is $|j\,\text{sgn}(f)| = 1$, an even function and the phase spectrum is $\angle j\,\text{sgn}(f) = \pi/2$ for $f \geq 0$ and $= -\pi/2$ for $f < 0$ an odd function.

Recall that a real time function always has an even magnitude spectrum and an odd phase spectrum.

(h) $jh(t) \longleftrightarrow j\,\text{sgn}(f)$.

(i) Note that

$$\frac{\mathrm{e}^{j2\pi f_c t} - \mathrm{e}^{-j2\pi f_c t}}{2j} \longleftrightarrow \sin(2\pi f_c t).$$

From (b) and (h), we get the block diagram of Fig. P2.42.

P2.48 In the frequency domain the lower single-sideband signal looks like:

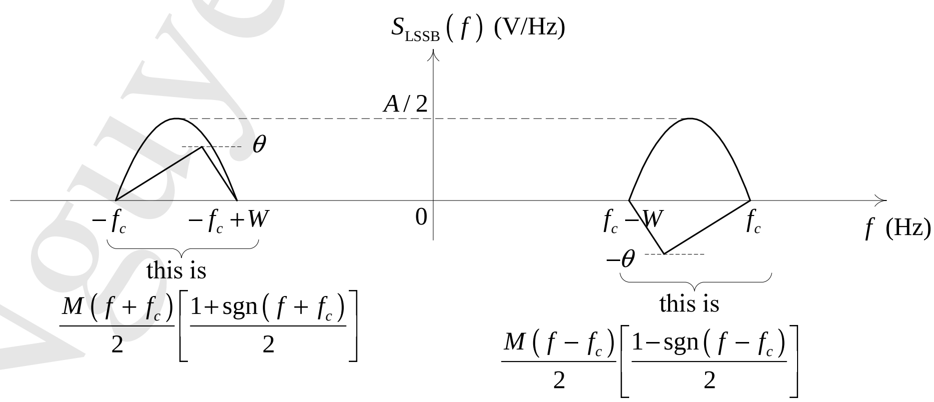


Figure 2.27

Therefore,

$$S_{\text{LSSB}}(f) = \frac{M(f+f_c)+M(f-f_c)}{2} + \frac{M(f+f_c)}{2}\frac{\text{sgn}(f+f_c)}{2} - \frac{M(f-f_c)}{2}\frac{\text{sgn}(f-f_c)}{2}$$

In the time domain:

$$\frac{m(t)}{2}\cos(2\pi f_c t) \longleftrightarrow \frac{M(f+f_c)+M(f-f_c)}{2}$$

Let $h(t) = \mathcal{F}^{-1}\{j\,\mathrm{sgn}(f)\}$. Recognize

$$\begin{aligned}
\left[\frac{m(t)}{2} * h(t)\right]\frac{\mathrm{e}^{-j2\pi f_c t}}{2j} &\longleftrightarrow \frac{M(f+f_c)}{2}\frac{j\,\mathrm{sgn}(f+f_c)}{2j}\\
\left[\frac{m(t)}{2} * h(t)\right]\frac{\mathrm{e}^{j2\pi f_c t}}{2j} &\longleftrightarrow \frac{M(f-f_c)}{2}\frac{j\,\mathrm{sgn}(f-f_c)}{2j}
\end{aligned}$$

Therefore,

$$\begin{aligned}
s_{\mathrm{LSSB}}(t) &= \frac{m(t)}{2}\cos(2\pi f_c t) + \left[\frac{m(t)}{2} * h(t)\right]\frac{\mathrm{e}^{-j2\pi f_c t}-\mathrm{e}^{j2\pi f_c t}}{2j}\\
&= \frac{m(t)}{2}\cos(2\pi f_c t) - \left[\frac{m(t)}{2} * h(t)\right]\sin(2\pi f_c t)
\end{aligned}$$

The above result means that the summer in Fig. 2.42 would now be a "subtractor".

P2.49 Nothing, except that there would be a carrier at $\pm f_c$.

P2.50

$$\begin{aligned}
h_a(t) &= -j\int_{-\infty}^{\infty}\mathrm{e}^{-a|f|}\mathrm{sgn}(f)\mathrm{e}^{j2\pi f t}\mathrm{d}f\\
&= -j\left[-\int_{-\infty}^{0}\mathrm{e}^{(a+j2\pi t)f}\mathrm{d}f + \int_{0}^{\infty}\mathrm{e}^{(-a+j2\pi t)f}\mathrm{d}f\right]\\
&= \frac{4\pi t}{-a^2-4\pi^2t^2} \overset{a\to 0}{=} -\frac{1}{\pi t}.
\end{aligned}$$

P2.51

$$\begin{aligned}
\int_{-\infty}^{\infty} s(t)\hat{s}(t)\mathrm{d}t &= \int_{-\infty}^{\infty} S(f)\hat{S}^*(f)\mathrm{d}f \qquad \text{(Parseval)}\\
&= \int_{-\infty}^{\infty} S(f)S^*(f)(-j\,\mathrm{sgn}(f))\mathrm{d}f\\
&= -j\int_{-\infty}^{\infty}\underbrace{|S(f)|^2}_{\text{even}}\underbrace{\mathrm{sgn}(f)}_{\text{odd}}\mathrm{d}f.
\end{aligned}$$

$|S(f)|^2\,\mathrm{sgn}(f)$ is odd $\therefore$ $\int_{-\infty}^{\infty} s(t)\hat{s}(t)\mathrm{d}t = 0$, i.e., they are orthogonal.

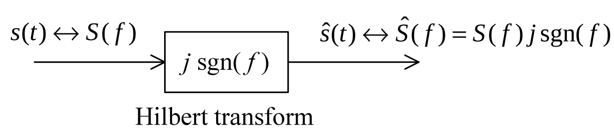


Figure 2.28

P2.52 We have established that $R(\tau) \longleftrightarrow |S(f)|^2$ are a Fourier transform pair. $\hat{S}(f) = j\,\text{sgn}(f)S(f)$, therefore $\left|\hat{S}(f)\right|^2 = |j\,\text{sgn}(f)|^2\,|S(f)|^2 = |S(f)|^2$, i.e., the autocorrelation functions are the same.

P2.53 In general,

$$\begin{aligned}
\hat{M}(f) &= j\,\text{sgn}(f)M(f) = j\left[\underbrace{V\frac{\delta(f-f_c)+\delta(f+f_c)}{2}}_{\mathcal{F}\{V\cos(2\pi f_c t)\}}\right]\text{sgn}(f) \\
&= Vj\left[\frac{\delta(f-f_c)\,\text{sgn}(f)+\delta(f+f_c)\,\text{sgn}(f)}{2}\right] \\
&= V\left[-\underbrace{\frac{\delta(f-f_c)\,\text{sgn}(f)-\delta(f+f_c)\,\text{sgn}(f)}{2j}}_{\sin(2\pi f_c t)}\right] \\
\Rightarrow \hat{m}(t) &= -V\sin(2\pi f_c t).
\end{aligned}$$

P2.54 (a)

$$\begin{aligned}
S(f) &= S_{\text{PB}}(f) - j\hat{S}_{\text{PB}}(f) = S_{\text{PB}}(f) - j(j\,\text{sgn}(f))\hat{S}_{\text{PB}}(f) \\
&= S_{\text{PB}}(f)\,(1+\text{sgn}(f)) = \begin{cases} 2S_{\text{PB}}(f), & f \geq 0 \\ 0, & f < 0 \end{cases}
\end{aligned}$$

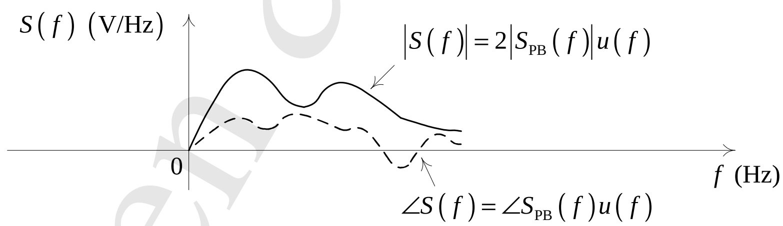


Figure 2.29

(b) Choose f_s as shown. Note that choice is somewhat arbitrary.

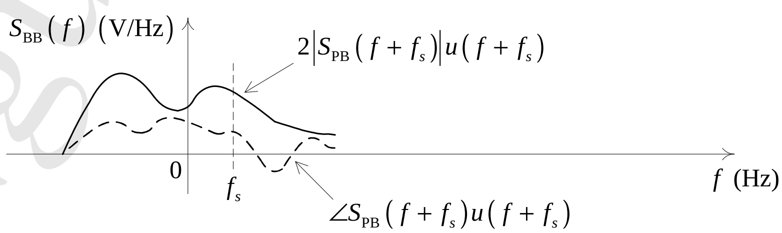


Figure 2.30

Note that $s_{\text{BB}}(t)$ is complex because the magnitude $|S_{\text{BB}}(f)|$ is not an even function and the phase is not an odd function, at least not in general.

(c) From (b) we have that $s_{\text{BB}}(t) = s(t)\mathrm{e}^{-j2\pi f_s t}$

$$\therefore\ \mathcal{R}\left\{s_{\text{BB}}(t)\mathrm{e}^{j2\pi f_s t}\right\} = \mathcal{R}\left\{s(t)\mathrm{e}^{-j2\pi f_s t}\mathrm{e}^{j2\pi f_s t}\right\} = \mathcal{R}\left\{s(t)\right\}.$$

But $\mathcal{R}\{s(t)\} = \mathcal{R}\{s_{\text{PB}}(t) - j\hat{s}_{\text{PB}}(t)\} = s_{\text{PB}}(t)$. Note that $\hat{s}_{\text{PB}}(t)$ is a real signal when $s_{\text{PB}}(t)$ is real.

$$\begin{aligned} s_{\text{PB}}(t) &= \mathcal{R}\left\{\left[s_{\text{BB}}^{[\text{real}]}(t) + j\, s_{\text{BB}}^{[\text{imag}]}(t)\right]\left[\cos(2\pi f_s t) + j\sin(2\pi f_s t)\right]\right\} \\ &= s_{\text{BB}}^{[\text{real}]}(t)\cos(2\pi f_s t) - s_{\text{BB}}^{[\text{imag}]}(t)\sin(2\pi f_s t). \end{aligned}$$

(d) If this axis of symmetry exists then when $S_{\text{PB}}(f)u(f)$ is shifted down by f_s, the resultant baseband spectrum will have magnitude spectrum that is even and a phase spectrum that is odd $\Rightarrow$ the baseband signal is real or the imaginary component $s_{\text{BB}}^{[\text{imag}]}(t)$ is zero.

P2.55 The spectrum $S(f) = S_{\text{USSB}}(f)u(f)$ looks like:

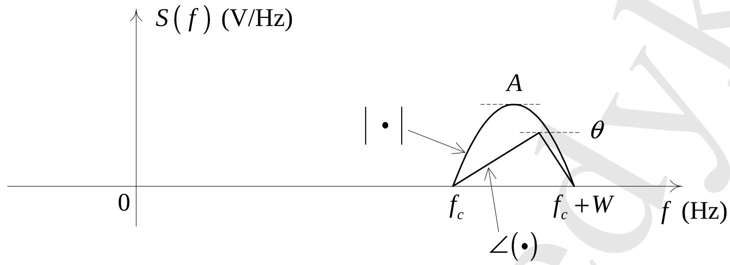


Figure 2.31

Let $s_{\text{BB}}(t) = s(t)\mathrm{e}^{-j2\pi f_s t}$ and choose f_s to be f_c. Then the picture looks as follows:

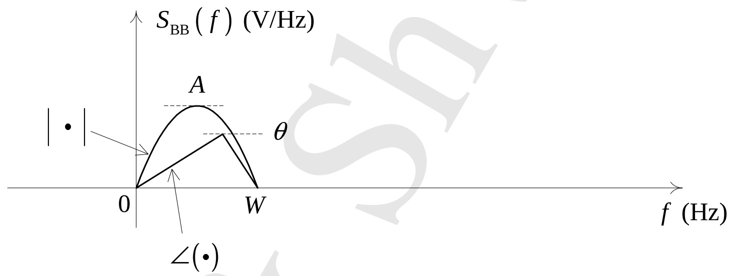


Figure 2.32

Note that $s_{\text{BB}}(t)$ is complex, i.e., has a real and imaginary component. The real component is modulated by $\cos(2\pi f_c t)$, the imaginary component by $\sin(2\pi f_c t)$ to produce the upper sideband signal, as shown in Fig 2.42 in the textbook.

Chapter 3

Probability Theory, Random Variables and Random Processes

P3.1 A random experiment consists of tossing two fair coins. Denote H="Head" and T="Tail".

(a) Obviously the sample space is Ω={HH, HT, TH, TT}. Since the coins are said to be fair, the implication is that each individual outcome is equally probably, namely the probability of each outcome is $\frac{1}{4}$.

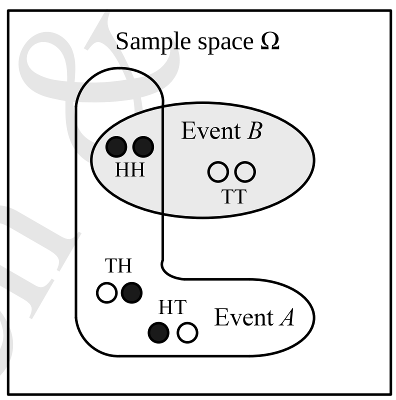


Figure 3.1: Sample space representation for the random experiment of tossing two coins.

(b) A="The event that at least one head shows"={HH, HT, TH}

$$P(A) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$$

B="The event that there is a match of two coins"={HH, TT}

$$P(B) = \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

(c) Using conditional probability:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\{\text{HH}\})}{P(B)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

$$P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{P(\{\text{HH}\})}{P(A)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

(d) Observe that $P(A\cap B) = P(\{\text{HH}\}) = \frac{1}{4}$ and $P(A)P(B) = \frac{3}{4}\times\frac{1}{2} = \frac{3}{8}$. Since $P(A\cap B) \neq P(A)P(B)$, A and B are not statistically independent.

P3.2 We write down all the provided probabilities. The test reliability specifies the conditional probability of $\mathbf{y}$ given $\mathbf{x}$, where $\mathbf{y}$ is the test result, $\mathbf{x}$ the test outcome:

$$\begin{array}{ll} P(\mathbf{y}=1|\mathbf{x}=1) = 0.95 & P(\mathbf{y}=1|\mathbf{x}=0) = 0.05 \\ P(\mathbf{y}=0|\mathbf{x}=1) = 0.05 & P(\mathbf{y}=0|\mathbf{x}=0) = 0.95; \end{array} \tag{3.1}$$

and the disease prevalence tells us about the marginal probability of $\mathbf{x}$:

$$P(\mathbf{x}=1) = 0.01 \qquad P(\mathbf{x}=0) = 0.99. \tag{3.2}$$

From the marginal $P(\mathbf{x})$ and the conditional probability $P(\mathbf{y}|\mathbf{x})$ we can deduce the joint probability $P(\mathbf{x},\mathbf{y}) = P(\mathbf{x})P(\mathbf{y}|\mathbf{x})$ and any other probabilities we are interested in. For example, by the sum rule, the marginal probability of $\mathbf{y}=1$ – the probability of getting a positive result – is

$$P(\mathbf{y}=1) = P(\mathbf{y}=1|\mathbf{x}=1)P(\mathbf{x}=1) + P(\mathbf{y}=1|\mathbf{x}=0)P(\mathbf{x}=0). \tag{3.3}$$

The person has received a positive result $\mathbf{y}=1$ and is interested in how plausible it is that she has the disease (i.e., that $\mathbf{x}=1$). The man in the street might be duped by the statement "the test is 95% reliable, so the person's positive result implies that there is a 95% chance that the person has the disease". But this is incorrect. The correct solution is found using the Bayes' theorem.

$$\begin{aligned} P(\mathbf{x}=1|\mathbf{y}=1) &= \frac{P(\mathbf{y}=1|\mathbf{x}=1)P(\mathbf{x}=1)}{P(\mathbf{y}=1|\mathbf{x}=1)P(\mathbf{x}=1)+P(\mathbf{y}=1|\mathbf{x}=0)P(\mathbf{x}=0)} && (3.4)\\ &= \frac{0.95\times 0.01}{0.95\times 0.01 + 0.05\times 0.99} && (3.5)\\ &= 0.16. && (3.6) \end{aligned}$$

Thus, in spite of the positive result, the probability that the person has the disease is only 16%!

A follow-up question is: Suppose that the doctor recommends a medical procedure that you have a 50/50 chance of surviving. Do you accept her recommendation? Implicit assumption is that disease is so nasty that you will not survive it. More interesting what should the probability of survival due to medical intervention or not be to tip your decision one way or another. How does the test reliability affect this, and on and on.

P3.3 An information source produces 0 and 1 with probabilities 0.6 and 0.4, respectively. The output of the source is transmitted via a channel that has a probability of error (turning a 1 into a 0 or a 0 into a 1) equal to 0.1 (see Fig. 3.2).

Define the following events:

$$\begin{aligned} E_1 &= \text{“Output of the channel is 1”} && (3.7)\\ E_2 &= \text{“Output of the source is 1”} && (3.8) \end{aligned}$$

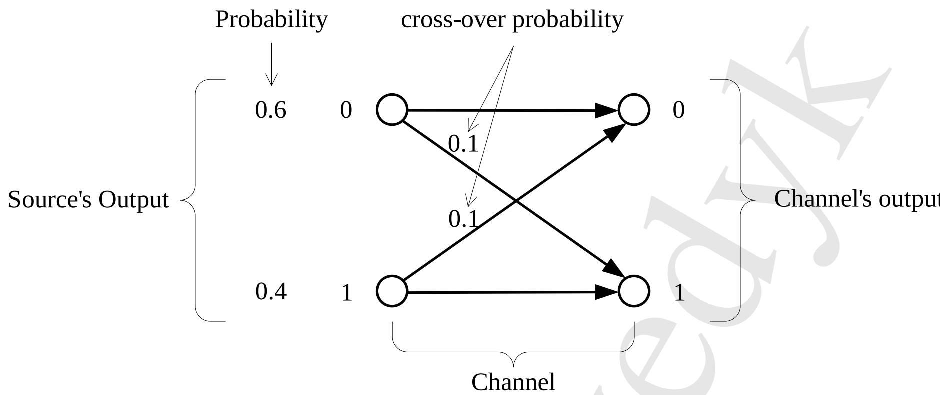


Figure 3.2

Obviously,

$$E_1^c = \text{"Output of the channel is 0"} \tag{3.9}$$
$$E_2^c = \text{"Output of the source is 0"} \tag{3.10}$$

Note that $P(E_2) = 0.4$ and $P(E_2^c) = 0.6$.

(a) Since what the channel does is independent from what the source does, the probability that "1 is observed at the output of the channel" can be calculated as follows:

$$\begin{aligned} P(E_1) &= P(\text{"Output of the source is 0"} \cap \text{"Channel makes error"}) \\ &+ P(\text{"Output of the source is 1"} \cap \text{"Channel makes no error"}) \\ &= P(\text{"Output of the source is 0"}) \times P(\text{"Channel makes error"}) \\ &+ P(\text{"Output of the source is 1"}) \times P(\text{"Channel makes no error"}) \\ &= 0.6 \times 0.1 + 0.4 \times (1 - 0.1) = 0.42 \end{aligned} \tag{3.11}$$

(b) The probability that "a 1 is the output of the source if at the output of the channel a 1 is observed" can be calculated using the *conditional probabilities* as follows:

$$\begin{aligned} P(E_2|E_1) &= \frac{P(E_2 \cap E_1)}{P(E_1)} \\ &= \frac{P(\text{"Output of the source is 1"} \cap \text{"Output of the channel is 1"})}{P(E_1)} \\ &= \frac{P(\text{"Output of the source is 1"} \cap \text{"Channel makes no error"})}{P(E_1)} \\ &= \frac{P(\text{"Output of the source is 1"}) \times P(\text{"Channel makes no error"})}{P(E_1)} \\ &= \frac{0.4 \times 0.9}{0.42} = \frac{0.36}{0.42} = 0.857 \end{aligned} \tag{3.12}$$

(c) The question in Part (b) is exactly the same as Problem 3.2 if $\mathbf{x}$ and $\mathbf{y}$ are used to describe the channel's input and output values, while the "reliability of the test" is considered as "the probability that the channel makes no error", i.e., defines the transition probability.

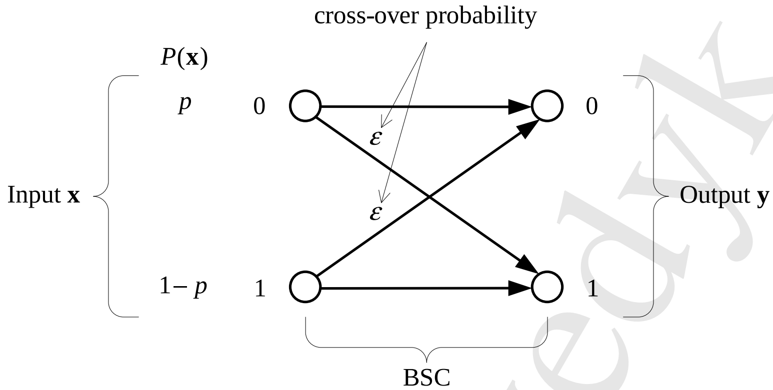


Figure 3.3

P3.4 Consider a binary symmetric channel (BSC) with cross-over probability ϵ, i.e., $P(\mathbf{y}=1|\mathbf{x}=0)=P(\mathbf{y}=0|\mathbf{x}=1)=\epsilon$, where $\mathbf{x}$ and $\mathbf{y}$ represent the binary input and output, respectively. The probability of a 0 transmitted is p (see Fig. 3.3).

(a)

$$\begin{aligned} P(\mathbf{x}=1|\mathbf{y}=1) &= \frac{P(\mathbf{x}=1,\mathbf{y}=1)}{P(\mathbf{y}=1)} \\ &= \frac{P(\mathbf{y}=1|\mathbf{x}=1)P(\mathbf{x}=1)}{P(\mathbf{y}=1|\mathbf{x}=1)P(\mathbf{x}=1)+P(\mathbf{y}=1|\mathbf{x}=0)P(\mathbf{x}=0)} \\ &= \frac{(1-\epsilon)(1-p)}{(1-\epsilon)(1-p)+\epsilon p} \end{aligned} \tag{3.13}$$

(b)

$$\begin{aligned} P[\text{error}] &= P[(\mathbf{y}=0,\mathbf{x}=1) \text{ or } (\mathbf{y}=1,\mathbf{x}=0] \\ &= P(\mathbf{y}=0|\mathbf{x}=1)P(\mathbf{x}=1)+P(\mathbf{y}=1|\mathbf{x}=0)P(\mathbf{x}=0) \\ &= \epsilon \end{aligned} \tag{3.14}$$

(c) There are $\binom{n}{k}=\frac{n!}{k!(n-k)!}$ possible sequences, each having k ones and $(n-k)$ zeros. Each sequence (or event if you wish) is mutually exclusive. Therefore

$$\begin{aligned} P[k \text{ 1's are received}] &= \binom{n}{k} P[\text{a sequence has } k \text{ 1's and } (n-k) \text{ 0's}] \\ &= \binom{n}{k} P[\text{there are } k \text{ 1's}]P[\text{there are } (n-k) \text{ 0's}] \\ &= \binom{n}{k} \left(P[\text{1 received}]\right)^k \left(P[\text{0 received}]\right)^{n-k}. \end{aligned} \tag{3.15}$$

The last two equalities in the above follow from the fact that the transmitted symbols are statistically independent. Now

$$\begin{aligned} P[\text{1 received}] &= p\epsilon + (1-p)(1-\epsilon) \\ P[\text{0 received}] &= 1-P[\text{1 received}] = 1-p\epsilon-(1-p)(1-\epsilon) = \epsilon(1-p)+p(1-\epsilon). \end{aligned}$$

$$\therefore P[k \text{ 1's are received}] = \binom{n}{k}[p\epsilon + (1-p)(1-\epsilon)]^k[\epsilon(1-p)+p(1-\epsilon)]^{n-k}.$$

(d) Since the messages are to be equally likely, it follows that $p = 1/2$. Note also that in contrast to (c) the n transmitted bits here are not at all statistically independent.

Assume that n is odd. A natural decision rule would be a majority rule one, i.e., if more 1's than 0's are received, decide that a 1 was transmitted and vice versa. If n is even there is the possibility that an equal numbers of 1's and 0's are received. If this happens, flip an unbiased coin to make a decision, otherwise rule is the same as when n is odd.

Let k be the number of received 1's. Then the decision rule can be written as:

$$k \underset{m_1}{\overset{m_2}{\gtrless}} \frac{n}{2}$$

where m_1 is one message (represented by n 0's) and m_2 is other message (represented by n 1's).

Therefore

$$\begin{aligned}
P[\text{error}] &= P[\{(m_1 \text{ decided}) \text{ and } (m_2 \text{ xmitted})\} \text{ or } \{(m_2 \text{ decided}) \text{ and } (m_1 \text{ xmitted})\}] \\
&= P[(m_1 \text{ decided})|(m_2 \text{ xmitted})]P[m_2 \text{ xmitted}] \\
&+ P[(m_2 \text{ decided})|(m_1 \text{ xmitted})]P[m_1 \text{ xmitted}] \\
&= \frac{1}{2}P\left[k < \frac{n}{2}\middle| n \text{ 1's xmitted}\right] + \frac{1}{2}P\left[k \geq \frac{n}{2}\middle| n \text{ 0's xmitted}\right] \\
&= \frac{1}{2}\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n}{k}(1-\epsilon)^k\epsilon^{n-k} + \frac{1}{2}\sum_{k=\lceil \frac{n}{2} \rceil}^{n}\binom{n}{k}\epsilon^k(1-\epsilon)^{n-k}
\end{aligned}$$

(e) As $n \to \infty$ the $P[\text{error}] \to 0$ (you might want to try justifying this mathematically). But as $n \to \infty$ the rate of transmission goes to zero. One could only transmit one message, if that.

Remark: The mapping in (d) is a *repetition code*. Thus a repetition code, if it is long enough, can always be detected correctly, but the efficiency is too low. Of course we do not want to use $n \to \infty$ time to just transmit one bit. Here we can see the tradeoff between the efficiency and the error probability. In practice, better codes that achieve the same error probability but use much less time exist.

P3.5 (a) Since $F_{\mathbf{x}}(x) = B$ for all $x > 10$ one has $1 = P(\mathbf{x} \leq \infty) = F_{\mathbf{x}}(\infty) = B$, i.e., $B = 1$. Moreover, $F_{\mathbf{x}}(x)$ reaches (or should reach) a value of 1 at $x = 10$. Therefore $A \times 10^3 = 1$ $\Rightarrow A = \frac{1}{10^3}$. So the cdf $F_{\mathbf{x}}(x)$ is as follows:

$$F_{\mathbf{x}}(x) = \begin{cases} 0, & x \leq 0 \\ 10^{-3}x^3, & 0 \leq x \leq 10 \\ 1, & x > 10 \end{cases} \tag{3.16}$$

(b) The pdf of $\mathbf{x}$ is simply:

$$f_{\mathbf{x}}(x) = \frac{\mathrm{d}F_{\mathbf{x}}(x)}{\mathrm{d}x} = \begin{cases} 0, & x \leq 0 \\ 3 \times 10^{-3}x^2, & 0 \leq x \leq 10 \\ 0, & x > 10 \end{cases} \tag{3.17}$$

Both the cdf and pdf are shown in Fig. 3.4.

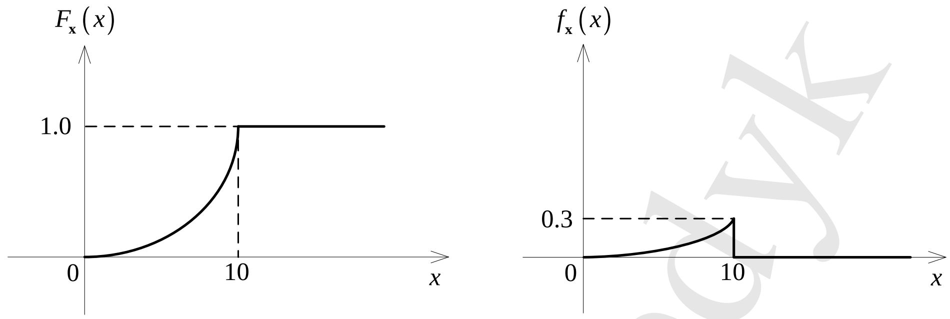


Figure 3.4: Plots of $F_{\mathbf{x}}(x)$ and $f_{\mathbf{x}}(x)$.

(c) The mean value of $\mathbf{x}$ is:

$$\begin{aligned} m_{\mathbf{x}} &= E\{\mathbf{x}\} = \int_{-\infty}^{+\infty} x f_{\mathbf{x}}(x)\mathrm{d}x \\ &= \int_0^{10} x f_{\mathbf{x}}(x)\mathrm{d}x = \int_0^{10} 3\times 10^{-3}x^3\mathrm{d}x = \frac{30}{4} = 7.5 \end{aligned}$$

The variance of $\mathbf{x}$ is:

$$\begin{aligned} \sigma_{\mathbf{x}}^2 &= \mathrm{var}(\mathbf{x}) = E\{(\mathbf{x}-m_{\mathbf{x}})^2\} = \int_{-\infty}^{+\infty}(x-m_{\mathbf{x}})^2 f_{\mathbf{x}}(x)\mathrm{d}x \\ &= \int_0^{10}(x-7.5)^2 f_{\mathbf{x}}(x)\mathrm{d}x = 3.75 \end{aligned}$$

(d)

$$P(3\le \mathbf{x}\le 7) = F_{\mathbf{x}}(7) - F_{\mathbf{x}}(3) = 0.316 \tag{3.18}$$

P3.6 To find $f_{\mathbf{y}}(y)$, we first find the cdf of $\mathbf{y}$, i.e., $F_{\mathbf{y}}(y)$. Consider the following cases for y (see Fig. 3.5):

(i) For $y < -b$. Note that $\mathbf{y}$ can only receive values in the range $-b \le y \le b$. Therefore the event $\mathbf{y} \le y$ is an *impossible event* and for this range of y one has $F_{\mathbf{y}}(y) = P(\mathbf{y}\le y) = 0$.

(ii) For $y = -b$. Then

$$\begin{aligned} F_{\mathbf{y}}(y) &= F_{\mathbf{y}}(-b) = P(\mathbf{y}\le -b) = P(\mathbf{y} < -b) + P(\mathbf{y} = -b) = P(\mathbf{y} = -b) \\ &= P(\mathbf{x}\le -b) = F_{\mathbf{x}}(-b) \\ &= \int_{-\infty}^{-b} f_{\mathbf{x}}(x)\mathrm{d}(x) = \int_{-\infty}^{-b}\frac{1}{\sqrt{2\pi\sigma^2}}\mathrm{e}^{-x^2/(2\sigma^2)}\mathrm{d}x \end{aligned} \tag{3.19}$$

(iii) For $y \ge b$. In contrast to case (i), for the range of y in this case the event $\mathbf{y}\le y$ is a *certain event* and therefore $F_{\mathbf{y}}(y) = P(\mathbf{y}\le y) = 1$.

(iv) For $-b < y < b$. For this range of y, the random variable $\mathbf{y}$ and $\mathbf{x}$ are the same. Thus $F_{\mathbf{y}}(y) = P(\mathbf{y}\le y) = P(\mathbf{x}\le y) = F_{\mathbf{x}}(y)$. This also implies that $f_{\mathbf{y}}(y) = f_{\mathbf{x}}(y) = \frac{1}{\sqrt{2\pi\sigma^2}}\mathrm{e}^{-y^2/(2\sigma^2)}$ for y in this range.

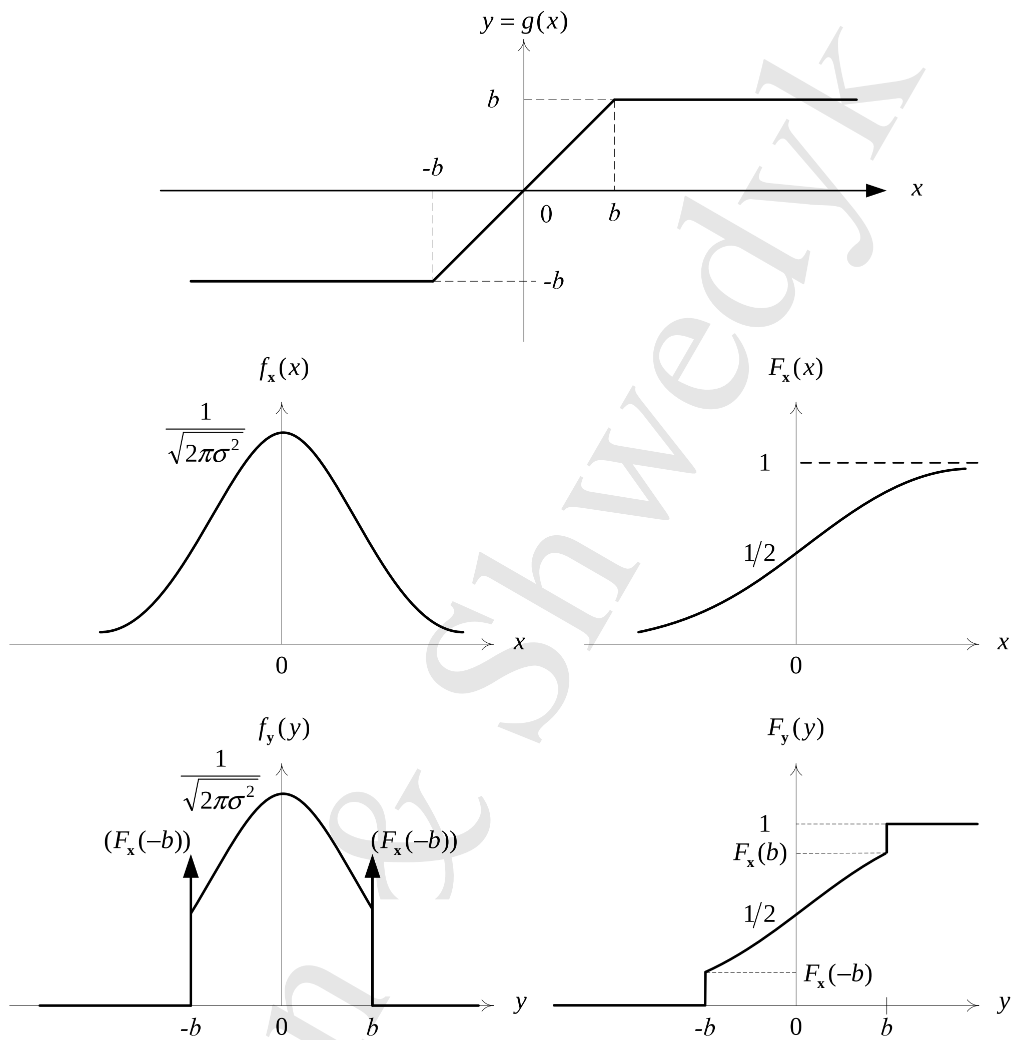


Figure 3.5

In summary,

$$F_{\mathbf{y}}(y) = \begin{cases} 0, & y < -b \\ F_{\mathbf{x}}(y), & -b \le y < b \\ 1, & y \ge b \end{cases} \tag{3.20}$$

Now, differentiating $F_{\mathbf{y}}(y)$ gives $f_{\mathbf{y}}(y)$, the pdf of $\mathbf{y}$:

$$f_{\mathbf{y}}(y) = \frac{\mathrm{d}F_{\mathbf{y}}(y)}{\mathrm{d}y} = \begin{cases} 0, & y < -b \\ F_{\mathbf{x}}(-b)\delta(y+b), & y = -b \\ f_{\mathbf{x}}(y) = \frac{1}{\sqrt{2\pi\sigma^2}}\mathrm{e}^{-y^2/(2\sigma^2)}, & -b < y < b \\ [1 - F_{\mathbf{x}}(b)]\delta(y-b) = F_{\mathbf{x}}(-b)\delta(y-b), & y = b \\ 0, & y > b \end{cases} \tag{3.21}$$

Note that the two delta functions are due to the discontinuities of $F_{\mathbf{y}}(y)$ at $y = -b$ and $y = b$. The strengths of these two delta functions are equal to the value of the "jump" of the cdf at the discontinuities, which is $F_{\mathbf{x}}(-b)$. Plots of $F_{\mathbf{x}}(x)$, $f_{\mathbf{x}}(x)$, $F_{\mathbf{y}}(y)$ and $f_{\mathbf{y}}(y)$ are provided in Fig. 3.5.

P3.7 It is straightforward enough to determine the mean $m_{\mathbf{x}}$ and variance $\sigma_{\mathbf{x}}^2$ of a random variable $\mathbf{x}$ from the definitions:

$$\begin{aligned} m_{\mathbf{x}} &= E\{\mathbf{x}\} = \int_{-\infty}^{\infty} x f_{\mathbf{x}}(x)\mathrm{d}x && (3.22) \\ \sigma_{\mathbf{x}}^2 &= E\{(\mathbf{x}-m_{\mathbf{x}})^2\} = E\{\mathbf{x}^2\} - m_{\mathbf{x}}^2 \\ &= \int_{-\infty}^{\infty} x^2 f_{\mathbf{x}}(x)\mathrm{d}x - m_{\mathbf{x}}^2. && (3.23) \end{aligned}$$

The only challenge perhaps is doing the integrations. However using the fact that the mean is the center of gravity (or centroid) of the probability "mass" density and that the variance is the variation around the mean can simplify (even eliminate sometimes) the necessity for integrating.

(a) The pdf looks as in Fig. 3.6(a). Obviously the centroid is $m_{\mathbf{x}} = \frac{1}{2}(a+b)$.

To find the variance, shift $f_{\mathbf{x}}(x)$ along the x-axis so that it lies symmetrically about $x=0$, i.e., work with the pdf in Fig. 3.6(b). Note that the shift does not affect the variance. Then

$$\sigma_{\mathbf{x}}^2 = \int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} x^2 \frac{1}{(b-a)}\mathrm{d}x = \frac{2}{(b-a)}\int_0^{\frac{b-a}{2}} x^2 \mathrm{d}x = \frac{1}{12}(b-a)^2$$

For special case of $a=-b$ then $m_{\mathbf{x}}=0$ and $\sigma_{\mathbf{x}}^2 = b^2/3$.

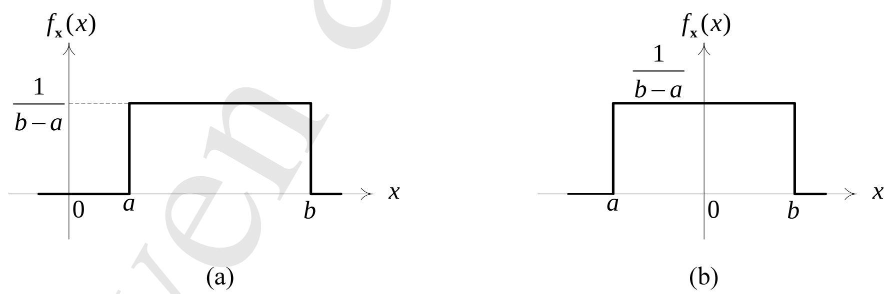


Figure 3.6: Uniform densities.

(b) For the Rayleigh density, we have no choice but to do the integrations. The following general results from G&R, p. 337 are useful:

$$\begin{aligned} \int_0^{\infty} x^{2n}\mathrm{e}^{-px^2}\mathrm{d}x &= \frac{(2n-1)!!}{2(2p)^n}\sqrt{\frac{\pi}{p}}, \text{ for } p>0 \quad \text{(Eqn. 3.461-2)} \\ \int_0^{\infty} x^{2n+1}\mathrm{e}^{-px^2}\mathrm{d}x &= \frac{n!}{2p^{n+1}}, \text{ for } p>0 \quad \text{(Eqn. 3.461-3)} \end{aligned}$$

These equations allow us to compute all the moments of a Rayleigh random variable, if we so desired. But here we are interested in only the 1st and 2nd moments.

The first moment is:

$$m_{\mathbf{x}} = \frac{1}{\sigma^2}\int_0^{\infty} x^2 \mathrm{e}^{-\frac{x^2}{2\sigma^2}}\mathrm{d}x = \frac{1}{\sigma^2}\frac{1!!}{2\left(2\frac{1}{2\sigma^2}\right)}\sqrt{\frac{\pi}{\frac{1}{2\sigma^2}}} = \sqrt{\frac{\pi}{2}}\sigma \approx 1.25\sigma.$$

The second moment is:

$$E\{\mathbf{x}^2\} = \frac{1}{\sigma^2}\int_0^\infty x^3 \mathrm{e}^{-\frac{x^2}{2\sigma^2}}\mathrm{d}x = \frac{1}{\sigma^2}\frac{1!}{2\left(\frac{1}{2\sigma^2}\right)^2}\sqrt{\frac{\pi}{\frac{1}{2\sigma^2}}} = 2\sigma^2.$$

Therefore the variance is

$$\sigma_{\mathbf{x}}^2 = E\{\mathbf{x}^2\} - m_{\mathbf{x}}^2 = 2\sigma^2 - \frac{\pi}{2}\sigma^2 = \left(2-\frac{\pi}{2}\right)\sigma^2 \approx 0.429\sigma^2.$$

Remark: The Rayleigh random variable is important in the study of fading channels (Chapter 10). As shall be seen there, the parameter σ^2 is also a variance (as the notation suggests), but of course not of the Rayleigh random variable but of the 2 underlying Gaussian random variables that make up the Rayleigh random variable.

(c) For the Laplacian density:

$$f_{\mathbf{x}}(x) = \frac{c}{2}\mathrm{e}^{-c|x|} \tag{3.24}$$

By inspection $f_{\mathbf{x}}(x)$ is an *even* function (symmetric about $x=0$), hence $m_{\mathbf{x}} = 0$. The variance of $\mathbf{x}$ is

$$\sigma_{\mathbf{x}}^2 = \frac{c}{2}\int_{-\infty}^{\infty} x^2\mathrm{e}^{-c|x|} = c\int_0^\infty x^2 \mathrm{e}^{-cx}\mathrm{d}x$$

Integrate by parts to obtain:

$$\int x^2\mathrm{e}^{-cx}\mathrm{d}x = -\frac{1}{c}x^2\mathrm{e}^{-cx} - \frac{2}{c^2}x\mathrm{e}^{-cx} - \frac{2}{c^3}\mathrm{e}^{-cx} \tag{3.25}$$

Thus

$$\int_0^\infty x^2\mathrm{e}^{-cx}\mathrm{d}x = \frac{2}{c^3} \;\Rightarrow\; \boxed{\sigma_{\mathbf{x}}^2 = \frac{2}{c^2}} \tag{3.26}$$

(d) $\mathbf{y}$ is discrete random variable, given by $\mathbf{y} = \sum_{i=1}^n \mathbf{x}_i$ where $\mathbf{x}_i$ are statistically independent and identically distributed random variables with the pmf: $P(\mathbf{x}_i = 1) = p$ and $P(\mathbf{x}_i = 0) = 1-p$.

First, the mean and variance of each random variable $\mathbf{x}_i$ can be found as:

$$\begin{aligned} m_{\mathbf{x}_i} &= E(\mathbf{x}_i) = p\times 1 + (1-p)\times 0 = p \quad (3.27)\\ E\{\mathbf{x}_i^2\} &= p\times 1^2 + (1-p)\times 0^2 = p \\ \sigma_{\mathbf{x}_i}^2 &= E\{\mathbf{x}_i^2\} - m_{\mathbf{x}_i}^2 = p - p^2 = p(1-p) \quad (3.28)\end{aligned}$$

Using linear property of the expectation operation $E\{\cdot\}$, one has:

$$E\{\mathbf{y}\} = E\left\{\sum_{i=1}^n \mathbf{x}_i\right\} = \sum_{i=1}^n E\{\mathbf{x}_i\} = np \tag{3.29}$$

To compute the variance of $\mathbf{y}$, first compute $E\{\mathbf{y}^2\}$. Write $\mathbf{y}^2$ as

$$Y^2 = \left(\sum_{i=1}^n \mathbf{x}_i\right)^2 = \sum_{i=1}^n\sum_{j=1}^n \mathbf{x}_i\mathbf{x}_j = \sum_{i=1}^n \mathbf{x}_i^2 + \sum_{i=1}^n\sum_{j=1,j\neq i}^n \mathbf{x}_i\mathbf{x}_j$$

Since $\mathbf{x}_i$, $i = 1, 2, \ldots, n$, are statistically independent, then

$$E\{\mathbf{x}_i\mathbf{x}_j\} = \begin{cases} p^2 & i \neq j \\ p & i = j \end{cases} \tag{3.30}$$

Using (3.30) and the linear property of $E\{\cdot\}$, $E\{\mathbf{y}^2\}$ can be computed as:

$$\begin{aligned} E\{\mathbf{y}^2\} &= E\left\{\sum_{i=1}^{n} \mathbf{x}_i^2\right\} + E\left\{\sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} \mathbf{x}_i\mathbf{x}_j\right\} = np + n(n-1)p^2 && (3.31) \\ \Rightarrow \sigma_{\mathbf{y}}^2 &= E\{\mathbf{y}^2\} - [E\{\mathbf{y}\}]^2 = np + n(n-1)p^2 - (np)^2 = np(1-p) && (3.32) \end{aligned}$$

P3.8 (a) $P(A \cup B) =$ area $A+$area $B-$common area (which is added twice). But area $A = P(A)$, area $B = P(B)$ and common area $= P(A \cap B)$. Hence

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

(b)

$$P(A \cup B \cup C) = P(A) + P(B \cup C) - P(A \cap (B \cup C),$$

where we consider $B \cup C$ as a single event. But $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Therefore

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) - P(A \cap C) + P(\underbrace{A \cap B \cap A \cap C}_{A \cap B \cap C}).$$

P3.9 (a) Given B then probability of A occurring is that it "lies" in the shaded area. Also B is the new sample space. Therefore

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Note that $P(A)$ can be written as

$$P(A) = \frac{P(A \cap \Omega)}{P(\Omega)}.$$

(b) Statistical independence can be expressed in 2 ways.

(i) $P(A|B) = P(A)$. In this case the interpretation is that the ratio of the common area ($P(A \cap B)$) to the new sample space area ($P(B)$) is the same as the ratio of the original area ($P(A)$) to the original sample space ($P(\Omega)$).

(ii) $P(A \cap B) = P(A)P(B)$. Here the common area $P(A \cap B)$ is equal to the product of the areas $P(A)$ and $P(B)$.

(c) No. One could say they are totally dependent. Knowledge of one occurring tells you a lot about the other occurring, i.e., $P(A|B) = 0$ and vice versa.

P3.10 Need to determine if the 3 axioms are satisfied. Given B all possible events now lie in the shaded area.

Axiom 1: P(any event) is certainly ≥ 0 and also ≤ 1.

Axiom 2: $P(B|B) = 1$.

Axiom 3: Holds. It is inherited from Ω.

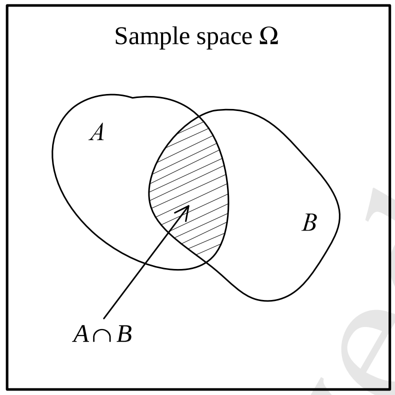


Figure 3.7

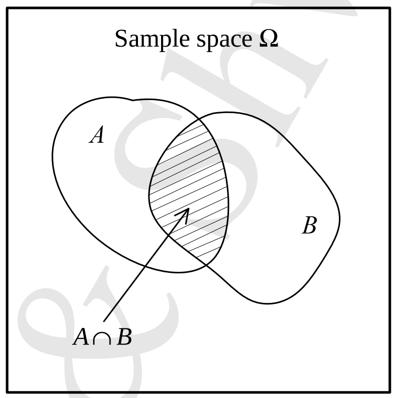


Figure 3.8

P3.11 (a) $P(A \cap B \cap C) = P(A)P(B)P(C)$, $P(A \cap B) = P(A)P(B)$, $P(A \cap C) = P(A)P(C)$; $P(B \cap C) = P(B)P(C)$.

(b) (i) All conditions of (a) satisfied, therefore statistically independent.

(ii) Note that $P(A \cap B \cap C) = 0 \neq P(A)P(B)P(C)$, therefore statistically dependent.

(iii) $P(B \cap C) = 1/8 \neq P(B)P(C) = (1/2)(1/2) = 1/4$, therefore statistically dependent.

(c) Yes. Consider the Venn diagram in Fig. 3.9. Let $a^2 = b$, where $a = P(A) = P(B) = P(C)$. Then $P(A \cap B \cap C) = 0$ while $P(A \cap B) = P(A \cap C) = P(B \cap C) = b = a^2 = P(A)P(B) = P(A)P(C) = P(B)P(C)$.

(d) Number of conditions is $\sum_{k=2}^{n} \binom{n}{k} = \sum_{k=2}^{n} \frac{n!}{k!(n-k)!}$.

P3.12 First calculate $P(A)$, $P(B)$ and $P(C)$ as follows:

$$\begin{aligned} P(A) &= 0.08 + 0.03 + 0.02 + 0.07 = 0.20 \\ P(B) &= 0.08 + 0.07 + 0.03 + 0.02 = 0.20 \\ P(C) &= 0.245 + 0.07 + 0.07 + 0.02 = 0.405 \end{aligned}$$

Now $P(A \cap B \cap C) = 0.02 \neq P(A)P(B)P(C) = (0.02)(0.02)(0.405) = 0.000162$. Therefore these events are statistically dependent.

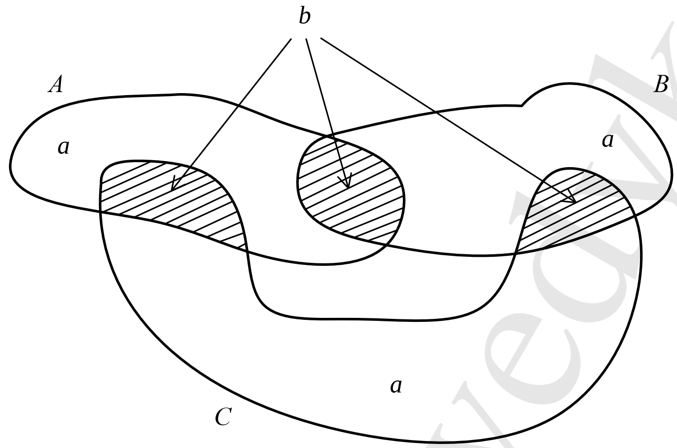


Figure 3.9

Next,

$$\begin{aligned}
P(A \cap B|C) &= \frac{P(A \cap B \cap C)}{P(C)} = \frac{0.02}{0.405} = \frac{4}{81} \\
P(A|C) &= \frac{P(A \cap C)}{P(C)} = \frac{0.09}{0.405} = \frac{2}{9} \\
P(B|C) &= \frac{P(B \cap C)}{P(C)} = \frac{0.09}{0.405} = \frac{2}{9} \\
P(A|C)P(B|C) &= \left(\frac{2}{9}\right)^2 = \frac{4}{81} = P(A \cap B|C).
\end{aligned}$$

Therefore the events A and B are conditionally independent.

Problems 3.3, 3.4, 3.13, 3.14 and 3.15 are important special cases of discrete-input discrete-output channel models. The general model is described in Fig. 3.10.

The quantities of interest are usually $P[\mathbf{y} = y_j]$, $P[\mathbf{x} = x_i|\mathbf{y} = y_j]$ and also $E\{\mathbf{xy}\}$. The general relationships for the quantities are:

$$P[\mathbf{y} = y_j] = \sum_{i=1}^{m} P[(\mathbf{y} = y_j) \cap (\mathbf{x} = x_i)] = \sum_{i=1}^{m} P[\mathbf{y} = y_j|\mathbf{x} = x_i]P[\mathbf{x} = x_i] = \sum_{i=1}^{m} P_{ij}P_i$$

$$P[\mathbf{x} = x_i|\mathbf{y} = y_j] = \frac{P[(\mathbf{x} = x_i) \cap (\mathbf{y} = y_j)]}{P[\mathbf{y} = y_j]} = \frac{P[\mathbf{y} = y_j|\mathbf{x} = x_i]P[\mathbf{x} = x_i]}{P[\mathbf{y} = y_j]} = \frac{P_{ij}P_i}{\sum_{k=1}^{m} P_{kj}P_k}$$

$$\begin{aligned}
E\{\mathbf{xy}\} &= \sum_{i=1}^{m}\sum_{j=1}^{n} x_i y_j P[(\mathbf{x} = x_i) \cap (\mathbf{y} = y_j)] \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} x_i y_j P[\mathbf{y} = y_j|\mathbf{x} = x_i)]P[\mathbf{x} = x_i] \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} x_i y_j P_{ij}P_i
\end{aligned}$$

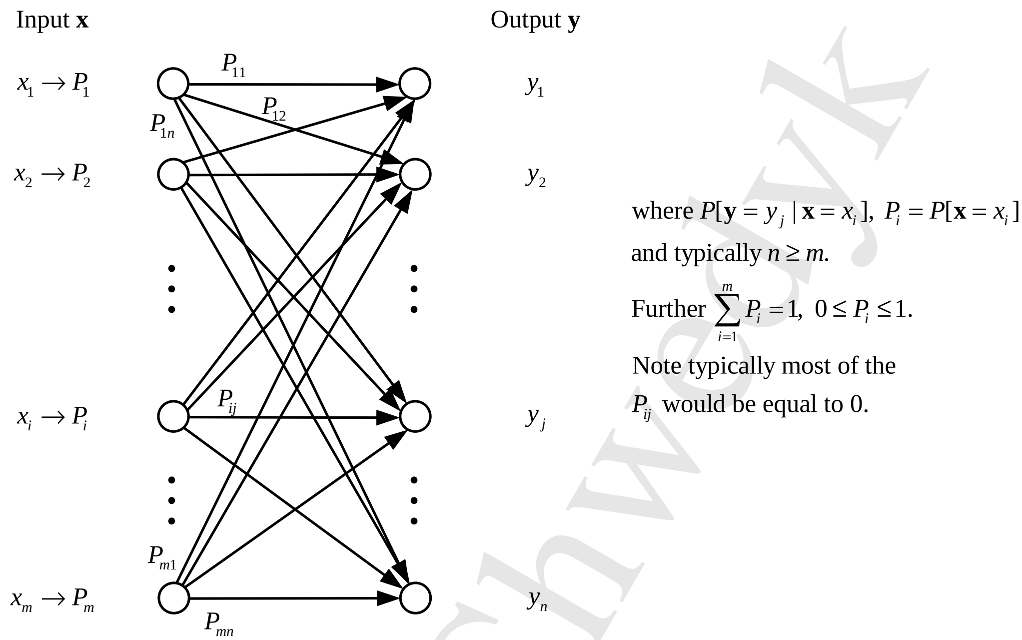


Figure 3.10

P3.13 Consider the "toy" channel model shown in Fig. 3.25 in the textbook.

(a) From the figure, we have $P(\mathbf{x} = -1) = 0.5$, $P(\mathbf{x} = -1) = 0.5$. All the transition probabilities of $\mathbf{y}$ given $\mathbf{x}$ are:

$$\begin{array}{ll} P(\mathbf{y}=1|\mathbf{x}=1)=0.4 & P(\mathbf{y}=1|\mathbf{x}=-1)=0.4 \\ P(\mathbf{y}=0|\mathbf{x}=1)=0.2 & P(\mathbf{y}=0|\mathbf{x}=-1)=0.2 \\ P(\mathbf{y}=-1|\mathbf{x}=1)=0.4 & P(\mathbf{y}=-1|\mathbf{x}=-1)=0.4; \end{array} \tag{3.33}$$

Then

$$\begin{aligned} P(\mathbf{y}=1) &= P(\mathbf{y}=1|\mathbf{x}=1)P(\mathbf{x}=1)+P(\mathbf{y}=1|\mathbf{x}=-1)P(\mathbf{x}=-1) \\ &= 0.4\times 0.5+0.4\times 0.5=0.4. \end{aligned} \tag{3.34}$$

Observe that $P(\mathbf{y} = 1) = P(\mathbf{y} = 1|\mathbf{x} = 1) = P(\mathbf{y} = 1|\mathbf{x} = -1) = 0.4$. Similarly, one can show that $P(\mathbf{y} = 0) = P(\mathbf{y} = 0|\mathbf{x} = 1) = P(\mathbf{y} = 0|\mathbf{x} = -1)$, and $P(\mathbf{y} = -1) = P(\mathbf{y} = -1|\mathbf{x} = 1) = P(\mathbf{y} = -1|\mathbf{x} = -1)$. Since the distribution of the output $\mathbf{y}$ does not depend on which value of the input $\mathbf{x}$ was transmitted, the two random variables $\mathbf{x}$ and $\mathbf{y}$ are statistically independent.

(b) We have

$$\begin{aligned} E\{\mathbf{x}\} &= x_1\times P(\mathbf{x}=x_1)+x_2\times P(\mathbf{x}=x_2) \\ &= 1\times 0.5+(-1)\times 0.5=0, \end{aligned} \tag{3.35}$$

and

$$\begin{aligned} E\{\mathbf{y}\} &= y_1\times P(\mathbf{y}=y_1)+y_2\times P(\mathbf{y}=y_2)+y_3\times P(\mathbf{y}=y_3) \\ &= 1\times 0.4+0\times 0.2+(-1)\times 0.4=0, \end{aligned} \tag{3.36}$$

Since $\mathbf{x}$ and $\mathbf{y}$ are statistically independent, one can compute $E\{\mathbf{xy}\}$ as

$$E\{\mathbf{xy}\} = E\{\mathbf{x}\}E\{\mathbf{y}\} = 0. \tag{3.37}$$

P3.14 Consider $P(\mathbf{y}=1) = \frac{1}{2}(0.3) + \frac{1}{2}(0.4) = 0.35$. But $P(\mathbf{y}=1|\mathbf{x}=1) = 0.3 \neq P[\mathbf{y}=1] \Rightarrow$ Statistical dependence.

The correlation is

$$\begin{aligned}
\sum_{i=1}^{2}\sum_{j=1}^{3} x_i y_j P_{ij} P_i &= \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{3} x_i y_j P_{ij} \\
&= \frac{1}{2}\left[\sum_{j=1}^{3} y_j P_{1j} - \sum_{j=1}^{3} y_j P_{2j}\right] \\
&= \frac{1}{2}[1(0.3)+0(0.3)-1(0.4)] - \frac{1}{2}[1(0.4)+0(0.1)-1(0.5)] \\
&= 0.
\end{aligned} \tag{3.38}$$

Therefore the random variables are uncorrelated.

Notes:

1) Uncorrelatedness does not mean (or imply) statistical independence. But statistical independence always implies uncorrelatedness.
2) One extremely important case where uncorrelatedness implies statistical independence is the case of jointly Gaussian random variables. Communication theory depends very crucially on this fact as shall be seen in later chapters.
3) Statistical independence can increase or decrease the conditional probability of an event. Two extreme cases are:
   (a) Mutually exclusive events, A, B then $P(A|B) = 0$.
   (b) An event B is contained in event A, i.e., $B \subset A$. Then $P(A|B) = 1$ where $P(A)$, $P(B) \neq 0$.

P3.15 Definitely statistically dependent since

$$P(\mathbf{y}=1) = \underbrace{\frac{1}{2}(1-\epsilon)}_{\approx 1/2} \neq P(\mathbf{y}=1|\mathbf{x}=1) = \underbrace{1-p-\epsilon.}_{\approx 1} \tag{3.39}$$

Also correlated because

$$\begin{aligned}
E\{\mathbf{xy}\} &= \frac{1}{2}\left[\sum_{j=1}^{3} y_j P_{1j} - \sum_{j=1}^{3} y_j P_{2j}\right] \\
&= \frac{1}{2}\{(1-p-\epsilon-p) - [p-(1-p-\epsilon)]\} \\
&= 1-2p-\epsilon > 0, \text{ for } p, \epsilon << 1.
\end{aligned} \tag{3.40}$$

P3.16 Consider two random variables $\mathbf{x}$ and $\mathbf{y}$ with means $m_{\mathbf{x}}$, $m_{\mathbf{y}}$ and variances $\sigma^2_{\mathbf{x}}$, $\sigma^2_{\mathbf{y}}$ respectively. Further let the two random variables be *uncorrelated*, which means that $E\{\mathbf{xy}\} = E\{\mathbf{x}\}E\{\mathbf{y}\} = m_{\mathbf{x}}m_{\mathbf{y}}$. Now "rotate" $\mathbf{x}$ and $\mathbf{y}$ to obtain new random variables $\mathbf{x}_R$ and $\mathbf{y}_R$ as follows:

$$\begin{bmatrix} \mathbf{x}_R \\ \mathbf{y}_R \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} (\cos\theta)\mathbf{x} + (\sin\theta)\mathbf{y} \\ (-\sin\theta)\mathbf{x} + (\cos\theta)\mathbf{y} \end{bmatrix} \tag{3.41}$$

where θ is some arbitrary angle.

First one has

$$\begin{aligned} \mathbf{x}_R\mathbf{y}_R &= [(\cos\theta)\mathbf{x} + (\sin\theta)\mathbf{y}]\,[-(\sin\theta)\mathbf{x} + (\cos\theta)\mathbf{y}] \\ &= -(\sin\theta\cos\theta)\mathbf{x}^2 - (\sin^2\theta)\mathbf{xy} + (\cos^2\theta)\mathbf{xy} + (\sin\theta\cos\theta)\mathbf{y}^2 \\ &= [\cos^2\theta - \sin^2\theta]\mathbf{xy} - \sin\theta\cos\theta[\mathbf{x}^2 - \mathbf{y}^2]. \end{aligned} \tag{3.42}$$

It follows that

$$\begin{aligned} E\{\mathbf{x}_R\mathbf{y}_R\} &= E\{[\cos^2\theta - \sin^2\theta]\mathbf{xy} - \sin\theta\cos\theta[\mathbf{x}^2 - \mathbf{y}^2]\} \\ &= \{\cos^2\theta - \sin^2\theta\}E\{\mathbf{xy}\} - \sin\theta\cos\theta[E\{\mathbf{x}^2\} - E\{\mathbf{y}^2\}]. \end{aligned}$$

But $E\{\mathbf{xy}\} = m_{\mathbf{x}}m_{\mathbf{y}}$, $E\{\mathbf{x}^2\} = m^2_{\mathbf{x}} + \sigma^2_{\mathbf{x}}$, $E\{\mathbf{y}^2\} = m^2_{\mathbf{y}} + \sigma^2_{\mathbf{y}}$. Therefore

$$E\{\mathbf{x}_R\mathbf{y}_R\} = [\cos^2\theta - \sin^2\theta]m_{\mathbf{x}}m_{\mathbf{y}} - \sin\theta\cos\theta[m^2_{\mathbf{x}} + \sigma^2_{\mathbf{x}} - m^2_{\mathbf{y}} - \sigma^2_{\mathbf{y}}]. \tag{3.43}$$

Now compute $E\{\mathbf{x}_R\}E\{\mathbf{y}_R\}$ as follows:

$$\begin{aligned} E\{\mathbf{x}_R\}E\{\mathbf{y}_R\} &= [(\cos\theta)m_{\mathbf{x}} + (\sin\theta)m_{\mathbf{y}}][-(\sin\theta)m_{\mathbf{x}} + (\cos\theta)m_{\mathbf{y}}] \\ &= -(\cos\theta\sin\theta)m^2_{\mathbf{x}} + \cos^2\theta m_{\mathbf{x}}m_{\mathbf{y}} - \sin^2\theta(m_{\mathbf{x}}m_{\mathbf{y}}) + (\sin\theta\cos\theta)m^2_{\mathbf{y}} \\ &= [\cos^2\theta - \sin^2\theta]m_{\mathbf{x}}m_{\mathbf{y}} - \sin\theta\cos\theta[m^2_{\mathbf{x}} - m^2_{\mathbf{y}}] \end{aligned} \tag{3.44}$$

Comparing (3.43) and (3.44) shows that the two random variables $\mathbf{x}_R$ and $\mathbf{y}_R$ are uncorrelated for any θ., i.e., $E\{\mathbf{x}_R\mathbf{y}_R\} = E\{\mathbf{x}_R\}E\{\mathbf{y}_R\}$, if and only if $\boxed{\sigma^2_{\mathbf{x}} = \sigma^2_{\mathbf{y}}}$.

Remarks:

1) Note that when one says uncorrelated one actually means *uncovarianced*. However, universally the terminology is to say uncorrelated.
2) A lot of algebra could be avoided by realizing that the mean values do not place any role as to whether the random variables are uncorrelated and therefore $m_{\mathbf{x}}$ and $m_{\mathbf{y}}$ could have been set to zero without loss of generality.
3) If the two random variables $\mathbf{x}$, $\mathbf{y}$ are correlated, then one can find a rotation angle, θ, that would make the random variables $\mathbf{x}_R$, $\mathbf{y}_R$ uncorrelated. However, the angle is determined by the correlation (and variances) of $\mathbf{x}$ and $\mathbf{y}$. Here the rotation angle is arbitrary and other considerations can be used to determine the rotation. This is exploited in Chapter 5.

P3.17 (a) $f_{\mathbf{x}}(x)\mathrm{d}x$

(b) $f_{\mathbf{y}}(y)\mathrm{d}y$

(c) Equal

(d) Replace $\mathbf{y}$ by $g(\mathbf{x})$ and $f_{\mathbf{y}}(y)\mathrm{d}y$ by $f_{\mathbf{x}}(x)\mathrm{d}x$ and sweep over x, i.e., from $x=-\infty$ to $x=+\infty$ to obtain the relationship.

P3.18 Start with the definition for the characteristic function:

$$\Phi_{\mathbf{x}}(f) = E\{\mathrm{e}^{j2\pi f\mathbf{x}}\} = \int_{-\infty}^{+\infty} \mathrm{e}^{j2\pi fx} f_{\mathbf{x}}(x)\mathrm{d}x. \tag{3.45}$$

Differentiate $\Phi_{\mathbf{x}}(f)$ n times:

$$\begin{aligned}\frac{\mathrm{d}^n\Phi_{\mathbf{x}}(f)}{\mathrm{d}f^n} &= \int_{-\infty}^{+\infty} (j2\pi x)^n \mathrm{e}^{j2\pi fx} f_{\mathbf{x}}(x)\mathrm{d}x \\ &= (j2\pi)^n \int_{-\infty}^{+\infty} x^n \mathrm{e}^{j2\pi fx} f_{\mathbf{x}}(x)\mathrm{d}x. \end{aligned} \tag{3.46}$$

Set $f=0$ and divide by $(j2\pi)^n$

$$\therefore \frac{1}{(j2\pi)^n}\frac{\mathrm{d}^n\Phi_{\mathbf{x}}(f)}{\mathrm{d}f^n}\bigg|_{f=0} = \int_{-\infty}^{+\infty} x^n f_{\mathbf{x}}(x)\mathrm{d}x = E\{x^n\}. \tag{3.47}$$

P3.19 Maclaurin expansion of $\mathrm{e}^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$.

$\therefore \mathrm{e}^{j2\pi fx} = \sum\limits_{n=0}^{\infty}\frac{(j2\pi f)^n x^n}{n!}$ and $\Phi_{\mathbf{x}}(f) = E\{\mathrm{e}^{j2\pi f\mathbf{x}}\} = \sum\limits_{n=0}^{\infty}\frac{(j2\pi f)^n}{n!}E\{\mathbf{x}^n\}$.

$\Rightarrow$ The characteristic function is completely determined by the moments of the random variable.

Note that expectation is a linear operation. Thus the expectation of a sum is the sum of the expectations.

P3.20 The characteristic function is the "Fourier transform" of the probability density function and everything that was discussed in Chapter 2 should carry over here. There are some differences that must be taken into account. In Chapter 2 we went from the time domain to the frequency domain and back whereas here we are going from the probability density domain to the frequency domain and back. But this is a trivial difference. Indeed the Fourier transform could be applied to a function of any independent variable, say temperature, length, etc..

The main (and only) difference between the Fourier transform in Chapter 2 and the characteristic function is the sign convention for f, i.e.,

$$\begin{array}{ccc} x(t) & \longleftrightarrow & X(f) = \int_{-\infty}^{+\infty} x(t)\mathrm{e}^{-j2\pi ft}\mathrm{d}t \\ \| & & \\ \int_{-\infty}^{+\infty} X(f)\mathrm{e}^{j2\pi ft}\mathrm{d}f & & \end{array} \tag{3.48}$$

$$
\begin{array}{ccc}
f_{\mathbf{x}}(x) & \longleftrightarrow & \Phi_{\mathbf{x}}(f) = \int\limits_{-\infty}^{+\infty} f_{\mathbf{x}}(x) \mathrm{e}^{j2\pi f x} \mathrm{d}x \\
\| & & \\
\int\limits_{-\infty}^{+\infty} \Phi_{\mathbf{x}}(f) \mathrm{e}^{-j2\pi f x} \mathrm{d}f & &
\end{array}
\tag{3.49}
$$

which shows that $f \to -f$.

(a) Example 2.17, Chapter 2, establishes that $V\mathrm{e}^{-at^2} \leftrightarrow V\sqrt{\frac{\pi}{a}}\mathrm{e}^{-\frac{\pi^2 f^2}{a}}$ are a Fourier transform pair. Here we need the characteristic function (or Fourier transform) of $\frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}$. Identify $V \equiv \frac{1}{\sqrt{2\pi}\sigma}, a = \frac{1}{2\sigma^2}$, substitute and do the algebra.

$$
\therefore \Phi_{\mathbf{x}}(f) = \frac{1}{\sqrt{2\pi}\sigma}\sqrt{\pi 2\sigma^2}\mathrm{e}^{-\pi^2(-f)^2 2\sigma^2} = \mathrm{e}^{-2\pi^2\sigma^2 f^2}. \tag{3.50}
$$

(b) Example 2.9, Chapter 2, states $V\mathrm{e}^{-a|t|} \leftrightarrow \frac{V2a}{a^2+4\pi^2 f^2}$.
Here we are concerned with $f_{\mathbf{x}}(x) = \frac{c}{2}\mathrm{e}^{-c|x|}$. Identify $V \equiv \frac{c}{2}, a \equiv c$. Then

$$
\therefore \Phi_{\mathbf{x}}(f) = \frac{(c/2)(2c)}{c^2 + 4\pi^2(-f)^2} = \frac{c^2}{c^2 + 4\pi^2 f^2}. \tag{3.51}
$$

(c) Example 2.11, Chapter 2, shows that a rectangular pulse of width T and height V has the Fourier transform, $VT\frac{\sin(\pi f T)}{\pi f T}$. Here we are dealing with a pdf of width $2A$, height $1/2A$, i.e., $T = 2A, V = 1/2A$.

$$
\therefore \Phi_{\mathbf{x}}(f) = \frac{\sin \pi(-f)2A}{\pi(-f)2A} = \frac{\sin(2\pi f A)}{(2\pi f A)}. \tag{3.52}
$$

(Note that both rectangular pulses are centered at the origin).

Remark: Because the pdf's were even functions the f convention did not matter. Is it possible to have a pdf that is odd?

P3.21 Obviously the odd moments are zero since the integrand is an odd function. Consider now the relationship

$$
\frac{1}{\sqrt{2\pi\sigma^2}} \int\limits_{-\infty}^{+\infty} \mathrm{e}^{-\frac{x^2}{2\sigma^2}} \mathrm{d}x = 1. \tag{3.53}
$$

Write this as

$$
\int\limits_{-\infty}^{+\infty} \mathrm{e}^{-\alpha x^2} = \frac{\sqrt{\pi}}{\sqrt{\alpha}}, \text{ where } \alpha \text{ is defined as } \alpha \equiv \frac{1}{2\sigma^2}. \tag{3.54}
$$

Differentiate both sides with respect to α:

$$
\frac{\mathrm{d}}{\mathrm{d}\alpha} : \int\limits_{-\infty}^{+\infty} -x^2 \mathrm{e}^{-\alpha x^2} \mathrm{d}x = -\frac{1}{2}\frac{\sqrt{\pi}}{\alpha^{3/2}} \tag{3.55}
$$

and again

$$\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}: \int_{-\infty}^{+\infty}(-x^2)(-x^2)\mathrm{e}^{-\alpha x^2}\mathrm{d}x = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{\sqrt{\pi}}{\alpha^{5/2}} \tag{3.56}$$

and again

$$\frac{\mathrm{d}^3}{\mathrm{d}\alpha^3}: \int_{-\infty}^{+\infty}(-x^2)^3\mathrm{e}^{-\alpha x^2}\mathrm{d}x = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\frac{\sqrt{\pi}}{\alpha^{7/2}}. \tag{3.57}$$

The pattern is such that after the nth differentiation

$$\frac{\mathrm{d}^n}{\mathrm{d}\alpha^n}: \int_{-\infty}^{+\infty}(-x^2)^n\mathrm{e}^{-\alpha x^2}\mathrm{d}x = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{2n-1}{2}\right)\frac{\sqrt{\pi}}{\alpha^{(2n+1)/2}}. \tag{3.58}$$

Note that the LHS has the form of $E\{\mathbf{x}^{2n}\}$ but some algebra is needed to see this

$$\int_{-\infty}^{+\infty}(-1)^n x^{2n}\mathrm{e}^{-\alpha x^2}\mathrm{d}x = (-1)^n\frac{[1\cdot 3\cdot 5\cdots(2n-1)]}{2^n}\frac{\sqrt{\pi}}{\alpha^n\alpha^{1/2}} \tag{3.59}$$

Now bring back the definition of α:

$$\int_{-\infty}^{+\infty} x^{2n}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}\mathrm{d}x = \frac{[1\cdot 3\cdot 5\cdots(2n-1)]}{2^n}\sqrt{\pi}2^n(\sigma^2)^n\sqrt{2}\sigma \tag{3.60}$$

Rearrange:

$$E\{x^{2n}\} = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty} x^{2n}\mathrm{e}^{-\alpha x^2}\mathrm{d}x = 1\cdot 3\cdot 5\cdots(2n-1)(\sigma^2)^n,\ n=0,1,2,\ldots \tag{3.61}$$

P3.22 (a) Let $\mathbf{y} = \mathbf{x} - m_{\mathbf{x}}$, i.e., $\mathbf{y}$ is a zero mean r.v., variance $\sigma^2_{\mathbf{x}}$. Then $\Phi_{\mathbf{y}}(f) = E\{\mathrm{e}^{j2\pi f\mathbf{y}}\} = E\{\mathrm{e}^{j2\pi f(\mathbf{x}-m_{\mathbf{x}})}\} = \mathrm{e}^{-j2\pi f m_{\mathbf{x}}}E\{\mathrm{e}^{j2\pi f\mathbf{x}}\}$ or $\Phi_{\mathbf{y}}(f) = \mathrm{e}^{-j2\pi f m_{\mathbf{x}}}\Phi_{\mathbf{x}}(f)$.

(b) $\Phi_{\mathbf{x}}(f) = \mathrm{e}^{j2\pi f m_{\mathbf{x}}}\mathrm{e}^{-j2\pi^2 f^2\sigma^2_{\mathbf{x}}}$.

P3.23 (a)

$$\begin{aligned} E\{\mathrm{e}^{j2\pi f(\mathbf{x}_1+\mathbf{x}_2)}\} &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{e}^{j2\pi f(x_1+x_2)}\underbrace{f_{\mathbf{x}_1\mathbf{x}_2}(x_1,x_2)}_{=f_{\mathbf{x}_1}(x_1)f_{\mathbf{x}_2}(x_2)}\mathrm{d}x_1\mathrm{d}x_2 \\ &= \underbrace{\int_{-\infty}^{+\infty}\mathrm{e}^{j2\pi f x_1}f_{\mathbf{x}_1}(x_1)\mathrm{d}x_1}_{\Phi_{\mathbf{x}_1}(f)}\cdot\underbrace{\int_{-\infty}^{+\infty}\mathrm{e}^{j2\pi f x_2}f_{\mathbf{x}_2}(x_2)\mathrm{d}x_2}_{\Phi_{\mathbf{x}_2}(f)}. \end{aligned} \tag{3.62}$$

Therefore $\Phi_{\mathbf{y}}(f) = \Phi_{\mathbf{x}_1}(f)\Phi_{\mathbf{x}_2}(f)$.

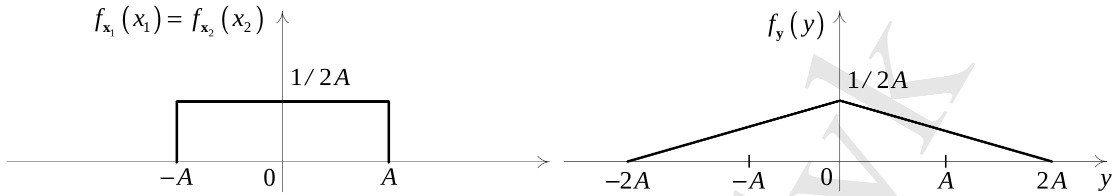


Figure 3.11: Plots of $f_{\mathbf{x}_1}(x_1)$ and $f_{\mathbf{y}}(y)$.

(b) Convolution $\Rightarrow f_{\mathbf{y}}(y) = f_{\mathbf{x}_1}(x_1) \circledast f_{\mathbf{x}_2}(x_2) = \int_{-\infty}^{+\infty} f_{\mathbf{x}_1}(\lambda) f_{\mathbf{x}_2}(y-\lambda)\mathrm{d}\lambda$. Convolving graphically gives $f_{\mathbf{y}}(y)$ as shown in Fig. 3.11.

(c) $\Phi_{\mathbf{y}}(f) = \prod_{k=1}^{n} \Phi_{\mathbf{x}_k}(x_k)$ or $f_{\mathbf{y}}(y) = f_{\mathbf{x}_1}(x_1) \circledast f_{\mathbf{x}_2}(x_2) \circledast \cdots \circledast f_{\mathbf{x}_n}(x_n)$.
Above is true as long as the $\mathbf{x}_k$'s are statistically independent, i.e., it does not depend on the random variables being Gaussian. However, if Gaussian, the characteristic function of the kth r.v. is:

$$\Phi_{\mathbf{x}_k}(f) = \mathrm{e}^{-j2\pi f m_{\mathbf{x}_k}} \mathrm{e}^{-2\pi^2 f^2 \sigma^2_{\mathbf{x}_k}} \tag{3.63}$$

$$\text{and } \Phi_{\mathbf{y}}(f) = \prod_{k=1}^{n} \mathrm{e}^{-j2\pi f m_{\mathbf{x}_k}} \mathrm{e}^{-2\pi^2 f^2 \sigma^2_{\mathbf{x}_k}} = \mathrm{e}^{-j2\pi f\left(\sum_{k=1}^{n} m_{\mathbf{x}_k}\right)} \mathrm{e}^{-2\pi^2 f^2 \left(\sum_{k=1}^{n} \sigma^2_{\mathbf{x}_k}\right)} \tag{3.64}$$

which is a characteristic function of a Gaussian r.v., with mean $m_{\mathbf{y}} = \sum_{k=1}^{n} m_{\mathbf{x}_k}$ and variance $\sigma^2_{\mathbf{y}} = \sum_{k=1}^{n} \sigma^2_{\mathbf{x}_k}$.

Note that the above result is easily generalized to a weighted linear sum, i.e., $\mathbf{y} = \sum_{k=1}^{n} a_k \mathbf{x}_k$. Then $\mathbf{y}$ is Gaussian, mean $m_{\mathbf{y}} = \sum_{k=1}^{n} a_k m_{\mathbf{x}_k}$, variance $\sigma^2_{\mathbf{y}} = \sum_{k=1}^{n} a_k^2 \sigma^2_{\mathbf{x}_k}$ where $\mathbf{x}_k$ are still statistically independent Gaussian random variables.

P3.24 (a) For $\mathbf{y}$ to lie in the region $(y, y+\mathrm{d}y]$ one of three mutually exclusive events has to occur: $\mathbf{x}$ has to fall in the region $[x_1 + \mathrm{d}x_1, x_1)$ or $(x_2, x_2 + \mathrm{d}x_2]$ or $[x_3 + \mathrm{d}x_3, x_3)$, i.e.,

$$\begin{aligned} P[y < \mathbf{y} \le y + \mathrm{d}y] &= P[x_1 + \mathrm{d}x_1 \le \mathbf{x}_1 < x_1] + P[x_2 < \mathbf{x}_2 \le x_2 + \mathrm{d}x_2] \\ &\quad + P[x_3 + \mathrm{d}x_3 \le \mathbf{x}_3 < x_3] \end{aligned} \tag{3.65}$$

In general $P[z < \mathbf{z} \le z + \mathrm{d}z] = f_{\mathbf{z}}(z)\mathrm{d}z$. Therefore:

$$f_{\mathbf{y}}\mathrm{d}y = f_{\mathbf{x}}(x_1)|\mathrm{d}x_1| + f_{\mathbf{x}}(x_2)\mathrm{d}x_2 + f_{\mathbf{x}}(x_3)|\mathrm{d}x_3|. \tag{3.66}$$

(b) Infinitesimals $\mathrm{d}x_1, \mathrm{d}x_3$ are negative quantities and the very least we should expect of a probability is for it to be ≥ 0.

(c) For the specific example we have

$$
\begin{aligned}
f_{\mathbf{y}}(y) &= \sum_{i=1}^{3} \frac{f_{\mathbf{x}}(x_i)|\mathrm{d}x_i|}{|\mathrm{d}y|} \\
&\quad \text{no harm in putting a magnitude sign on a positive quantity, } \mathrm{d}y. \\
&= \sum_{i=1}^{3} \frac{f_{\mathbf{x}}(x_i)}{\left|\frac{\mathrm{d}y}{\mathrm{d}x_i}\right|} = \sum_{i=1}^{3} \frac{f_{\mathbf{x}}(x_i)}{\left|\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=x_i}} \qquad (3.67)
\end{aligned}
$$

More generally $f_{\mathbf{y}}(y) = \sum_i \frac{f_{\mathbf{x}}(x_i)}{\left|\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=x_i}}$ where the x_i's are the *real* roots of $g(x) = y$ for the specific value of y.

P3.25 Consider the graph of $g(x)$ in Fig. 3.12.

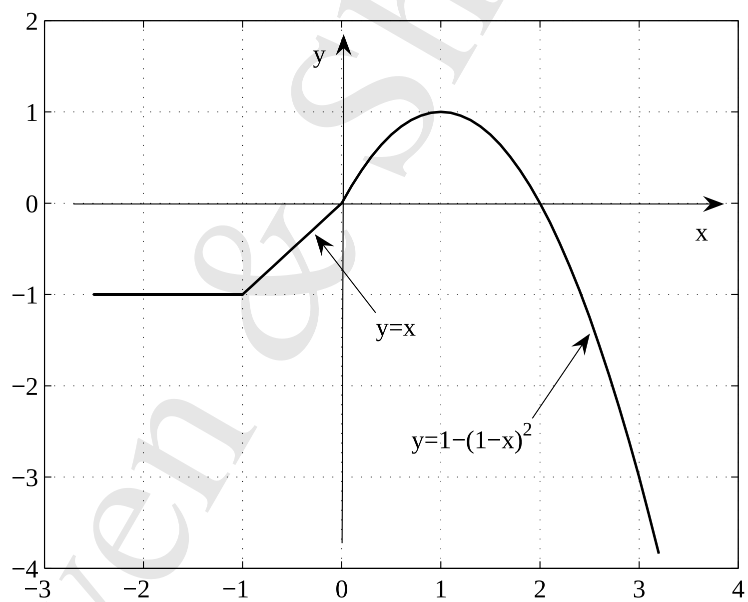


Figure 3.12

Inspection shows that:

(a) For y in the range $(-\infty, -1)$ there is only one real root, namely the positive root of $1-(1-x)^2 = y \Rightarrow x_r = 1+\sqrt{1-y}$, $-\infty < y < -1$.

(b) At $y = -1$ there are an infinite number of roots (actually a noncountable infinity) which implies that $P(\mathbf{y} = -1)$ is nonzero. Indeed $P(\mathbf{y} = -1) = P(-\infty < \mathbf{x} < -1) = \int_{-\infty}^{-1} f_{\mathbf{x}}(x)\mathrm{d}x = \frac{1}{2}\mathrm{e}^{-1}$. Therefore $f_{\mathbf{y}}(y)$ has an impulse at $y = -1$ of this strength.

(c) For $-1 < y \leq 0$ there are two roots to the equation $g(x) = y$, one from the relationship $x = y$; the other from the relationship $1-(1-x)^2 = y$ (again the positive root). The roots are therefore $x_1 = y$, $x_2 = 1+\sqrt{1-y}$.

(d) For $0 < y < 1$ there again are two roots from $1-(1-x)^2 = y$ which are $x_{1,2} = 1 \pm \sqrt{1-y}$.

(e) Finally for $y > 1$ there are no root $\Rightarrow f_{\mathbf{y}}(y) = 0$.

As always

$$f_{\mathbf{y}}(y) = \sum_{x_r:\text{ the roots of } g(x)=y} \frac{f_{\mathbf{x}}(x_r)}{\left|\frac{\mathrm{d}g(x)}{\mathrm{d}x}\right|_{x_r}}. \text{ Here } \frac{\mathrm{d}g(x)}{\mathrm{d}x} = 2(1-x). \tag{3.68}$$

Therefore:

(a) $-\infty < y < -1$: $x_r = 1+\sqrt{1-y} \Rightarrow f_{\mathbf{y}}(y) = \frac{1}{2}\frac{\mathrm{e}^{-|1+\sqrt{1-y}|}}{|2(-\sqrt{1-y})|}$.

(b) $y=-1$: $f_{\mathbf{y}}(y) = \frac{1}{2}\mathrm{e}^{-1}\delta(y+1)$.

(c) $-1 < y < 0$: $f_{\mathbf{y}}(y) = \frac{1}{2}\frac{\mathrm{e}^{-|1+\sqrt{1-y}|}}{|2(-\sqrt{1-y})|} + \frac{1}{2}\mathrm{e}^{-|y|}$.

(d) $0 < y < 1$: $f_{\mathbf{y}}(y) = \frac{1}{2}\frac{\mathrm{e}^{-|1+\sqrt{1-y}|}}{|2(-\sqrt{1-y})|} + \frac{1}{2}\frac{\mathrm{e}^{-|1-\sqrt{1-y}|}}{|2(-\sqrt{1-y})|}$.

(e) $y > 1$: $f_{\mathbf{y}}(y) = 0$.

A plot of the pdf is shown in Fig. 3.13

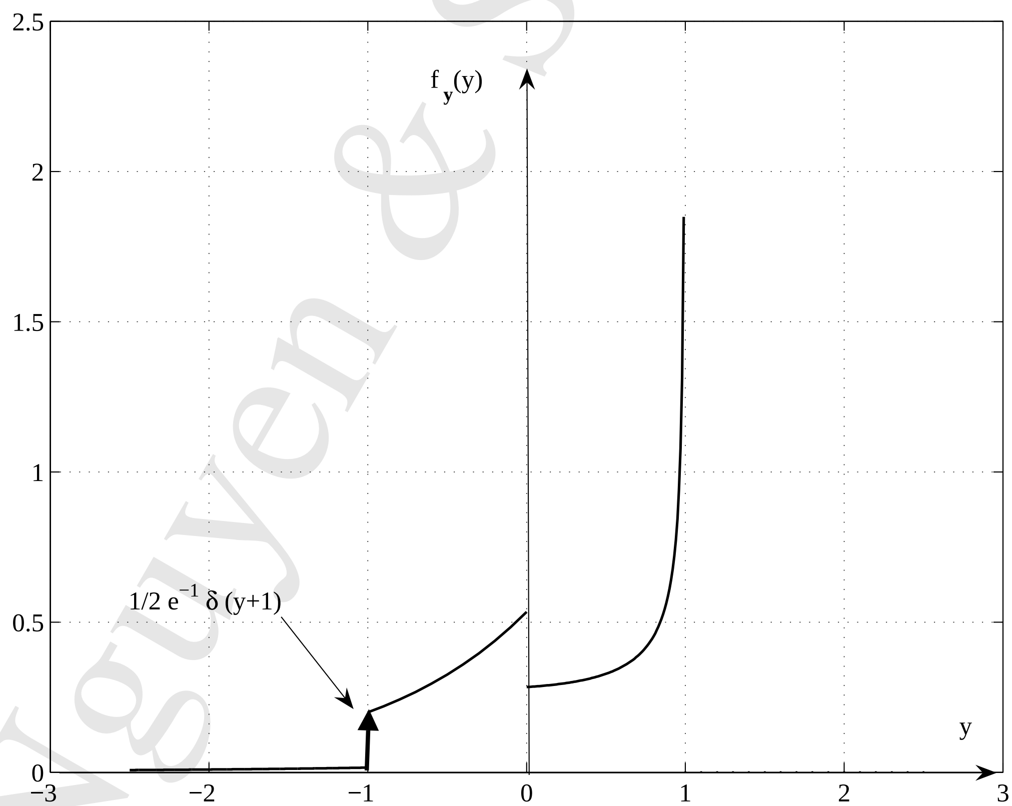


Figure 3.13

P3.26 (a) $y = g(\theta) = \cos(\alpha+\theta)$. Consider the graph of the nonlinearity $g(\theta)$ in Fig. 3.14. The roots are $\theta_1 = \cos^{-1} y - \alpha$; $\theta_2 = \pi - \alpha + d = 2\pi - 2\alpha - \theta_1 = 2\pi - \alpha - \cos^{-1} y$. $\frac{\mathrm{d}y}{\mathrm{d}\theta} = -\sin(\alpha+\theta)$.

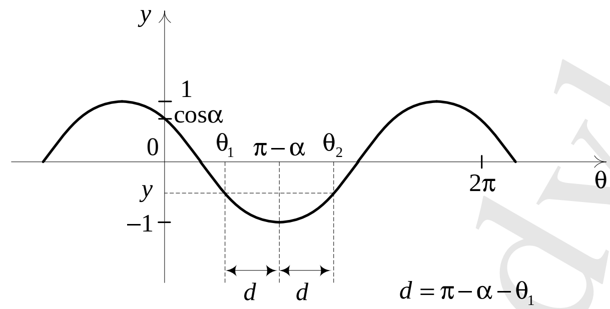


Figure 3.14: Plot of the nonlinearity $y = g(\theta) = \cos(\alpha + \theta)$.

$$\therefore f_{\mathbf{y}}(y) = \frac{f_{\boldsymbol{\theta}}(\theta)|_{\theta_1}}{\left|\frac{\mathrm{d}y}{\mathrm{d}\theta}\Big|_{\theta_1}\right|} + \frac{f_{\boldsymbol{\theta}}(\theta)|_{\theta_2}}{\left|\frac{\mathrm{d}y}{\mathrm{d}\theta}\Big|_{\theta_2}\right|}. \tag{3.69}$$

Now $f_{\boldsymbol{\theta}}(\theta)|_{\theta_1} = f_{\boldsymbol{\theta}}(\theta)|_{\theta_2} = \frac{1}{2\pi}$ ($\boldsymbol{\theta}$ is uniform over $(0, 2\pi]$).

$$\frac{\mathrm{d}y}{\mathrm{d}\theta}\bigg|_{\theta_1} = -\sin(\alpha + \cos^{-1} y - \alpha) = -\sin(\cos^{-1} y) = -\sqrt{1-y^2} \tag{3.70}$$

$$\frac{\mathrm{d}y}{\mathrm{d}\theta}\bigg|_{\theta_2} = -\sin(\alpha + 2\pi - \alpha - \cos^{-1} y) = \sin(\cos^{-1} y) = \sqrt{1-y^2} \tag{3.71}$$

Therefore

$$f_{\mathbf{y}}(y) = \begin{cases} \frac{1}{\pi\sqrt{1-y^2}}, & -1 \le y \le 1 \\ 0, & \text{elsewhere} \end{cases} \tag{3.72}$$

which plots as in Fig. 3.15.
As a check, we know

$$\int_{-\infty}^{+\infty} f_{\mathbf{y}}(y)\mathrm{d}y = \int_{-1}^{1} \frac{1}{\pi\sqrt{1-y^2}}\mathrm{d}y \stackrel{?}{=} 1 \tag{3.73}$$

$$\text{or } \int_{0}^{1} \frac{1}{\sqrt{1-y^2}}\mathrm{d}y \stackrel{?}{=} \frac{\pi}{2}. \tag{3.74}$$

The LHS is $\sin^{-1}(y)\Big|_0^1 = \sin^{-1}(1) - \sin^{-1}(0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$. So the question mark can be removed.
No, it *does not* depend on α.

(b) A sketch of the relationship $y = \cos(\theta + \alpha)$ where $0 \le \theta \le \pi/9$ (see Fig. 3.16) shows that there is only one real root at $\theta_r = \cos^{-1} y - \alpha$ and that y lies in the range $\cos(\alpha + \pi/9) \le y \le \cos\alpha)$.

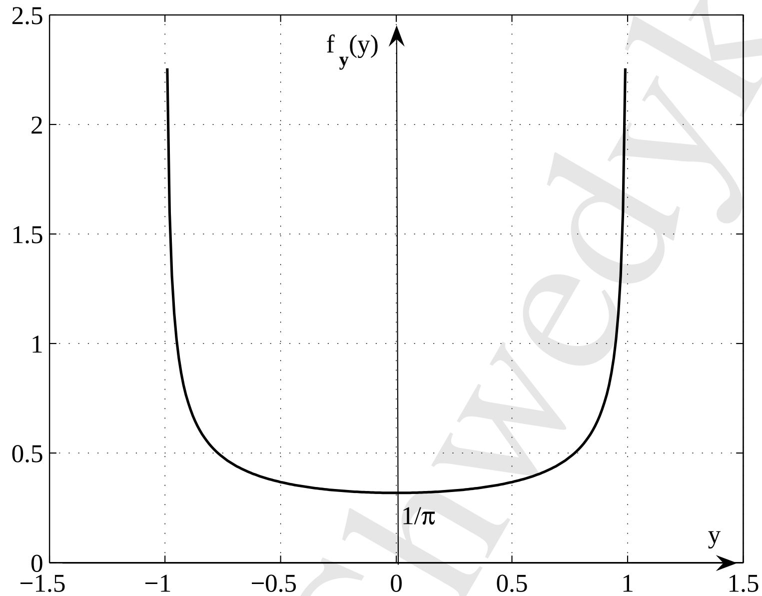


Figure 3.15

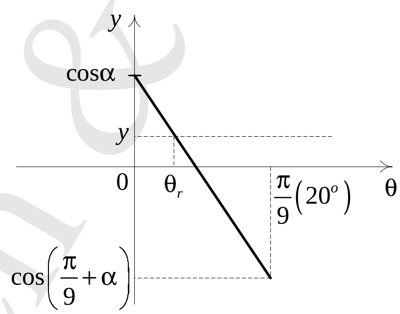


Figure 3.16

Again $\left.\frac{\mathrm{d}y}{\mathrm{d}\theta}\right|_{\theta=\theta_r} = -\sin(\alpha + \cos^{-1} y - \alpha) = \frac{1}{\sqrt{1-y^2}}$.

Now $f_{\boldsymbol{\theta}}(\theta) = \frac{9}{\pi}\left[u(0) - u(\theta - \frac{\pi}{9})\right]$ and therefore $\left. f_{\boldsymbol{\theta}}(\theta)\right|_{\theta_r} = \frac{9}{\pi}\left[u(0) - u(\cos^{-1} y - \alpha - \frac{\pi}{9})\right]$ where $\cos(\alpha + \pi/9) \le y \le \cos\alpha$.

Therefore

$$f_{\mathbf{y}}(y) = \frac{\frac{9}{\pi}\left[u(0) - u(\cos^{-1} y - \alpha - \frac{\pi}{9})\right]}{\sqrt{1-y^2}} \left\{ u[\cos(\alpha + \frac{\pi}{9}) - u(\cos\alpha)] \right\}, \tag{3.75}$$

which depends on α.

In communication models α is invariably $2\pi f t$, i.e., $\mathbf{y} = \cos(2\pi f t + \boldsymbol{\theta})$. The above shows that if $\boldsymbol{\theta}$ is uniform over $[0, 2\pi)$ then the random process is stationary (at least 1st order). However if it is restricted then in general it becomes nonstationary.

P3.27 (a) ≥ 0

(b) Since it is always ≥ 0 the function $g(\lambda) = E\{(\mathbf{x}+\lambda\mathbf{y})^2\}$ cannot cross the λ axis, i.e.,

$$g(\lambda) = E\{\mathbf{x}^2\} + 2\lambda E\{\mathbf{xy}\} + \lambda^2 E\{\mathbf{y}^2\} \geq 0. \tag{3.76}$$

This means that the equation $g(\lambda) = 0$ has two complex roots at

$$\lambda_{1,2} = \frac{-2E\{\mathbf{xy}\} \pm \sqrt{4E^2\{\mathbf{xy}\} - 4E\{\mathbf{x}^2\}E\{\mathbf{y}^2\}}}{2E\{\mathbf{y}^2\}}. \tag{3.77}$$

(Recall from high school that the roots of $ax^2+bx+c=0$ are at $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$).

For the roots to be complex the expression under the square root must be ≤ 0. This requires that $E^2\{\mathbf{xy}\} \leq E\{\mathbf{x}^2\}E\{\mathbf{y}^2\}$ or $E\{\mathbf{xy}\} \leq \sqrt{E\{\mathbf{x}^2\}E\{\mathbf{y}^2\}}$.

(c) Equality holds when $\mathbf{y} = k\mathbf{x}$, i.e., there is a linear relationship between the random variables $\mathbf{x}, \mathbf{y}$. (Statisticians are wont to call this a linear regression).

(d) Consider the random variables $\mathbf{x} - m_{\mathbf{x}}$ and $\mathbf{y} - m_{\mathbf{y}}$. Then from (b) we have

$$E\{(\mathbf{x}-m_{\mathbf{x}})(\mathbf{y}-m_{\mathbf{y}})\} \leq \sqrt{E\{(\mathbf{x}-m_{\mathbf{x}})^2\}E\{(\mathbf{y}-m_{\mathbf{y}})^2\}} = \sqrt{\sigma_{\mathbf{x}}^2\sigma_{\mathbf{y}}^2} = \sigma_{\mathbf{x}}\sigma_{\mathbf{y}}$$

$$\text{or } \underbrace{\frac{E\{(\mathbf{x}-m_{\mathbf{x}})(\mathbf{y}-m_{\mathbf{y}})\}}{\sigma_{\mathbf{x}}\sigma_{\mathbf{y}}}}_{=\rho_{\mathbf{x},\mathbf{y}}} \leq 1.$$

But the expression in (b) should properly be written as

$$-\sqrt{E\{\mathbf{x}^2\}E\{\mathbf{y}^2\}} \leq E\{\mathbf{xy}\} \leq \sqrt{E\{\mathbf{x}^2\}E\{\mathbf{y}^2\}} \tag{3.78}$$

$$\left(\text{or } |E\{\mathbf{xy}\}| \leq \sqrt{E\{\mathbf{x}^2\}E\{\mathbf{y}^2\}}\right) \tag{3.79}$$

from which it follows that $-1 \leq \rho \leq 1$.

(e)

$$|E\{\mathbf{x}(t)\mathbf{x}(t+\tau)\}| \leq \sqrt{E\{\mathbf{x}^2(t)\}E\{\mathbf{x}^2(t+\tau)\}} \tag{3.80}$$

$$\text{or } |R_{\mathbf{x}}(\tau)| \leq \sqrt{R_{\mathbf{x}}(0)R_{\mathbf{x}}(0)} = R_{\mathbf{x}}(0) \tag{3.81}$$

where it is assumed that the process is stationary.

P3.28 (a)

$$P(y < \mathbf{y} \leq y+\Delta y | x < \mathbf{x} \leq x+\Delta x) = \frac{P(y<\mathbf{y}\leq y+\Delta y, x<\mathbf{x}\leq x+\Delta x)}{P(x<\mathbf{x}\leq x+\Delta x)}. \tag{3.82}$$

(b) Assuming continuous random variables the expression becomes $\frac{0}{0}$.

(c) Rewrite the expression in (a) as:

$$\frac{P(y<\mathbf{y}\le y+\Delta y|x<\mathbf{x}\le x+\Delta x)}{\Delta y}=\frac{P(y<\mathbf{y}\le y+\Delta y,x<\mathbf{x}\le x+\Delta x)}{\Delta x\Delta y\frac{P(x<\mathbf{x}\le x+\Delta x)}{\Delta x}}. \tag{3.83}$$

As $\Delta x,\Delta y\to 0$ the LHS becomes $f_{\mathbf{y}}(y|\mathbf{x}=x)$ (more appropriately $f_{\mathbf{y}|\mathbf{x}}(y|\mathbf{x}=x)$) and the RHS becomes $\frac{f_{\mathbf{x}\mathbf{y}}(x,y)}{f_{\mathbf{x}}(x)}$.

P3.29

$$f_{\mathbf{y}}(y|x)=\frac{f_{\mathbf{x},\mathbf{y}}(x,y)}{f_{\mathbf{x}}(x)}\quad \begin{array}{l}\longrightarrow \text{ a quadratic in } \mathbf{x},\mathbf{y}\\ \longrightarrow \text{ a quadratic in } \mathbf{x}\end{array}\Bigg\}\begin{array}{l}\Rightarrow \text{ a quadratic in } \mathbf{y}.\\ \text{Therefore Gaussian.}\end{array} \tag{3.84}$$

By symmetry $f_{\mathbf{x}}(x|y)$ is also quadratic.

P3.30 Consider

$$\begin{aligned} P(A|r<\mathbf{r}\le r+\Delta r) &= \frac{P(r<\mathbf{r}\le r+\Delta r,A)}{P(r<\mathbf{r}\le r+\Delta r)}\\ &= \frac{P(r<\mathbf{r}\le r+\Delta r|A)P(A)}{P(r<\mathbf{r}\le r+\Delta r)}\\ &= \frac{\frac{P(r<\mathbf{r}\le r+\Delta r|A)}{\Delta r}P(A)}{\frac{P(r<\mathbf{r}\le r+\Delta r)}{\Delta r}}. \end{aligned} \tag{3.85}$$

Let $\Delta r\to 0$. Therefore $P(A|\mathbf{r}=r)=\frac{f(r|A)P(A)}{f_{\mathbf{r}}(r)}$.

P3.31 To be a valid (not necessarily useful) pdf, a function must satisfy two, and only two, conditions: (i) the area under it must be equal to one, and (ii) it must always be ≥ 0 (nonnegative). Therefore:

(a) (i) Satisfies both conditions. It is a non-stationary Gaussian random process (or random variable for any fixed t), whose mean is $\text{sgn}(t)$ and variance is σ^2.

(ii) No, neither condition is satisfied. $f_{\mathbf{x}}(0)<0$ for $t<0$. Also $\int_{-\infty}^{\infty}f_{\mathbf{x}}(x;t)\mathrm{d}x=\text{sgn}(t)\neq 1$.

(b) Answered above.

Note that the functions are considered to be functions of x; t is looked upon as a parameter.

P3.32 A random process is generated as follows: $\mathbf{x}(t)=\mathrm{e}^{-\mathbf{a}|t|}$, where $\mathbf{a}$ is a random variable with pdf $f_{\mathbf{a}}(a)=u(a)-u(a-1)$ (1/seconds).

(a) Sketches of several members of the ensemble are shown in Fig. 3.17. They are generated with the Matlab codes below. Note that the Matlab function `rand` generates a random number according to a uniform distribution over $[0,1]$.

```
t=[-2:0.001:2];
for i=1:4
    a=rand(1,1);
    x=exp(-a*abs(t));
    subplot(4,1,i);plot(t,x,'linewidth',1.5);
```

```
    tname=['{\bfa}=',num2str(a)];
    title(tname,'FontName','Times New Roman','FontSize',16);
    grid on;
    xlabel('\itt','FontName','Times New Roman','FontSize',16);
    set(gca,'FontSize',16,'XGrid','on','YGrid','on','GridLineStyle',':',...
    'MinorGridLineStyle','none','FontName','Times New Roman');
    ylim([0,1.1]);
end
```

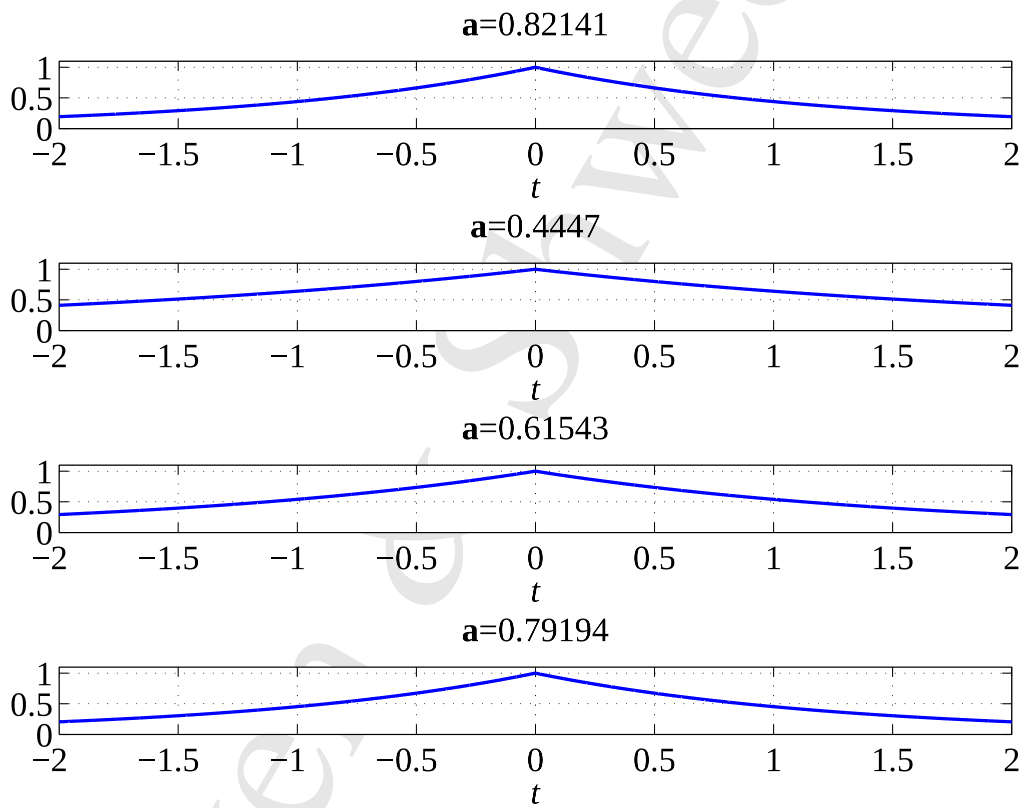


Figure 3.17: Several member functions of the random process $\mathbf{x}(t) = \mathrm{e}^{-\mathbf{a}|t|}$.

(b) Since $\mathbf{a}$ ranges between 0 and 1, for a specific time, t, the random variable $\mathbf{x}(t)$ ranges from $\mathrm{e}^{-|t|}$ (when $\mathbf{a} = 1$) to 1 (when $\mathbf{a} = 0$).

(c) Again, similar to P3.31, we consider $\mathbf{x}$ to be a function of $\mathbf{a}$ with t a parameter. The mean and mean-squared values of $\mathbf{x}(t)$ are:

$$E\{\mathbf{x}(t)\} = \int_{-\infty}^{\infty} x(t) f_{\mathbf{a}}(a)\mathrm{d}a = \int_{0}^{1} \mathrm{e}^{-a|t|}\mathrm{d}a = \left.\frac{\mathrm{e}^{-a|t|}}{-|t|}\right|_0^1 = \frac{1}{|t|}\left(1 - \mathrm{e}^{-|t|}\right). \tag{3.86}$$

$$E\{\mathbf{x}^2(t)\} = \int_{-\infty}^{\infty} x^2(t) f_{\mathbf{a}}(a)\mathrm{d}a = \int_{0}^{1} \mathrm{e}^{-2a|t|}\mathrm{d}a = \left.\frac{\mathrm{e}^{-2a|t|}}{-2|t|}\right|_0^1 = \frac{1}{2|t|}\left(1 - \mathrm{e}^{-2|t|}\right). \tag{3.87}$$

Note that at $t=0$ the mean value is $E\{\mathbf{x}(t)\} = \frac{0}{0} \overset{\text{by L'Hospitale}}{=} \left.\frac{\mathrm{e}^t}{1}\right|_{t=0} = 1$ (expected intuitively), the MSV is $E\{\mathbf{x}^2(t)\} = 1$. Therefore the variance is 0, again expected intuitively. This is not true, of course, at other values of t (except $t = \pm\infty$).

(d) We now determine the first-order pdf of $\mathbf{x}(t)$.
For $x \le \mathrm{e}^{-|t|}$ or $x > 1$, we have $f_{\mathbf{x}(t)}(x;t) = 0$.
For $\mathrm{e}^{-|t|} < x \le 1$, the equation $x = g(a) = \mathrm{e}^{-a|t|}$ has only one solution $a = -\frac{1}{|t|}\ln x, \quad \mathrm{e}^{-|t|} < x \le 1$. Furthermore

$$\left| \left.\frac{\mathrm{d}g(a)}{\mathrm{d}a}\right|_{a=-\frac{1}{|t|}\ln x} \right| = \left| \left. -|t|\mathrm{e}^{-a|t|}\right|_{a=-\frac{1}{|t|}\ln x} \right| = |t|x. \tag{3.88}$$

Therefore

$$f_{\mathbf{x}(t)}(x;t) = \frac{f_{\mathbf{a}}(a = -\frac{1}{|t|}\ln x)}{\left| \left.\frac{\mathrm{d}g(a)}{\mathrm{d}a}\right|_{a=-\frac{1}{|t|}\ln x} \right|} = \frac{1}{|t|x}. \tag{3.89}$$

To conclude:

$$f_{\mathbf{x}(t)}(x;t) = \begin{cases} \frac{1}{|t|x}, & \mathrm{e}^{-|t|} < x \le 1 \\ 0, & \text{otherwise} \end{cases} \tag{3.90}$$

Are the 2 conditions for this to be a valid pdf satisfied?

P3.33 (a) The transfer function of the filter is $H(f) = \frac{1}{1+j2\pi fRC}$. The 3-dB frequency is given by $f_{3\text{ dB}} = \frac{1}{2\pi RC}$. For $f_{3\text{ dB}} = 10$ kHz and $C_{\text{nom}} = 1.5 \times 10^{-9}$ F the resistor value is $R_{\text{nom}} = 10.6 \times 10^3 \approx 10$ kΩ.

(b) The time constant $\boldsymbol{\tau} = \boldsymbol{RC}$ is given by $\boldsymbol{\tau} = (R_{\text{nom}} + \boldsymbol{\Delta R})(C_{\text{nom}} + \boldsymbol{\Delta C}) \approx R_{\text{nom}}C_{\text{nom}} + C_{\text{nom}}\boldsymbol{\Delta R} + R_{\text{nom}}\boldsymbol{\Delta C}$ (where the term $\boldsymbol{\Delta R \Delta C}$ is ignored).
The random variables $\boldsymbol{\Delta R}$ and $\boldsymbol{\Delta C}$ have pdfs shown in Fig. 3.18.

Figure 3.18

The random variables $\mathbf{x} = C_{\text{nom}}\boldsymbol{\Delta R}$ and $\mathbf{y} = R_{\text{nom}}\boldsymbol{\Delta C}$ have the pdfs shown in Fig. 3.19:
Because $\mathbf{x}$ and $\mathbf{y}$ are statistically independent, the pdf of the random variable $\mathbf{z} = \mathbf{x} + \mathbf{y}$ is the convolution of $f_{\mathbf{x}}(x)$ and $f_{\mathbf{y}}(y)$. It looks as in Fig. 3.20.
Finally, the pdf of $\boldsymbol{RC}$ is a shifted version of $f_{\mathbf{z}}(z)$ and it is shown in Fig. 3.21.

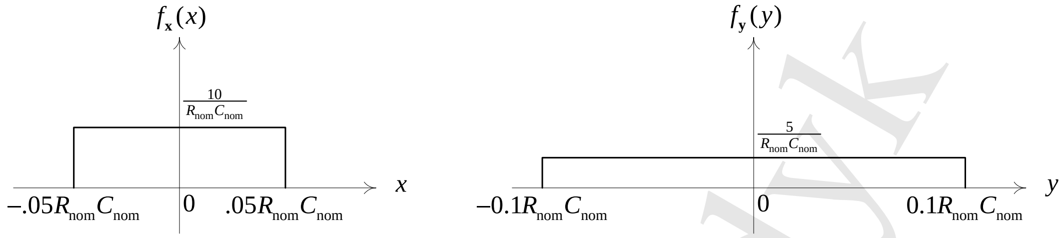


Figure 3.19

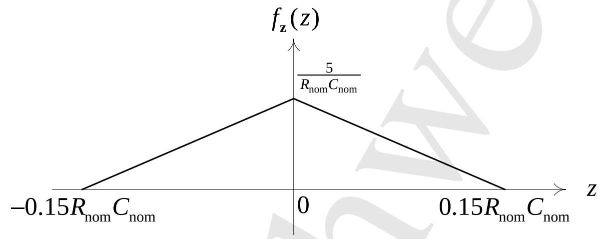


Figure 3.20

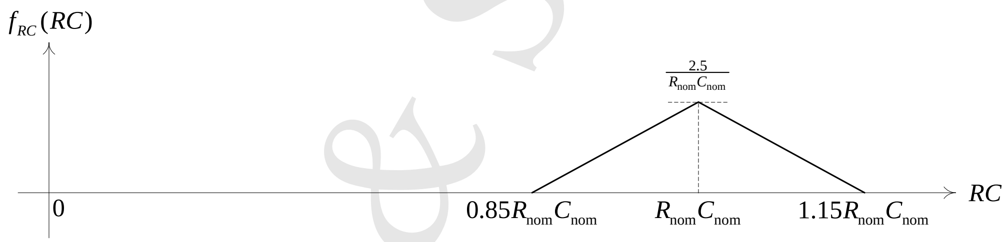


Figure 3.21

(c) What is meant by average impulse response? Simplest is to let $R = R_{\text{nom}}$ and $C = C_{\text{nom}}$ and let this be the average impulse response. More appropriately it should be called the nominal impulse response.

Otherwise find the statistical average of $\mathbf{h}(t)$:

$$\begin{aligned} E\{\mathbf{h}(t)\} &= \iint \frac{1}{RC}\mathrm{e}^{-\frac{t}{RC}} \underbrace{f_{\boldsymbol{RC}}(R,C)}_{=f_R(R)\cdot f_C(C)} \mathrm{d}R\mathrm{d}C \\ &= \int_{C=0.9C_{\text{nom}}}^{C=1.1C_{\text{nom}}} \frac{5}{C_{\text{nom}}}\mathrm{d}C \frac{1}{C}\int_{R=0.95R_{\text{nom}}}^{R=1.05R_{\text{nom}}} \mathrm{d}R\frac{10}{R_{\text{nom}}}\frac{1}{R}\mathrm{e}^{-\frac{t}{RC}} \end{aligned}$$

Let $\lambda \equiv \frac{1}{R}$. Then $\mathrm{d}\lambda = -\frac{1}{R^2}\mathrm{d}R$, or $\mathrm{d}R = -\frac{1}{\lambda^2}\mathrm{d}\lambda$. The inner integral becomes:

$$\frac{10}{R_{\text{nom}}}\int_{\lambda=\frac{1}{1.05R_{\text{nom}}}}^{\lambda=\frac{1}{0.95R_{\text{nom}}}} \frac{1}{\lambda}\mathrm{e}^{-\lambda a}\mathrm{d}\lambda = \frac{10}{R_{\text{nom}}}\left[\mathrm{Ei}(-a\lambda)\Big|_{\lambda=\frac{1}{0.95R_{\text{nom}}}} - \mathrm{Ei}(-a\lambda)\Big|_{\lambda=\frac{1}{1.05R_{\text{nom}}}}\right],$$

where $a \equiv \frac{t}{C}$ and the $\mathrm{Ei}(\cdot)$ function is defined as $\mathrm{Ei}(x) = -\int_{-x}^{\infty}\frac{\mathrm{e}^{-t}}{t}\mathrm{d}t$.

Define the worst case impulse responses to occur when R and C to be fall at either the extreme upper value of extreme lower value, i.e., $R = 1.05R_{\text{nom}}$ and $C = 1.1C_{\text{nom}}$, or $R = 0.95R_{\text{nom}}$ and $C = 0.9C_{\text{nom}}$

(d) Let $\mathbf{x} \equiv \frac{1}{\boldsymbol{RC}}$. Then the impulse response is given by:

$$\mathbf{h}(t) = \mathbf{x}e^{-\mathbf{x}t}u(t).$$

To find the first-order pdf, treat t as a parameter. Therefore we have a nonlinear mapping from random variable $\mathbf{x}$ to random variable $\mathbf{h}$:

$$\mathbf{h} = \mathbf{x}e^{-\mathbf{x}t}$$

(of course the parameter t is restricted to be ≥ 0).

Now given $f_{\mathbf{x}}(x)$ try to determine $f_{\mathbf{h}}(h)$. But the question is how to find the roots of $xe^{-xt} = h$ in an explicit form?

(e) The 3-dB bandwidth is given by $\mathbf{f}_{3\text{ dB}} = \frac{1}{2\pi RC}$. The mean value is

$$\begin{aligned}
E\{\mathbf{f}_{3\text{ dB}}\} &= E\left\{\frac{1}{2\pi RC}\right\} = \frac{1}{2\pi}E\left\{\frac{1}{R}\right\}E\left\{\frac{1}{C}\right\} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{R}f_{\mathbf{R}}(R)\mathrm{d}R\int_{-\infty}^{\infty}\frac{1}{C}f_{\mathbf{C}}(C)\mathrm{d}R \\
&= \frac{1}{2\pi}\frac{10}{R_{\text{nom}}}\left(\ln R\Big|_{0.95R_{\text{nom}}}^{1.05R_{\text{nom}}}\right)\frac{5}{C_{\text{nom}}}\left(\ln C\Big|_{0.95R_{\text{nom}}}^{1.05R_{\text{nom}}}\right) \\
&= \frac{1}{2\pi}\frac{10}{R_{\text{nom}}}(0.1)\frac{5}{C_{\text{nom}}}(0.201) \\
&= \frac{1.005}{2\pi R_{\text{nom}}C_{\text{nom}}}
\end{aligned}$$

To find the variance, determine the MSV of $\mathbf{f}_{3\text{ dB}}$, i.e., $E\{\mathbf{f}_{3\text{ dB}}^2\}$:

$$\begin{aligned}
E\{\mathbf{f}_{3\text{ dB}}^2\} &= \frac{1}{4\pi^2}E\left\{\frac{1}{R^2}\right\}E\left\{\frac{1}{C^2}\right\} = \frac{1}{4\pi^2}\frac{1.003}{R_{\text{nom}}^2}\left(\frac{1.010}{C_{\text{nom}}^2}\right) \\
&= \frac{1.013}{4\pi^2R_{\text{nom}}^2C_{\text{nom}}^2}
\end{aligned}$$

The variance is

$$E\{\mathbf{f}_{3\text{ dB}}^2\} - (E\{\mathbf{f}_{3\text{ dB}}\})^2 = \frac{0.003}{4\pi^2R_{\text{nom}}^2C_{\text{nom}}^2} = 0.003f_{\text{3-dB nominal}}^2$$

or the standard deviation is $0.055f_{\text{3-dB nominal}}$.

P3.34 (a) Check the two conditions for it to be a valid pdf.

The first condition: Is $f_{\mathbf{x}}(x;t) \geq 0$ for all x and t? The answer is YES.

The second condition: Is the area equal to one?

$$\begin{aligned}
\int_{-\infty}^{\infty}f_{\mathbf{x}}(x;t)\mathrm{d}x &= \int_{-\infty}^{\infty}\frac{|t|}{2}e^{-|x|/|t|}\mathrm{d}x = \frac{|t|}{2}\left[2\int_{0}^{\infty}e^{-|x|/|t|}\mathrm{d}x\right] \\
&= |t|\left(-|t|e^{-x/|t|}\Big|_0^{\infty}\right) = |t|^2 = t^2.
\end{aligned}$$

So the answer is NO.

Therefore pdf is not valid. But $\frac{|t|}{2}e^{-|t||x|}$ is a valid pdf. As is $\frac{1}{2|t|}e^{-|x|/|t|}$.

(b) pdf is not valid.

(c) $f_{\mathbf{x}}(x;t) = 0$ at $t = 0$.

P3.35 (a) For any time $t > 0$ (where the time origin is when you start the measurement), the pdf is a Gaussian distribution with zero mean and standard deviation $\sigma(t) = [1-\mathrm{e}^{-10t}]u(t) > 0$. Hence it is a valid pdf but time-varying.

(b) For $t > 0$, the mean is zero, while the variance is $\sigma^2(t) = [1 - \mathrm{e}^{-10t}]^2 u(t)$, which is time varying.

(c) Fig. 3.22 plots $\sigma^2(t)$. Observe that $\sigma^2(t)$ closely approaches 1 when $t = 0.5$ sec (500 msec). At this point in time the first-order pdf does not change much and the process can be considered first-order stationary.

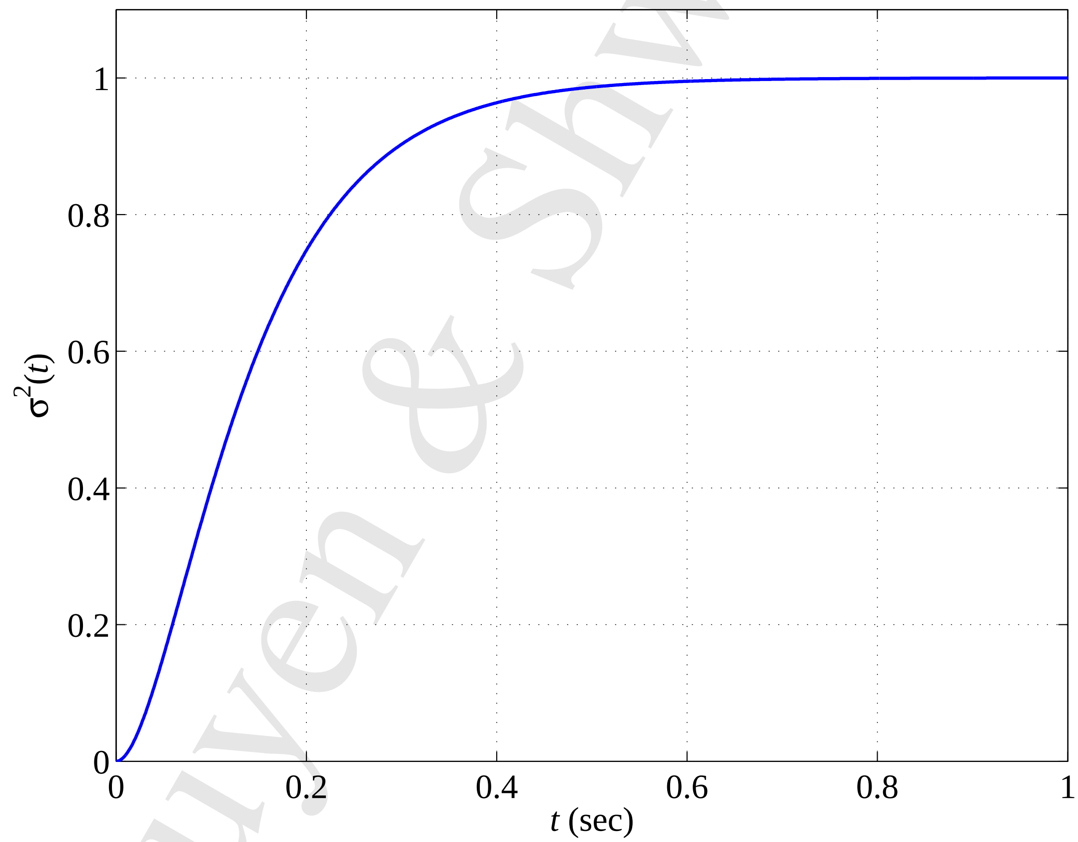


Figure 3.22: Plot of $\sigma^2(t)$.

P3.36 The mean value of $\mathbf{x}$ is:

$$E\{\mathbf{x}\} = E\left\{\int_0^{T_b} \mathbf{w}(t)g(t)\mathrm{d}t\right\} = \int_0^{T_b} E\{\mathbf{w}(t)\}g(t)\mathrm{d}t = 0. \tag{3.91}$$

The variance of $\mathbf{x}$ is (since mean is zero, it equals the MSV):

$$
\begin{aligned}
\sigma_{\mathbf{x}}^2 &= E\left\{\mathbf{x}^2\right\} = E\left\{\int_{t=0}^{T_b} \mathbf{w}(t)g(t)\mathrm{d}t \int_{\lambda=0}^{T_b} \mathbf{w}(\lambda)g(\lambda)\mathrm{d}\lambda\right\} \\
&= \int_{t=0}^{T_b} \mathrm{d}t g(t) \underbrace{\int_{\lambda=0}^{T_b} \mathrm{d}\lambda g(\lambda) \underbrace{E\left\{\mathbf{w}(t)\mathbf{w}(\lambda)\right\}}_{\frac{N_0}{2}\delta(t-\lambda)}}_{\frac{N_0}{2}g(t)} \\
&= \frac{N_0}{2}\int_0^{T_b} g^2(t)\mathrm{d}t = \frac{N_0 E_g}{2}. \qquad (3.92)
\end{aligned}
$$

Note: Expectation is in essence an integration operation. Therefore in interchanging the expectation and integration (w.r.t. time) operations we are simply changing the order of integration.

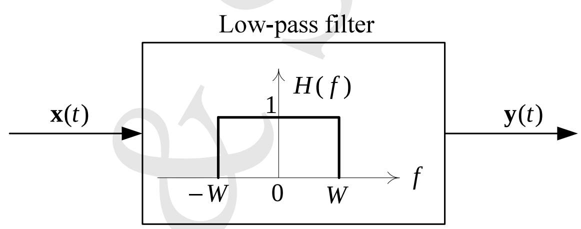


Figure 3.23: Passing a random process through a low-pass filter.

P3.37 (a) The power spectral density (PSD) of $\mathbf{y}(t)$ is

$$
S_{\mathbf{y}}(f) = S_{\mathbf{x}}(f)\cdot|H(f)|^2 = \begin{cases} \frac{N_0}{2}, & |f| \le W \\ 0, & \text{otherwise} \end{cases} \qquad (3.93)
$$

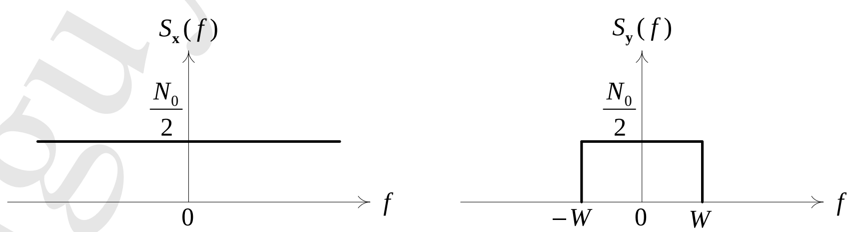


Figure 3.24

(b) The autocorrelation can be found as the inverse Fourier transform of the PSD:

$$R_{\mathbf{y}}(\tau) = \mathcal{F}^{-1}\{S_{\mathbf{y}}(f)\} = \int_{-\infty}^{+\infty} S_{\mathbf{y}}(f)\mathrm{e}^{j2\pi f\tau}\mathrm{d}f \tag{3.94}$$

$$= \int_{-W}^{W} \frac{N_0}{2}\mathrm{e}^{j2\pi f\tau}\mathrm{d}f \tag{3.95}$$

$$= \frac{N_0}{4\pi\tau}\int_{-W}^{W} \mathrm{e}^{j2\pi f\tau}(2\pi\tau)\mathrm{d}f \overset{x=2\pi\tau f}{=} \frac{N_0}{4\pi\tau}\int_{-2\pi W\tau}^{2\pi W\tau} \mathrm{e}^{jx}\mathrm{d}x \tag{3.96}$$

$$= \frac{N_0}{4\pi\tau}\cdot 2\sin(2\pi W\tau) = \frac{N_0\sin(2\pi W\tau)}{2\pi\tau} \tag{3.97}$$

$$= N_0 \cdot W \cdot \mathrm{sinc}(2W\tau) \tag{3.98}$$

The (normalized) autocorrelation function $R_{\mathbf{y}}(\tau)$ is sketched in Fig. 3.25.

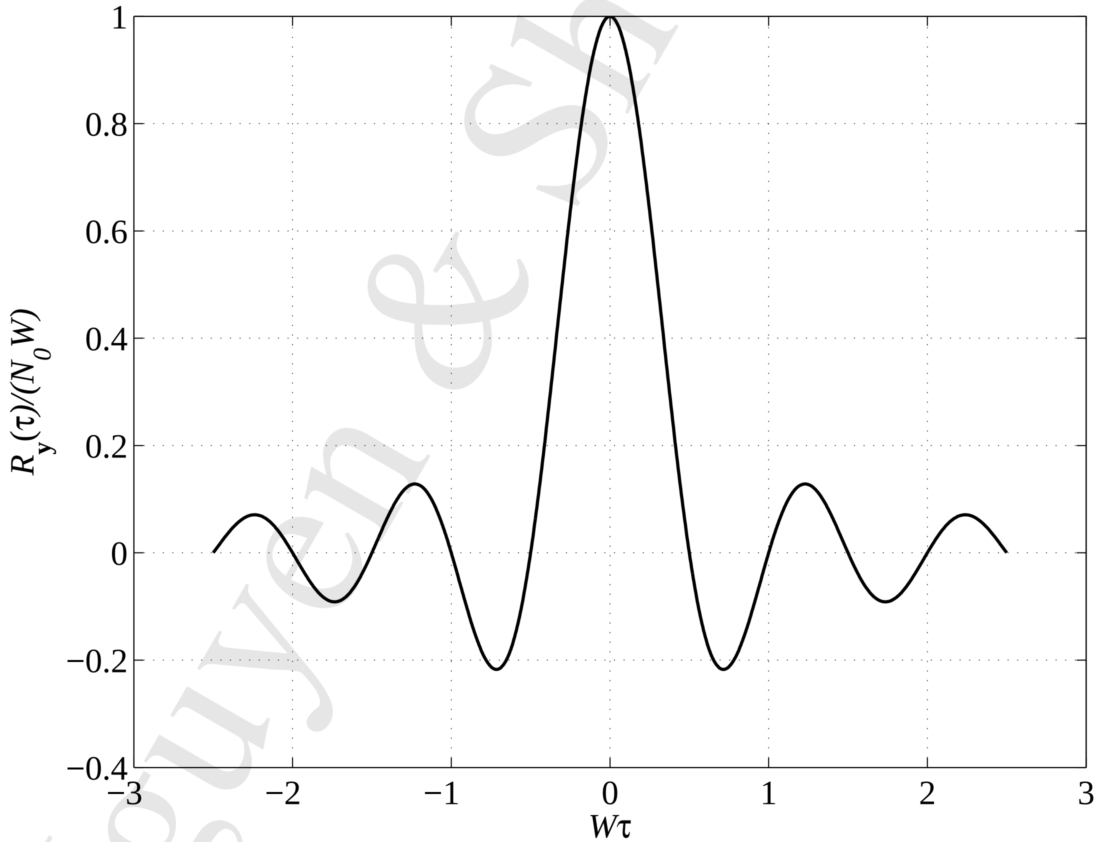


Figure 3.25: Autocorrelation function $R_{\mathbf{y}}(\tau)$.

(c) The DC level of $\mathbf{y}(t)$ is

$$m_{\mathbf{y}} = E\{\mathbf{y}(t)\} = m_{\mathbf{x}} \cdot H(0) = 0 \cdot 1 = 0 \tag{3.99}$$

The average power of $\mathbf{y}(t)$ is

$$\sigma_{\mathbf{y}}^2 = E\{\mathbf{y}^2(t)\} = R_{\mathbf{y}}(0) = N_0 W \tag{3.100}$$

(d) Since the input $\mathbf{x}(t)$ is a Gaussian process, the output $\mathbf{y}(t)$ is also a Gaussian process and the samples $\mathbf{y}(kT_s)$ are Gaussian random variables. For Gaussian random variables, statistical independence is equivalent to uncorrelatedness. Thus one needs to find the smallest value for τ (or T_s) so that the autocorrelation is zero. From the graph of $R_{\mathbf{y}}(\tau)$, the answer is:

$$\tau_{\min} = (T_s)_{\min} = \frac{1}{2W} \tag{3.101}$$

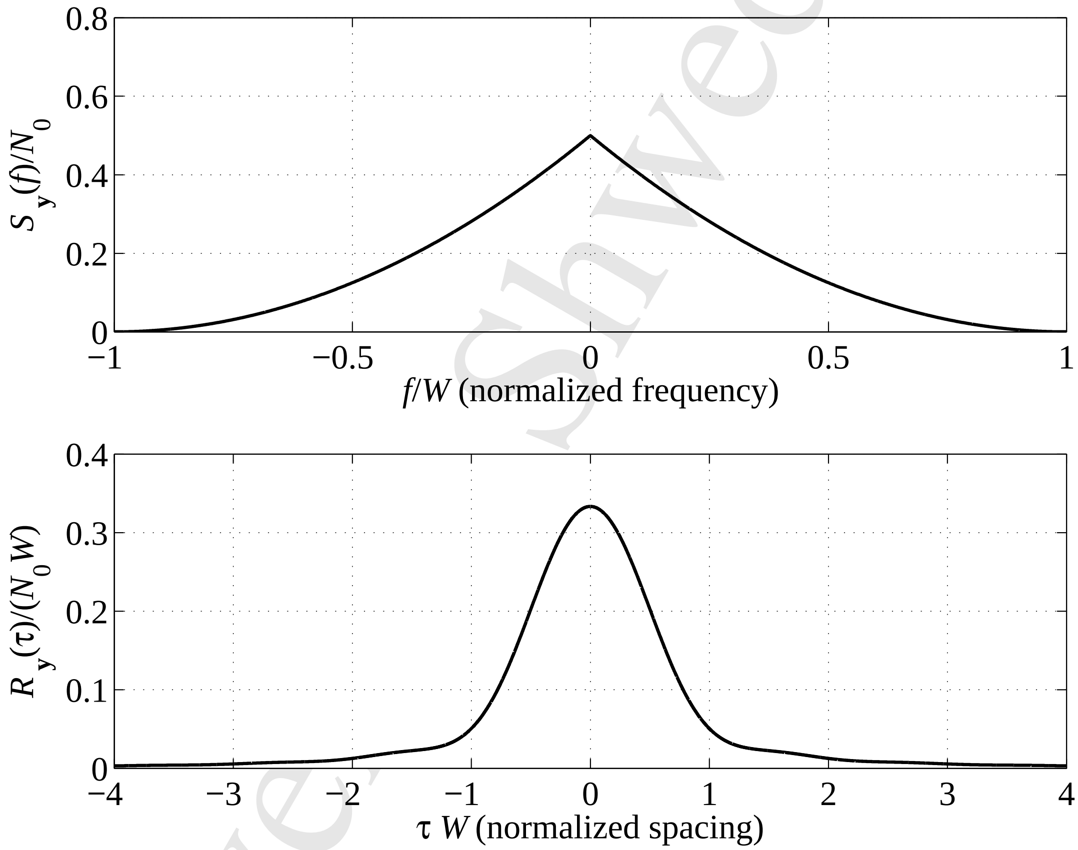


Figure 3.26: Plot of power spectral density and autocorrelation of the output process in P3.38.

P3.38 (a) As usual $S_{\mathbf{y}}(f) = |H(f)|^2 \frac{N_0}{2}$. See Fig. 3.26 for the plot of $S_{\mathbf{y}}(f)$.

(b)

$$\begin{aligned} R_{\mathbf{y}}(\tau) &= \mathcal{F}^{-1}\{S_{\mathbf{y}}(f)\} = \frac{N_0}{2}\int_{-W}^{W}\left(1-\frac{|f|}{W}\right)^2 \mathrm{e}^{j2\pi f\tau}\mathrm{d}f \\ &= N_0\int_0^W \left(1-\frac{|f|}{W}\right)^2 \cos(2\pi f\tau)\mathrm{d}f \\ &= \frac{2N_0}{W(2\pi\tau)^2}\left[1-\frac{\sin(2\pi W\tau)}{2\pi W\tau}\right] = \frac{2N_0}{W(2\pi\tau)^2}\left[1-\operatorname{sinc}(2W\tau)\right] \end{aligned}$$

See Fig. 3.26 for the plot of $R_{\mathbf{y}}(\tau)$.

(c) DC level is zero. Average power is $N_0\int_0^W\left(1-\frac{|f|}{W}\right)^2\cos(2\pi f\tau)\mathrm{d}f = \frac{N_0W}{3}$ (watts), which can also be found from $R_{\mathbf{y}}(0)$.

(d) Since $R_{\mathbf{y}}(\tau) > 0$ for $|\tau| < \infty$, there is no finite value of sampling period to yield statistically independent noise samples.

P3.39 (a) The output noise power is $N_0 W$ watts.
The input signal PSD is

$$\mathcal{F}\{R_{\mathbf{s}}(\tau)\} = K\int_{-\infty}^{\infty} \mathrm{e}^{-a|\tau|}\mathrm{e}^{-j2\pi f\tau}\mathrm{d}\tau = \frac{2aK}{a^2+4\pi^2 f^2} \text{ (watts/Hz)}$$

Output signal power is

$$\int_{-W}^{W} \frac{2aK}{a^2+4\pi^2 f^2}\mathrm{d}f = \frac{2K}{a\pi}\tan^{-1}\left(\frac{2\pi W}{a}\right).$$

Therefore

$$\mathrm{SNR}(W) = \frac{2K}{a\pi}\frac{\tan^{-1}\left(\frac{2\pi W}{a}\right)}{N_0 W}.$$

(b) To find the maximum,

$$\frac{\mathrm{dSNR}(W)}{\mathrm{d}W} = \frac{2K}{a\pi N_0}\left[-\frac{\tan^{-1}\left(\frac{2\pi W}{a}\right)}{W^2} + \frac{2\pi}{aW\sqrt{1+\frac{4\pi^2W^2}{a^2}}}\right] = 0,$$

which gives

$$\sqrt{1+\frac{4\pi^2W^2}{a^2}}\tan^{-1}\left(\frac{2\pi W}{a}\right) = \frac{2\pi W}{a}.$$

Let $x \equiv \frac{2\pi W}{a}$. Then one needs to solve $\sqrt{1+x^2}\tan^{-1}(x) = x$. The only solution is $x = 0$, i.e., $W = 0$. This solution could also been obtained by recognizing that, since the noise PSD is flat and the signal power decays as $1/f^2$, increasing the filter bandwidth can only reduces the SNR.

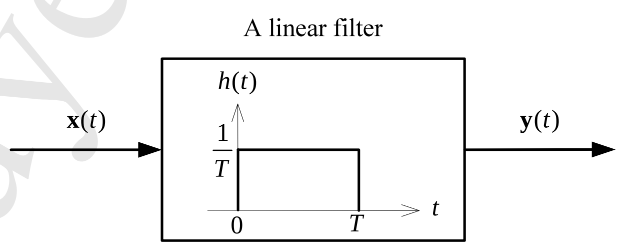


Figure 3.27: System under consideration in Problem 3.40.

P3.40 (a) The filter is time-invariant (linearity does not matter). The input noise process $\mathbf{x}(t)$ is WSS. The output noise process $\mathbf{y}(t)$ therefore is also WSS.

(b) The frequency response of the filter in Fig. 3.27 is:

$$\begin{aligned}
H(f) &= \int_{-\infty}^{\infty} h(t)\mathrm{e}^{-j2\pi ft}\mathrm{d}t = \int_0^T \frac{1}{T}\mathrm{e}^{-j2\pi ft}\mathrm{d}t \\
&= \frac{1}{T}\frac{1}{-j2\pi f}\,\mathrm{e}^{-j2\pi ft}\Big|_0^T = -\frac{1}{j2\pi fT}\left[\mathrm{e}^{-j2\pi fT}-1\right] \\
&= -\frac{\mathrm{e}^{-j\pi fT}-\mathrm{e}^{j\pi fT}}{j2\pi fT}\mathrm{e}^{-j\pi fT} \\
&= \frac{j2\sin(\pi fT)}{j2\pi fT}\mathrm{e}^{-j\pi fT} = \frac{\sin(\pi fT)}{\pi fT}\mathrm{e}^{-j\pi fT} = \mathrm{sinc}(fT)\mathrm{e}^{-j\pi fT}
\end{aligned}$$

If $m_{\mathbf{x}}$ is the DC component in $\mathbf{x}(t)$, then the DC component in $\mathbf{y}(t)$ is

$$m_{\mathbf{y}} = m_{\mathbf{x}} \cdot H(0) = m_{\mathbf{x}} \cdot \mathrm{sinc}(0) = m_{\mathbf{x}}$$

(c) If $\mathbf{x}(t)$ is a zero-mean, *white* noise process with power spectral density $N_0/2$, the power spectral density of $\mathbf{y}(t)$ is given by:

$$S_{\mathbf{y}}(f) = S_{\mathbf{x}}(f)\cdot|H(f)|^2 = \frac{N_0}{2}\left[\frac{\sin(\pi fT)}{\pi fT}\right]^2 = \frac{N_0}{2}\mathrm{sinc}^2(fT)$$

(d) The frequency components that are not present in the output process are the components that make $S_{\mathbf{y}}(f) = 0$. Obviously, these components correspond to $\pi fT = k\pi$, or $f = k/T$, $k = \pm1, \pm2, \ldots$.

P3.41 Consider the system in Fig. 3.28.

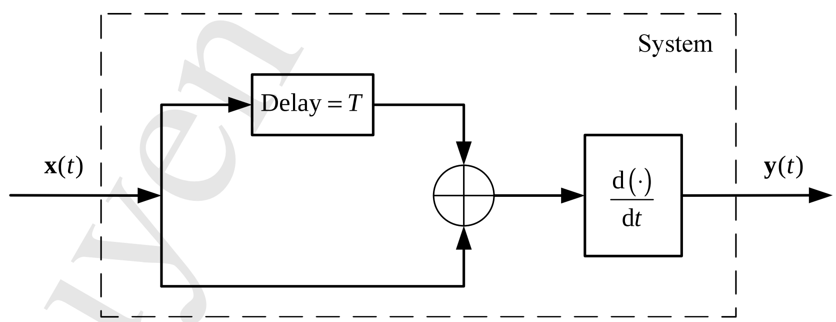


Figure 3.28: The system under consideration in Problem 3.41.

(a) Let the input $x(t)$ and output $y(t)$ be deterministic signals. Then

$$\begin{aligned}
y(t) &= \frac{\mathrm{d}}{\mathrm{d}t}[x(t)+x(t-T)] \\
&= \frac{\mathrm{d}}{\mathrm{d}t}x(t) + \frac{\mathrm{d}}{\mathrm{d}t}x(t-T) \qquad (3.102)
\end{aligned}$$

Obviously, the system is LTI. Since the response of an LTI system to a stationary process is also a stationary process, it follows that $\mathbf{y}(t)$ is a wide-sense stationary process.

(b) Rewrite (3.102) as

$$y(t) = \frac{\mathrm{d}}{\mathrm{d}t}[x(t) + x(t-T)]$$

Taking the Fourier transform of both sides and using the differentiating and shifting properties of FT gives:

$$Y(f) = (j2\pi f)[1 + \mathrm{e}^{-j2\pi fT}]X(f)$$

Thus, the frequency response of the system is:

$$H(f) = \frac{Y(f)}{X(f)} = (j2\pi f)[1 + \mathrm{e}^{-j2\pi fT}]$$

(c) Now consider the input $\mathbf{x}(t)$ being a white noise process with PSD of $\frac{N_0}{2} \Rightarrow S_{\mathbf{x}}(f) = \frac{N_0}{2}, \forall f$. The PSD of $\mathbf{y}(t)$ is:

$$S_{\mathbf{y}}(f) = S_{\mathbf{x}}(f) \cdot |H(f)|^2 = \frac{N_0}{2}|H(f)|^2$$

Since $|H(f)|^2 = 16\pi^2 f^2 \cos^2(\pi fT)$, it follows that

$$S_{\mathbf{y}}(f) = 8N_0\pi^2 f^2 \cos^2(\pi fT)$$

The above (normalized) PSD is plotted in Fig. 3.29.

(d) Here we need to find frequency components such that $S_{\mathbf{y}}(f) = 0$:

$$\begin{aligned} S_{\mathbf{y}}(f) = 8N_0\pi^2 f^2 \cos^2(\pi fT) = 0 \quad &\Leftrightarrow \quad \begin{cases} f = 0 \\ \cos(\pi fT) = 0 \end{cases} \\ &\Leftrightarrow \quad \begin{cases} f = 0 \\ \pi fT = \frac{\pi}{2} + k\pi; k = 0, \pm 1, \pm 2, \ldots \end{cases} \\ &\Leftrightarrow \quad \begin{cases} f = 0 \\ f = \frac{2k+1}{2T}; k = 0, \pm 1, \pm 2, \ldots \end{cases} \end{aligned}$$

P3.42 (a) The PSD of $\mathbf{m}(t)$ is found by taking the Fourier transform of the autocorrelation function $R_{\mathbf{m}}(\tau)$:

$$\begin{aligned} S_{\mathbf{m}}(f) &= \mathcal{F}\{R_{\mathbf{m}}(\tau)\} = \int_{-\infty}^{\infty} R_{\mathbf{m}}(\tau)\mathrm{e}^{-j2\pi f\tau}\mathrm{d}\tau = \int_{-\infty}^{\infty} A\mathrm{e}^{-|\tau|}\mathrm{e}^{-j2\pi f\tau}\mathrm{d}\tau \\ &= \int_{-\infty}^{0} A\mathrm{e}^{(1-j2\pi f)\tau}\mathrm{d}\tau + \int_{0}^{\infty} A\mathrm{e}^{-(1+j2\pi f)\tau}\mathrm{d}\tau \\ &= \frac{A}{1-j2\pi f}\mathrm{e}^{(1-j2\pi f)\tau}\Big|_{-\infty}^{0} + \frac{A}{-(1+j2\pi f)}\mathrm{e}^{-(1+j2\pi f)\tau}\Big|_{0}^{\infty} \\ &= \frac{A}{1-j2\pi f} + \frac{A}{1+j2\pi f} = \frac{2A}{1+4\pi^2 f^2} \end{aligned} \qquad (3.103)$$

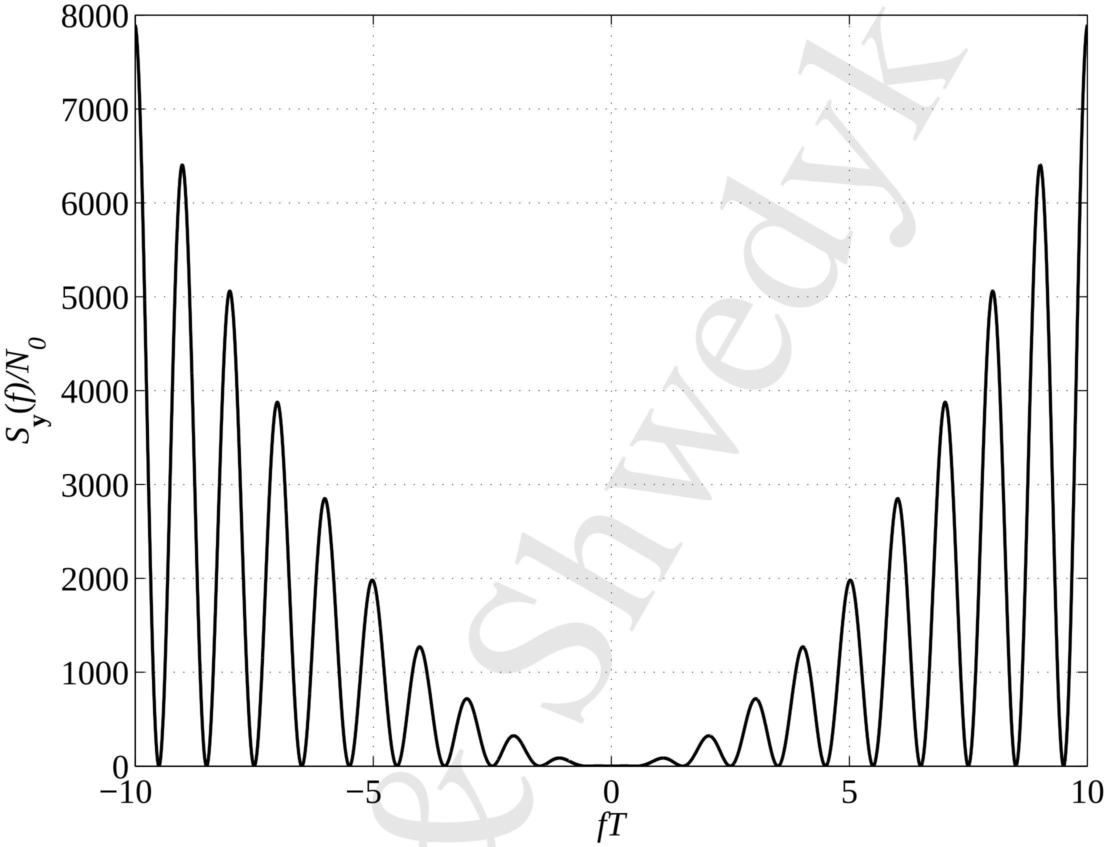


Figure 3.29: Plot of $S_{\mathbf{y}}(f)$.

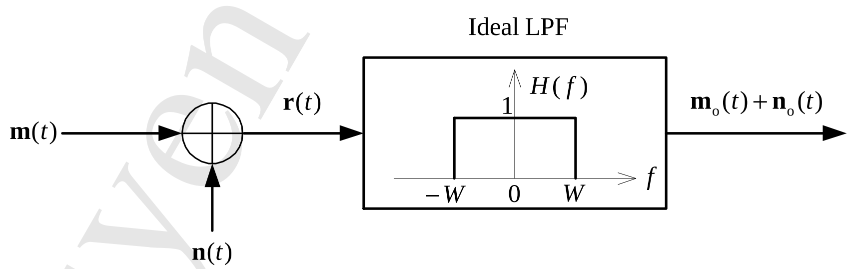


Figure 3.30: System under consideration in Problem 3.42.

(b) Since the filter is an ideal LPF of bandwidth W, the PSD of the output message is:

$$S_{\mathbf{m}_o}(f) = \begin{cases} S_{\mathbf{m}}(f) & ; \quad -W \le f \le W \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{3.104}$$

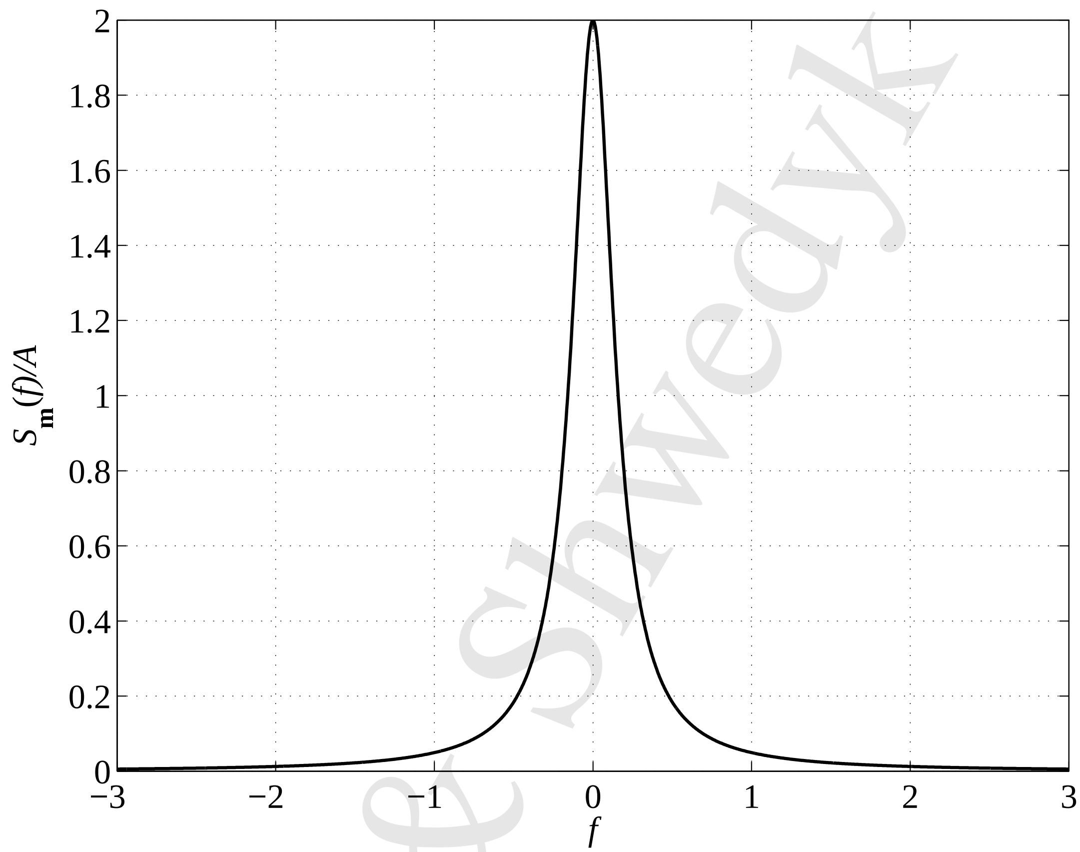


Figure 3.31: The power spectral density $S_{\mathbf{m}}(f)$.

Thus, the power of the output message is:

$$\begin{aligned} P_{\mathbf{m}_o} &= \int_{-\infty}^{\infty} S_{\mathbf{m}_o}(f)\mathrm{d}f = \int_{-W}^{W} S_{\mathbf{m}}(f)\mathrm{d}f \\ &= \int_{-W}^{W} \frac{2A}{1+4\pi^2 f^2}\mathrm{d}f = \frac{2A}{4\pi^2} \times 2 \times \int_{0}^{W} \frac{1}{(\frac{1}{2\pi}) + f^2}\mathrm{d}f \\ &= \frac{A}{\pi^2}(2\pi)\tan^{-1}(2\pi f)\Big|_0^W = \frac{2A}{\pi}\tan^{-1}(2\pi W). \end{aligned} \tag{3.105}$$

To find the power percentages at the filter output, note that the power of the input message is $P_m = R_{\mathbf{m}}(0) = A$. Therefore

W (Hz)	$P_{\mathbf{m}_o}$	Percentage $= P_{\mathbf{m}_o}/A$
10	$0.9899A$	98.99%
50	$0.9979A$	99.79%
100	$0.9989A$	99.89%

(c) Since the PSD of the output noise is:

$$S_{\mathbf{n}_o}(f) = \begin{cases} \frac{N_0}{2} & ; \quad -W \le f \le W \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{3.106}$$

Then the noise power at the output of the filter is:

$$P_{\mathbf{n}_o} = \int_{-W}^{W} \frac{N_0}{2} \mathrm{d}f = \frac{N_0}{2} \cdot 2W = N_0 W. \tag{3.107}$$

(d) Let $A = 4 \times 10^{-3}$ watts, $W = 4$ kHz and $N_0 = 10^{-8}$ watts/Hz. Determine the SNR in dB at the filter output.

$$\Rightarrow P_{\mathbf{m}_o} = \frac{2A}{\pi} \tan^{-1}(2\pi W) \approx A.$$

This is because the bandwidth $W = 4$ kHz is large enough to practically pass all the message power.

$$P_{\mathbf{n}_o} = N_0 W = 10^{-8} \times 4 \times 10^3 = 4 \times 10^5.$$

$$\text{SNR} = 10\log_{10} \frac{P_{\mathbf{m}_o}}{P_{\mathbf{n}_o}} = 10\log_{10} \frac{4 \times 10^{-3}}{4 \times 10^{-5}} = 20\text{dB}.$$

P3.43

$$\begin{aligned} R_{\mathbf{yw}}(\tau) &= E\{\mathbf{y}(t)\mathbf{w}(t+\tau)\} = E\left\{\left[\int_{-\infty}^{\infty} h(\lambda)[\mathbf{s}(t-\lambda) + \mathbf{w}(t-\lambda)]\mathrm{d}\lambda\right]\mathbf{w}(t+\tau)\right\} \\ &= \int_{-\infty}^{\infty} h(\lambda)\left[\underbrace{E\{\mathbf{s}(t-\lambda)\mathbf{w}(t+\tau)\}}_{=0 \text{ (they are uncorrelated)}} + \underbrace{E\{\mathbf{w}(t-\lambda)\mathbf{w}(t+\tau)\}}_{=\frac{N_0}{2}\delta(\tau+\lambda)}\right]\mathrm{d}\lambda \\ &= \frac{N_0}{2} h(-\tau) \end{aligned}$$

Chapter 4

Sampling and Quantization

P4.1 The spectrum of the analog signal is given by

$$S(f) = \frac{5[\delta(f-500)+\delta(f+500)]}{2} + \frac{2[\delta(f-1800)+\delta(f+1800)]}{2} \quad \text{(volts/Hz)}. \tag{4.1}$$

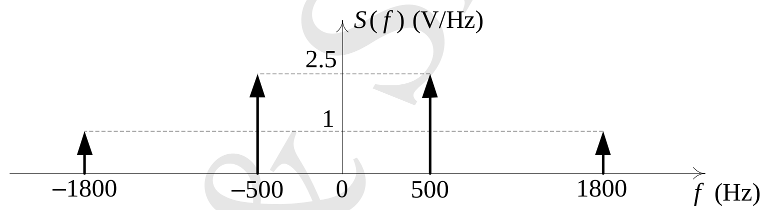


Figure 4.1

The sampled spectrum $S_{\text{sampled}}(f)$ is the sum of all the shifted versions by multiples of f_s (and scaled by $1/T_s = f_s$ which we will assume is understood). $S_{\text{sampled}}(f)$ plots as follows:

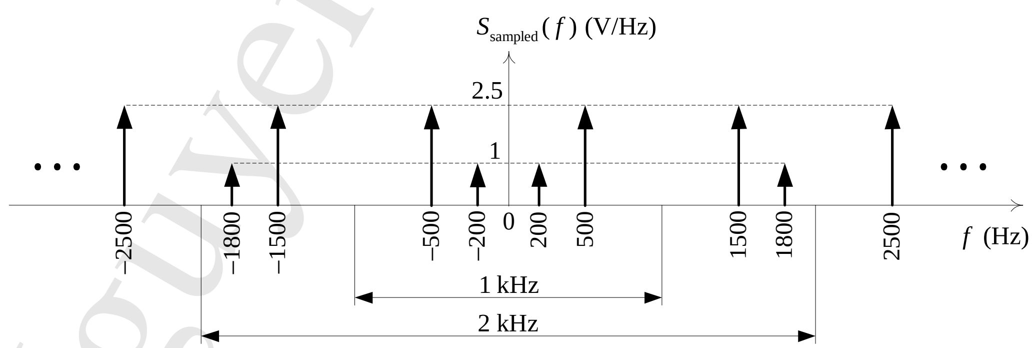


Figure 4.2

(a) Output spectrum is

$$\underbrace{2.5[\delta(f-500)+\delta(f+500)]}_{\text{This is due to the analog signal}} + \underbrace{[\delta(f-200)+\delta(f+200)]}_{\text{This is an alias - due to undersampling}} . \tag{4.2}$$

Output time signal is

$$5\cos\left(2\pi(500)t\right) + 2\cos\left(2\pi(200)t\right). \tag{4.3}$$

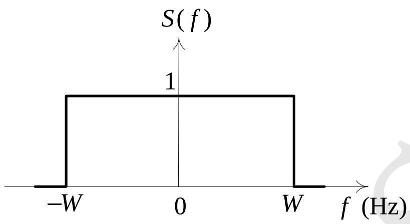
(b) Output spectrum is

$$\underbrace{2.5[\delta(f-500)+\delta(f+500)]+[\delta(f-1800)+\delta(f+1800)]}_{\text{This is due to the analog signal}}$$
$$+\underbrace{[\delta(f-200)+\delta(f+200)]+2.5[\delta(f-1500)+\delta(f+1500)]}_{\text{Aliases}}. \quad (4.4)$$

Output time signal is

$$5\cos\left(2\pi(500)t\right)+2\cos\left(2\pi(1800)t\right)+2\cos\left(2\pi(200)t\right)+5\cos\left(2\pi(1500)t\right). \quad (4.5)$$

P4.2 First we have the following transform pair:



$$\leftrightarrow s(t)=\int_{-W}^{W} e^{j2\pi ft}\mathrm{d}f=\frac{e^{j2\pi Wt}-e^{-j2\pi Wt}}{j2\pi t}=\frac{\sin 2\pi Wt}{\pi t}$$

Figure 4.3

Therefore

$$\int_{-\infty}^{\infty}\underbrace{\sqrt{T_s}\frac{\sin 2\pi W(t-nT_s)}{\pi(t-nT_s)}}_{s_n(t)}\mathrm{e}^{-j2\pi ft}\mathrm{d}t \overset{\lambda=t-nT_s}{=} \int_{-\infty}^{\infty}\sqrt{T_s}\frac{\sin 2\pi W\lambda}{\pi\lambda}\mathrm{e}^{-j2\pi f(\lambda+nT_s)}\mathrm{d}\lambda$$
$$= \sqrt{T_s}S(f)\mathrm{e}^{-j2\pi fnT_s}. \quad (4.6)$$

$$\begin{aligned}
\int_{-\infty}^{\infty}s_n(t)s_m^*(t)\mathrm{d}t &= \int_{-\infty}^{\infty}T_s\underbrace{S(f)}_{\substack{\| \\ u(f+W)-u(f-W)=S^*(f)}}\mathrm{e}^{-j2\pi fnT_s}S(f)^*\mathrm{e}^{j2\pi fnT_s}\mathrm{d}f \\
&= T_s\int_{-W}^{W}\mathrm{e}^{-j2\pi f(n-m)T_s}\mathrm{d}f \\
&= T_s\frac{\mathrm{e}^{j2\pi(n-m)WT_s}-\mathrm{e}^{-j2WT_s}}{j2\pi(n-m)T_s} \overset{2WT_s=1}{=} \frac{\sin(\pi(n-m))}{\pi(n-m)} \quad (4.7)\\
&= \begin{cases} 0 & n\neq m \quad \text{(orthogonal)} \\ 1 & n=m \quad \text{(normal)}\end{cases}
\end{aligned}$$

P4.3 (a)

$$R_{\mathbf{w}_{\text{out}}}(\tau)=\mathcal{F}^{-1}\left\{S_{\mathbf{w}_{\text{out}}}(f)\right\}=N_0W\frac{\sin(2\pi W\tau)}{2\pi W\tau} \quad \text{(watts)}. \quad (4.8)$$

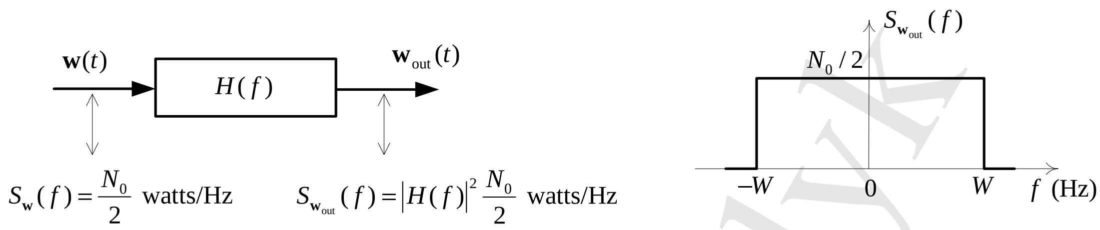


Figure 4.4

(b) No. It depends on the transfer function, $H(f)$, and the input PSD. However if the input is a Gaussian process then so is the output since it was obtained by the linear operations – essentially the output is a weighted linear sum of the input, $\mathbf{w}(t)$.

(c) At $\tau = kT_s$:

$$\begin{aligned} R_{\mathbf{w}_{\text{out}}}(kT_s) &= N_0 W \frac{\sin(2\pi W kT_s)}{2\pi W kT_s} \overset{WT_s=1}{=} N_0 W \frac{\sin(2\pi k)}{2\pi k} \\ &= 0, \quad k \neq 0. \end{aligned} \tag{4.9}$$

The above means that the samples are uncorrelated. And because they are Gaussian, they are statistically independent.

(d) If increased then definitely correlated, except for very special cases such as if T_s is halved then the set of alternate samples would be uncorrelated, etc. If decreased then if T_s is increased by an integer multiple the samples are uncorrelated, otherwise correlated.

Statistically independent depends on the process being Gaussian and samples being uncorrelated. If not Gaussian, nothing can be said about statistical independence even if uncorrelated.

P4.4
- 8-bit $\Rightarrow$ 256 levels $\Rightarrow$ Step size $= 2/256 = 7.81$ mV.
- 12-bit $\Rightarrow$ 4096 levels $\Rightarrow$ Step size $= 2/4096 = 0.488$ mV.
- 16-bit $\Rightarrow$ 65,536 levels $\Rightarrow$ Step size $= 2/2^{16} = 30.5\mu$V.

P4.5 3-bit $\Rightarrow$ 8 levels; 4 on each side of zero.

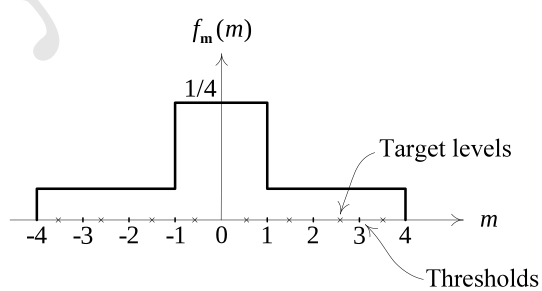


Figure 4.5

By inspection the uniform quantizer is the optimum one.

Signal power is:

$$\begin{aligned}\sigma_{\mathbf{m}}^2 &= \int_{-4}^{4} m^2 f_{\mathbf{m}}(m)\mathrm{d}m = 2\int_{0}^{4} m^2 f_{\mathbf{m}}(m)\mathrm{d}m \\ &= 2\left[\frac{1}{4}\frac{m^3}{3}\Big|_0^1 + \frac{1}{12}\frac{m^3}{3}\Big|_0^1\right] = \frac{11}{3}\ \text{(watts)}. \end{aligned} \tag{4.10}$$

Quantization noise power:

$$\begin{aligned}\sigma_{\mathbf{q}}^2 &= 2\Bigg[\int_0^1 \underbrace{\left(m-\frac{1}{2}\right)^2\frac{1}{4}\mathrm{d}m}_{\text{Change variables}\to\lambda=m-1/2} + \int_1^2 \underbrace{\left(m-\frac{3}{2}\right)^2\frac{1}{12}\mathrm{d}m}_{\lambda=m-3/2} \\ &\quad + \int_2^3 \underbrace{\left(m-\frac{5}{2}\right)^2\frac{1}{12}\mathrm{d}m}_{\lambda=m-5/2} + \int_3^4 \underbrace{\left(m-\frac{7}{2}\right)^2\frac{1}{12}\mathrm{d}m}_{\lambda=m-7/2}\Bigg] \\ &= 2\left[\frac{1}{4}\int_{-1/2}^{1/2}\lambda^2\mathrm{d}\lambda + \frac{1}{12}\int_{-1/2}^{1/2}\lambda^2\mathrm{d}\lambda + \frac{1}{12}\int_{-1/2}^{1/2}\lambda^2\mathrm{d}\lambda + \frac{1}{12}\int_{-1/2}^{1/2}\lambda^2\mathrm{d}\lambda\right] \\ &= \int_{-1/2}^{1/2}\lambda^2\mathrm{d}\lambda = 2\int_0^1 \lambda^2\mathrm{d}\lambda = \frac{2}{3}\ \text{(watts)}. \end{aligned} \tag{4.11}$$

Therefore

$$\text{SNR}_{\text{q}} = \frac{\sigma_{\mathbf{m}}^2}{\sigma_{\mathbf{q}}^2} = \frac{11/3}{2/3} = 5.5 \quad \text{or} \quad 10\log_{10}5.5 = 7.4\text{dB}.$$

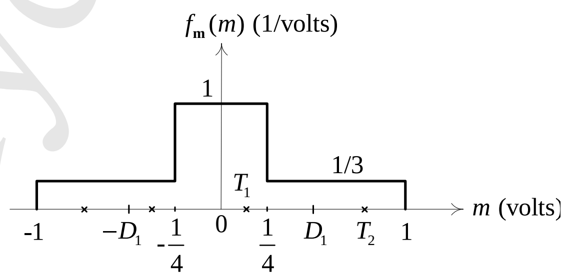


Figure 4.6

P4.6

$$T_1 = \frac{\int_0^{D_1} m f_{\mathbf{m}}(m)\mathrm{d}m}{\int_0^{D_1} f_{\mathbf{m}}(m)\mathrm{d}m} = \frac{\int_0^{1/4} m\mathrm{d}m + \frac{1}{3}\int_{1/4}^{D_1} m\mathrm{d}m}{\frac{1}{4} + \left(D_1 - \frac{1}{4}\right)\frac{1}{3}} = \frac{1+8D_1^2}{8+16D_1} \tag{4.12}$$

$$T_2 = \frac{\int_{D_1}^{1} m f_{\mathbf{m}}(m)\mathrm{d}m}{\int_{D_1}^{1} f_{\mathbf{m}}(m)\mathrm{d}m} = \frac{1-D_1^2}{2(1-D_1)} = \frac{1+D_1}{2} \tag{4.13}$$

$$D_1 = \frac{T_1+T_2}{2} \tag{4.14}$$

$$\therefore 2D_1 = \frac{1+8D_1^2}{8+16D_1} + \frac{1+D_1}{2} \Rightarrow 4D_1^2 - D_1 + \frac{5}{4} = 0 \Rightarrow D_1 = 0.4478 \tag{4.15}$$

$$T_1 = 0.1717; \quad T_2 = 0.7239. \tag{4.16}$$

P4.7 Regardless of the quantizer used, the signal power is:

$$\sigma_{\mathbf{m}}^2 = E\{\mathbf{m}^2\} = \int_{-1}^{1} m^2 \underbrace{(1-|m|)}_{f_{\mathbf{m}}(m)} \mathrm{d}m = 2\int_0^1 m^2(1-m)\mathrm{d}m = \frac{1}{6} \text{ watts} \tag{4.17}$$

Rather than solving the 1-bit, 2-bit and 3-bit quantizers separately, we first set up general expressions for the $\sigma_{\mathbf{q}}^2$, D_i and T_i in terms of n, the number of bits used for quantization. Before this, we observe that the pdf is symmetrical about zero and that the number of levels is even ($=2^n$). Therefore the decision boundaries and target levels shall be symmetrical about zero $\Rightarrow$ need only consider the positive m axis. The picture looks as follows:

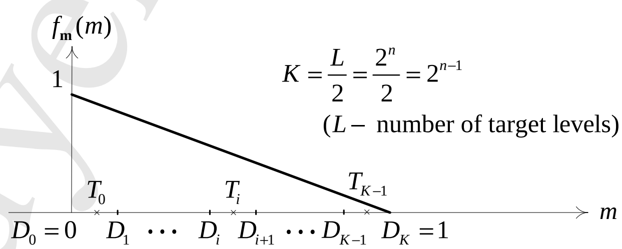


Figure 4.7

General expressions for the uniform quantizer:

The decision intervals $D_{i+1} - D_i$ are equal and are $= 1/2^{n-1} = 1/K$.

The target levels T_i are in the middle of the interval, i.e., $T_i = (D_i + D_{i+1})/2,\ i = 0, \ldots, K-1$, or another way of expressing T_i is $T_i = (i+1/2)\Delta,\ i = 0, \ldots, K-1, \Delta = D_{i+1} - D_i = 1/2^{n-1}$.

The quantization noise power is given by:

$$
\begin{aligned}
\sigma_{\mathbf{q}}^2 &= 2\sum_{i=0}^{K-1}\int_{D_i}^{D_{i+1}}(m-T_i)^2(1-m)\mathrm{d}m \overset{\lambda=m-T_i}{=} 2\sum_{i=0}^{K-1}\int_{-1/2^n}^{1/2^n}\lambda^2(1-\lambda-T_i)\mathrm{d}\lambda \\
&= 2\sum_{i=0}^{K-1}\left[(1-T_i)\int_{-1/2^n}^{1/2^n}\lambda^2\mathrm{d}\lambda - \underbrace{\int_{-1/2^n}^{1/2^n}\lambda^3\mathrm{d}\lambda}_{=0}\right] = 4\sum_{i=0}^{K-1}(1-T_i)\left(\frac{1}{3(2^{3n})}\right) \\
&= \frac{1}{3(2^{3n-2})}\left[\sum_{i=0}^{K-1}(1-T_i)\right]. \qquad (4.18)
\end{aligned}
$$

Consider

$$
\begin{aligned}
\sum_{i=0}^{K-1}(1-T_i) &= K-\sum_{i=0}^{K-1}\left(i+\frac{1}{2}\right)\Delta = K-\Delta\left[\frac{(K-1)K}{2}+\frac{K}{2}\right] \\
&= K-\frac{\Delta K}{2}K \overset{\Delta K=1}{=} \frac{K}{2} = 2^{n-2} \qquad (4.19) \\
\therefore\ \sigma_{\mathbf{q}}^2 &= \frac{1}{3(2^{3n-2})}2^{n-2} = \frac{1}{3(2^{2n})}\ \text{watts.} \qquad (4.20)
\end{aligned}
$$

It is now easy to find $\mathrm{SNR_q}$ for any number of bits:

$$
\mathrm{SNR_q} = \frac{\sigma_{\mathbf{m}}^2}{\sigma_{\mathbf{q}}^2} = \frac{3(2^{2n})}{6} = 2^{2n-1} \quad \text{or} \quad (2n-1)\log_{10}2 = (2n-1)3\ \text{dB}. \qquad (4.21)
$$

n	1	2	3	4
$\mathrm{SNR_q}$	2	8	32	124
$\mathrm{SNR_q}$(dB)	3	9	15	21

<u>Turning our attention to the optimum quantizer</u>

The equations for the decision boundaries are easy to write. They are:

$$
D_i = \frac{T_{i-1}+T_i}{2},\ i=1,2,\dots,K-1=2^{n-1}-1. \qquad (4.22)
$$

For the target levels, consider the generic interval as shown:

The centroid of this "probability mass" is:

$$
\begin{aligned}
T_i &= \frac{\int_{D_i}^{D_{i+1}} m(1-m)\mathrm{d}m}{\int_{D_i}^{D_{i+1}}\underbrace{(1-m)}_{f_{\mathbf{m}}(m)}\mathrm{d}m} = \frac{\frac{D_{i+1}^2-D_i^2}{2}-\frac{D_{i+1}^3-D_i^3}{3}}{(D_{i+1}-D_i)-\frac{D_{i+1}^2-D_i^2}{2}} = \frac{\frac{D_{i+1}+D_i}{2}-\frac{D_{i+1}^2+D_{i+1}D_i+D_i^2}{3}}{1-\frac{D_{i+1}+D_i}{2}} \\
&= \frac{3(D_{i+1}+D_i)-2(D_{i+1}^2+D_{i+1}D_i+D_i^2)}{3\left[2-(D_{i+1}+D_i)\right]},\quad i=0,1,\dots,K-1 \qquad (4.23)
\end{aligned}
$$

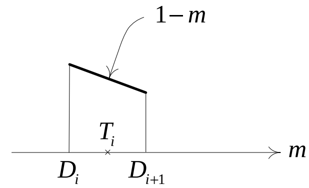


Figure 4.8

(Note that $D_0 = 0$, $D_K = 1$).

The above set of $K + K - 1 = 2^n - 1$ nonlinear algebraic equations needs to be solved for the decision boundaries and target levels. For $n = 1$ & 2 this can be done reasonably simply.

1-bit optimum quantizer

Only the target level, T_0, needs to be determined ($D_0 = 0, D_1 = 1$)

$$T_0 = \frac{3(D_1 + D_0) - 2(D_1^2 + D_1 D_0 + D_0^2)}{3\left[2 - (D_1 + D_0)\right]} = \frac{3-2}{3} = \frac{1}{3}. \tag{4.24}$$

2-bit optimum quantizer

Have $2^2 - 1 = 3$ equations in T_0, T_1, D_1 with $D_0 = 0, D_2 = 1$.

$$\begin{aligned}
D_1 &= \frac{T_0 + T_1}{2} \text{ or } 2D_1 = T_0 + T_1 && (4.25)\\
T_0 &= \frac{3(D_1 + D_0) - 2(D_1^2 + D_1 D_0 + D_0^2)}{3\left[2 - (D_1 + D_0)\right]} = \frac{3D_1 - 2D_1^2}{3(2 - D_1)} && (4.26)\\
T_1 &= \frac{3(D_2 + D_1) - 2(D_2^2 + D_2 D_1 + D_1^2)}{3\left[2 - (D_2 + D_1)\right]} \\
&= \frac{3(1 + D_1) - 2(1 + D_1 + D_1^2)}{3\left[2 - (1 + D_1)\right]} = \frac{1 + D_1 - 2D_1^2}{3(1 - D_1)} && (4.27)\\
\Rightarrow 2D_1 &= \frac{3D_1 - 2D_1^2}{3(2 - D_1)} + \frac{1 + D_1 - 2D_1^2}{3(1 - D_1)}. && (4.28)
\end{aligned}$$

This simplifies to $D_1^3 - 4D_1^2 + 4D_1 - 1 = 0$. Use either `roots` function in Matlab or observe that $D_1 = 1$ is a root and therefore the polynomial factors as $(D_1 - 1)(D_1^2 - 3D_1 + 1)$. The roots of the quadratic are: $\frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2}$.

Choosing $\frac{3-\sqrt{5}}{2}$ (since D_1 must lie in $[0, 1]$) we have $D_1 = 0.382$ and $T_0 = 0.176, T_1 = 0.588$.

3-bit optimum quantizer

There are $2^3 - 1 = 7$ equations in the unknowns $T_0, D_1, T_1, D_2, T_2, D_3, T_3$. In particular, 3 for the decision boundaries: $D_1 = \frac{T_0+T_1}{2}, D_2 = \frac{T_1+T_2}{2}, D_3 = \frac{T_2+T_3}{2}$, and 4 for the target levels:

$(D_0 = 0, D_4 = 1)$

$$T_0 = \frac{3D_1 - 2D_1^2}{3(2 - D_1)} \tag{4.29}$$

$$T_1 = \frac{3(D_2 + D_1) - 2(D_2^2 + D_2 D_1 + D_1^2)}{3[2 - (D_2 + D_1)]} \tag{4.30}$$

$$T_2 = \frac{3(D_3 + D_2) - 2(D_3^2 + D_3 D_2 + D_2^2)}{3[2 - (D_3 + D_2)]} \tag{4.31}$$

$$T_3 = \frac{1 + D_3 - 2D_3^2}{3(1 - D_3)}. \tag{4.32}$$

To solve the above set of nonlinear algebraic equations the function `fsolve` in the optimization toolbox of Matlab was used. The obtained solution is

$D_1 = 0.1879;\ D_2 = 0.3920;\ D_3 = 0.6243;$

$T_0 = 0.0907;\ T_1 = 0.2851;\ T_2 = 0.4990;\ T_3 = 0.7495.$

Turning to the quantization noise power, $\sigma_{\mathbf{q}}^2$, for the optimum quantizer we derive at first a general expression for the noise power when m falls in the region D_i, D_{i+1}. It is

$$\begin{aligned}\sigma_{\mathbf{q}}^2(i) &= \int_{D_i}^{D_{i+1}} (m - T_i)^2 (1 - m)\mathrm{d}m \overset{\lambda = m - T_i}{=} (1 - T_i) \int_{D_i - T_i}^{D_{i+1} - T_i} \lambda^2 \mathrm{d}\lambda - \int_{D_i - T_i}^{D_{i+1} - T_i} \lambda^3 \mathrm{d}\lambda \\ &= \frac{(1 - T_i)}{3}\left[(D_{i+1} - T_i)^3 - (D_i - T_i)^3\right] + \frac{1}{4}\left[(D_i - T_i)^4 - (D_{i+1} - T_i)^4\right]\end{aligned} \tag{4.33}$$

and $\sigma_{\mathbf{q}}^2 = 2\sum_{i=0}^{K-1} \sigma_{\mathbf{q}}^2(i)$.

Programming this in Matlab gives:

2-bit quantizer: $\sigma_{\mathbf{q}}^2 = 0.0155$;

3-bit quantizer: $\sigma_{\mathbf{q}}^2 = 0.0041$;

The quantization signal-to-noise ratios are therefore:

	1-bit	2-bit	3-bit
$\mathrm{SNR_q}$	$\frac{(1/6)}{(1/18)} = 3$	$\frac{(1/6)}{0.0155} = 10.75$	$\frac{(1/6)}{0.0041} = 40.65$
$\mathrm{SNR_q}$(dB)	4.77	10.3	16.1

Remark: Compared to the uniform quantizer, the optimum quantizer gives the biggest performance improvement at 1 bit. As the number of bits increases the performance improvement becomes less and less $\Rightarrow$ the optimum quantizer tends to be a uniform quantizer for n (number of bits) "large".

Matlab code

(i) Thresholds and decision levels (2-bit quantizer)

```
function F=quantizer(x)
F=[x(1) - (1 + 8*x(3)^2)/(8 + 16*x(3)); x(2) - 0.5*(1 + x(3));
x(3) - 0.5*(x(1) + x(2))] % x_1 = T_1; x(2)=T_2; x(3)=D_1
x0=[0.25 0.75 0.5];
% initial guess - basically uniform quantizer values
options = optimset('Display','iter')
[x, fval]=fsolve(@quantizer, x0, options)
```

(ii)

```
sigmaq=0; K=   ;
D_vec=[0, D_1, D_2,...,D_{K-1}, 1];
T=[T_0,...,T_{K-1}];
for i=1:1:K
    a=1-T(i); b=D_vec(i+1)-T(i); c=D_vec(i)-T(i);
    sigmaq= a*(b^3-c^3)/3 + 0.25*(c^4 - b^4);
end
sigmaq=2*sigmaq;
```

P4.8 (a) Since $f_{\mathbf{m}}(m)$ is symmetric about zero, one only needs to compute the decision and target values for the positive m axis:

$$T_l = \frac{\int_{D_l}^{D_{l+1}} m f_{\mathbf{m}}(m)\mathrm{d}m}{\int_{D_l}^{D_{l+1}} f_{\mathbf{m}}(m)\mathrm{d}m} \tag{4.34}$$

$$= \frac{(c/2)\int_{D_l}^{D_{l+1}} m\exp(-cm)\mathrm{d}m}{(c/2)\int_{D_l}^{D_{l+1}} \exp(-cm)\mathrm{d}m} = \frac{\int_{D_l}^{D_{l+1}} m\exp(-cm)\mathrm{d}m}{\int_{D_l}^{D_{l+1}} \exp(-cm)\mathrm{d}m} \tag{4.35}$$

Applying integral by parts to obtain the following:

$$\int \exp(-cm)\mathrm{d}m = -\frac{1}{c}\exp(-cm) \tag{4.36}$$

$$\int m\exp(-cm)\mathrm{d}m = -\frac{1}{c}m\exp(-cm) - \frac{1}{c^2}\exp(-cm) \tag{4.37}$$

Thus

$$T_l = \frac{D_{l+1}\exp(-cD_{l+1}) - D_l\exp(-cD_l)}{\exp(-cD_{l+1}) - \exp(-cD_l)} + \frac{1}{c} \tag{4.38}$$

The average signal power is

$$\sigma_{\mathbf{m}}^2 = \int_{-\infty}^{+\infty} m^2 f_{\mathbf{m}}(m)\mathrm{d}m = c\int_0^{+\infty} m^2\exp(-cm)\mathrm{d}m$$

$$= \frac{2}{c^2}. \tag{4.39}$$

The following integrals may be useful when computing the average quantization noise power:

$$\begin{aligned}\mathcal{I}_1(a,b) &= \int_a^b m^2\exp(-cm)\mathrm{d}m \\ &= -\left[\exp(-cm)\left(\frac{m^2}{c}+\frac{2m}{c^2}+\frac{2}{c^3}\right)\right]\Bigg|_a^b, \qquad (4.40)\\ \mathcal{I}_2(a,b) &= \int_a^b m\exp(-cm)\mathrm{d}m \\ &= -\left[\exp(-cm)\left(\frac{m}{c}+\frac{1}{c^2}\right)\right]\Bigg|_a^b. \qquad (4.41)\end{aligned}$$

(b) One-bit uniform quantizer: $L=2, D_0=0, D_1=+\infty$, one need to find T_0. Since $\mathbf{m}_{\max}\to+\infty$, one can assume that $\mathbf{m}_{\max}$ will not exceed a value p with the probability ϵ and then compute D_l, T_l corresponding to p and ϵ. Therefore,

$$\epsilon = \int_0^p \frac{1}{2}c\exp(-cm)\mathrm{d}m = \frac{1}{2}\left[1-\exp(-cp)\right] \qquad (4.42)$$

$$\Rightarrow\ p = -\frac{1}{c}\ln(1-2\epsilon). \qquad (4.43)$$

Then the target value for this case will be

$$T_0 = -\frac{1}{2c}\ln(1-2\epsilon). \qquad (4.44)$$

The average quantization noise power is:

$$\begin{aligned}\sigma_{\mathbf{q}}^2 &= 2\int_0^\infty (m-T_0)^2 f_{\mathbf{m}}(m)\mathrm{d}m \\ &= c\int_0^\infty (m-T_0)^2\exp(-cm)\mathrm{d}m \\ &= c\left[\mathcal{I}_1(0,+\infty)-2T_0\mathcal{I}_2(0,+\infty)+\frac{T_0^2}{c}\right] \\ &= \frac{1}{c^2}\left[2+\ln(1-2\epsilon)+\frac{1}{4}\ln^2(1-2\epsilon)\right]. \qquad (4.45)\end{aligned}$$

The quantization signal-to-noise ratio is therefore:

$$\mathrm{SNR_q} = \frac{\sigma_{\mathbf{m}}^2}{\sigma_{\mathbf{q}}^2} = \frac{2}{2+\ln(1-2\epsilon)+\frac{1}{4}\ln^2(1-2\epsilon)}. \qquad (4.46)$$

(c) One-bit optimum quantizer: $L=2, D_0=0$ and $D_1=+\infty$. Use (4.38) to find T_0:

$$\begin{aligned}T_0 &= \frac{D_1\exp(-cD_1)-D_0\exp(-cD_0)}{\exp(-cD_1)-\exp(-cD_0)}+\frac{1}{c} \\ &= \frac{D_1\exp(-cD_1)}{\exp(-cD_1)-1}+\frac{1}{c} \\ &= \frac{1}{c} \qquad (4.47)\end{aligned}$$

Thus the quantizer maps all the positive values of the sampled message to $\frac{1}{c}$ and all the negative values to $-\frac{1}{c}$.
Similar to (4.45), the quantization noise power is

$$\begin{aligned}\sigma_{\mathbf{q}}^2 &= 2\int_0^{\infty}(m-T_0)^2 f_{\mathbf{m}}(m)\mathrm{d}m \\ &= c\int_0^{\infty}(m-T_0)^2\exp(-cm)\mathrm{d}m \\ &= c\left[\mathcal{I}_1(0,+\infty)-2T_0\mathcal{I}_2(0,+\infty)+\frac{T_0^2}{c}\right] \\ &= \frac{1}{c^2}. \end{aligned} \tag{4.48}$$

The quantization signal-to-noise ratio is therefore:

$$\mathrm{SNR_q} = \frac{\sigma_{\mathbf{m}}^2}{\sigma_{\mathbf{q}}^2} = \frac{2/c^2}{1/c^2} = 2. \tag{4.49}$$

(d) Two-bit uniform quantizer: Similar to the one-bit uniform quantizer, one has $L=4, D_0=0, D_2=-\frac{1}{c}\ln(1-2\epsilon)$. Thus, $D_1=-\frac{1}{2c}\ln(1-2\epsilon)$, $T_0=-\frac{1}{4c}\ln(1-2\epsilon)$, $T_1=-\frac{3}{4c}\ln(1-2\epsilon)$.
Two-bit optimum quantizer: $L=4, D_0=0, D_2=+\infty \Rightarrow$ Need to find T_0, T_1 and D_1. Using (4.38) one has:

$$T_0 = \frac{D_1\exp(-cD_1)}{\exp(-cD_1)-1}+\frac{1}{c} \tag{4.50}$$

$$T_1 = D_1+\frac{1}{c} \tag{4.51}$$

Next, the equation to solve for D_1 is:

$$\begin{aligned} D_1 &= \frac{T_0+T_1}{2} \\ &= \frac{1}{c}\left[D_1+\frac{D_1\exp(-cD_1)}{\exp(-cD_1)-1}\right]+\frac{1}{c} \end{aligned} \tag{4.52}$$

Let $x=cD_1$, then after some manipulations, the above equation becomes:

$$x = 2-2\mathrm{e}^{-x} \tag{4.53}$$

Equation (4.53) can be solved by trial and error. One method is to plot $2-2\mathrm{e}^{-x}$ and x versus x and look at the intersection point (see Fig. 4.9). The solution is $x=1.59 \Rightarrow D_1=\frac{1.59}{c} \Rightarrow T_1=D_1+\frac{1}{c}=\frac{2.59}{c}$ and $T_0=D_1-\frac{1}{c}=\frac{0.59}{c}$.
For these two quantizers, the average quantization noise power can be computed as

$$\sigma_{\mathbf{q}}^2 = 2\left[\sigma_{\mathbf{q}}^2(0)+\sigma_{\mathbf{q}}^2(1)\right], \tag{4.54}$$

where

$$\sigma_{\mathbf{q}}^2(0) = \frac{c}{2}\left[\mathcal{I}_1(0,D_1)-2T_0\mathcal{I}_2(0,D_1)+\frac{T_0^2}{c}\left(1-\exp(-cD_1)\right)\right], \tag{4.55}$$

$$\sigma_{\mathbf{q}}^2(1) = \frac{c}{2}\left[\mathcal{I}_1(D_1,+\infty)-2T_1\mathcal{I}_2(D_1,+\infty)+\frac{T_1^2}{c}\exp(-cD_1)\right], \tag{4.56}$$

and the quantization signal-to-noise ratio can be found by $\mathrm{SNR_q}=\sigma_{\mathbf{m}}^2/\sigma_{\mathbf{q}}^2$.

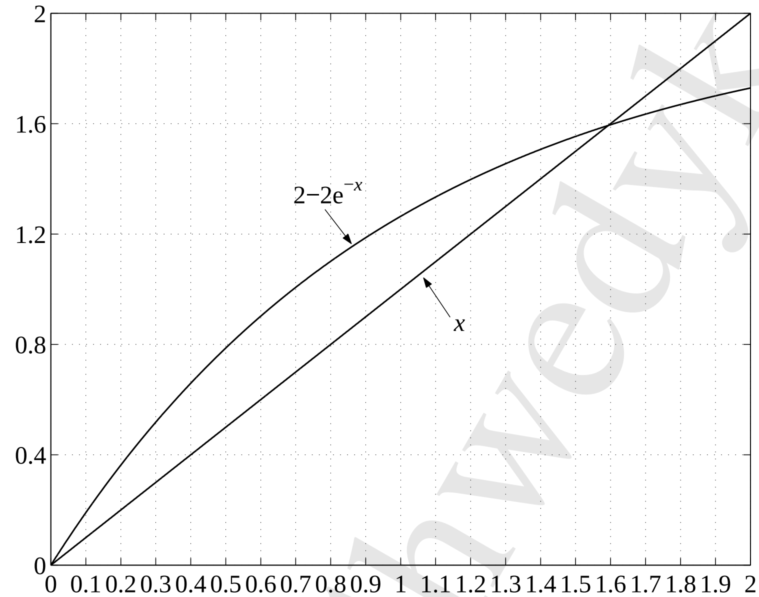


Figure 4.9: Plots of $2 - 2e^{-x}$ and x versus x.

P4.9 **To be added.**

P4.10 **To be added.**

P4.11 By inspection the optimum quantizer is a uniform quantizer with decision boundaries at $k\Delta, (k+1)\Delta$, etc., and a target level of $k\Delta + \Delta/2$. With these values the target level is the center of gravity of the probability mass density and the decision boundary is the arithmetic of the two adjacent target values, i.e., the optimum equations are satisfied.

Note: This implies that as the number of quantization bits, R, increases the approximation becomes better and better and that the uniform quantizer's performance approaches that of an optimum quantizer. Given the manufacturing simplicity of a uniform quantizer there is little or no profit going to an optimum quantizer.

P4.12 Ignoring for the moment the range of $\mathbf{m}_{\text{out}}$, the first observation is that for $f_{\mathbf{m}_{\text{out}}}(m_{\text{out}})$ to be uniform the RHS of (P4.5) must be a constant, $\frac{\mathrm{d}g(m)}{\mathrm{d}m} = f_{\mathbf{m}}(m) \Rightarrow$ that $g(m)$ is the cumulative distribution function of $\mathbf{m}(t)$, i.e., $g(m) = F_{\mathbf{m}}(m)$. $F_{\mathbf{m}}(m)$ is monotonic and ranges from 0 (at $-\infty$) to 1 (at $+\infty$). Therefore $f_{\mathbf{m}_{\text{out}}}(m_{\text{out}})$ is uniform over $[0, 1]$.

Pass $\mathbf{m}_{\text{out}}(t)$ through a linear (mathematically more appropriately called affine) transformation which shifts it to lie in the $[-1, 1]$ range.

$$\mathbf{y}(t) = 2\left[\mathbf{m}_{\text{out}}(t) - \frac{1}{2}\right] \tag{4.57}$$

$$f_{\mathbf{y}}(y) = \frac{1}{2}\left[u(y+1) - u(y-1)\right]. \tag{4.58}$$

Refuses, because the quantizer in all likelihood would not be robust enough. It would be sensitive to the assumed pdf model for $\mathbf{m}(t)$.

P4.13 Crest factor F is defined as

$$F = \frac{\text{Peak value of the signal}}{\text{RMS value of the signal}} = \frac{m_{\max}}{\sigma_{\mathbf{m}}} \tag{4.59}$$

(a) A sinusoid with peak amplitude of $m_{\max}$: $V_{\text{RMS}} = m_{\max}/\sqrt{2} \Rightarrow \boxed{F = \sqrt{2}}$.

(b) A square wave of period T and amplitude range $[-m_{\max}, m_{\max}]$: $V_{\text{RMS}} = m_{\max} \Rightarrow \boxed{F = 1}$.

(c) Uniform over the amplitude range $[-m_{\max}, m_{\max}]$: $\sigma_{\mathbf{m}}^2 = E\{\mathbf{m}^2\} = 2\int_0^{m_{\max}} m^2 \frac{1}{2m_{\max}} \mathrm{d}m = \frac{m_{\max}^2}{3} \Rightarrow V_{\text{RMS}} = \frac{m_{\max}}{\sqrt{3}} \Rightarrow \boxed{F = \sqrt{3} = 1.73}$.

(d) Zero-mean Gaussian with variance $\sigma_{\mathbf{m}}^2$: $V_{\text{RMS}} = \sigma_{\mathbf{m}}$. Equation to solve for $m_{\max}$ is

$$\frac{1}{\sqrt{2\pi}\sigma_{\mathbf{m}}} \int_0^{m_{\max}} \exp\left(-\frac{m^2}{2\sigma_{\mathbf{m}}^2}\right) \mathrm{d}m = \frac{0.99}{2} \tag{4.60}$$

By changing the variable $t = \frac{m}{\sqrt{2}\sigma_{\mathbf{m}}}$ the above can be rewritten as

$$\begin{aligned}
&\frac{2}{\sqrt{\pi}} \int_0^{\frac{m_{\max}}{\sqrt{2}\sigma_{\mathbf{m}}}} \mathrm{e}^{-t^2} \mathrm{d}t = 0.99 \Rightarrow \operatorname{erf}\left(\frac{m_{\max}}{\sqrt{2}\sigma_{\mathbf{m}}}\right) = 0.99 \\
\Rightarrow\; & \frac{m_{\max}}{\sqrt{2}\sigma_{\mathbf{m}}} = \operatorname{erf}^{-1}(0.99) = 1.8214 \Rightarrow m_{\max} = \sqrt{2} \times 1.8214 \times \sigma_{\mathbf{m}} \\
\Rightarrow\; & \boxed{F = 2.5758}
\end{aligned} \tag{4.61}$$

(e) Zero-mean Laplacian: $\sigma_{\mathbf{m}}^2 = E\{\mathbf{m}^2\} = \frac{2}{c^2} \Rightarrow \sigma_{\mathbf{m}} = \frac{\sqrt{2}}{c}$. Now solve for $m_{\max}$:

$$\begin{aligned}
&\frac{c}{2} \int_0^{m_{\max}} \exp(-cm)\, \mathrm{d}m = \frac{0.99}{2} \Rightarrow -\exp(-cm)\Big|_0^{m_{\max}} = 0.99 \\
\Rightarrow\; & 1 - \exp(-cm_{\max}) = 0.99 \Rightarrow m_{\max} = \frac{-\ln(0.01)}{c} = \frac{4.6052}{c} \\
\Rightarrow\; & \boxed{F = \frac{4.6052}{\sqrt{2}} = 3.2563}
\end{aligned} \tag{4.62}$$

(f) Zero-mean Gamma. The pdf is

$$f_{\mathbf{m}}(m) = \sqrt{\frac{k}{4\pi|m|}} \exp(-k|m|) \tag{4.63}$$

$$\sigma_{\mathbf{m}}^2 = 2\int_0^{\infty} m^2 \sqrt{\frac{k}{4\pi m}} \exp(-km) \mathrm{d}m = \frac{\sqrt{k}}{\sqrt{\pi}} \int_0^{\infty} m^{3/2} \exp(-km) \mathrm{d}m \tag{4.64}$$

Integrate by parts to obtain:

$$
\begin{aligned}
\int_0^\infty m^{3/2}\exp(-km)\mathrm{d}m &= \underbrace{-\frac{1}{k}m^{3/2}\exp(-km)\Big|_0^\infty}_{=0} + \frac{3}{2k}\int_0^\infty m^{1/2}\exp(-km)\mathrm{d}m \\
&= -\frac{3}{2k^2}\left[\underbrace{m^{1/2}\exp(-km)\Big|_0^\infty}_{=0} - \frac{1}{2}\int_0^\infty \frac{\exp(-km)}{\sqrt{m}}\mathrm{d}m\right] \\
&= \frac{3}{4k^2}\int_0^\infty \frac{\exp(-km)}{\sqrt{m}}\mathrm{d}m \overset{t=\sqrt{km}}{=} \frac{3\sqrt{\pi}}{4k^2\sqrt{k}}\frac{2}{\sqrt{\pi}}\int_0^\infty \mathrm{e}^{-t^2}\mathrm{d}t \\
&= \frac{3\sqrt{\pi}}{4k^2\sqrt{k}} \times \underbrace{\mathrm{erf}(\infty)}_{=1} = \frac{3\sqrt{\pi}}{4k^2\sqrt{k}} \qquad (4.65)
\end{aligned}
$$

Combining (4.64) and (4.65) yields $\sigma_{\mathbf{m}}^2 = \frac{3}{4k^2} \Rightarrow \sigma_{\mathbf{m}} = \frac{\sqrt{3}}{2k}$.

Now $m_{\max}$ is found as follows:

$$
\begin{aligned}
&\frac{\sqrt{k}}{2\sqrt{\pi}}\int_0^{m_{\max}} \frac{\exp(-km)}{\sqrt{m}}\mathrm{d}m = \frac{0.99}{2} \overset{t=\sqrt{km}}{\Rightarrow} \frac{1}{\sqrt{\pi}}\int_0^{\sqrt{km_{\max}}} \mathrm{e}^{-t^2}\mathrm{d}t = 0.99 \\
\Rightarrow\quad & \mathrm{erf}\left(\sqrt{km_{\max}}\right) = 0.99 \Rightarrow \sqrt{km_{\max}} = \mathrm{erf}^{-1}(0.99) \\
\Rightarrow\quad & m_{\max} = \frac{\left[\mathrm{erf}^{-1}(0.99)\right]^2}{k} = \frac{3.3174}{k} \qquad (4.66)
\end{aligned}
$$

$$
\boxed{F = \frac{3.3174 \times 2}{\sqrt{3}} = 3.8307} \qquad (4.67)
$$

Therefore if one was to list the signals from least "peaked" to most "peaked", they are:

square wave, sinusoid, uniform, Gaussian, Laplacian, Gamma,

which, from either the waveshapes or the pdfs, makes some intuitive sense.

P4.14 (a) Sinusoid: $m_{\max}\cos(2\pi f_r t) \Rightarrow T = 1/f_r$. Then over half a period the average value of $|m(t)|$ is

$$
= \frac{1}{(T/2)}\int_{-T/4}^{T/4} m_{\max}\cos(2\pi f_r t)\mathrm{d}t = \frac{2}{T}m_{\max}\int_{-1/(4f_r)}^{1/(4f_r)} \cos(2\pi f_r t)\mathrm{d}t = \frac{2m_{\max}}{\pi} \qquad (4.68)
$$

$$
V_{\mathrm{RMS}} = \frac{m_{\max}}{\sqrt{2}} \Rightarrow \frac{E\{|m(t)|\}}{V_{\mathrm{RMS}}} = \frac{2\sqrt{2}}{\pi} = 0.9 \qquad (4.69)
$$

(b)

$$
E\{|m(t)|\} = m_{\max},\ V_{\mathrm{RMS}} = m_{\max} \ \Rightarrow\ \frac{E\{|m(t)|\}}{V_{\mathrm{RMS}}} = 1 \qquad (4.70)
$$

(c)

$$E\{|\mathbf{m}|\} = \int_{-m_{\max}}^{m_{\max}} |m| \frac{1}{2m_{\max}} \mathrm{d}m = \frac{m_{\max}}{2} \tag{4.71}$$

$$\sigma_{\mathbf{m}} = \frac{m_{\max}}{\sqrt{3}} \tag{4.72}$$

Therefore $\frac{E\{|\mathbf{m}|\}}{\sigma_{\mathbf{m}}} = \frac{\sqrt{3}}{2} = 0.87$.

(d)

$$E\{|\mathbf{m}|\} = \frac{1}{\sqrt{2\pi}\sigma_{\mathbf{m}}} \int_{-\infty}^{\infty} |m| \mathrm{e}^{-\frac{m^2}{2\sigma_{\mathbf{m}}^2}} \mathrm{d}m \overset{\lambda = \frac{m^2}{2\sigma_{\mathbf{m}}^2}}{=} \frac{2\sigma_{\mathbf{m}}}{\sqrt{2\pi}} \int_0^{\infty} \mathrm{e}^{-\lambda} \mathrm{d}\lambda = \sqrt{\frac{2}{\pi}} \sigma_{\mathbf{m}} \tag{4.73}$$

$$\therefore \frac{E\{|\mathbf{m}|\}}{\sigma_{\mathbf{m}}} = \sqrt{\frac{2}{\pi}} = 0.8 \tag{4.74}$$

(e)

$$E\{|\mathbf{m}|\} = \frac{c}{2} \int_{-\infty}^{\infty} |m| \mathrm{e}^{-c|m|} \mathrm{d}m = \frac{1}{c} \tag{4.75}$$

$$\sigma_{\mathbf{m}} = \frac{\sqrt{2}}{c} \text{ (from P 4.13e) } \Rightarrow \frac{E\{|\mathbf{m}|\}}{\sigma_{\mathbf{m}}} = \frac{1}{\sqrt{2}} = 0.707 \tag{4.76}$$

(f)

$$\begin{aligned} E\{|\mathbf{m}|\} &= \sqrt{\frac{k}{4\pi}} \int_{-\infty}^{\infty} \frac{|m|}{|m|^{1/2}} \mathrm{e}^{-k|m|} \mathrm{d}m = \sqrt{\frac{k}{\pi}} \int_0^{\infty} m^{1/2} \mathrm{e}^{-km} \mathrm{d}m \\ &= \sqrt{\frac{k}{\pi}} \frac{\Gamma(\frac{3}{2})}{(k)^{3/2}} = \frac{1}{2k} \end{aligned} \tag{4.77}$$

$$\sigma_{\mathbf{m}} = \frac{\sqrt{3}}{2k} \text{ (from P4.13e) } \Rightarrow \frac{E\{|\mathbf{m}|\}}{\sigma_{\mathbf{m}}} = \frac{1}{\sqrt{3}} = 0.58 \tag{4.78}$$

Listing the parameters from largest to smallest results in:

square wave, sinusoid, uniform, Gaussian, Laplacian, Gamma.

The above is the same order as in P4.13, but the parameter values are now decreasing instead of increasing.

Note: The relationship can be used (and has been used) to design an RMS meter. The block diagram looks as in Fig. 4.10. The advantage of this approach is a "true" RMS voltmeter requires a squaring circuit whereas here only a fullwave rectifier is needed. The disadvantage is that the voltmeter is good only as an RMS voltmeter for the specific signal that the signal is calibrated.

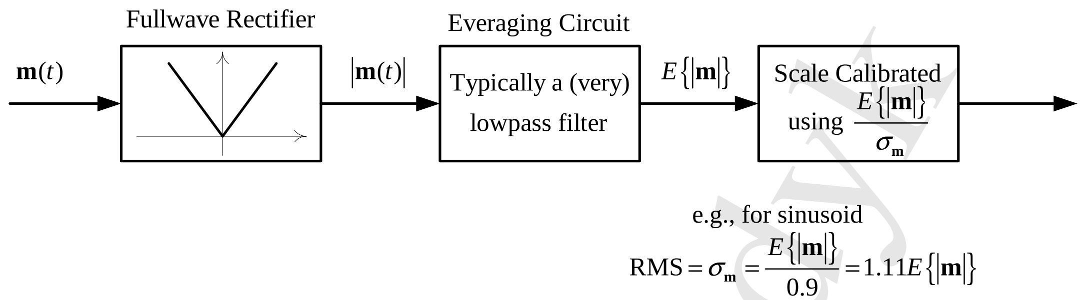


Figure 4.10: Block diagram of an RMS meter.

P4.15 Eqn. (4.46) states that

$$\text{SNR}_{\text{q}}(\sigma_n^2) = \frac{3L^2\mu^2}{\ln^2(1+\mu)} \frac{\sigma_n^2}{\left(1 + 2\mu\sigma_n \frac{E\{|\mathbf{m}|\}}{\sigma_{\mathbf{m}}} + \mu^2\sigma_n^2\right)}. \tag{4.79}$$

From P4.14 the signal parameter $\frac{E\{|\mathbf{m}|\}}{\sigma_{\mathbf{m}}}$ is as follows:

- Gaussian: $\sqrt{\frac{2}{\pi}}$;
- Laplacian: $\frac{1}{\sqrt{2}}$;
- Gamma: $\frac{1}{\sqrt{3}}$;
- Uniform: $\frac{\sqrt{3}}{2}$.

Matlab code for the plot:

```
L=256; % 8-bit quantizer
mu=255; % mu-law parameter
SPar= sqrt(2/pi); % appropriate signal parameter, entered is for Gaussian
K=3*L^2*mu^2/((log(1+mu))^2);
SPB =[-100:1:0]; % signal power from -100 to 0 dB, 1 dB increments
SP=10.^(SPB/10); % signal power
SNRq=K*SP./(1 + 2*mu*SPar*sqrt(SP) + mu^2*SP);
plot(SPB,10*log10(SNRq)); % plot SNRq in dB
hold;
%(repeat for next signal model)
```

From the plots it can be seen that the quantizer is quite insensitive to variations in input signal power over a wide range (≈ 40 dB) and also insensitive to the actual pdf model of the input signal - Both desirable properties.

P4.16 Consider positive m. Note that nonlinearity is an odd function. Therefore the derivative shall

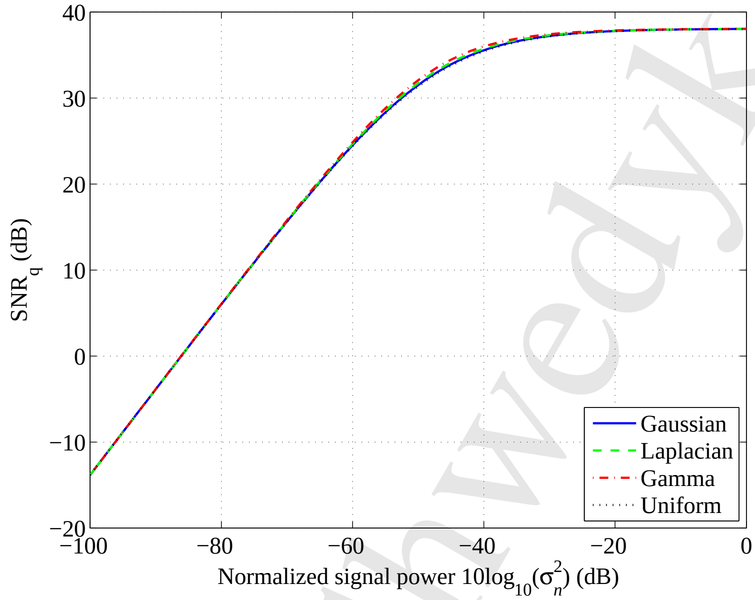


Figure 4.11: Plots of $\mathrm{SNR_q}$ for different signal models with 8-bit μ-law quantizer ($L = 256$, $\mu = 255$).

be an even function.

$$\text{For } 0 < m \leq \frac{m_{\max}}{A}: \quad \frac{\mathrm{d}y}{\mathrm{d}m} = \frac{A y_{\max}}{m_{\max}(1+\ln A)} \tag{4.80}$$

$$\text{For } \frac{m_{\max}}{A} < m \leq m_{\max}: \quad \frac{\mathrm{d}y}{\mathrm{d}m} = \frac{y_{\max}}{m(1+\ln A)} \tag{4.81}$$

$$\therefore \left(\frac{\mathrm{d}y}{\mathrm{d}m}\right)^2 = \begin{cases} \frac{A^2 y_{\max}^2}{m_{\max}^2(1+\ln A)^2}, & 0 < m \leq \frac{m_{\max}}{A} \\ \frac{y_{\max}^2}{m^2(1+\ln A)^2}, & \frac{m_{\max}}{A} < m \leq m_{\max} \end{cases} \tag{4.82}$$

$$
\begin{aligned}
N_{\mathrm{q}} &= \frac{y_{\max}^2}{3L^2}\int_{-m_{\max}}^{m_{\max}} \frac{f_{\mathbf{m}}(m)}{\left(\frac{\mathrm{d}y}{\mathrm{d}m}\right)^2}\mathrm{d}m \\
&= \frac{y_{\max}^2}{3L^2}\Bigg[\int_{-m_{\max}}^{-m_{\max}/A} \frac{m^2(1+\ln A)^2}{y_{\max}^2} f_{\mathbf{m}}(m)\mathrm{d}m + \int_{-m_{\max}/A}^{m_{\max}/A} \frac{m_{\max}^2(1+\ln A)^2}{A^2 y_{\max}^2} f_{\mathbf{m}}(m)\mathrm{d}m \\
&\quad + \int_{m_{\max}/A}^{m_{\max}} \frac{m^2(1+\ln A)^2}{y_{\max}^2} f_{\mathbf{m}}(m)\mathrm{d}m\Bigg] \\
&= \frac{(1+\ln A)^2}{3L^2}\Bigg[\int_{-m_{\max}}^{-m_{\max}/A} m^2 f_{\mathbf{m}}(m)\mathrm{d}m + \int_{m_{\max}/A}^{m_{\max}} m^2 f_{\mathbf{m}}(m)\mathrm{d}m + \frac{m_{\max}^2}{A^2}\underbrace{\int_{-m_{\max}/A}^{m_{\max}/A} f_{\mathbf{m}}(m)\mathrm{d}m}_{=P\left(-\frac{m_{\max}}{A}\le\mathbf{m}\le\frac{m_{\max}}{A}\right)}\Bigg] \\
&= \frac{(1+\ln A)^2}{3L^2}\left\{\sigma_{\mathbf{m}}^2 - \int_{-m_{\max}/A}^{m_{\max}/A} m^2 f_{\mathbf{m}}(m)\mathrm{d}m + \frac{m_{\max}^2}{A^2}P\left(-\frac{m_{\max}}{A}\le\mathbf{m}\le\frac{m_{\max}}{A}\right)\right\}
\end{aligned}
$$

$$
\therefore \mathrm{SNR}_{\mathrm{q}} = \frac{\frac{3L^2}{(1+\ln A)^2}\sigma_{\mathbf{m}}^2}{\sigma_{\mathbf{m}}^2 + \left(\frac{m_{\max}^2}{A^2}P\left(-\frac{m_{\max}}{A}\le\mathbf{m}\le\frac{m_{\max}}{A}\right) - \int_{-m_{\max}/A}^{m_{\max}/A} m^2 f_{\mathbf{m}}(m)\mathrm{d}m\right)} \tag{4.83}
$$

The term $\frac{m_{\max}^2}{A^2}P\left(-\frac{m_{\max}}{A}\le\mathbf{m}\le\frac{m_{\max}}{A}\right) - \int_{-m_{\max}/A}^{m_{\max}/A} m^2 f_{\mathbf{m}}(m)\mathrm{d}m$ depends on the ratio $m_{\max}/A$ and on the pdf model assumed for $f_{\mathbf{m}}(m)$. In a well-engineered system $m_{\max}/A \ll 1$ and therefore for high $\sigma_{\mathbf{m}}^2$ the $\mathrm{SNR}_{\mathrm{q}} = \frac{3L^2}{(1+\ln A)^2}$ (analogous to μ-law behavior for large μ). In essence, it appears that both μ-law and A-law are equally (perhaps similarly is a better word) robust to the assumed signal model. Plots of $\mathrm{SNR}_{\mathrm{q}}$ are shown below for Laplacian and uniform pdf models since these are fairly easy to handle analytically.

For uniform and Laplacian densities the 2 "correction" factors are:

(a) Uniform:

$$
P\left(-\frac{m_{\max}}{A}\le\mathbf{m}\le\frac{m_{\max}}{A}\right) = \frac{1}{2m_{\max}}\int_{-m_{\max}/A}^{m_{\max}/A}\mathrm{d}m = \frac{1}{A} \tag{4.84}
$$

$$
\int_{-m_{\max}/A}^{m_{\max}/A} m^2 f_{\mathbf{m}}(m)\mathrm{d}m = \frac{1}{m_{\max}}\int_{0}^{m_{\max}/A} m^2\mathrm{d}m = \frac{m_{\max}^2}{3A^3} = \frac{\sigma_{\mathbf{m}}^2}{A^3}. \tag{4.85}
$$

(b) Laplacian:

First define $m_{\max}$ to be the value that captures $\epsilon\%$ (percent) of the probability, given by $2\int_{-m_{\max}}^{m_{\max}} \frac{c}{2}\mathrm{e}^{-c|m|}\mathrm{d}m \Rightarrow cm_{\max} = -\ln(1-\epsilon/100)$. Then

$$P\left(-\frac{m_{\max}}{A} \le \mathbf{m} \le \frac{m_{\max}}{A}\right) = 2\int_0^{m_{\max}/A} \frac{c}{2}\mathrm{e}^{-cm}\mathrm{d}m = 1-\mathrm{e}^{-\frac{cm_{\max}}{A}} \tag{4.86}$$

and

$$\begin{aligned}\int_{-m_{\max}/A}^{m_{\max}/A} m^2\left(\frac{c}{2}\mathrm{e}^{-c|m|}\right)\mathrm{d}m &= c\int_0^{m_{\max}/A} m^2\mathrm{e}^{-cm}\mathrm{d}m \\ &= \frac{1}{c^2}\left\{2-\mathrm{e}^{-\frac{cm_{\max}}{A}}\left[\left(\frac{cm_{\max}}{A}\right)^2+2\frac{cm_{\max}}{A}+2\right]\right\}.\end{aligned} \tag{4.87}$$

But $\sigma_{\mathbf{m}}^2 = 2/c^2$ for Laplacian. Therefore

$$\int_{-m_{\max}/A}^{m_{\max}/A} m^2\left(\frac{c}{2}\mathrm{e}^{-c|m|}\right)\mathrm{d}m = \frac{\sigma_{\mathbf{m}}^2}{2}\left\{2-\mathrm{e}^{-\frac{cm_{\max}}{A}}\left[\left(\frac{cm_{\max}}{A}\right)^2+2\frac{cm_{\max}}{A}+2\right]\right\} \tag{4.88}$$

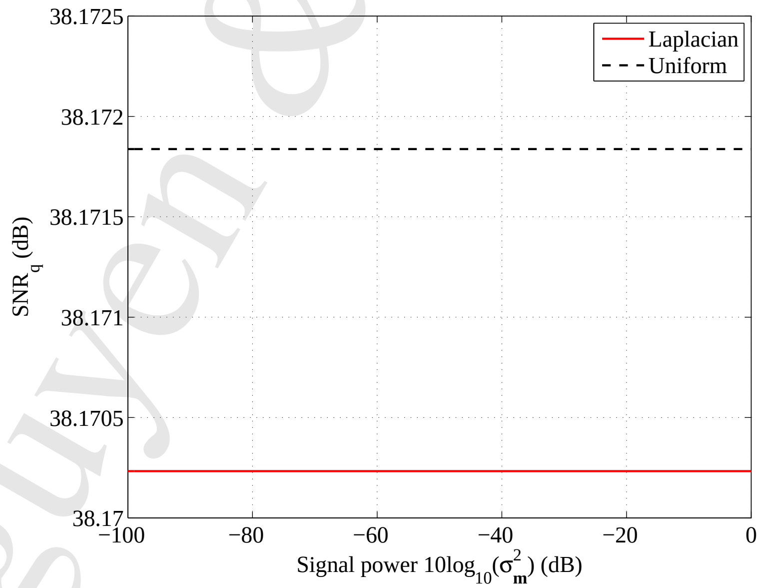


Figure 4.12: Plots of SNR_q for different signal models with 8-bit A-law quantizer ($L = 256$, $A = 87.6$, $\epsilon = 0.9999$ for Laplacian pdf).

Plots in Fig. 4.12 show that A-law quantizer is more robust to μ-law quantizer, both over input signal power and pdf models.

P4.17 The μ-law nonlinearity can be visualized in 2 different ways as illustrated in Fig. 4.13.

Consider (a) which is more of an engineering model (model (b) is left as a further exercise).

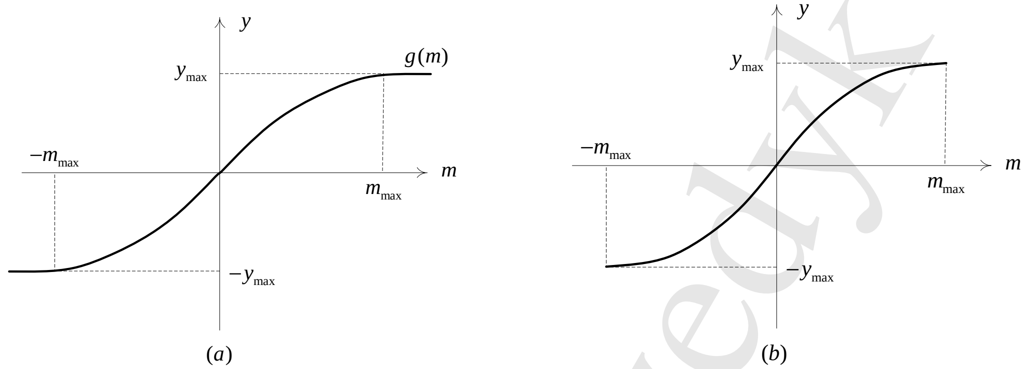


Figure 4.13: Visualization of μ-law nonlinearity.

The pdf, $f_{\mathbf{y}}(y)$, shall have two impulses, one at $y = y_{\max}$, the other at $y = -y_{\max}$ each of strength $P(m_{\max} \le \mathbf{m} < \infty)$.

In the range $-m_{\max} \le m \le m_{\max}$, there is only one root to the equation $g(m) = y$. Solve this for $0 \le y \le y_{\max}$. Note that because of the symmetry in $f_{\mathbf{m}}(m)$ & $g(m)$, $f_{\mathbf{y}}(y)$ is an even function.

$$\frac{y_{\max}}{\ln(1+\mu)}\left[1+\frac{\mu m}{m_{\max}}\right] = y \Rightarrow m_{\text{root}} = \frac{m_{\max}}{\mu}\left[(1+\mu)^{y/y_{\max}} - 1\right] \tag{4.89}$$

$$\left|\frac{\mathrm{d}y}{\mathrm{d}m}\right|_{m_{\text{root}}} = \frac{y_{\max}}{\ln(1+\mu)}\frac{1}{\left(1+\frac{\mu m}{m_{\max}}\right)}\frac{\mu}{m_{\max}}\Bigg|_{m_{\text{root}}} = \frac{y_{\max}\mu}{m_{\max}\ln(1+\mu)}\frac{1}{(1+\mu)^{y/y_{\max}}} \tag{4.90}$$

$$f_{\mathbf{y}}(y) = \frac{f_{\mathbf{m}}(m)|_{m_{\text{root}}}}{\left|\frac{\mathrm{d}y}{\mathrm{d}m}\right|_{m_{\text{root}}}} = \frac{m_{\max}\ln(1+\mu)}{\sqrt{2\pi}y_{\max}\mu}(1+\mu)^{y/y_{\max}}\mathrm{e}^{-\frac{1}{2}\left[(1+\mu)^{y/y_{\max}}-1\right]^2} \tag{4.91}$$

where $0 \le y \le y_{\max}$ and $f_{\mathbf{m}}(m) = \mathcal{N}(0,1)$.

Since $f_{\mathbf{y}}(y) = f_{\mathbf{y}}(-y)$ we have:

$$f_{\mathbf{y}}(y) = \begin{cases} \frac{m_{\max}\ln(1+\mu)}{\sqrt{2\pi}y_{\max}\mu}(1+\mu)^{|y|/y_{\max}}\mathrm{e}^{-\frac{1}{2}\left[(1+\mu)^{|y|/y_{\max}}-1\right]^2}, & -y_{\max} \le y \le y_{\max} \\ P(-\infty < \mathbf{m} \le -m_{\max})\delta(y+y_{\max}) + P(m_{\max} \le \mathbf{m} < \infty)\delta(y-y_{\max}) & \\ = 0, \quad \text{elsewhere} & \end{cases} \tag{4.92}$$

To plot let $m_{\max}$ be such that 99% of pdf is "captured", which gives $m_{\max} = 2.576$ and $P(-\infty < \mathbf{m} \le -m_{\max}) = P(m_{\max} \le \mathbf{m} < \infty) = 0.005$. Let $y_{\max} = 1$. The plot is shown in Fig. 4.14, which does not seem to confirm the "intuition".

P4.18 Consider $x > 0$, $\quad y(x) = (1+S)\frac{x}{1+Sx}$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1+S}{(1+Sx)^2} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1+S}{(1+S|x|)^2}, \quad |x| < 1 \tag{4.93}$$

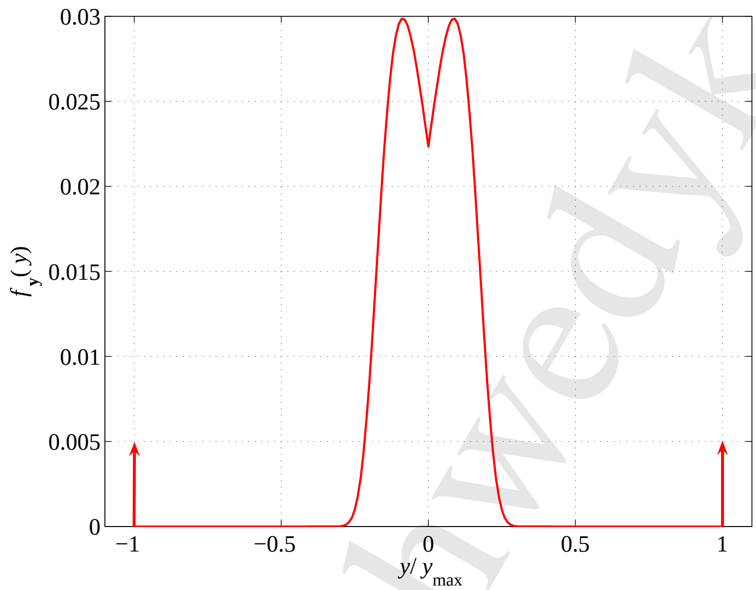


Figure 4.14: Pdf plot of the output of the μ-law compressor when input is Gaussian.

$$N_{\mathrm{q}} = \frac{1}{3L^2(1+S)^2}\int_{-1}^{1}(1+S|x|)^4 f_{\mathbf{x}}(x)\mathrm{d}x, \quad \text{where } \mathbf{x} = \frac{\mathbf{m}}{m_{\max}}. \tag{4.94}$$

In terms of the unnormalized variable m, one has

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}m}\frac{\mathrm{d}m}{\mathrm{d}x} = m_{\max}\frac{\mathrm{d}y}{\mathrm{d}m} \quad \text{or} \quad \frac{\mathrm{d}y}{\mathrm{d}m} = \frac{1}{m_{\max}}\frac{\mathrm{d}y}{\mathrm{d}x}. \tag{4.95}$$

Then

$$\begin{aligned}
N_{\mathrm{q}} &= \frac{m_{\max}^2}{3L^2(1+S)^2}\int_{-m_{\max}}^{m_{\max}}\left(1+S\frac{|m|}{m_{\max}}\right)^4 f_{\mathbf{m}}(m)\mathrm{d}m \\
&= \frac{m_{\max}^2}{3L^2(1+S)^2}\int_{-m_{\max}}^{m_{\max}}\left(1+4S\frac{|m|}{m_{\max}}+6S^2\frac{|m|^2}{m_{\max}^2}+4S^3\frac{|m|^3}{m_{\max}^3}+S^4\frac{|m|^4}{m_{\max}^4}\right) f_{\mathbf{m}}(m)\mathrm{d}m \\
&= \frac{m_{\max}^2}{3L^2(1+S)^2}\left(1+4S\frac{E\{|\mathbf{m}|\}}{m_{\max}}+6S^2\frac{E\{|\mathbf{m}|^2\}}{m_{\max}^2}+4S^3\frac{E\{|\mathbf{m}|^3\}}{m_{\max}^3}+S^4\frac{E\{|\mathbf{m}|^4\}}{m_{\max}^4}\right)
\end{aligned}$$

To write this in terms of $\sigma_n^2 = \sigma_{\mathbf{m}}^2/m_{\max}^2$ note that

$$\frac{E\{|\mathbf{m}|\}}{m_{\max}} = \frac{\sigma_{\mathbf{m}}}{m_{\max}}\frac{E\{|\mathbf{m}|\}}{\sigma_{\mathbf{m}}} = \sqrt{\sigma_n^2}\frac{E\{|\mathbf{m}|\}}{\sigma_{\mathbf{m}}} \quad \text{(as seen in P4.15, Eqn. (P4.9))} \tag{4.96}$$

$$\frac{E\{|\mathbf{m}|^2\}}{m_{\max}^2} = \sigma_n^2 \tag{4.97}$$

$$\frac{E\{|\mathbf{m}|^3\}}{m_{\max}^3} = \frac{\sigma_{\mathbf{m}}^3}{m_{\max}^3}\frac{E\{|\mathbf{m}|^3\}}{\sigma_{\mathbf{m}}^3} = (\sigma_n^2)^{3/2}\frac{E\{|\mathbf{m}|^3\}}{\sigma_{\mathbf{m}}^3} \tag{4.98}$$

$$\frac{E\{|\mathbf{m}|^4\}}{m_{\max}^4} = \frac{\sigma_{\mathbf{m}}^4}{m_{\max}^4}\frac{E\{|\mathbf{m}|^4\}}{\sigma_{\mathbf{m}}^4} = (\sigma_n^2)^2\frac{E\{|\mathbf{m}|^4\}}{\sigma_{\mathbf{m}}^4} \tag{4.99}$$

$$\therefore \text{SNR}_\text{q} = \frac{3L^2(1+S)^2\sigma_n^2}{1 + 4S\sqrt{\sigma_n^2}\frac{E\{|\mathbf{m}|\}}{\sigma_\mathbf{m}} + 6S^2\sigma_n^2 + 4S^3(\sigma_n^2)^{3/2}\frac{E\{|\mathbf{m}|^3\}}{\sigma_\mathbf{m}^3} + S^4\sigma_n^4\frac{E\{|\mathbf{m}|^4\}}{\sigma_\mathbf{m}^4}} \tag{4.100}$$

For a zero-mean Gaussian signal with power level $\sigma_\mathbf{m}^2$ the quantities are:

$$\frac{E\{|\mathbf{m}|\}}{\sigma_\mathbf{m}} = \sqrt{\frac{2}{\pi}}; \quad \frac{E\{|\mathbf{m}|^3\}}{\sigma_\mathbf{m}^3} = 2\sqrt{\frac{2}{\pi}}; \quad \frac{E\{|\mathbf{m}|^4\}}{\sigma_\mathbf{m}^4} = 3. \tag{4.101}$$

Aside: If $\mathbf{x}$ is $\mathcal{N}(0,\sigma^2)$ then

$$E\{\mathbf{x}^n\} = \begin{cases} 1\cdot 3\cdots(n-1)\sigma^n, & n \text{ even} \\ 0, & n \text{ odd} \end{cases} \tag{4.102}$$

$$E\{|\mathbf{x}|^n\} = \begin{cases} 1\cdot 3\cdots(n-1)\sigma^n, & n = 2k \\ \sqrt{\frac{2}{\pi}}2^k k!\sigma^{2k+1}, & n = 2k+1 \end{cases} \tag{4.103}$$

Plots for $S = 5, 10, 50, 100$ are shown in Fig. 4.15. Note that the performance is not that attractive in that the SNR_q varies considerably with σ_n^2. Some reflection leads one to conclude that this is due to the "higher" order moments presented in the N_q expression, namely σ_n^3, σ_n^4. One likes no moments higher than σ_n^2 because then $\text{SNR}_\text{q} \to$ constant as σ_n^2 becomes larger. The bottom line is that any old saturating nonlinearity is not suitable in a compander. It has to be one that results in a σ_n^2 term in N_q and this is what $\ln(\cdot)$ nonlinearities of the μ-law and A-law give you.

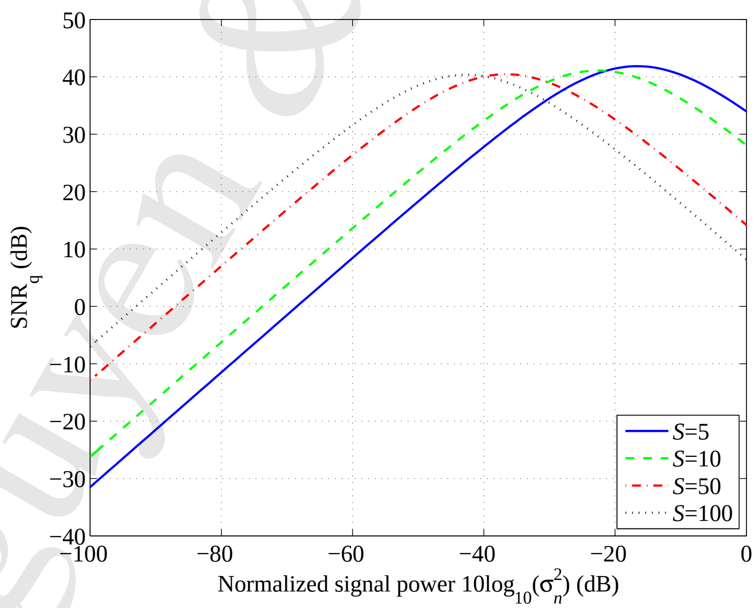


Figure 4.15: Plots of SNR_q for 8-bit S-law quantizer with Gaussian input signal.

P4.19 Since we would like the mapping to be unique then the number of possible mappings from the $L = 2^R$ target levels to the R-bit sequences is $L!$ which for $R = 2, 4, 8, 16$ bits are $4! = 24$; $16! = 2.0923 \times 10^13$; $256! =$ Matlab says inf; $2^{16}! =$ more than $256!$ but still countable. Though some of these mappings may be considered to be trivially different, there is still a humongous number of them.

P4.20 Let $P[\text{bit error}] = P_e$. Then $E\{\mathbf{q}^2\} = E\{\mathbf{q}^2|\text{no bit error}\}(1-P_e) + E\{\mathbf{q}^2|\text{1 bit error}\}P_e$.

Assume a uniform quantizer and also a uniform pdf for the quantization error (Eqn. (4.21)). Then $E\{\mathbf{q}^2|\text{no bit error}\}$ is that of Equation (4.22) and it is $\Delta^2/12$ (watts).

With one bit error the quantization error, $\mathbf{q}$, shall still have a uniform pdf but with a nonzero mean. The mean value depends on which bit is in error and the bit sequence considered. The picture looks as shown in Fig. 4.16.

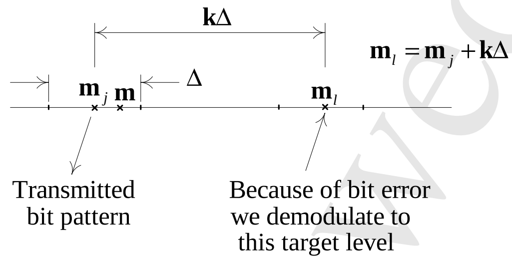


Figure 4.16

$$\begin{aligned}
\therefore E\{(\mathbf{m}-\mathbf{m}_l)^2\} &= E\{(\mathbf{m}-\mathbf{m}_j-\mathbf{k}\Delta)^2\} \\
&= E\{(\mathbf{m}-\mathbf{m}_j)^2\} - E\{2\mathbf{k}\Delta(\mathbf{m}-\mathbf{m}_j)\} + E\{\mathbf{k}^2\Delta^2\} \quad (4.104)
\end{aligned}$$

$$\text{Now } E\{(\mathbf{m}-\mathbf{m}_j)^2\} = \frac{\Delta^2}{12} \quad \text{(i.e., as if no bit error occurred)} \quad (4.105)$$

$$E\{(\mathbf{m}-\mathbf{m}_j)\} = 0 \quad (4.106)$$

$$E\{\mathbf{k}^2\Delta^2\} = \sum_k k^2\Delta^2 P[\mathbf{k}] \quad (4.107)$$

$$\begin{aligned}
\therefore E\{\mathbf{q}^2\} &= \sigma_{\mathbf{q}}^2(1-P_e) + \left(\sigma_{\mathbf{q}}^2 + \sum_k^2 k\Delta^2 P[\mathbf{k}]\right) P_e \\
&= \sigma_{\mathbf{q}}^2 + P_e \sum_k k^2\Delta^2 P[\mathbf{k}]. \quad (4.108)
\end{aligned}$$

So now it is a matter of determining the k profile along with the value of $P[\mathbf{k}]$. We do this for the three mappings of Fig. 4.19 and $L = 8$.

Bit pattern	000	001	010	011	100	101	110	111
NBC	$\{1,2,4\}$	$\{1,2,4\}$	$\{1,2,4\}$	$\{1,2,4\}$	$\{1,2,4\}$	$\{1,2,4\}$	$\{1,2,4\}$	$\{1,2,4\}$
Gray	$\{1,3,7\}$	$\{1,1,5\}$	$\{1,3,1\}$	$\{1,1,3\}$	$\{1,3,7\}$	$\{1,1,5\}$	$\{1,3,1\}$	$\{1,1,3\}$
FBC	$\underbrace{\{1,2,1\}}_{\text{values of }\mathbf{k}}$	$\{1,1,3\}$	$\{1,2,5\}$	$\{1,2,7\}$	$\{1,2,1\}$	$\{1,2,3\}$	$\{1,2,5\}$	$\{1,2,7\}$

Now assuming that each bit sequence is equally probable and that which bit is in error is equally probable, the $P[\mathbf{k}]$ is as follows:

NBC: $P[\mathbf{k}=1] = 1/3$; $P[\mathbf{k}=2] = 1/3$; $P[\mathbf{k}=4] = 1/3$
Gray: $P[\mathbf{k}=1] = 14/24 = 7/12$; $P[\mathbf{k}=3] = 6/24 = 3/12$; $P[\mathbf{k}=5,7] = 2/24 = 1/12$
FBC: $P[\mathbf{k}=1] = 11/24$; $P[\mathbf{k}=2] = 7/24$; $P[\mathbf{k}=3,5,7] = 2/24 = 1/12$

Therefore the quantization noise variance is:

NBC: $E\{\mathbf{q}^2\} = \sigma_{\mathbf{q}}^2 + \frac{1}{3}[\Delta^2 + 4\Delta^2 + 16\Delta^2]P_{\mathrm{e}} = \Delta^2[\frac{1}{12} + 7P_{\mathrm{e}}]$.

Gray: $E\{\mathbf{q}^2\} = \sigma_{\mathbf{q}}^2 + [\frac{7}{12}\Delta^2 + \frac{3}{12}(9\Delta^2) + \frac{1}{12}(25\Delta^2) + \frac{1}{12}(25\Delta^2)]P_{\mathrm{e}} = \Delta^2[\frac{1}{12} + 9P_{\mathrm{e}}]$.

FBC: $E\{\mathbf{q}^2\} = \sigma_{\mathbf{q}}^2 + [\frac{1}{24}\Delta^2 + \frac{7}{24}(4\Delta^2) + \frac{2}{24}(9\Delta^2) + \frac{2}{24}(25\Delta^2) + \frac{2}{24}(49\Delta^2)]P_{\mathrm{e}} = \Delta^2[\frac{1}{12} + 8.46P_{\mathrm{e}}]$.

If $P_{\mathrm{e}} \leq 10^{-3}$ the effect is negligible.

P4.21 (a) $\Delta = \frac{2V_{\max}}{256} = \frac{V_{\max}}{128}$. (Note: Here the optimum quantizer is a uniform quantizer).
$\sigma_{\mathbf{m}}^2 = R_{\mathbf{m}}(\tau = 0) = 1$ (watts)
Since $\sigma_{\mathbf{m}}^2 = \frac{V_{\max}^2}{3} \Rightarrow V_{\max}^2 = 3$.
$\sigma_{\mathbf{q}}^2 = \frac{\Delta^2}{12} = \frac{V_{\max}^2}{12(2^7)^2} = \frac{1}{2^{16}}$.

(b) $\frac{\partial}{\partial\alpha}\left[E\left\{[\mathbf{m}(k) - \alpha\mathbf{m}(k-1)]^2\right\}\right] = 0 \Rightarrow \alpha = \frac{R_{\mathbf{m}}(1)}{R_{\mathbf{m}}(0)} = \frac{R_{\mathbf{m}}(T_s)}{R_{\mathbf{m}}(0)} = \mathrm{e}^{-1/100}$.

(c) $\mathbf{y}(k) = \mathbf{m}(k) - \alpha\mathbf{m}(k-1) = \mathbf{m}(k) - \mathrm{e}^{-1/100}\mathbf{m}(k-1)$.
$E\{\mathbf{y}(k)\} = 0; \quad E\{\mathbf{y}^2(k)\} = (1+\alpha^2)\underbrace{R_{\mathbf{m}}(0)}_{=1} - 2\alpha\underbrace{R_{\mathbf{m}}(1)}_{=\alpha} = 1 - \alpha^2 = 0.0198.$

$\mathbf{y}'(kT_s)$ has a mean of zero and a variance of 0.0198. Because it is uniform we know that the variance is $(y'_{\max})^2/12 \Rightarrow y'_{\max} = 0.48745$.

$$\therefore \sigma_{\mathbf{q}}^2\Big|_{\text{diff. quan.}} = \frac{(y'_{\max})^2}{3 \cdot 2^{16}} = \frac{0.2376}{3 \cdot 2^{16}} \tag{4.109}$$

$$\sigma_{\mathbf{q}}^2\Big|_{\text{no diff. quan.}} = \frac{1}{2^{16}} \tag{4.110}$$

$$\frac{\sigma_{\mathbf{q}}^2\Big|_{\text{diff. quan.}}}{\sigma_{\mathbf{q}}^2\Big|_{\text{no diff. quan.}}} = \frac{0.2376}{3} = 0.0792 = -11 \text{ dB}. \tag{4.111}$$

P4.22 The USB drive can store 8×10^9 bits.

(a) 4 kHz speech, 8 bits per sample.
The Nyquist sampling rate is 8 kHz, giving a bit rate of $8 \times 8,000 = 64,000$ bits/sec.

$$T_D = \frac{8 \times 10^9 \text{ bits}}{64,000 \text{ bits/sec}} = 1.25 \times 10^5 \text{ seconds} = 34.7 \text{ hrs} = 1.446 \text{ days}$$

(b) 22 kHz audio signal, stereo, 16 bits per sample in each stereo channel.
Have $44,000$ samples/sec $\times$ 16 bits/sample $\times$ 2 channels.

$$T_D = \frac{8 \times 10^9 \text{ bits}}{44,000 \times 16 \times 2 \text{ bits/sec}} = 5.682 \times 10^3 \text{ seconds} = 94.7 \text{ minutes} = 1.58 \text{ hrs}$$

(c) 5 MHz video signal, 12 bits per sample, combined with the audio signal above.
Have 10×10^6 samples/sec $\times$ 12 bits/sample $+ 44,000 \times 16 \times 2$bits/sec.

$$T_D = \frac{8 \times 10^9 \text{ bits}}{120 \times 10^6 + 1.408 \times 10^6} = \frac{8 \times 10^9 \text{ bits}}{121.408 \times 10^6 \text{ bits/sec}} = 65.9 \text{ seconds} = 1.1 \text{ minutes}$$

(d) Digital surveillance video, 1024×768 pixels per frame, 8 bits per pixel, 1 frame per second.

Have (1024×768) pixels/frame $\times$ 8 bits/pixel $\times$ 1 frame/sec $= 6,291,456$ bits/sec.

$$T_D = \frac{8 \times 10^9 \text{ bits}}{6,291,456 \text{ bits/sec}} = 1.272 \times 10^3 \text{ seconds} = 21.20 \text{ minutes}$$

P4.23 We divide the range $[-m_{\max}, m_{\max}]$ into 2^R equally spaced quantization regions, each of width $\Delta = 2m_{\max}/2^R$. If we place a target level in the middle of each quantization region, the maximum value of the quantization noise is $q_{\max} = \Delta/2 = m_{\max}/2^R$. The PSNR_q is then given by

$$\text{PSNR}_\text{q} \quad = \quad 20 \log_{10}\left(\frac{m_{\max}}{m_{\max}/2^R}\right) = 20 \log_{10}\left(2^R\right) = 6.02R \text{ (dB)} \tag{4.112}$$

To achieve distortion D requires R at least $D/6.02$, so the smallest value of R is

$$R = \lceil D/6.02 \rceil \text{ bits} \tag{4.113}$$

Note that with this definition, the 6dB/bit rule-of-thumb hold as well.

Chapter 5

Optimum Receiver for Binary Data Transmission

P5.1 First consider

$$
\begin{aligned}
\int_0^{T_b} [\phi_2'(t)]^2 \mathrm{d}t &= \int_0^{T_b} \left[\frac{s_2(t)}{\sqrt{E_2}} - \rho\phi_1(t)\right]^2 \mathrm{d}t \\
&= \underbrace{\frac{1}{E_2} \underbrace{\int_0^{T_b} s_2^2(t)\mathrm{d}t}_{=E_2}}_{=1} - 2\rho \underbrace{\int_0^{T_b} \frac{s_2(t)}{\sqrt{E_2}}\phi_1(t)\mathrm{d}t}_{=\underbrace{\frac{1}{\sqrt{E_2}\sqrt{E_1}}\int_0^{T_b} s_2(t)s_1(t)\mathrm{d}t}_{=\rho}} + \rho^2 \int_0^{T_b} \phi_1^2(t)\mathrm{d}t \\
&= 1 - 2\rho^2 + \rho^2 = 1 - \rho^2.
\end{aligned}
$$

$$
\therefore \phi_2(t) = \frac{1}{\sqrt{1-\rho^2}}\phi_2'(t) = \frac{1}{\sqrt{1-\rho^2}}\left[\frac{s_2(t)}{\sqrt{E2}} - \rho \underbrace{\phi_1(t)}_{\frac{s_1(t)}{\sqrt{E_1}}}\right] = \frac{1}{\sqrt{1-\rho^2}}\left[\frac{s_2(t)}{\sqrt{E2}} - \rho\frac{s_1(t)}{\sqrt{E_1}}\right].
$$

P5.2 (a)

$$
\begin{bmatrix} \hat{\phi}_1(t) \\ \hat{\phi}_2(t) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix} \tag{5.1}
$$

$$
\begin{aligned}
\hat{\phi}_1(t) &= \cos\theta\phi_1(t) + \sin\theta\phi_2(t), \\
\hat{\phi}_2(t) &= -\sin\theta\phi_1(t) + \cos\theta\phi_2(t),
\end{aligned} \tag{5.2}
$$

$$
\hat{\phi}_1(t)\cdot\hat{\phi}_2(t) = -\sin\theta\cos\theta\phi_1^2(t) + \sin\theta\cos\theta\phi_2^2(t) + [\cos^2\theta - \sin^2\theta]\phi_1(t)\phi_2(t),
$$

$$
\begin{aligned}
\int_0^{T_b} \hat{\phi}_1(t)\hat{\phi}_2(t)\mathrm{d}t &= -\sin\theta\cos\theta \underbrace{\int_0^{T_b} \phi_1^2(t)\mathrm{d}t}_{=1} + \sin\theta\cos\theta \underbrace{\int_0^{T_b} \phi_2^2(t)\mathrm{d}t}_{=1} \\
&\quad + \left[\cos^2\theta - \sin^2\theta\right] \underbrace{\int_0^{T_b} \phi_1(t)\phi_2(t)\mathrm{d}t}_{=0} \\
&= -\sin\theta\cos\theta + \sin\theta\cos\theta = 0
\end{aligned} \tag{5.3}
$$

Therefore $\{\hat{\phi}_1(t), \hat{\phi}_2(t)\}$ are orthogonal. To see if they are normal, consider first

$$\begin{aligned}\int_0^{T_b} \hat{\phi}_1^2(t)\mathrm{d}t &= \int_0^{T_b} [\cos\theta\phi_1(t) + \sin\theta\phi_2(t)]^2\mathrm{d}t \\ &= \cos^2\theta \underbrace{\int_0^{T_b} \phi_1^2(t)\mathrm{d}t}_{=1} + 2\cos\theta\sin\theta \underbrace{\int_0^{T_b} \phi_1(t)\phi_2(t)\mathrm{d}t}_{=0} + \sin^2\theta \underbrace{\int_0^{T_b} \phi_2^2(t)\mathrm{d}t}_{=1} \\ &= \cos^2\theta + \sin^2\theta = 1. \end{aligned} \tag{5.4}$$

Similar derivation shows that $\hat{\phi}_2(t)$ also has unit energy. Therefore $\{\hat{\phi}_1(t), \hat{\phi}_2(t)\}$ form an orthonormal set.

What if the minus sign is put elsewhere, i.e., $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ or $\begin{bmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Are $\{\phi_1(t), \phi_2(t)\}$ still orthonormal? If so does the matrix represent just a rotation?

(b) Need to find θ such that $\int_0^{T_b} \hat{\phi}_1(t)[s_1(t) - s_2(t)]\mathrm{d}t = 0$:

$$\begin{aligned}\int_0^{T_b} \hat{\phi}_1(t)\,[s_1(t) - s_2(t)]\,\mathrm{d}t &= \int_0^{T_b} [\cos\theta\phi_1(t) + \sin\theta\phi_2(t)]\,[s_1(t) - s_2(t)]\,\mathrm{d}t \\ &= \cos\theta\left[\int_0^{T_b} s_1(t)\phi_1(t)\mathrm{d}t - \int_0^{T_b} s_2(t)\phi_1(t)\mathrm{d}t\right] \\ &\quad + \sin\theta\left[\int_0^{T_b} s_1(t)\phi_2(t)\mathrm{d}t - \int_0^{T_b} s_2(t)\phi_2(t)\mathrm{d}t\right] \\ &= \cos\theta(s_{11} - s_{21}) + \sin\theta(s_{12} - s_{22}) = 0. \end{aligned} \tag{5.5}$$

$$\begin{aligned}\sin\theta(s_{12} - s_{22}) &= \cos\theta(s_{21} - s_{11}) \\ \tan\theta &= \frac{s_{21} - s_{11}}{s_{12} - s_{22}} \\ \theta &= \tan^{-1}\frac{s_{21} - s_{11}}{s_{12} - s_{22}}. \end{aligned} \tag{5.6}$$

Of course θ is determined by the signal set. You must know the signal set in order to rotate the original basis set as desired: $\hat{\phi}_2(t)$ is perpendicular to the line joining $s_1(t)$ and $s_2(t)$.

The above result of θ can also be obtained geometrically (see the signal space diagram on Fig. 5.1). Use the following fact from geometry: Given the equation of a straight line $y = mx + b$, then the perpendicular to the line has a slope of $-\frac{1}{m}$.

The slope of the line joining $s_1(t)$ and $s_2(t)$ is $\frac{s_{22}-s_{12}}{s_{21}-s_{11}}$. Therefore the slope of $\hat{\phi}_1(t)$ is $\tan\theta = -\frac{s_{21}-s_{11}}{s_{22}-s_{12}}$, or $\theta = \tan^{-1}\frac{s_{21}-s_{11}}{s_{12}-s_{22}}$.

Since we choose θ such that

$$\int_0^{T_b} \hat{\phi}_1(t)\,[s_1(t) - s_2(t)]\,\mathrm{d}t = 0, \tag{5.7}$$

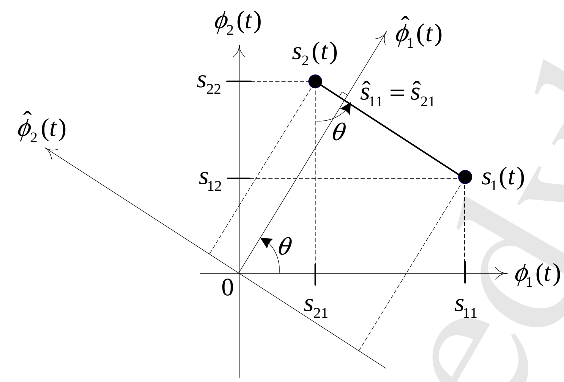


Figure 5.1: Determining the rotation angle θ.

It follows that

$$\int_0^{T_b} s_1(t)\hat{\phi}_1(t)\mathrm{d}t = \int_0^{T_b} s_2(t)\hat{\phi}_1(t)\mathrm{d}t$$
$$\hat{s}_{11} = \hat{s}_{21}, \tag{5.8}$$

i.e., the components of $s_1(t)$ and $s_2(t)$ along $\hat{\phi}_1(t)$ are the same.

P5.3 Area $= \dfrac{1}{\sqrt{2\pi}\sigma}\displaystyle\int_T^{\infty} \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}\mathrm{d}x \underset{(\therefore \mathrm{d}\lambda = \frac{\mathrm{d}x}{\sigma})}{\overset{(\lambda = \frac{x-\mu}{\sigma})}{=}} \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{\frac{T-\mu}{\sigma}}^{\infty} \mathrm{e}^{-\frac{\lambda^2}{2}}\mathrm{d}\lambda = Q\left(\dfrac{T-\mu}{\sigma}\right)$.

As shown in Fig. P5.40 of the text, $T-\mu > 0$ (i.e., $T > \mu$) but result is true even if $T < \mu$. Of course the argument of $Q(x)$ is now negative and one would use the relationship $Q(-x) = 1 - Q(x)$.

Stated in English, *the area is a Q function whose argument is the distance from the mean to the threshold divided by the RMS value (standard deviation).*

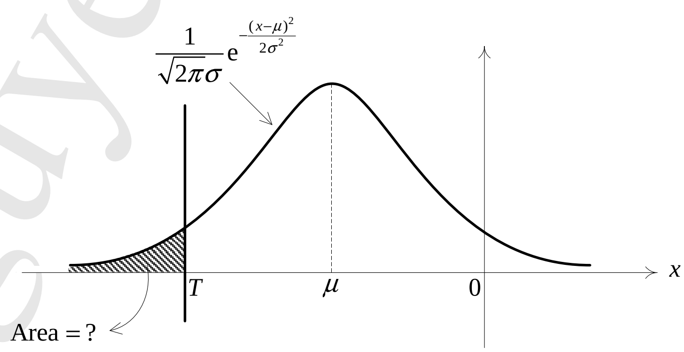


Figure 5.2: Writing the shaded area in terms of the Q function.

Of course one must pay attention on which side of the mean the threshold lies and write the expression accordingly. For instance in terms of the Q function what is the shaded area in Fig. 5.2?

P5.4 (a) Since $s_i(t) = s_{i1}\phi_1(t) + s_{i2}\phi_2(t)$, $i = 1, 2$, one has:

$$\begin{aligned}
E_i &= \int_0^{T_b} s_i^2(t)\mathrm{d}t = \int_0^{T_b} [s_{i1}\phi_1(t) + s_{i2}\phi_2(t)]^2 \mathrm{d}t \\
&= \int_0^{T_b} [s_{i1}^2\phi_1^2(t) + 2s_{i1}s_{i2}\phi_1(t)\phi_2(t) + s_{i2}^2\phi_2^2(t)]\mathrm{d}t \\
&= s_{i1}^2 \underbrace{\int_0^{T_b} \phi_1^2(t)\mathrm{d}t}_{=1} + 2s_{i1}s_{i2} \underbrace{\int_0^{T_b} \phi_1(t)\phi_2(t)\mathrm{d}t}_{=0} + s_{i2}^2 \underbrace{\int_0^{T_b} \phi_2^2(t)\mathrm{d}t}_{=1} \\
&= s_{i1}^2 + s_{i2}^2
\end{aligned}$$

(b) Now let

$$\phi_1(t) = \begin{cases} \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t), & 0 \le t \le T_b \\ 0, & \text{otherwise} \end{cases} \tag{5.9}$$

and

$$\phi_2(t) = \begin{cases} \sqrt{\frac{2}{T_b}}\sin(2\pi f_c t), & 0 \le t \le T_b \\ 0, & \text{otherwise} \end{cases} \tag{5.10}$$

Compute

$$\begin{aligned}
\int_0^{T_b} \phi_1(t)\phi_2(t) &= \frac{2}{T_b}\int_0^{T_b} \cos(2\pi f_c t)\sin(2\pi f_c t)\mathrm{d}t \\
&= \frac{1}{T_b}\int_0^{T_b} \sin(4\pi f_c t)\mathrm{d}t = -\frac{1}{4T_b\pi f_c}\cos(4\pi f_c t)\Big|_0^{T_b} \\
&= -\frac{1}{4T_b\pi f_c}[\cos(4\pi f_c T_b) - 1]
\end{aligned} \tag{5.11}$$

Thus, $\phi_1(t)$ and $\phi_2(t)$ are orthogonal if $\cos(4\pi f_c T_b) = 1 \Rightarrow 4\pi f_c T_b = 2k\pi \Rightarrow f_c = \frac{k}{2T_b}$, where k is a positive integer ($k = 1, 2, \ldots$). There must be an integer number of *half cycles* of the cos and sin in the interval $[0, T_b]$ for them to be orthogonal over the interval of T_b seconds. No other frequency suffices.
Finally, $(f_c)_{\min} = \frac{1}{2T_b}$ when $k = 1$. It is graphically illustrated in Fig. 5.3.

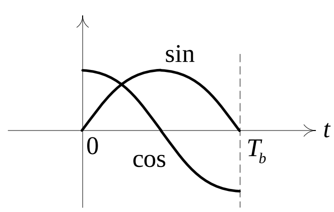


Figure 5.3: Orthogonality of sin and cos with a minimum frequency.

P5.5 Consider the following two signals $s_1(t)$ and $s_2(t)$ (this is the same signal set considered in Example 5.5):

$$\begin{aligned}
s_1(t) &= V\cos(2\pi f_c t) \\
s_2(t) &= V\cos(2\pi f_c t + \theta), \quad 0 \le t \le T_b,\ f_c = \frac{k}{2T_b},\ k \text{ integer}
\end{aligned} \tag{5.12}$$

(a) The energies of two signals can be computed as:

$$\begin{aligned} E_1 &= \int_0^{T_b} s_1^2(t)\mathrm{d}t = \int_0^{T_b} V^2\cos^2(2\pi f_c t)\mathrm{d}t \\ &= V^2\int_0^{T_b}\frac{[1+\cos(4\pi f_c t)]}{2}\mathrm{d}t = \frac{V^2}{2}\left[\int_0^{T_b}\mathrm{d}t + \int_0^{T_b}\cos(4\pi f_c t)\mathrm{d}t\right] \\ &= \frac{V^2 T_b}{2} \end{aligned} \quad (5.13)$$

$$\begin{aligned} E_2 &= \int_0^{T_b} s_2^2(t)\mathrm{d}t = \int_0^{T_b} V^2\cos^2(2\pi f_c t+\theta)\mathrm{d}t \\ &= V^2\int_0^{T_b}\frac{[1+\cos(4\pi f_c t+2\theta)]}{2}\mathrm{d}t = \frac{V^2}{2}\left[\int_0^{T_b}\mathrm{d}t + \int_0^{T_b}\cos(4\pi f_c t+2\theta)\mathrm{d}t\right] \\ &= \frac{V^2 T_b}{2} \end{aligned} \quad (5.14)$$

where we have used the fact that $\cos(4\pi f_c t+\beta)$ is a periodic function with a fundamental period of T_b/k (regardless of the value of the phase β) and an integration over multiple periods of this function equals to zero.

The two signals $s_1(t)$ and $s_2(t)$ have unit energy if $E_1 = E_2 = \frac{V^2T_b}{2} = 1$. Therefore, $V = \sqrt{\frac{2}{T_b}}$.

Remark: In communications we invariably take f_c to be an integer multiple of $\frac{1}{T_b}$ typically, sometimes $\frac{1}{2T_b}$ so the results for E_1 and E_2 hold (and those in P5.4). Typically f_c is very large, on the order of 10^6 (MHz) or 10^9 (GHz) so that even if there aren't an integer number of cycles (or half cycles) in the time interval, for engineering purposes E_1 and E_2 still are closely approximated by $V^2T_b/2$.

(b) Since $E_1 = E_2 = E$, the correlation coefficient of the two signals is

$$\begin{aligned} \rho &= \frac{1}{E}\int_0^{T_b} s_1(t)s_2(t)\mathrm{d}t = \frac{V^2}{E}\int_0^{T_b}\cos(2\pi f_c t)\cos(2\pi f_c t+\theta)\mathrm{d}t \\ &= \frac{V^2}{2E}\int_0^{T_b}\cos\theta\mathrm{d}t + \underbrace{\int_0^{T_b}\cos(4\pi f_c t+\theta)\mathrm{d}t}_{=0 \text{ since } f_c = \frac{k}{2T_b}} = \frac{V^2T_b}{2E}\cos\theta = \cos\theta \end{aligned} \quad (5.15)$$

Two signals are orthogonal when the correlation coefficient $\rho = 0$, which is equivalent to $\cos\theta = 0$. So, $\theta = k\frac{\pi}{2}$, $k = \pm1, \pm2, \pm3, \ldots$

(c) Plot of ρ as a function of θ is shown in Fig. 5.4.

(d) $d = \sqrt{2E}\sqrt{1-\rho} \Rightarrow d_{\max} = \sqrt{2E}\sqrt{1-(-1)} = 2\sqrt{E}$ when $\rho = -1$, or $\theta = \pi$ (antipodal signals).

To relate θ of part (c) to that of P5.4b note when $\theta = -\pi/2$ then $s_2(t) = V\cos(2\pi f_c t - \pi/2) = V\sin(2\pi f_c t)$, i.e., except for a scale factor it is $\phi_2(t)$ of P5.4b. On the other hand $\theta = \pi/2$ results in $-\phi_2(t)$ of P5.4b. The signal space of the signal set is shown in Fig. 5.12 of the text and it is reproduced in Fig. 5.5.

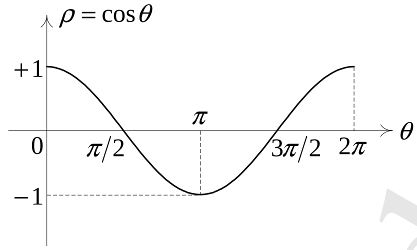


Figure 5.4: Plot of $\rho = \cos\theta$.

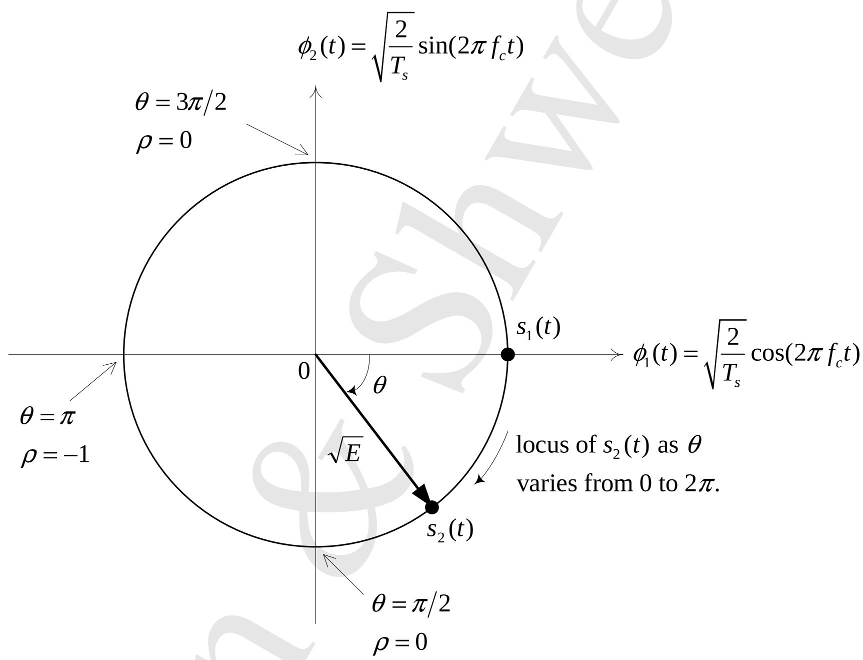


Figure 5.5: Signal space representation of Problem 5.5.

P5.6 (a)

$$\begin{aligned}
\rho &= \frac{1}{\sqrt{E_1}\sqrt{E_2}}\int_0^{T_b} V_1V_2\cos(2\pi(f_c-\Delta f/2)t)\cos(2\pi(f_c+\Delta f/2)t)\mathrm{d}t \\
&= \frac{V_1V_2}{2\sqrt{E_1}\sqrt{E_2}}\Bigg[\underbrace{\int_0^{T_b}\cos(2\pi(2f_c)t)\mathrm{d}t}_{=0\text{ (integer number of cycles in }T_b)} + \underbrace{\int_0^{T_b}\cos(2\pi\Delta f t)\mathrm{d}t}_{=\frac{\sin(2\pi\Delta f T_b)}{2\pi\Delta f}}\Bigg] \\
&\qquad \text{since } \cos x\cos y = \frac{\cos(x+y)+\cos(x-y)}{2} \\
&= \frac{V_1V_2T_b}{2\sqrt{E_1}\sqrt{E_2}}\frac{\sin(2\pi\Delta f T_b)}{2\pi\Delta f T_b} \qquad (5.16)
\end{aligned}$$

But $\frac{V_1\sqrt{T_b}}{\sqrt{2}} = \sqrt{E_1} = \sqrt{E}$ and $\frac{V_2\sqrt{T_b}}{\sqrt{E_2}} = \sqrt{E}$. Therefore

$$\rho(\Delta f) = \frac{\sin(2\pi\Delta f T_b)}{2\pi\Delta f T_b}.$$

Plot of ρ versus the normalized frequency $\Delta f T_b$ is shown in Fig. 5.6.

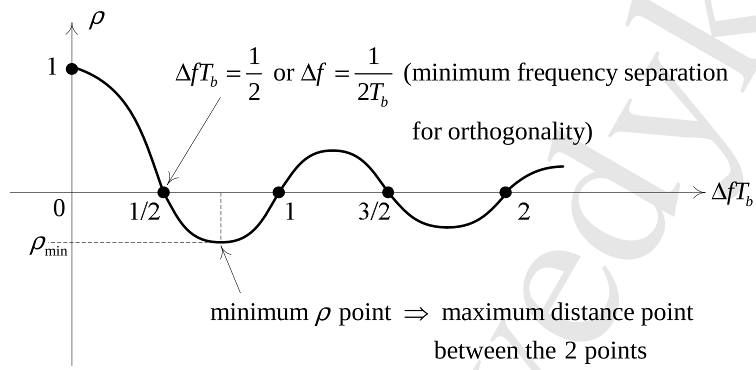


Figure 5.6: Plotting of $\rho(\Delta f)$.

(b) One can differentiate ρ with respect to the (normalized) variable $\Delta f T_b$ to find where the minimum occurs. From the plot of ρ we expect it to be between 0.5 and 0.75. Formally let $x \equiv 2\pi\Delta f T_b$. Then $\rho(x) = \frac{\sin x}{x}$ and $\frac{\mathrm{d}\rho(x)}{\mathrm{d}x} = \frac{\cos x}{x} - \frac{\sin x}{x^2} = 0$ (for maximum/minimum points) or $x = \tan x$. Solve this numerically.

Another approach (still numerical) is to set up a table of values of $\frac{\sin x}{x}$ between $x = 2\pi \times 0.65$ and $2\pi \times 0.75$ in increments of say 0.01 and pick out value of x where the minimum occurs.

The simplest approach is to trust the answer and check it by seeing if $x \stackrel{?}{=} \tan x$ at $x = 2\pi \times 0.715$ (within acceptable engineering numerical accuracy), i.e., $2\pi \times 0.715 \stackrel{?}{=} \tan(2\pi \times 0.715)$.

For $\Delta f T_b = 0.715$, $\rho = -0.2172$. Therefore $d = \sqrt{2E(1.2172)}$.

When $\rho = 0$, $d = \sqrt{2E}$. The distance has increased by a factor of 1.1033.

A more relevant way to "quantify" this increase is to state that for $\rho = 0$ the energy E needs to be increased by 1.2172 to achieve the same distance between the 2 signals as in the $\rho = -0.2172$ case. This is a $10\log_{10}(1.2172) = 0.85$ dB increase in energy (& hence power).

Nonetheless for other reasons, namely synchronization, Δf is chosen so that $\rho = 0$.

Remarks:

(i) When the frequency separation Δf is chosen to be $\frac{1}{2T_b}$ the two signals are said to be "coherently" orthogonal, when Δf is chosen to be $k\frac{1}{T_b}$ $(k \geq 2)$ they are called "noncoherently" orthogonal. Chapter 7 discusses this further.

(ii) To obtain the signal space representation is not that straightforward, except for $\rho = 0$. All that can be said of the top of one's head is that the signal space is 2–dimensional, and that the 2 signals lie at a distance of $\sqrt{E}$ (because of how we adjust V_1, V_2) from the origin. One basis function can be chosen to be, say $\phi_1(t) = \frac{s_1(t)}{\sqrt{E}}$. The other basis function is a function of Δf.

P5.7 Using the Gram-Schmidt procedure, construct an orthonormal basis for the space of quadratic polynomials $\{a_2t^2 + a_1t + a_0; a_0, a_1, a_2 \in \mathbb{R}\}$ over the interval $-1 \leq t \leq 1$.

It is simple to see that the equivalent problem is to find an orthonormal basis for three signals $s_1(t) = 1$, $s_2(t) = t$ and $s_3(t) = t^2$ over the interval $-1 \le t \le 1$.

The first orthonormal basis function is simply $\phi_1(t) = \frac{1}{\sqrt{2}}$ since the energy of $s_1(t) = 1$ over the interval $-1 \le t \le 1$ is 2.

Next observe that $s_2(t) = t$ is an odd function, while $\phi_1(t)$ is an even function. It follows that $s_2(t)$ is already orthogonal to $\phi_1(t)$. Thus, we set:

$$\phi_2(t) = \frac{s_2(t)}{\sqrt{\int_{-1}^{1} s_2^2(t)\mathrm{d}t}} = \frac{t}{\sqrt{\int_{-1}^{1} t^2 \mathrm{d}t}} = t\sqrt{3/2}$$

Next we project $s_3(t) = t^2$ on the subspace spanned by $\phi_1(t)$ and $\phi_2(t)$. It is clear that since $s_3(t)$ is an even function and $\phi_2(t)$ is an odd function, their product is odd, and we have $s_{32} = \int_{-1}^{1} s_3(t)\phi_2(t)\mathrm{d}t = 0$. On the other hand,

$$s_{31} = \int_{-1}^{1} \frac{1}{\sqrt{2}} t^2 \mathrm{d}t = \frac{2}{3\sqrt{2}}$$

The orthogonal signal is given by

$$\phi_3'(t) = s_3(t) - \frac{2}{3\sqrt{2}}\phi_1(t) - 0 \cdot \phi_2(t) = t^2 - \frac{1}{3}$$

Finally, the third orthonormal basis function is

$$\phi_3(t) = \left(t^2 - \frac{1}{3}\right) / \sqrt{8/45} = \sqrt{\frac{45}{8}} \left(t^2 - \frac{1}{3}\right) = \sqrt{\frac{5}{8}}(3t^2 - 1).$$

In summary,

$$\left\{ \sqrt{\frac{1}{2}}, \sqrt{\frac{3}{2}}t, \sqrt{\frac{5}{8}}(3t^2 - 1) \right\}$$

is an orthonormal basis set for the quadratic functions over $[-1, 1]$. The set is plotted in Fig. 5.7.

To find the coefficients of any quadratic polynomial, $p(t) = a_2 t^2 + a_1 t + a_0$, in terms of $\phi_1(t)$, $\phi_2(t)$ and $\phi_3(t)$ we find the projections of $p(t)$ onto the 3 basis functions, i.e.,

$$p(t) = p_1\phi_1(t) + p_2\phi_2(t) + p_3\phi_3(t)$$

where

$$p_1 = \int_{-1}^{1} p(t)\phi_1(t)\mathrm{d}t = \int_{-1}^{1} [a_2 t^2 + a_1 t + a_0] \frac{1}{\sqrt{2}} \mathrm{d}t = \sqrt{\frac{2}{3}} a_2 + \sqrt{2} a_0 \tag{5.17}$$

$$p_2 = \int_{-1}^{1} p(t)\phi_2(t)\mathrm{d}t = \int_{-1}^{1} [a_2 t^2 + a_1 t + a_0] \sqrt{\frac{3}{2}} t \mathrm{d}t = \sqrt{\frac{2}{3}} a_1 \tag{5.18}$$

$$p_3 = \int_{-1}^{1} p(t)\phi_1(t)\mathrm{d}t = \int_{-1}^{1} [a_2 t^2 + a_1 t + a_0] \left[\sqrt{\frac{3}{8}}(3t^2 - 1) \right] \mathrm{d}t = \frac{4}{5}\sqrt{\frac{3}{8}} a_2. \tag{5.19}$$

One, of course, does not achieve very much by approximating a quadratic polynomial by another quadratic polynomial. Of more practical interest is the situation where we are given

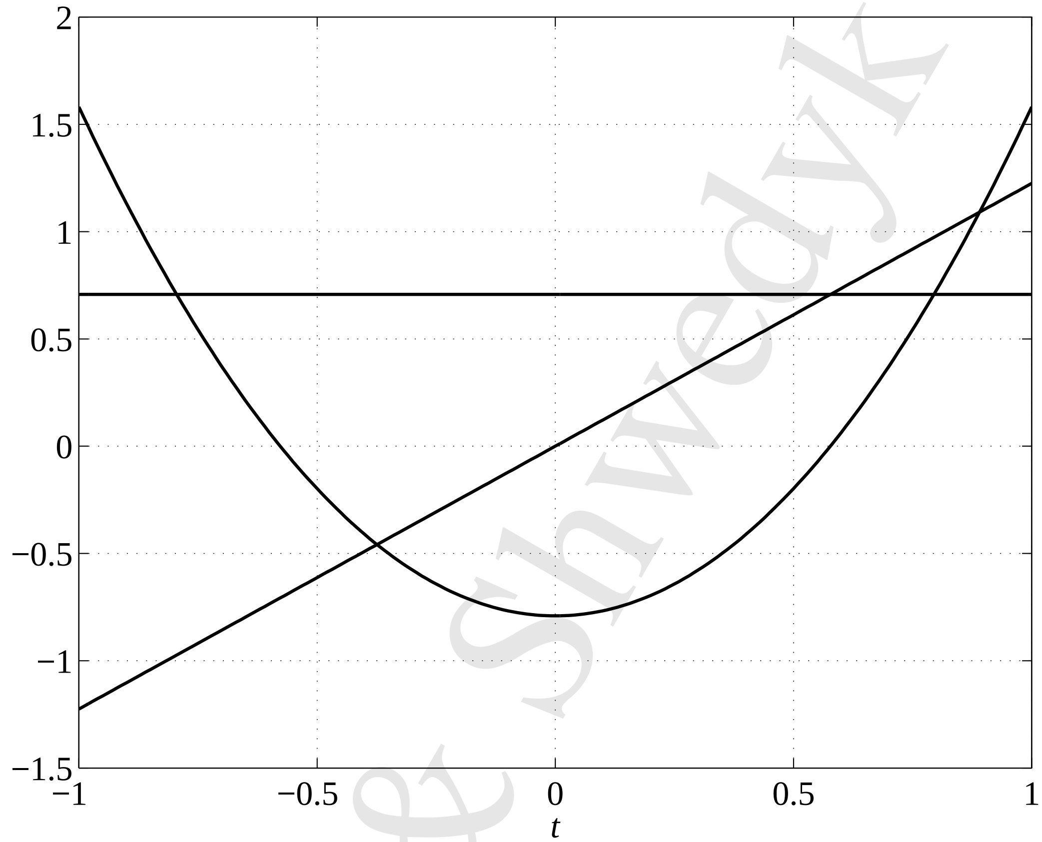


Figure 5.7: Orthonormal basis set for quadratic polynomials.

a more general time function, say $s(t)$ of finite energy, and wish to approximate it by a quadratic polynomial over the interval of $[-1,1]$. Then the set of $\{\phi_1(t), \phi_2(t), \phi_3(t)\}$ found above can be used to approximate $s(t)$ where the coefficients in the approximation $s(t) \approx \hat{s}(t) = s_1\phi_1(t) + s_2\phi_2(t) + s_3\phi_3(t)$ are found by:

$$s_1 = \int_{-1}^{1} s(t)\phi_1(t)\mathrm{d}t,\ s_2 = \int_{-1}^{1} s(t)\phi_2(t)\mathrm{d}t,\ s_3 = \int_{-1}^{1} s(t)\phi_3(t)\mathrm{d}t.$$

Furthermore the mean-squared error $[s(t) - \hat{s}(t)]^2$ is minimum (see P5.9).

Remark: The 3 polynomials $\{\phi_1(t), \phi_2(t), \phi_3(t)\}$ are normalized versions of the first three members of what are referred to as Legendre polynomials. These are orthogonal polynomials over $[-1,1]$ that are solutions of Legendre's differential equations:

$$(1-t^2)\frac{\mathrm{d}^2 p_n(t)}{\mathrm{d}t^2} - 2t\frac{\mathrm{d}p_n(t)}{\mathrm{d}t} + n(n+1)p_n(t) = 0.$$

The nth member is given by $p_n(t) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}t^n}(t^2-1)^n$ (Rodrique's formula) and

$$\int_{-\infty}^{\infty} p_n(t)p_m(t)\mathrm{d}t = \begin{cases} \frac{2}{2n+1}, & n = m \\ 0, & n \neq m \end{cases}.$$

P5.8 Orthogonal :

$$\int_0^T s_i(t)s_j(t)\mathrm{d}t = 0,\ (i \neq j)$$

Equal energy:

$$E = \int_0^T s_1^2(t)\mathrm{d}t = \cdots = \int_0^T s_M^2(t)\mathrm{d}t.$$

First, let us find the energy, say in $s_m(t)$:

$$\begin{aligned}
\hat{E} &= \int_0^T \hat{s}_m^2(t)\mathrm{d}t = \int_0^T \left[s_m(t) - \frac{1}{M}\sum_{k=1}^{M} s_k(t)\right]^2 \mathrm{d}t \\
&= \int_0^T s_m^2(t) - \int_0^T \frac{2}{M} s_m(t)\sum_{k=1}^{M} s_k(t)\mathrm{d}t + \int_0^T \frac{1}{M^2}\left[\sum_{k=1}^{M} s_k(t)\right]^2 \mathrm{d}t \\
&= E - \frac{2}{M}\int_0^T s_m(t)\left[s_1(t) + \ldots + s_M(t)\right]\mathrm{d}t \\
&+ \frac{1}{M^2}\left\{\int_0^T \left[s_1^2(t) + \ldots + s_M^2(t) + s_1(t)s_2(t) + \ldots + s_{M-1}(t)s_M(t)\right]\mathrm{d}t\right\} \\
&= E - \frac{2}{M}\int_0^T \left[s_1(t)s_m(t) + \ldots + s_m^2(t) + \ldots + s_m(t)s_M(t)\right]\mathrm{d}t \\
&+ \frac{1}{M^2}\left\{\int_0^T s_1^2(t)\mathrm{d}t + \ldots + \int_0^T s_M^2(t)\mathrm{d}t + \int_0^T s_1(t)s_2(t)\mathrm{d}t + \ldots + \int_0^T s_{M-1}(t)s_M(t)\mathrm{d}t\right\} \\
&= E - \frac{2}{M}E + \frac{1}{M^2}(ME + 0) = E - \frac{1}{M}E = \frac{E}{M}(M-1)
\end{aligned}$$

Now let us find the correlation coefficient:

$$\begin{aligned}
\rho_{mn} &= \int_0^T \frac{\hat{s}_m(t)\hat{s}_n(t)}{\hat{E}}\mathrm{d}t \\
&= \frac{1}{\hat{E}}\int_0^T \left[s_m(t) - \frac{1}{M}\sum_{k=1}^{M} s_k(t)\right]\left[s_n(t) - \frac{1}{M}\sum_{k=1}^{M} s_k(t)\right]\mathrm{d}t \\
&= \frac{1}{\hat{E}}\int_0^T \left[s_m(t)s_n(t) - \frac{1}{M}s_m(t)\sum_{k=1}^{M} s_k(t) - \frac{1}{M}s_n(t)\sum_{k=1}^{M} s_k(t) + \left(\frac{1}{M}\sum_{k=1}^{M} s_k(t)\right)^2\right]\mathrm{d}t \\
&= \frac{1}{\hat{E}}\int_0^T s_m(t)s_n(t)\mathrm{d}t - \frac{1}{M}\int_0^T s_m(t)\sum_{k=1}^{M} s_k(t)\mathrm{d}t - \frac{1}{M}\int_0^T s_n(t)\sum_{k=1}^{M} s_k(t)\mathrm{d}t \\
&+ \frac{1}{M^2}\int_0^T \left[\sum_{k=1}^{M} s_k(t)\right]^2 \mathrm{d}t \\
&= \frac{1}{\hat{E}}\left[0 - \frac{1}{M}E - \frac{1}{M}E + \frac{1}{M^2}ME\right] = \frac{1}{\hat{E}}\left(-\frac{E}{M}\right) = \frac{-E/M}{E/M(M-1)} = \frac{1}{1-M}
\end{aligned}$$

The signal space plots for $M = 2, 3, 4$ are as follows.

<u>$M = 2$</u>: It is easy to show that $\hat{s}_1(t) = \frac{s_1(t)-s_2(t)}{2}$ and $\hat{s}_2(t) = \frac{s_2(t)-s_1(t)}{2} = -\hat{s}_1(t)$, i.e., the signals are antipodal with energy $\hat{E} = E/2$.

Note that

$$\sum_{m=1}^{M} \hat{s}_m(t) = \sum_{m=1}^{M} s_m(t) - \frac{1}{M}\sum_{m=1}^{M}\left(\sum_{k=1}^{M} s_k(t)\right) = \sum_{m=1}^{M} s_m(t) - \frac{1}{M}\sum_{k=1}^{M} s_k(t)\underbrace{\left(\sum_{m=1}^{M} 1\right)}_{=M} = 0.$$

Therefore the signals $\{\hat{s}_m(t)\}_{m=1}^{M}$ are linearly independent since any one of them is a linear combination of the remaining $M-1$ of them. They therefore lie in an $(M-1)$-dimensional signal space at distance $\sqrt{\frac{M-1}{M}}$ from the origin. Further, since they are equally correlated the angle between any 2 vectors from the origin to the signal points is the same for each pair. Putting this all together we have for:

<u>$M=3$</u>: A 2-dimensional signal space. Signal points are at distance $\sqrt{\frac{2}{3}E}$ from origin, laying at a angle spacing of 120°, i.e., on vertices of an equilateral triangle.

<u>$M=4$</u>: Lying in a 3-dimensional space, distance of $\sqrt{\frac{3}{4}E}$, on the vertices of a regular tetrahedron.

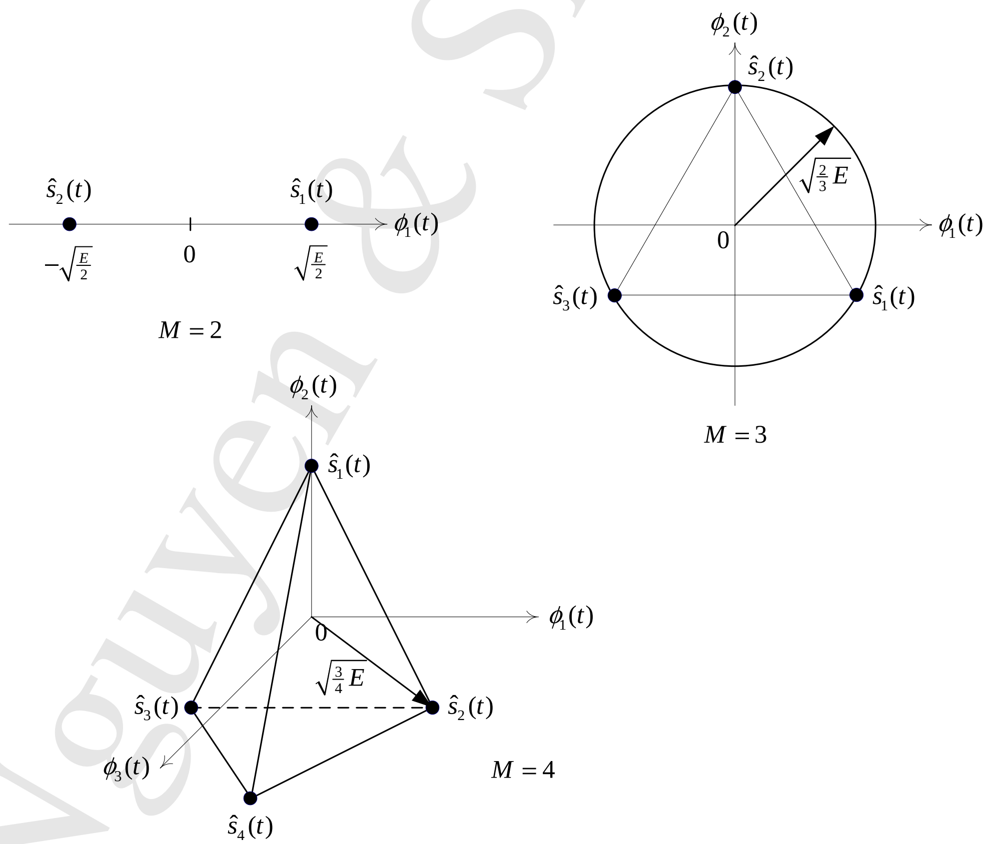


Figure 5.8

P5.9 (a) The approximation error is simply $e(t) = s(t) - \hat{s}(t) = s(t) - \sum_{k=1}^{N} s_k\phi_k(t)$. Thus the energy of the estimation error is:

$$\begin{aligned}
E_e &= \int_{-\infty}^{+\infty}\left[s(t)-\sum_{k=1}^{N}s_k\phi_k(t)\right]^2\mathrm{d}t \\
&= E_s - 2\int_{-\infty}^{+\infty}s(t)\hat{s}(t)\mathrm{d}t + \int_{-\infty}^{+\infty}\left[s_1\phi_1(t)+\ldots+s_N\phi_N(t)\right]^2\mathrm{d}t \\
&= E_s - 2\int_{-\infty}^{+\infty}s(t)\hat{s}(t)\mathrm{d}t + \int_{-\infty}^{+\infty}[s_1^2\phi_1^2(t)+ \\
&+ \ldots + s_N^2\phi_N^2(t) + s_1s_2\phi_1(t)\phi_2(t)+\ldots+s_{N-1}s_N\phi_{N-1}(t)\phi_N(t)]\mathrm{d}t \\
&= E_s - 2\int_{-\infty}^{+\infty}s(t)\hat{s}(t)\mathrm{d}t + \int_{-\infty}^{+\infty}\sum_{k=1}^{N}s_k^2\phi_k^2(t)\mathrm{d}t + 0 \\
&= E_s - 2\int_{-\infty}^{+\infty}s(t)\hat{s}(t)\mathrm{d}t + \sum_{k=1}^{N}\int_{-\infty}^{+\infty}s_k^2\phi_k^2(t)\mathrm{d}t \\
&= E_s - 2\int_{-\infty}^{+\infty}s(t)\hat{s}(t)\mathrm{d}t + \sum_{k=1}^{N}s_k^2 \\
&= E_s - 2\sum_{k=1}^{N}s_k\int_{-\infty}^{+\infty}s(t)\phi_k(t)\mathrm{d}t + \sum_{k=1}^{N}s_k^2 \qquad (5.20)
\end{aligned}$$

Obviously E_e is a function of s_k and therefore s_k can be chosen to minimize E_e. To find the optimal s_k, differentiate E_e with respect to each s_k and set the result to zero:

$$\begin{aligned}
\frac{\mathrm{d}E_e}{\mathrm{d}s_k} &= 0 - \frac{2\mathrm{d}\left[\sum_{k=1}^{N}s_k\int_{-\infty}^{+\infty}s(t)\phi_k(t)\mathrm{d}t\right]}{\mathrm{d}s_k} + \frac{\mathrm{d}\left[\sum_{k=1}^{N}s_k^2\right]}{\mathrm{d}s_k} \\
&= -2\int_{-\infty}^{+\infty}s(t)\phi_k(t)\mathrm{d}t + 2s_k = 0 \\
\Rightarrow s_k &= \int_{-\infty}^{+\infty}s(t)\phi_k(t)\mathrm{d}t
\end{aligned}$$

(b) Substituting $\int_{-\infty}^{+\infty}s(t)\phi_k(t)\mathrm{d}t = s_k$ into (5.20), the minimum mean square approximation error is given by:

$$\begin{aligned}
(E_e)_{\min} &= E_s - 2\sum_{k=1}^{N}s_k\int_{-\infty}^{+\infty}s(t)\phi_k(t)\mathrm{d}t + \sum_{k=1}^{N}s_k^2 \\
&= E_s - 2\sum_{k=1}^{N}s_k^2 + \sum_{k=1}^{N}s_k^2 = E_s - \sum_{k=1}^{N}s_k^2 \text{ (joules)}
\end{aligned}$$

Basically the energy of the approximation error is the signal energy minus the energy in the approximation signal.

P5.10 (a) The coefficients s_{ij}, $i,j\in\{1,2\}$ are found as follows:

$$\begin{aligned}
s_{11} &= \int_0^1 s_1(t)\phi_1(t)\mathrm{d}t = 1,\ s_{12} = \int_0^1 s_1(t)\phi_2(t)\mathrm{d}t = 1, \\
s_{21} &= \int_0^1 s_2(t)\phi_1(t)\mathrm{d}t = 1,\ s_{22} = \int_0^1 s_2(t)\phi_2(t)\mathrm{d}t = -1.
\end{aligned}$$

Note that these results can also be found by inspecting that $s_1(t) = \phi_1(t) + \phi_2(t)$ and $s_2(t) = \phi_1(t) - \phi_2(t)$.

(b)

$$\begin{bmatrix} \phi_1^R(t) \\ \phi_2^R(t) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix} \tag{5.21}$$

$$\phi_1^R(t) = (\cos\theta)\phi_1(t) + (\sin\theta)\phi_2(t) \overset{\theta=60^0}{=} \frac{1}{2}\phi_1(t) + \frac{\sqrt{3}}{2}\phi_2(t)$$

$$\phi_2^R(t) = -(\sin\theta)\phi_1(t) + (\cos\theta)\phi_2(t) \overset{\theta=60^0}{=} -\frac{\sqrt{3}}{2}\phi_1(t) + \frac{1}{2}\phi_2(t) \tag{5.22}$$

The time waveforms of $\phi_1^R(t)$ and $\phi_2^R(t)$ for $\theta = 60^0$ are shown in Fig. 5.9.

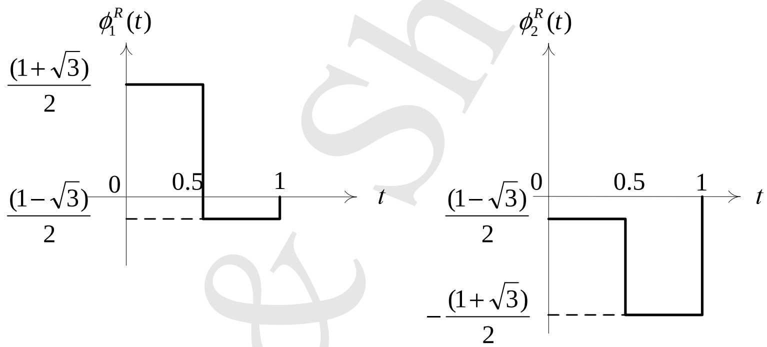


Figure 5.9: Plots of $\phi_1^R(t)$ and $\phi_2^R(t)$.

(c) Determine the new set of coefficients s_{ij}^R, $i, j \in \{1, 2\}$ in the representation:

$$\begin{bmatrix} s_1(t) \\ s_2(t) \end{bmatrix} = \begin{bmatrix} s_{11}^R & s_{12}^R \\ s_{21}^R & s_{22}^R \end{bmatrix} \begin{bmatrix} \phi_1^R(t) \\ \phi_2^R(t) \end{bmatrix} \tag{5.23}$$

- One method is a straightforward calculation: $s_{ij}^R = \int_0^1 s_i(t)\phi_j^R(t)\mathrm{d}t$. Thus

$$\begin{aligned} s_{11}^R &= \frac{1}{2}\left(2 \times \frac{1+\sqrt{3}}{2}\right) = \frac{1+\sqrt{3}}{2},\ s_{12}^R = \frac{1}{2}\left(2 \times \frac{1-\sqrt{3}}{2}\right) = \frac{1-\sqrt{3}}{2} \\ s_{21}^R &= \frac{1}{2}\left(2 \times \frac{1-\sqrt{3}}{2}\right) = \frac{1-\sqrt{3}}{2},\ s_{21}^R = \frac{1}{2}\left(2 \times \frac{-(1+\sqrt{3})}{2}\right) = -\frac{1+\sqrt{3}}{2}. \end{aligned}$$

- A second method is based on s_{ij} and θ directly:

$$\begin{aligned} \begin{bmatrix} s_1(t) \\ s_2(t) \end{bmatrix} &= \begin{bmatrix} s_{11}^R & s_{12}^R \\ s_{21}^R & s_{22}^R \end{bmatrix} \begin{bmatrix} \phi_1^R(t) \\ \phi_2^R(t) \end{bmatrix} = \begin{bmatrix} s_{11}^R & s_{12}^R \\ s_{21}^R & s_{22}^R \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix} \\ &= \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix} \end{aligned}$$

Therefore
$$\begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} = \begin{bmatrix} s_{11}^R & s_{12}^R \\ s_{21}^R & s_{22}^R \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{aligned} \Rightarrow \begin{bmatrix} s_{11}^R & s_{12}^R \\ s_{21}^R & s_{22}^R \end{bmatrix} &= \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^{-1} \\ &= \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\ &\overset{\theta=60^0}{=} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}+1}{2} & \frac{1-\sqrt{3}}{2} \\ \frac{1-\sqrt{3}}{2} & -\frac{\sqrt{3}+1}{2} \end{bmatrix} \end{aligned}$$

(d) Geometrical picture for both the basis function sets (original and rotated) and for the two signal points is shown in Fig. 5.10. *Note:* As seen from the geometrical picture,

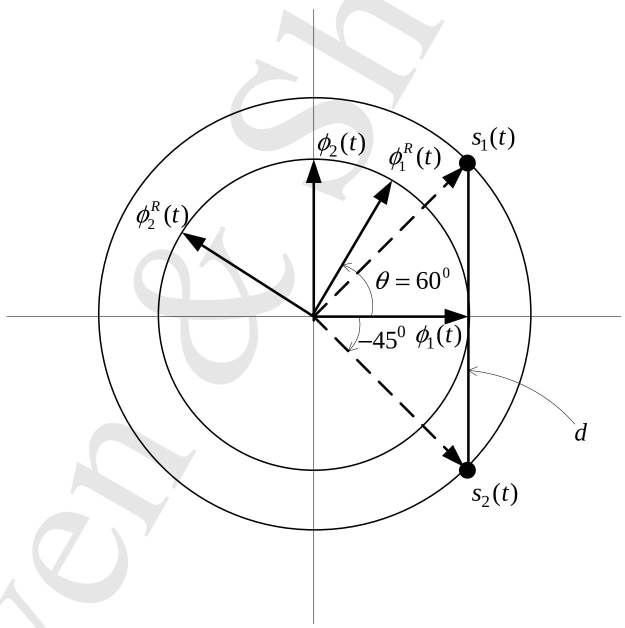


Figure 5.10: Geometrical representation of the basis function sets (original and rotated) and the two signals.

$s_{11}^R = -s_{22}^R$ and $s_{12}^R = s_{21}^R$. This is seen in (c) above and is true for any θ. But do not jump to conclusions that this is true all the time. It comes about because of the locations of $s_1(t)$, $s_2(t)$ in the signal space.

(e) Determine the distance d between the two signals $s_1(t)$ and $s_2(t)$ in two ways:

(i) Algebraically:

$$d^2 = \int_0^1 [s_1(t) - s_2(t)]^2 \mathrm{d}t = \int_0^{0.5} 2^2 \mathrm{d}t + \int_{0.5}^1 2^2 \mathrm{d}t = 4 \Rightarrow \; d = 2 \tag{5.24}$$

(ii) Geometrically: From the signal space plot of (d) one has $d = \sqrt{2+2} = 2$ (Note that both signal have energy $E_1 = E_2 = 2$).

(f) Thought $\phi_1(t)$ and $\phi_2(t)$ can represent the two given signals, they are by no means a complete basis set because they cannot represent an arbitrary, finite-energy signal defined on the time interval $[0,1]$. As a start to complete the basis, we wish to plot the next two possible orthonormal functions $\phi_3(t)$ and $\phi_4(t)$. Obviously there are may possible choices for $\phi_3(t)$ and $\phi_4(t)$. One simple choice is shown below.

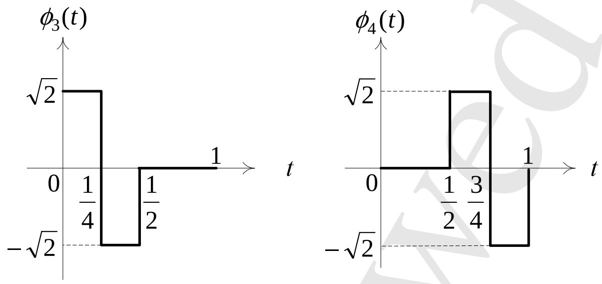


Figure 5.11: An example of $\phi_3(t)$ and $\phi_4(t)$.

P5.11 (a) The first orthonormal basis function is chosen, somewhat arbitrarily, as $\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \frac{1}{\sqrt{2}} s_1(t)$.

Next $s_2(t)$ is already orthogonal to $\phi_1(t)$. Thus, we set:

$$\phi_2(t) = \frac{s_2(t)}{\sqrt{\int_0^2 s_2^2(t)\mathrm{d}t}} = \begin{cases} \frac{1}{\sqrt{2}}, & 0 \le t \le 1 \\ -\frac{1}{\sqrt{2}}, & 1 \le t \le 2 \\ 0, & \text{elsewhere} \end{cases} \tag{5.25}$$

Then project $s_3(t)$ onto $\phi_1(t)$ and $\phi_2(t)$ to have

$$s_{31} = \int_0^2 \frac{1}{\sqrt{2}}\mathrm{d}t = \sqrt{2}, \qquad s_{32} = \int_0^1 \frac{1}{\sqrt{2}}\mathrm{d}t - \int_1^2 \frac{1}{\sqrt{2}}\mathrm{d}t = 0$$

The third orthogonal signal is given by

$$\phi_3'(t) = s_3(t) - \sqrt{2}\phi_1(t) - 0 \cdot \phi_2(t) = \begin{cases} -1, & 2 \le t \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

Since the energy of $\phi_3'(t)$ equal 1, one sets $\phi_3(t) = \phi_3'(t)$.

Finally, the fourth orthonormal basis function is obtained by fist projecting $s_4(t)$ onto $\phi_1(t)$, $\phi_2(t)$, $\phi_3(t)$:

$$s_{41} = \int_0^2 \frac{1}{\sqrt{2}} \times (-1)\mathrm{d}t = -\sqrt{2}, \quad s_{42} = \int_0^1 \frac{1}{\sqrt{2}} \times (-1)\mathrm{d}t + \int_1^2 \frac{-1}{\sqrt{2}} \times (-1)\mathrm{d}t = 0$$

$$s_{43} = \int_2^3 -1 \times (-1)\mathrm{d}t = 1$$

The orthogonal signal is given by

$$\phi_4'(t) = s_4(t) + \sqrt{2}\phi_1(t) - 0\phi_2(t) - \phi_3(t) = 0$$

In conclusion, we need only 3 orthonormal basis functions to represent the signal set. They are plotted in Fig. 5.12.

$$\phi_1(t) = \begin{cases} \frac{1}{\sqrt{2}}, & 0 \le t \le 2 \\ 0, & \text{elsewhere} \end{cases}, \quad \phi_2(t) = \begin{cases} \frac{1}{\sqrt{2}}, & 0 \le t \le 1 \\ -\frac{1}{\sqrt{2}}, & 1 \le t \le 2 \\ 0, & \text{elsewhere} \end{cases}, \quad \phi_3(t) = \begin{cases} 1, & 2 \le t \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

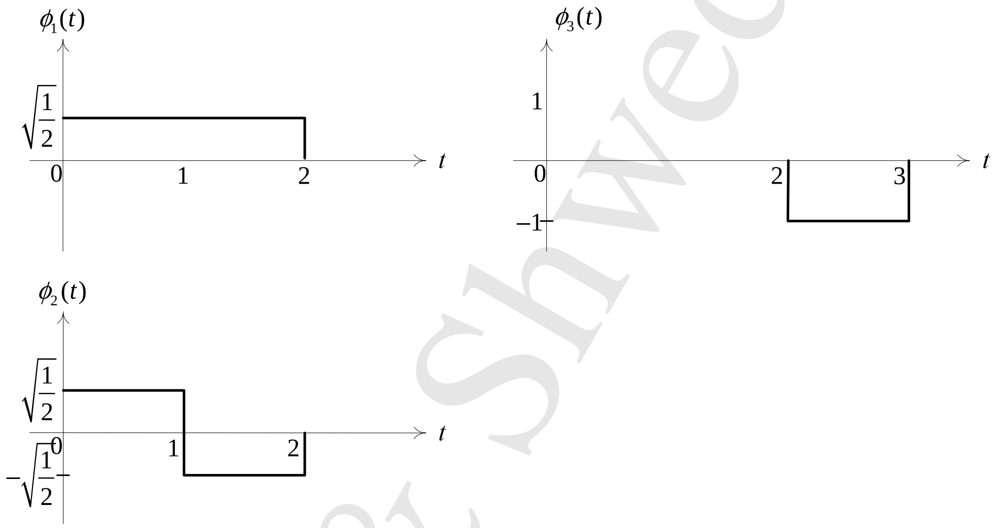


Figure 5.12: Three basis functions.

In terms of the above basis functions, the four signals are:

$$\begin{bmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \\ s_4(t) \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & 1 \\ \sqrt{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \\ \phi_3(t) \end{bmatrix}$$

(b) First the 3 functions are certainly orthogonal (they are what one calles “time orthogonal” since they do not overlap in time) and obviously each of them has unit energy. The question is do they represent the 4 time functions *exactly*. Well,

$$\begin{aligned} s_1(t) &= v_1(t) + v_2(t) + 0 \cdot v_3(t), \quad s_2(t) = v_1(t) - v_2(t) + 0 \cdot v_3(t) \\ s_3(t) &= v_1(t) + v_2(t) - v_3(t), \quad s_4(t) = -v_1(t) - v_2(t) - v_3(t) \end{aligned}$$

(c) Geometrical representation of the four signals is plotted in Fig. 5.13.

P5.12 The signal set is given as:

$$\begin{aligned} s_1(t) &= \begin{cases} \sqrt{3}A\cos\left(\frac{2\pi t}{T_b}\right), & 0 \le t \le T_b \\ 0, & \text{otherwise} \end{cases} \\ s_2(t) &= \begin{cases} A\sin\left(\frac{2\pi t}{T_b}\right), & 0 \le t \le T_b \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

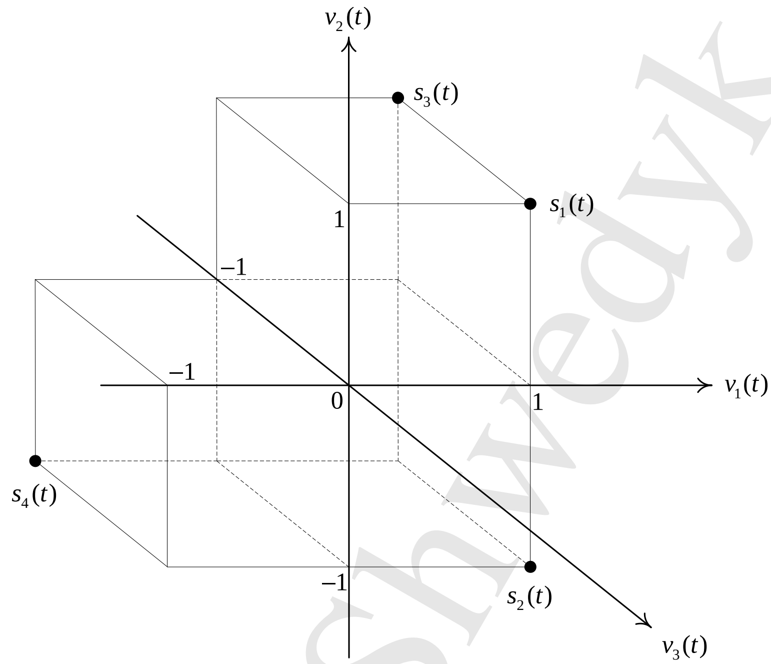


Figure 5.13: Geometrical representation of the 4 signals by $\{v_1(t), v_2(t), v_3(t)\}$.

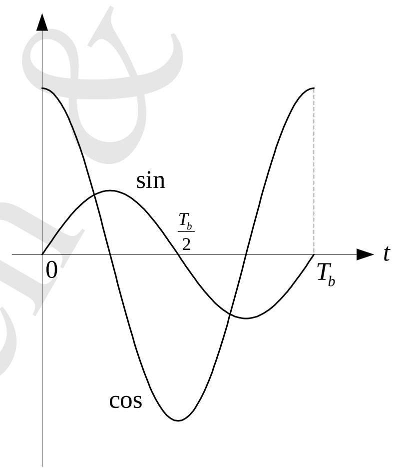


Figure 5.14

(a) There is one cycle of the cos and one cycle of the sin in the interval $[0, T_b]$. Graphically:

Obviously the area of the product is zero, regardless of the amplitudes of the sine and cos. Therefore $s_1(t)$ and $s_2(t)$ are orthogonal.

Show mathematically:

$$\begin{aligned}\int_0^{T_b} s_1(t)s_2(t)\mathrm{d}t &= \sqrt{3}A^2\int_0^{T_b}\cos\left(\frac{2\pi t}{T_b}\right)\sin\left(\frac{2\pi t}{T_b}\right)\mathrm{d}t\\ &= \frac{\sqrt{3}}{2}A^2\int_0^{T_b}\sin\left(\frac{4\pi t}{T_b}\right)\mathrm{d}t = -\frac{\sqrt{3}}{2}A^2\frac{T_b}{4\pi}\cos\left(\frac{4\pi t}{T_b}\right)\Bigg|_0^{T_b} = 0\end{aligned}$$

The energies of the two signals are:

$$\begin{aligned}E_1 &= \int_0^{T_b} s_1^2(t)\mathrm{d}t = \left(\frac{\sqrt{3}A}{\sqrt{2}}\right)^2 T_b = \frac{3A^2T_b}{2}\\ E_2 &= \int_0^{T_b} s_2^2(t)\mathrm{d}t = \left(\frac{A}{\sqrt{2}}\right)^2 T_b = \frac{A^2T_b}{2}\end{aligned}$$

Simply choose the basis functions $\{\phi_1(t), \phi_2(t)\}$ to be scaled versions of $s_1(t)$ and $s_2(t)$:

$$\begin{aligned}\phi_1(t) &= \frac{s_1(t)}{\sqrt{E_1}} = \sqrt{\frac{2}{T_b}}\cos\left(\frac{2\pi t}{T_b}\right); \qquad 0 \le t \le T_b\\ \phi_2(t) &= \frac{s_2(t)}{\sqrt{E_2}} = \sqrt{\frac{2}{T_b}}\sin\left(\frac{2\pi t}{T_b}\right)); \qquad 0 \le t \le T_b\end{aligned}$$

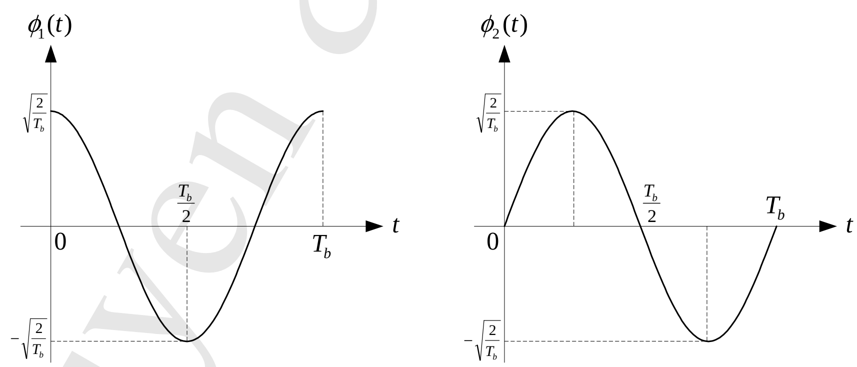


Figure 5.15

(b) The optimum decision rule for the minimum distance receiver is:

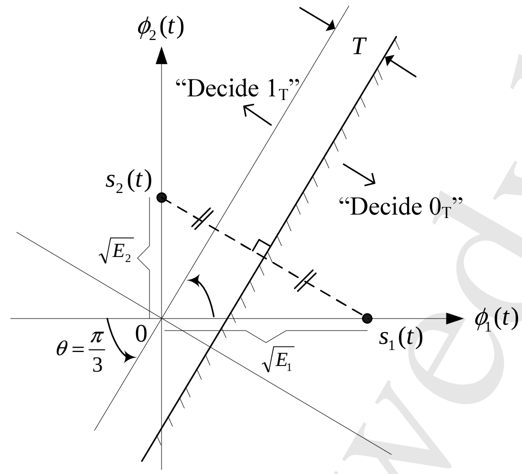


Figure 5.16

$$
\begin{aligned}
& (r_1-s_{21})^2+(r_2-s_{22})^2 \underset{\text{“}1_D\text{”}}{\overset{\text{“}0_D\text{”}}{\gtreqless}} (r_1-s_{11})^2+(r_2-s_{12})^2 \\
\Leftrightarrow\ & r_1^2+(r_2-\sqrt{E_2})^2 \underset{\text{“}1_D\text{”}}{\overset{\text{“}0_D\text{”}}{\gtreqless}} (r_1-\sqrt{E_1})^2+r_2^2 \\
\Leftrightarrow\ & -2r_2\sqrt{E_2}+E_2 \underset{\text{“}1_D\text{”}}{\overset{\text{“}0_D\text{”}}{\gtreqless}} -2r_1\sqrt{E_1}+E_1 \\
\Leftrightarrow\ & -2r_2A\sqrt{\frac{T_b}{2}}+\frac{A^2T_b}{2}+2r_1A\sqrt{\frac{3T_b}{2}}-\frac{3A^2T_b}{2} \underset{\text{“}1_D\text{”}}{\overset{\text{“}0_D\text{”}}{\gtreqless}} 0 \\
\Leftrightarrow\ & \sqrt{3}r_1-r_2-\frac{A\sqrt{T_b}}{\sqrt{2}} \underset{\text{“}1_D\text{”}}{\overset{\text{“}0_D\text{”}}{\gtreqless}} 0
\end{aligned}
$$

(c) From the signal space diagram, the squared distance between the two signals is:

$$
d_{21}^2 = E_1+E_2=\sqrt{\frac{3A^2T_b}{2}}+\frac{A^2T_b}{2}=2A^2T_b=2T_b \qquad \text{(when } A=1 \text{ volt)}
$$

It follows that the bit error rate bit is:

$$
\begin{aligned}
P[\text{error}] &= Q\left(\frac{d_{21}/2}{\sqrt{N_0/2}}\right)=Q\left(\frac{\sqrt{2T_b}/2}{\sqrt{N_0/2}}\right)=Q\left(\sqrt{\frac{T_b}{N_0}}\right)\le 10^{-6} \\
\Rightarrow \sqrt{\frac{T_b}{N_0}} &\ge 4.75 \Rightarrow T_b \ge (4.75)^2 N_0 \\
\Rightarrow r_b=\frac{1}{T_b} &\le \frac{1}{(4.75)^2N_0}=\frac{1}{(4.75)^2.10^{-8}} \\
&= 4.43\times 10^6 \text{ bits/sec } = 4.43 \text{ Mbps}
\end{aligned}
$$

(d) To implement the optimum receiver that uses only matched filter, one needs to rotate the basis functions as shown in the signal space diagram to obtain $\hat{\phi}_1(t)$ and $\hat{\phi}_2(t)$: $\hat{\phi}_1(t)$ is perpendicular to the line joining $s_1(t)$ and $s_2(t)$. Since

$$\tan(\theta) = \frac{\sqrt{E_1}}{\sqrt{E_2}} = \sqrt{\frac{3A^2T_b/2}{A^2T_b/2}} = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$$

It follows that

$$\begin{aligned}
\hat{\phi}_2(t) &= -\sin\theta\phi_1(t) + \cos\theta\phi_2(t) \\
&= -\sqrt{\frac{2}{T_b}}\cos\left(\frac{2\pi t}{T_b}\right)\sin\theta + \sqrt{\frac{2}{T_b}}\sin\left(\frac{2\pi t}{T_b}\right)\cos\theta \\
&= \sqrt{\frac{2}{T_b}}\sin\left(\frac{2\pi t}{T_b} - \theta\right) = \sqrt{\frac{2}{T_b}}\sin\left(\frac{2\pi t}{T_b} - \frac{\pi}{3}\right)
\end{aligned}$$

The block diagram of the optimum receiver is shown below:

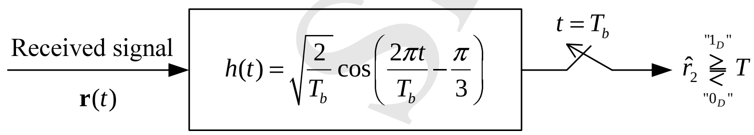


Figure 5.17

The impulse response of the filter is found as:

$$\begin{aligned}
h(t) &= \hat{\phi}_2(T_b - t) = \sqrt{\frac{2}{T_b}}\sin\left(\frac{2\pi(T_b - t)}{T_b} - \frac{\pi}{3}\right) \\
&= \sqrt{\frac{2}{T_b}}\sin\left(-\frac{2\pi t}{T_b} + \frac{5\pi}{3}\right) = \sqrt{\frac{2}{T_b}}\cos\left(\frac{2\pi t}{T_b} - \frac{\pi}{3}\right)
\end{aligned}$$

The threshold can be found from the signal space diagram to be:

$$T = -\left[\sqrt{E_1}\sin\left(\frac{\pi}{3}\right) - \frac{d_{21}}{2}\right] = -\left[\sqrt{3}A\sqrt{\frac{T_b}{2}}\frac{\sqrt{3}}{2} - A\sqrt{2T_b}\frac{1}{2}\right] = -\frac{A}{2}\sqrt{\frac{T_b}{2}}$$

(e) The average energy of the two signals $s_1(t)$ and $s_2(t)$ is :

$$E_b = \frac{E_1 + E_2}{2} = A^2T_b$$

For a given (fixed) energy per bit E_b, antipodal signalling achieves the smallest error probability amongst all binary signalling schemes. This is because the distance between the two signals in antipodal signalling is the largest (again, for a fixed E_b). Thus, the two signals can be modified as:

$$\begin{aligned}
\hat{s}_1(t) &= -\hat{s}_2(t) = \sqrt{E_b}\phi_1(t) \\
&= A\sqrt{T_b}\sqrt{\frac{2}{T_b}}\cos\left(\frac{2\pi t}{T_b}\right) = \sqrt{2}A\cos\left(\frac{2\pi t}{T_b}\right), \quad 0 \le t \le T_b
\end{aligned}$$

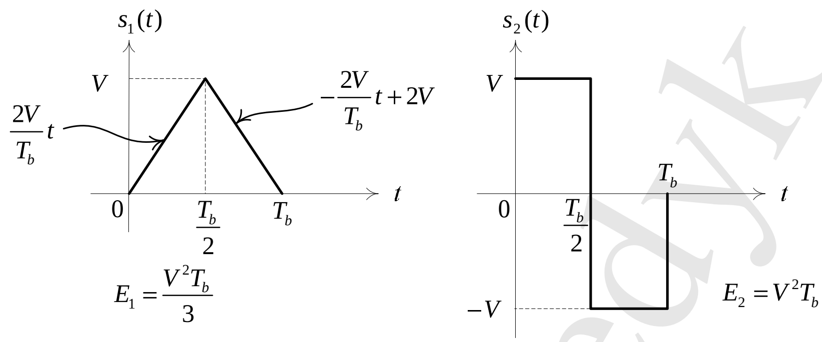


Figure 5.18

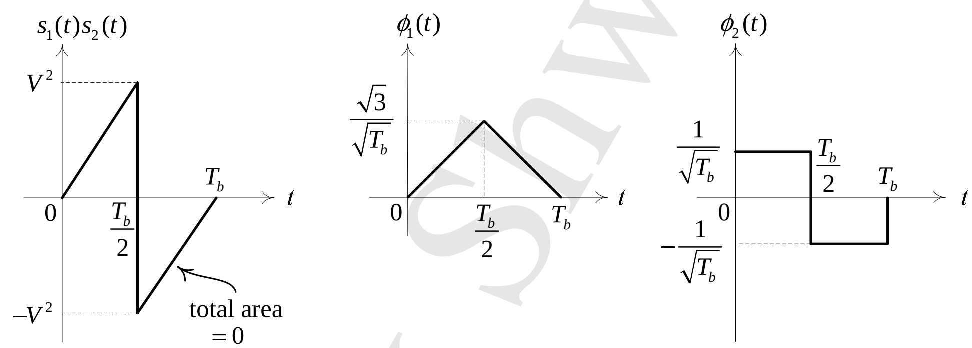


Figure 5.19

P5.13 (a) To show $s_1(t)$ is orthogonal to $s_2(t)$, one needs to show that $\int_0^{T_b} s_1(t)s_2(t)\mathrm{d}t = 0$. The above is quite obvious by plotting $s_1(t)s_2(t)$.
Since $s_1(t)$ and $s_2(t)$ are orthogonal, choose:

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \frac{\sqrt{3}}{V\sqrt{T_b}}s_1(t); \qquad \phi_2(t) = \frac{s_2(t)}{\sqrt{E_2}} = \frac{1}{V\sqrt{T_b}}s_2(t)$$

Plots of $\phi_1(t)$ and $\phi_2(t)$ are shown above.

(b) Plots of signal space diagram and the optimum decision regions are shown below:

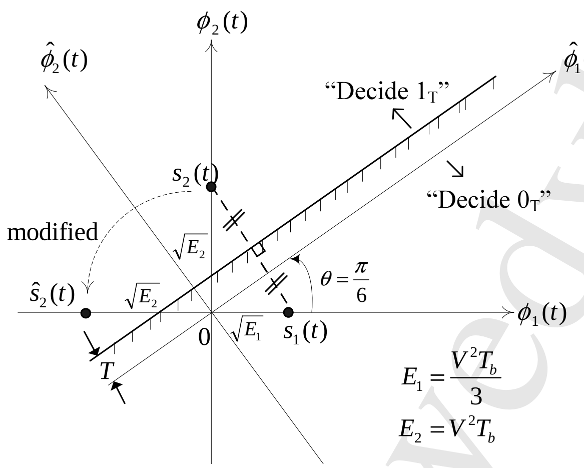


Figure 5.20

The optimum decision rule is that of the minimum distance rule:

$$
\begin{aligned}
& (r_1 - s_{21})^2 + (r_2 - s_{22})^2 \underset{\text{“}1_D\text{”}}{\overset{\text{“}0_D\text{”}}{\gtreqless}} (r_1 - s_{11})^2 + (r_2 - s_{12})^2 \\
\Leftrightarrow\ & r_1^2 + (r_2 - \sqrt{E_2})^2 \underset{\text{“}1_D\text{”}}{\overset{\text{“}0_D\text{”}}{\gtreqless}} (r_1 - \sqrt{E_1})^2 + r_2^2 \\
\Leftrightarrow\ & -2r_2\sqrt{E_2} + E_2 \underset{\text{“}1_D\text{”}}{\overset{\text{“}0_D\text{”}}{\gtreqless}} -2r_1\sqrt{E_1} + E_1 \\
\Leftrightarrow\ & -2r_2 V\sqrt{T_b} + V^2 T_b + 2r_1 V\sqrt{\frac{T_b}{3}} - \frac{V^2 T_b}{3} \underset{\text{“}1_D\text{”}}{\overset{\text{“}0_D\text{”}}{\gtreqless}} 0 \\
\Leftrightarrow\ & \frac{r_1}{\sqrt{3}} - r_2 + \frac{V\sqrt{T_b}}{3} \underset{\text{“}1_D\text{”}}{\overset{\text{“}0_D\text{”}}{\gtreqless}} 0
\end{aligned}
$$

(c) The squared distance between the two signals:

$$
d_{21}^2 \;=\; E_1 + E_2 = \frac{V^2 T_b}{3} + V^2 T_b = \frac{4V^2 T_b}{3} = \frac{4T_b}{3} \qquad \text{(when } V = 1 \text{ volt)}
$$

The error probability of the minimum distance receiver is:

$$
\begin{aligned}
P[\text{error}] &= Q\left(\frac{d_{21}/2}{\sqrt{N_0/2}}\right) = Q\left(\frac{2\sqrt{T_b}/2\sqrt{3}}{\sqrt{N_0/2}}\right) = Q\left(\sqrt{\frac{T_b}{N_0}}\cdot\frac{\sqrt{2}}{\sqrt{3}}\right) \leq 10^{-6} \\
\Rightarrow \sqrt{\frac{T_b}{N_0}}\cdot\frac{\sqrt{2}}{\sqrt{3}} &\geq 4.75 \Rightarrow T_b \geq (4.75)^2 \cdot \frac{3}{2} N_0 \\
\Rightarrow r_b = \frac{1}{T_b} &\leq \frac{2}{(4.75)^2 \times 3 \times N_0} = \frac{1}{(4.75)^2 \times 1.5 \times 10^{-8}} \\
&= 2.955 \times 10^6 \text{ bits/sec } = 2.955 \text{ Mbps}
\end{aligned}
$$

(d) Need to rotate $\phi_1(t)$ and $\phi_2(t)$ to obtain $\hat{\phi}_1(t)$ and $\hat{\phi}_2(t)$ as shown in the signal space diagram: $\hat{\phi}_1(t)$ is perpendicular to the line joining $s_1(t)$ and $s_2(t)$. Since

$$\tan(\theta) = \frac{\sqrt{E_1}}{\sqrt{E_2}} = \frac{V\sqrt{T_b}/\sqrt{3}}{V\sqrt{T_b}} = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6} \Rightarrow \sin\theta = \frac{1}{2};\ \cos\theta = \frac{\sqrt{3}}{2}$$

To realize the optimum receiver that uses only one matched filter, one needs to obtain $\hat{\phi}_2(t)$ and the threshold T.

$$\begin{aligned}\hat{\phi}_2(t) &= -\sin\theta\phi_1(t) + \cos\theta\phi_2(t)\\ &= -\frac{1}{2}\phi_1(t) + \frac{\sqrt{3}}{2}\phi_2(t)\end{aligned}$$

the filter's impulse response is $h(t) = \hat{\phi}_2(T_b - t)$. Both $\hat{\phi}_2(t)$ and $h(t)$ are plotted below

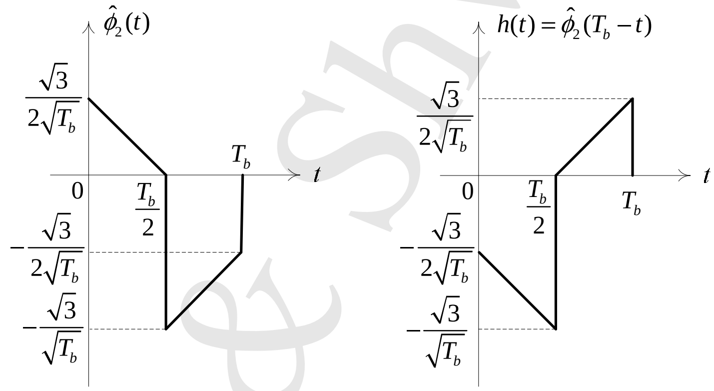


Figure 5.21

The block diagram of the optimum receiver:

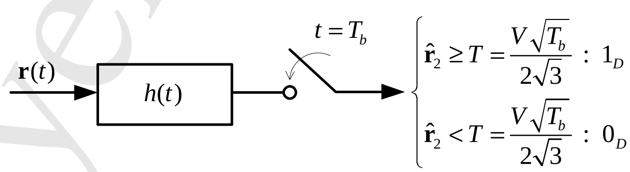


Figure 5.22

where the threshold can be found from the signal space diagram to be:

$$T = \frac{d_{21}}{2} - \sqrt{E_1}\sin\left(\frac{\pi}{6}\right) = \frac{2V\sqrt{T_b}}{2\sqrt{3}} - \frac{V\sqrt{T_b}}{\sqrt{3}} \cdot \frac{1}{2} = \frac{V\sqrt{T_b}}{2\sqrt{3}}$$

Note: One can also make use of the result obtained in Question 2 to come up with the following implementation:

$$\int_0^{T_b} \mathbf{r}(t)[s_2(t) - s_1(t)]\mathrm{d}t \underset{\text{“}0_D\text{”}}{\overset{\text{“}1_D\text{”}}{\gtrless}} \frac{E_2 - E_1}{2}$$

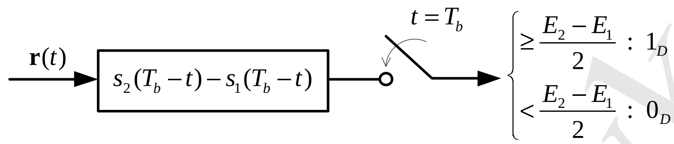


Figure 5.23

(e) Need to modify $s_2(t)$ so that the Euclidean distance d_{21} is maximized. Since the energy of $s_2(t)$ cannot be changed $\Rightarrow$ move $s_2(t)$ to the point $(-\sqrt{E_2}, 0)$ so that the maximized distance is

$$d_{21} = \sqrt{E_2} + \sqrt{E_1} = \left(\frac{\sqrt{3}+1}{\sqrt{3}}\right) V\sqrt{T_b}$$

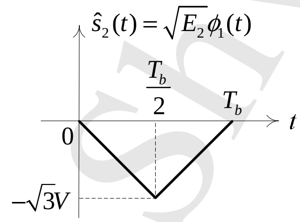


Figure 5.24

P5.14 (a) The two signals are (time) orthogonal with equal energy $E = \frac{A^2T}{2}$ (joules). The orthogonal basis is $\phi_1(t) = \frac{s_1(t)}{\sqrt{E}}$ & $\phi_2(t) = \frac{s_2(t)}{\sqrt{E}}$.

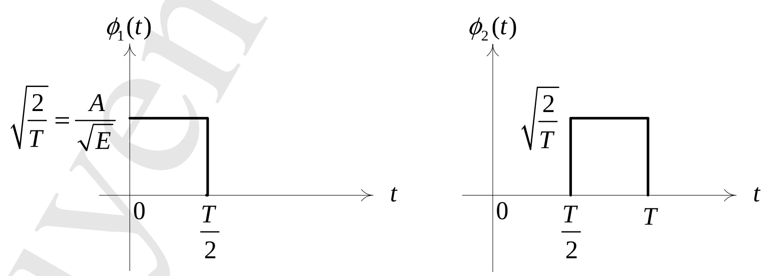


Figure 5.25

(b)+(c) See above plots.

(d) $P[\text{bit error}] = Q\left(\sqrt{\frac{E}{N_0}}\right) = Q\left(\sqrt{\frac{A^2T_b}{2N_0}}\right) = Q\left(\sqrt{\frac{1}{2r_b\times 10^{-3}}}\right) = 10^{-6}$

$\Rightarrow \frac{1}{\sqrt{2}}\sqrt{\frac{1}{2r_b\times 10^{-3}}} = \text{erfc}^{-1}\left(2\times 10^{-6}\right) \Rightarrow r_b = \frac{1}{(4\times 10^{-3})\left[\text{erfc}^{-1}(2\times 10^{-6})\right]^2}$.

(e) Rotate the orthogonal basis by 45^0 $\curvearrowleft$ and look at projection along $\phi_2^R(t) = \frac{s_2(t)-s_1(t)}{\text{normalizing factor}}$.

(f) We want $s_2(t)$ as far away from $s_1(t)$ as possible. Therefore make it antipodal, i.e., $s_2(t) = -s_1(t)$.

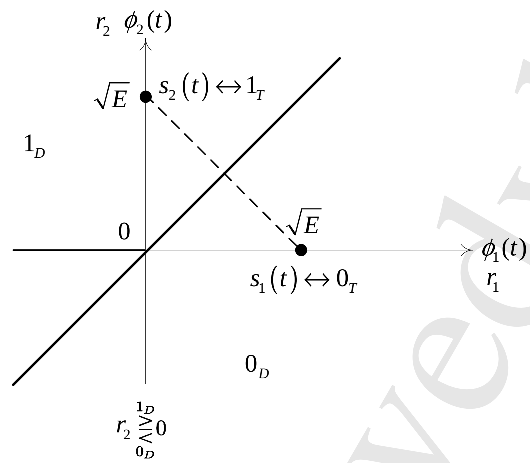


Figure 5.26

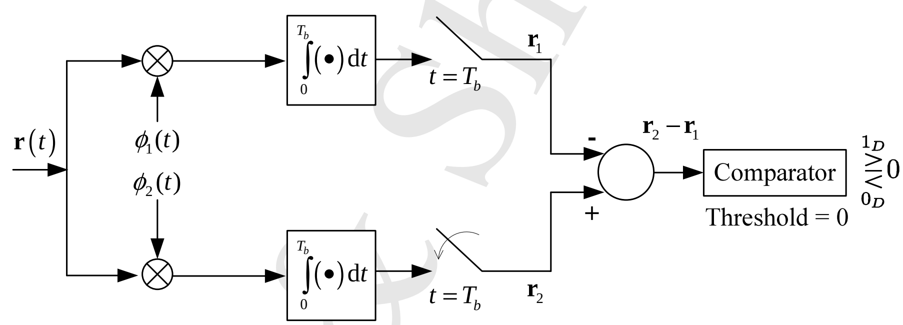


Figure 5.27

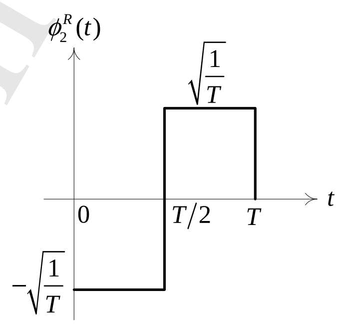


Figure 5.28

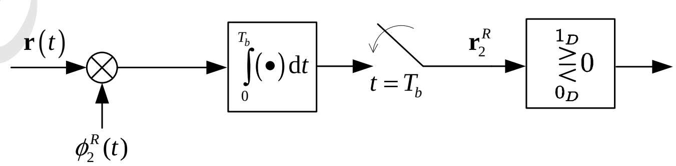


Figure 5.29

P5.15 The noise, $\mathbf{n}(t) = \mathbf{n}$, is simply a DC level but the amplitude of the level is random and is

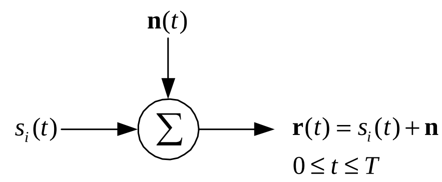


Figure 5.30: A binary communication system with additive noise.

Gaussian distributed with a probability density function given by

$$f_{\mathbf{n}}(n) = \frac{1}{\sqrt{2\pi N_0}} \exp\left(-\frac{n^2}{2N_0}\right) \tag{5.26}$$

The received signal in the first bit interval can be written as:

$$\mathbf{r}(t) = s_i(t) + \mathbf{n}(t) = s_i(t) + \mathbf{n},\ 1 \le t \le T_b \tag{5.27}$$

(a) The autocorrelation function of the noise $n(t)$ is

$$R_{\mathbf{n}}(\tau) = E\{\mathbf{n}(t)\mathbf{n}(t+\tau)\} = E\{\mathbf{n} \times \mathbf{n}\} = E\{\mathbf{n}^2\} = N_0,$$

Since $R_{\mathbf{n}}(\tau) = N_0$ for every τ, it follows that any two noise samples, no matter how far they are, are correlated. In fact, because the noise is modeled as a random DC level, knowing the value of the noise at one time instant gives a full knowledge about the noise at any other time instants.

The power spectral density of the noise is

$$S_{\mathbf{n}}(f) = \mathcal{F}\{R_{\mathbf{n}}(\tau)\} = \mathcal{F}\{N_0\} = N_0\delta(f)$$

Note that $S_{\mathbf{n}}(f) \neq 0$ only when $f = 0$, or $S_{\mathbf{n}}(f) = 0$ for $f \neq 0$. This is of course due to the noise being modeled as a random DC level!

(b) Consider the following signal set and the receiver (as usual, $s_1(t)$ is used for the transmission of bit "0" and $s_2(t)$ is for the transmission of bit "1"). In essence, the system

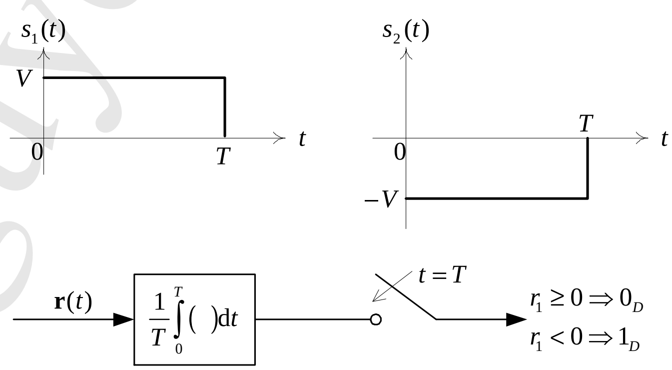


Figure 5.31

uses antipodal signalling. The decision variable is $\mathbf{r}_1 = \frac{1}{T}\int_0^T \mathbf{r}(t)\mathrm{d}t$. More specifically,

$$\begin{cases} \mathbf{r}_1 = V + \mathbf{n}_1 & \text{if "0" was transmitted} \\ \mathbf{r}_1 = -V + \mathbf{n}_1 & \text{if "1" was transmitted} \end{cases}$$

where the noise $\mathbf{n}_1$ is Gaussian, zero-mean and has a variance of N_0/T. Therefore the error probability of the receiver is:

$$P[\text{error}] = Q\left(\frac{2V/2}{\sqrt{N_0/T}}\right) = Q\left(\sqrt{\frac{V^2T}{N_0}}\right)$$

(c) Consider the signal set plotted in Fig. 5.32.

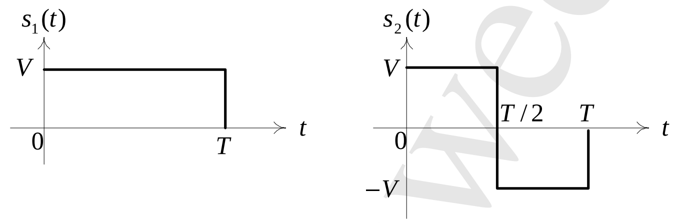


Figure 5.32: A signal set.

To achieve a probability of zero, the receiver needs to remove the noise completely. There are many possibilities. Perhaps the simplest is to take 2 samples of $\mathbf{r}(t)$, at $\mathbf{r}(t_1)$ and $\mathbf{r}(t_2)$, where $0 < t_1 < T/2$ and $T/2 < t_2 < T$, i.e., they are in the first half and second half of the bit interval, respectively. The decision rule is as follows:

If sample $\mathbf{r}(t_1)$ equals $\mathbf{r}(t_2)$, then decide $s_1(t)$.

If sample $\mathbf{r}(t_1)$ is not equal to $\mathbf{r}(t_2)$, then decide $s_2(t)$ (note that $\mathbf{r}(t_1) > \mathbf{r}(t_2)$).

When $s_1(t)$ is transmitted, $\mathbf{r}(t) = V + \mathbf{n}$, $0 \le t \le T \Rightarrow \mathbf{r}(t_1) = \mathbf{r}(t_2)$.

When $s_2(t)$ is transmitted, then

$$\mathbf{r}(t) = \begin{cases} V + \mathbf{n}, & 0 \le t \le T/2 \\ -V + \mathbf{n}, & T/2 \le t \le T \end{cases} \Rightarrow \mathbf{r}(t_1) \neq \mathbf{r}(t_2).$$

Therefore it is possible to determine whether $s_1(t)$ or $s_2(t)$ was transmitted perfectly. Another receiver implementation is as follows:

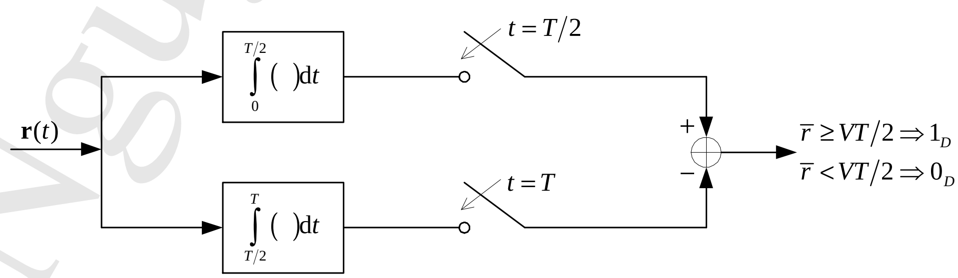


Figure 5.33

The noise process in this problem is an extreme example of colored noise. Using the signal space ideas developed in this chapter one can see that the noise lies along one axis only, namely that given by $\phi_1(t) = \frac{s_1(t)}{\sqrt{E}}$. It does not have any component along $\phi_2(t) = \frac{s_2(t)}{\sqrt{E}}$. This is illustrated in Fig. 5.34. This allows you to determine which signal is sent exactly. In this context, another receiver implementation is shown in Fig. 5.34.

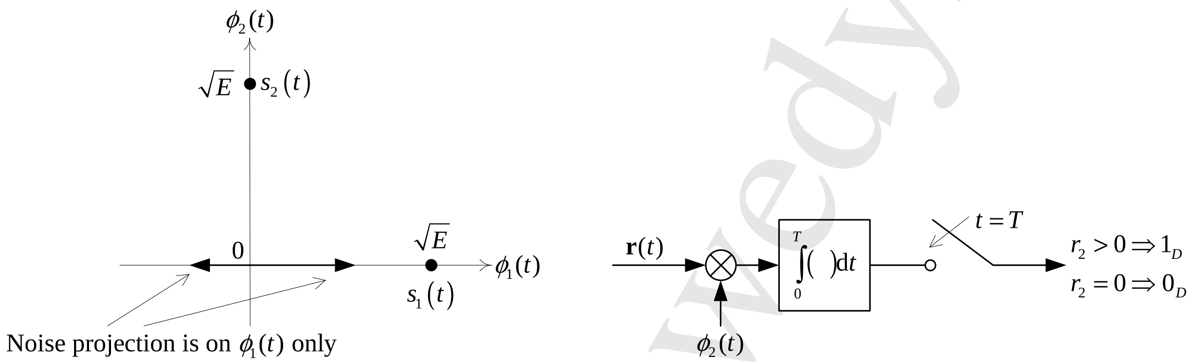


Figure 5.34: Noise projection and another receiver implementation.

P5.16 (*Matched Filter*)

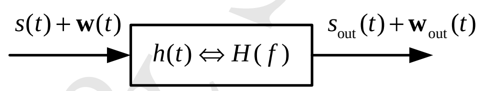


Figure 5.35

(a) The response of the filter to the input signal component can be written as

$$s_{\text{out}}(t) = \mathcal{F}^{-1}\{S(f)H(f)\} = \int_{-\infty}^{\infty} S(f)H(f)\mathrm{e}^{j2\pi ft}\mathrm{d}f$$

$$\Rightarrow \quad s_{\text{out}}(t_0) = \int_{-\infty}^{\infty} S(f)H(f)\mathrm{e}^{j2\pi ft_0}\mathrm{d}f \;\Rightarrow\; [s_{\text{out}}(t_0)]^2 = \left[\int_{-\infty}^{\infty} S(f)H(f)\mathrm{e}^{j2\pi ft_0}\mathrm{d}f\right]^2 \tag{5.28}$$

(b) Since the input noise is white with power spectral density (PSD) $N_0/2$, it follows that the PSD of the output noise is $\frac{N_0}{2}|H(f)|^2$. The total power of the output noise $P_{\mathbf{w}_{\text{out}}}$ is thus given by,

$$P_{\mathbf{w}_{\text{out}}} = \int_{-\infty}^{\infty} \frac{N_0}{2}|H(f)|^2\mathrm{d}f = \frac{N_0}{2}\int_{-\infty}^{\infty}|H(f)|^2\mathrm{d}f \tag{5.29}$$

From (5.28) and (5.29) the SNR_{out} can be written as,

$$\text{SNR}_{\text{out}} = \frac{[s_{\text{out}}(t_0)]^2}{P_{\mathbf{w}_{\text{out}}}} = \frac{\left[\int_{-\infty}^{\infty} S(f)H(f)\mathrm{e}^{j2\pi ft_0}\mathrm{d}f\right]^2}{\frac{N_0}{2}\int_{-\infty}^{\infty}|H(f)|^2\mathrm{d}f} \tag{5.30}$$

(c) Now $\int_{-\infty}^{\infty}[a(t)+\lambda b(t)]^2 \mathrm{d}t \geq 0$ (always). Considered as a quadratic in λ:

$$\alpha\lambda^2 + \beta\lambda + \gamma \geq 0 \tag{5.31}$$

with coefficients α, β and γ given as:

$$\alpha = \int_{-\infty}^{\infty} b^2(t)\mathrm{d}t \tag{5.32}$$

$$\beta = 2\int_{-\infty}^{\infty} a(t)b(t)\mathrm{d}t\lambda \tag{5.33}$$

$$\gamma = \int_{-\infty}^{\infty} a^2(t)\mathrm{d}t \tag{5.34}$$

The above inequality requires that the quadratic $\alpha\lambda^2+\beta\lambda+\gamma$ cannot have any real roots, otherwise it was cross the real axis for some value of λ (actually at two points) and would be less than 0. From the formula for the roots this means that $\Delta = \beta^2 - 4\alpha\gamma < 0$, which, upon substituting for $\alpha,\ \beta,\ \gamma$ is

$$\left[\int_{-\infty}^{\infty} a(t)b(t)\mathrm{d}t\right]^2 \leq \int_{-\infty}^{\infty} a^2(t)\mathrm{d}t \int_{-\infty}^{\infty} b^2(t)\mathrm{d}t \tag{5.35}$$

which is known as Cauchy-Schwartz inequality. The equality holds when $a(t) = Kb(t)$, where K is an arbitrary (real) scaling factor, i.e., the shapes of $a(t)$ and $b(t)$ are the same.

(d) Using Parseval's relationship, $\displaystyle\int_{-\infty}^{\infty} a(t)b(t)\mathrm{d}t = \int_{-\infty}^{\infty} A(f)B^*(f)\mathrm{d}f$ (see P2.26, page 69), 5.35 becomes

$$\left[\int_{-\infty}^{\infty} A(f)B^*(f)\mathrm{d}f\right]^2 \leq \int_{-\infty}^{\infty} |A(f)|^2\mathrm{d}f \int_{-\infty}^{\infty} |B(f)|^2(f)\mathrm{d}f \tag{5.36}$$

where equality holds when $A(f) = KB(f)$, K an arbitrary (real) scaling constant.

(e) Now let $A(f) = H(f)$ and $B^*(f) = S(f)\mathrm{e}^{j2\pi f t_0}$, then (5.36) becomes:

$$\left[\int_{-\infty}^{\infty} H(f)S(f)\mathrm{e}^{j2\pi f t_0}\mathrm{d}f\right]^2 \leq \int_{-\infty}^{\infty} |H(f)|^2\mathrm{d}f \int_{-\infty}^{\infty} |S(f)|^2(f)\mathrm{d}f \tag{5.37}$$

Combining (5.30) and (5.37) gives:

$$\mathrm{SNR_{out}} \leq \frac{\int_{-\infty}^{\infty} |H(f)|^2\mathrm{d}f \int_{-\infty}^{\infty} |S(f)|^2(f)\mathrm{d}f}{\frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2\mathrm{d}f} = \int_{-\infty}^{\infty} \frac{|S(f)|^2}{N_0/2}\mathrm{d}f \tag{5.38}$$

(f) The maximum value of $\mathrm{SNR_{out}}$ is achieved by choosing $A(f) = KB(f)$, i.e., $H(f) = KS^*(f)\mathrm{e}^{-j2\pi f t_0}$. Finally, the corresponding impulse response of the filter is

$$h(t) = Ks(t_0 - t) \tag{5.39}$$

where K is any real number. Note that to arrive at the above relation, the following two properties of the Fourier transform have been used ($x(t)$ is a real function here):

1. Time shift: $x(t-t_0) \Leftrightarrow X(f)\mathrm{e}^{-j2\pi f t_0}$.

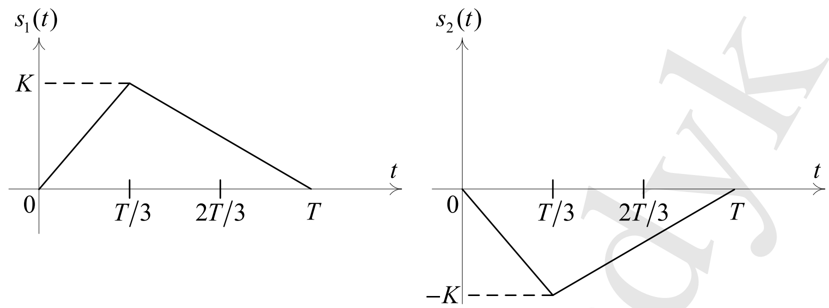


Figure 5.36: Signal set matched to the filter's impulse response.

2. Time reversal: $x(-t) \;\Leftrightarrow\; X^*(f)$.

In our situation we take t_0 to be T_b, the bit duration. Therefore $h(t) = Ks(T_b - t)$.

P5.17 (a) The signal set would be $\pm Kh(T_b - t)$ where K is a scaling factor that determines the transmitted energy (or average power). They plot as in Fig. 5.36. This makes $h(t)$ a matched filter to the signal set.

(b) Assume $K = 1$. The transmitted energy of either signal is

$$E = \int_0^{T_b} s_1^2(t)\mathrm{d}t = \int_0^{T_b/3}\left(\frac{3}{T_b}t\right)^2 \mathrm{d}t + \int_{T_b/3}^{T_b}\left[-\frac{3}{2T_b}t + \frac{3}{2}\right]^2 \mathrm{d}t = \frac{T_b}{3} \quad \text{(joules)}$$

$$P[\text{bit error}] = Q\left(\sqrt{\frac{2E}{N_0}}\right) = Q\left(\sqrt{\frac{2T_b}{3N_0}}\right) = Q\left(\sqrt{\frac{2}{3r_b \times 10^{-9}}}\right) \le 10^{-3}$$

$$r_b \le \frac{2 \times 10^9}{3[Q^{-1}(10^{-3})]^2} = 6.98 \times 10^7 \text{ bits/second} = 69.8 \text{ Mbps}$$

P5.18 (*Antipodal signalling*)

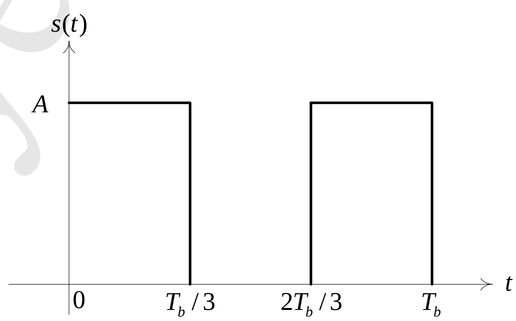


Figure 5.37

(a) The impulse response of the filter *matched* to $s(t)$ is

$$h(t) = s(T_b - t) = s(t) \tag{5.40}$$

where to arrive at the last equality we recognize that $s(t)$ is even about $t = T_b/2$. Therefore, the impulse response $h(t)$ looks exactly the same as $s(t)$.

(b) The output of the matched filter when $s(t)$ is applied at the input is

$$y(t) = s(t) * h(t) = \int_0^t s(\lambda)s(t-\lambda)\mathrm{d}\lambda$$
$$= \begin{cases} 0 & t < 0 \\ A^2 t & 0 \le t < T_b/3 \\ A^2(2T_b/3 - t) & T_b/3 \le t < 2T_b/3 \\ 2A^2(t - 2T_b/3) & 2T_b/3 \le t < T_b \\ 2A^2(4T_b/3 - t) & T_b \le t < 4T_b/3 \\ A^2(t - 4T_b/3) & 4T_b/3 \le t < 5T_b/3 \\ A^2(2T_b - t) & 5T_b/3 \le t < 2T_b \\ 0 & 6 \le t \end{cases} \tag{5.41}$$

The output $y(t)$ is sketched in Fig. 5.38. Observe that the output peaks at the sampling time $t = T_b = 3$. This is of course not a coincidence but due to the property of the matched filter, namely maximizing the signal-to-noise ratio at the sampling instant (see P5.16).

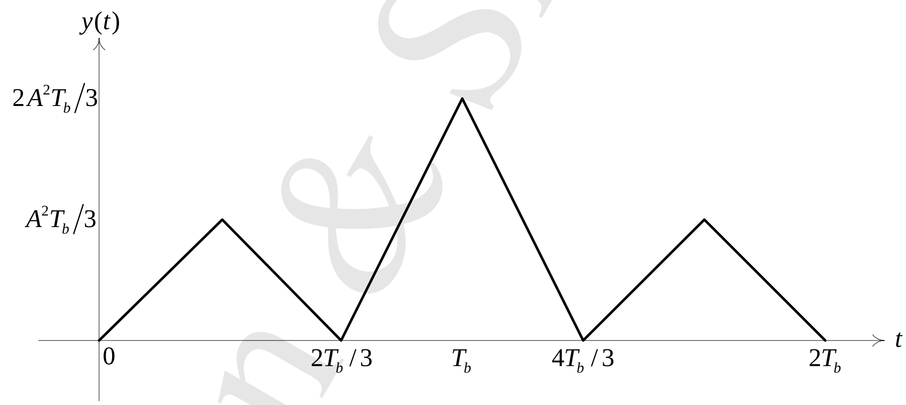


Figure 5.38: Output of the matched filter: only signal $s(t)$ is present at the input.

(c) The output of the matched filter when $-s(t)$ is applied at the input is the negative version of the waveform in Fig. 5.38. This is simply because $[-s(t)] * h(t) = -[s(t) * h(t)]$.

(d) To compute the SNR, the variance (i.e., average power) of the noise at the output of the filter at $t = T_b$ needs to be found. This can be accomplished as follows.

Approach 1: At the output of the matched filter and for $t = T_b$ the noise is

$$\begin{aligned} \mathbf{n} &= \int_0^{T_b} \mathbf{n}(\lambda)h(T_b - \lambda)\mathrm{d}\lambda \\ &= \int_0^{T_b} \mathbf{n}(\lambda)s(T_b - (T_b - \lambda))\mathrm{d}\lambda = \int_0^{T_b} \mathbf{n}(\lambda)s(\lambda)\mathrm{d}\lambda \end{aligned} \tag{5.42}$$

The variance of the noise is

$$\begin{aligned} \sigma_{\mathbf{n}}^2 &= E\left[\int_0^{T_b}\int_0^{T_b} \mathbf{n}(\lambda)\mathbf{n}(v)s(\lambda)s(v)\mathrm{d}\lambda\mathrm{d}v\right] = \int_0^{T_b}\int_0^{T_b} E\{\mathbf{n}(\lambda)\mathbf{n}(v)\}s(\lambda)s(v)\mathrm{d}\lambda\mathrm{d}v \\ &= \frac{N_0}{2}\int_0^{T_b}\int_0^{T_b} \delta(\lambda - v)s(\lambda)s(v)\mathrm{d}\lambda\mathrm{d}v = \frac{N_0}{2}\int_0^{T_b} s^2(\lambda)\mathrm{d}\lambda = N_0A^2\frac{T_b}{3} \end{aligned} \tag{5.43}$$

Approach 2: The PSD of the noise at the output of the matched filter is $\frac{N_0}{2}|H(f)|^2 = \frac{N_0}{2}|S(f)|^2$. Then the noise power is

$$\sigma_{\mathbf{n}}^2 = \int_{-\infty}^{\infty} \frac{N_0}{2}|S(f)|^2 \mathrm{d}f = \frac{N_0}{2}\int_0^{T_b} s^2(\lambda)\mathrm{d}\lambda = N_0 A^2 \frac{T_b}{3} \tag{5.44}$$

The signal-to-noise ratio (SNR) at the output of the matched filter at the sampling instant is given by

$$\mathrm{SNR} = \frac{[\pm y(T_b)]^2}{\sigma_{\mathbf{n}}^2} = \frac{(2A^2T_b/3)^2}{N_0A^2\frac{T_b}{3}} = \frac{4A^2T_b}{3N_0} \tag{5.45}$$

(e) For antipodal signalling, the distance between the two signals is $d_{21} = 2\sqrt{E} = 2\sqrt{\frac{2A^2T_b}{3}}$. Therefore the probability of bit error is

$$P[\text{error}] = Q\left(\frac{d_{21}/2}{\sqrt{N_0/2}}\right) = Q\left(\frac{\sqrt{\frac{2A^2T_b}{3}}}{\sqrt{N_0/2}}\right) = Q\left(\sqrt{\frac{4A^2T_b}{3N_0}}\right) \tag{5.46}$$

Finally, it follows from (5.46) and (5.45) that

$$P[\text{error}] = Q\left(\sqrt{\mathrm{SNR}}\right) \tag{5.47}$$

Equation (5.47) holds for general antipodal signalling, i.e., regardless of what $s(t)$ is used. The relationship in (5.47) also clearly shows that for antipodal signalling over an AWGN channel **maximizing the SNR at the output of a receiving filter is the same as minimizing the probability of error!**

(f) Plot of $P[\text{error}]$ as a function of SNR is shown in Fig. 5.39. The minimum SNR to achieve a probability of error of 10^{-6} is

$$\mathrm{SNR} = [Q^{-1}(10^{-6})]^2 = 22.595 = 13.54 \text{ (dB)} \tag{5.48}$$

P5.19 (*Bandwidth*) Define W to be the bandwidth of the signal $s(t)$ if $\epsilon\%$ of the total energy of $s(t)$ is contained inside the band $[-W, W]$:

$$\frac{\int_{-W}^{W}|S(f)|^2\mathrm{d}f}{\int_{-\infty}^{\infty}|S(f)|^2\mathrm{d}f} = \frac{2\int_0^W |S(f)|^2\mathrm{d}f}{E} = \frac{\epsilon}{100} \tag{5.49}$$

(a) (i) Rectangular pulse: $s_{(\mathrm{i})}(t) = \sqrt{\frac{1}{T_b}}$, $0 \le t \le T_b$.

$$\begin{aligned} S_{(\mathrm{i})}(f) &= \frac{1}{\sqrt{T_b}}\int_0^{T_b} \mathrm{e}^{-j2\pi ft}\mathrm{d}t = \frac{T_b}{\sqrt{T_b}}\mathrm{e}^{-j\pi fT_b}\frac{\mathrm{e}^{j\pi fT_b} - \mathrm{e}^{-j\pi fT_b}}{2j\pi fT_b} \\ &= \sqrt{T_b}\mathrm{e}^{-j\pi fT_b}\frac{\sin(\pi fT_b)}{(\pi fT_b)}. \end{aligned}$$

Then

$$\left|S_{(\mathrm{i})}(f)\right|^2 = T_b\frac{\sin^2(\pi fT_b)}{\pi^2(fT_b)^2}.$$

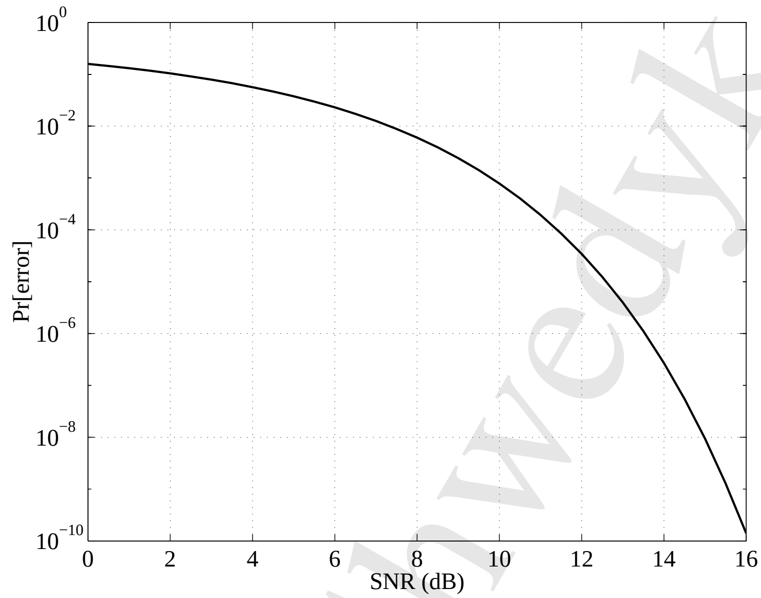


Figure 5.39: Plot of $P[\text{error}]$ as a function of SNR.

(ii) Half-sine: $s_{(\text{ii})}(t) = \sqrt{\frac{2}{T_b}}\sin\left(\frac{\pi t}{T_b}\right)$, $0 \le t \le T_b$. To find $S_{(\text{ii})}(f)$, write $s_{(\text{ii})}(t)$ as $s_{(\text{ii})}(t) = \sqrt{2}s_{(\text{i})}(t)\sin\left(2\pi\left(\frac{1}{2T_b}\right)\right)$. Then $S_{(\text{ii})}(f) = \sqrt{2}S_{(\text{i})}(f) * \mathcal{F}\left\{\sin\left(2\pi\left(\frac{1}{2T_b}\right)\right)\right\}$. But

$$\mathcal{F}\left\{\sin\left(2\pi\left(\frac{1}{2T_b}\right)\right)\right\} = \frac{\delta(f-1/2T_b)-\delta(f+1/2T_b)}{2j} = S_{\sin}(f)$$

and

$$S_{(\text{i})}(f) * S_{\sin}(f) = \int_{-\infty}^{\infty} S_{(\text{i})}(\lambda)S_{\sin}(f-\lambda)\mathrm{d}\lambda$$

$$= \frac{\sqrt{T_b}}{2j}\int_{-\infty}^{\infty} \mathrm{e}^{-j\pi\lambda T_b}\frac{\sin(\pi\lambda T_b)}{(\pi\lambda T_b)}\left[\frac{\delta(f-\lambda-1/2T_b)-\delta(f-\lambda+1/2T_b)}{2j}\right]\mathrm{d}\lambda$$

Using the sifting property of the impulse function we have

$$S_{(\text{ii})}(f) = \frac{\sqrt{T_b}\sqrt{2}}{2j}$$

$$\left[\mathrm{e}^{-j\pi\left(f-\frac{1}{2T_b}\right)T_b}\frac{\sin\left(\pi\left(f-\frac{1}{2T_b}\right)T_b\right)}{\pi\left(f-\frac{1}{2T_b}\right)T_b} - \mathrm{e}^{-j\pi\left(f+\frac{1}{2T_b}\right)T_b}\frac{\sin\left(\pi\left(f+\frac{1}{2T_b}\right)T_b\right)}{\pi\left(f+\frac{1}{2T_b}\right)T_b}\right]$$

Now $\mathrm{e}^{-j\pi\left(f-\frac{1}{2T_b}\right)T_b} = \mathrm{e}^{-j\pi fT_b}\mathrm{e}^{j\pi/2} = j\mathrm{e}^{-j\pi fT_b}$ and $\mathrm{e}^{-j\pi\left(f+\frac{1}{2T_b}\right)T_b} = -j\mathrm{e}^{-j\pi fT_b}$,

$\sin\left(\pi\left(f \pm \frac{1}{2T_b}\right)T_b\right) = \sin\left(\pi f T_b \pm \frac{\pi}{2}\right) = \pm\cos(\pi f T_b)$. Therefore

$$\begin{aligned} S_{(\text{ii})}(f) &= \frac{\sqrt{T_b}\sqrt{2}}{2j}\frac{j\mathrm{e}^{-j\pi f T_b}}{\pi T_b}\cos(\pi f T_b)\underbrace{\left[\frac{-1}{f-\frac{1}{2T_b}}+\frac{1}{f+\frac{1}{2T_b}}\right]}_{=\frac{-1}{T_b\left(f^2-\frac{1}{4T_b^2}\right)}=-\frac{4T_b^2}{T_b(4f^2T_b^2-1)}} \\ &= -\frac{2\sqrt{2}\sqrt{T_b}}{\pi}\frac{\cos(\pi f T_b)}{(4f^2T_b^2-1)}\mathrm{e}^{-j\pi f T_b} \end{aligned}$$

Then

$$\left|S_{(\text{ii})}(f)\right|^2 = \frac{8T_b}{\pi^2}\frac{\cos^2(\pi f T_b)}{(4(fT_b)^2-1)^2}. \tag{5.50}$$

(iii) Raised cosine: $s_{(\text{iii})}(t) = \sqrt{\frac{2}{3T_b}}\left[1-\cos\left(\frac{2\pi t}{T_b}\right)\right]$, $0 \le t \le T_b$. It can be written as $s_{(\text{iii})}(t) = \sqrt{\frac{2}{3}}\left[s_{(\text{i})}(t) - s_{(\text{i})}(t)\cos\left(\frac{2\pi t}{T_b}\right)\right]$. Therefore,

$$S_{(\text{iii})}(f) = \sqrt{\frac{2}{3}}\left[S_{(\text{i})}(f) - S_{(\text{i})}(f) * \mathcal{F}\left\{\cos\left(\frac{2\pi t}{T_b}\right)\right\}\right]$$

But $\mathcal{F}\left\{\cos\left(\frac{2\pi t}{T_b}\right)\right\} = \frac{\delta(f-1/T_b)+\delta(f+1/T_b)}{2}$. Again the impulse functions just sift (or if you wish shift) the spectrum of $S_{(\text{i})}(f)$ to $S_{(\text{i})}\left(f-\frac{1}{T_b}\right)$ and $S_{(\text{i})}\left(f+\frac{1}{T_b}\right)$. Therefore

$$S_{(\text{iii})}(f) = \sqrt{\frac{2}{3}}\left[S_{(\text{i})}(f) - \frac{S_{(\text{i})}\left(f-\frac{1}{T_b}\right)+S_{(\text{i})}\left(f+\frac{1}{T_b}\right)}{2}\right]$$

Doing all the necessary algebra, one obtains:

$$S_{(\text{iii})}(f) = \frac{\sqrt{2T_b}}{\sqrt{3}\pi}\frac{\sin(\pi f T_b)}{(fT_b)[1-(fT_b)^2]}\mathrm{e}^{-j\pi f T_b}. \tag{5.51}$$

and

$$\left|S_{(\text{iii})}(f)\right|^2 = \frac{2T_b}{3\pi^2}\frac{\sin^2(\pi f T_b)}{(fT_b)^2[1-(fT_b)^2]^2}. \tag{5.52}$$

Substituting the above into (5.49) and changing the variable $\lambda = fT_b$ one has the following equations to solve for WT_b:

(i) Rectangular pulse:

$$2\int_0^W \frac{T_b\sin^2(\pi f T_b)}{(\pi f T_b)^2}\mathrm{d}f = 2\int_0^{WT_b}\frac{\sin^2(\pi\lambda)}{(\pi\lambda)^2}\mathrm{d}\lambda = \frac{\epsilon}{100}. \tag{5.53}$$

(ii) Half-sine:

$$2\int_0^W \frac{8}{\pi^2}\frac{T_b\cos^2(\pi f T_b)}{[1-4(fT_b)^2]^2}\mathrm{d}f = \frac{16}{\pi^2}\int_0^{WT_b}\frac{\cos^2(\pi\lambda)}{(1-4\lambda^2)^2}\mathrm{d}\lambda = \frac{\epsilon}{100} \tag{5.54}$$

(iii) Raised cosine:

$$2\int_0^W \frac{2T_b}{3\pi^2}\left[\frac{\sin(\pi f T_b)}{(fT_b)[1-(fT_b)^2]}\right]^2 \mathrm{d}f = \frac{4}{3\pi^2}\int_0^{WT_b}\frac{\sin^2(\pi\lambda)}{\lambda^2(1-\lambda^2)^2}\mathrm{d}\lambda = \frac{\epsilon}{100} \qquad (5.55)$$

(b) The above equations can only be solved numerically. Using the numerical integration routine in MATLAB (`quad` or `quad8` in MATLAB 5.3 and `quadl` in MATLAB 6.0), the following table is obtained.

Table 5.1: Values of WT_b

ϵ%	Rectangular	Half-Sine	Raised Cosine
90.0%	0.8487	0.7769	0.9501
95.0%	2.0740	0.9116	1.1146
99.0%	10.2860	1.1820	1.4093
99.9%	31.1677	2.7355	1.7290

Note that the bandwidth is normalized in terms of the signalling rate $r_b = 1/T_b$ (bits/second). Asymptotically, as f becomes very large, the energy spectral densities behave as follows: Rectangular: $1/f^2$; Half-sine: $1/f^4$ and Raised cosine: $1/f^6$.

Thus the raised cosine decays the fastest. This is related to the number of times a signal can be differentiated before impulses appear. In the above, impulses appear in the rectangular pulse after the first derivative, in the half-sine after the second derivative and in the raised cosine after the third derivative. Thus one would feel that the raised cosine is the smoothest, then the half-sine and finally the rectangular pulse. This agrees with the plot of all three signals in Figure 5.40.

In the frequency domain, this translates to the smoother signal occupies the lesser amount of frequency axis (though this may also depends on the value of ϵ used for bandwidth definition. Note from Table 5.1 that though the raised cosine's spectral density decays (asymptotically) the fastest, it does not start to occupying "less frequency axis" until the energy bandwidth criterion is 99.9%, a value that is probably not of engineering interest. A criterion of 95% or 99% would be more realistic values for engineering design and analysis.

Figures 5.41 and 5.42 plot the energy spectral densities of all three signals in linear and logarithmic scales, respectively. It can be seen that the faster the density decays with f, the wider the main lobe. This is reasonable because all three signals have the same energy (of 1 joule). Thus the energy of the respective signals has to wide up someplace on the frequency axis. If not at the higher frequency then the main lobe will do.

P5.20 (a) When the two bits are equally likely (uniform source), the error performance of a digital binary communications system only depends on the Euclidean distance between the two signals. This distance is $2\sqrt{E}$ for antipodal signalling, where $\sqrt{E}$ is the energy of $s(t)$.

It follows that, for the two communication systems to have the same error performance, the energies of the two waveforms in Figure 5.43 have to be the same. That is:

$$A^2T = \frac{B^2T}{3} \;\Rightarrow\; \boxed{B = A\sqrt{3}}$$

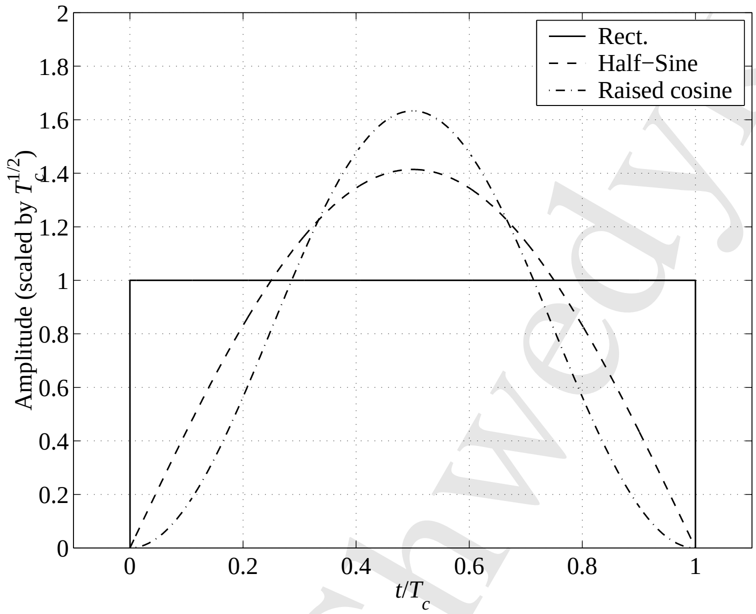


Figure 5.40: Plot of three signals in the time domain.

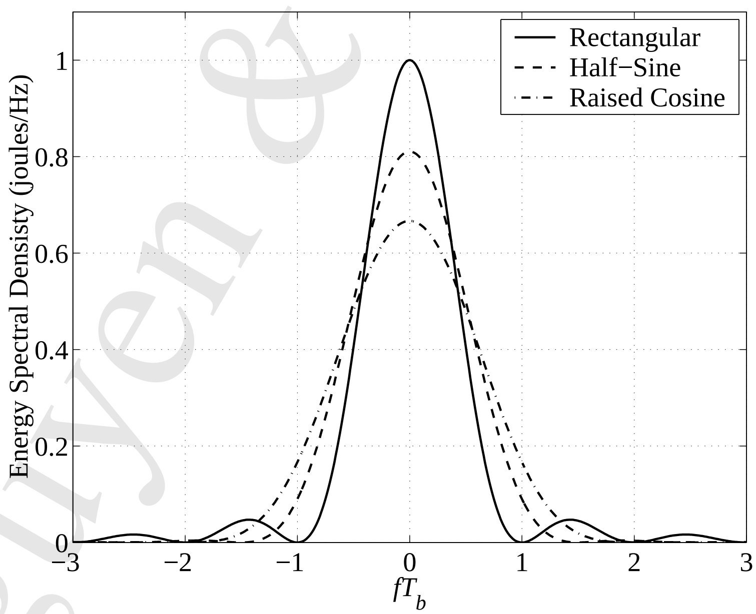


Figure 5.41: Plot of Energy Spectral Densities of three signals.

(b) Define W to be the bandwidth of the signal $s(t)$ if 95% of the total energy of $s(t)$ is contained inside the band $[-W, W]$. The bandwidth of the rectangular signal has been found in P5.19 to be $W_{(\mathrm{i})} = 2.07/T$.

For the triangular signal, to find the bandwidth $W_{(\mathrm{ii})}$ we first find $S_{(\mathrm{ii})}(f)$. To this end, differentiate $s_{(\mathrm{ii})}(t)$ twice, find the Fourier transform of the resulting signal and divide

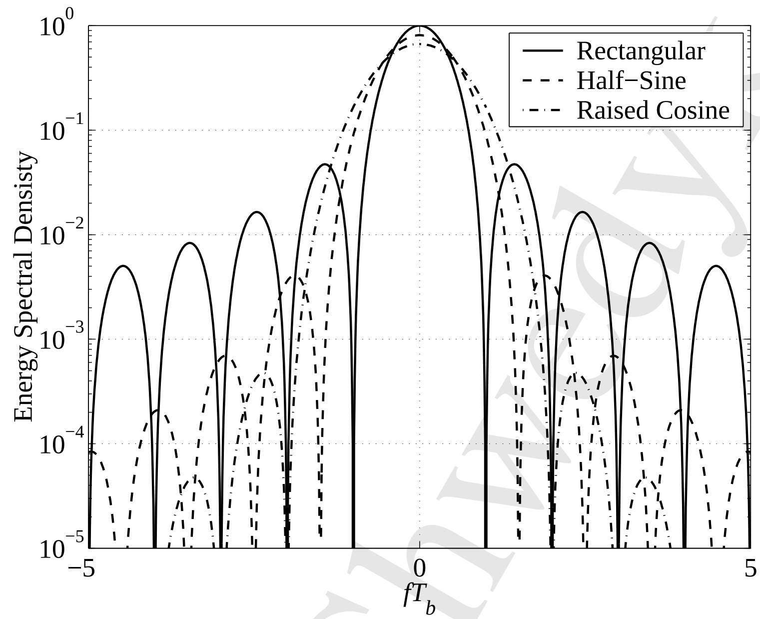


Figure 5.42: Plot of Energy Spectral Densities of three signals.

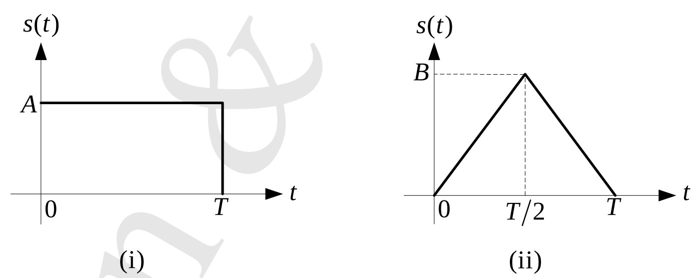


Figure 5.43: Two waveforms for $s(t)$.

by $(j2\pi f)^2$.

Since

$$\frac{\mathrm{d}^2 s_{\text{(ii)}}(t)}{\mathrm{d}t^2} = \frac{2B}{T}\delta(t) - \frac{4B}{T}\delta\left(t - \frac{T}{2}\right) + \frac{2B}{T}\delta\left(t - T\right)$$

and

$$\mathcal{F}\left\{\frac{\mathrm{d}^2 s_{\text{(ii)}}(t)}{\mathrm{d}t^2}\right\} = \frac{2B}{T} - \frac{4B}{T}\mathrm{e}^{-j\pi f T} + \frac{2B}{T}\mathrm{e}^{-j2\pi f T}$$

$$\begin{aligned}
S_{\text{(ii)}}(f) &= \frac{\mathcal{F}\left\{\frac{\mathrm{d}^2 s_{\text{(ii)}}(t)}{\mathrm{d}t^2}\right\}}{(j2\pi f)^2} = \frac{\frac{2B}{T}\left(1 + \mathrm{e}^{-j2\pi f T}\right) - \frac{4B}{T}\mathrm{e}^{-j\pi f T}}{(j2\pi f)^2} \\
&= \frac{\frac{4B}{T}\mathrm{e}^{-j\pi f T}\left(\frac{\mathrm{e}^{j\pi f T} + \mathrm{e}^{-j\pi f T}}{2}\right) - \frac{4B}{T}\mathrm{e}^{-j\pi f T}}{(j2\pi f)^2} \\
&= \frac{4B}{T}\frac{\cos(\pi f T) - 1}{(j\pi f)^2}
\end{aligned}$$

Then

$$\left|S_{(\text{ii})}(f)\right|^2 = 16B^2\frac{(\cos(\pi fT)-1)^2}{\pi^2 f^2T^2} \Rightarrow \frac{\left|S_{(\text{ii})}(f)\right|^2}{E_{(\text{ii})}} = \frac{48T}{\pi^2}\frac{(\cos(\pi fT)-1)^2}{f^2T^2}$$

So the equation to find $W_{(\text{ii})}$ is

$$\begin{aligned} &\frac{96T}{\pi^2}\int_0^{W_{(\text{ii})}} \frac{(\cos(\pi fT)-1)^2}{f^2T^2}\mathrm{d}f = \frac{\epsilon}{100} = 0.95 \\ \Leftrightarrow\quad &\frac{96}{\pi^2}\int_0^{W_{(\text{ii})}T} \frac{(\cos(\pi\lambda)-1)^2}{\lambda^2}\mathrm{d}\lambda = 0.95 \end{aligned}$$

which gives $W_{(\text{ii})} = 1.00/T$.

Since both systems have the same bit rate of 2 Mbps, the bit duration in both system is $T = 1/(2\times 10^6) = 0.5\times 10^{-6}$ seconds.

The required bandwidths of the two systems are therefore $W_{(\text{i})} = 2.07/(0.5\times 10^{-6}) = 4.14\times 10^6$ Hz $= 4.14$ MHz and $W_{(\text{ii})} = 1.00/(0.5\times 10^{-6}) = 2.00\times 10^6$ Hz $= 1.82$ MHz.

Clearly, when the two systems have the same error performance, system-(ii) is preferred since it requires less transmission bandwidth.

(c) Consider the system that uses $s(t)$ in Fig. 5.43-(i). Then the error probability is

$$P[\text{error}] = Q\left(\sqrt{\frac{2E}{N_0}}\right) = Q\left(\sqrt{\frac{2A^2T}{N_0}}\right) = 10^{-6}$$

$$\Rightarrow\quad \frac{2A^2T}{N_0} = [Q^{-1}(10^{-6})]^2 = 4.75^2$$

$$\Rightarrow\quad A = \left(\frac{4.75^2 N_0}{2T}\right)^{1/2} = \left(\frac{4.75^2\times 2\times 10^{-8}}{2\times 0.5\times 10^{-6}}\right)^{1/2} = 0.6718 \text{ (volts)}$$

P5.21 Consider a diversity system in Fig. 5.44.

(a) The decision variable r can be written as:

$$\begin{aligned} \mathbf{r} &= K\mathbf{r}_A + \mathbf{r}_B = \pm(KV_A + V_B) + (K\mathbf{w}_A + \mathbf{w}_B) \\ &= \pm V + \mathbf{w} \end{aligned}$$

Note that the signal components always have the same sign in both branches. In the above one has

$$\begin{aligned} V = KV_A + V_B &: \quad \text{total signal component} \\ \mathbf{w} = K\mathbf{w}_A + \mathbf{w}_B &: \quad \text{total noise component} \end{aligned}$$

Both $\mathbf{w}_A$ and $\mathbf{w}_B$ are zero-mean, with variances σ_A^2 and σ_B^2 respectively; furthermore they are uncorrelated, thus, the noise $\mathbf{w}$ is zero-mean Gaussian random variable with variance $\sigma^2 = K^2\sigma_A^2 + \sigma_B^2$.

Rewrite the decision variable $\mathbf{r}$ as:

$$\begin{cases} \mathbf{r} = V + \mathbf{w} & \text{if “1” was transmitted} \\ \mathbf{r} = -V + \mathbf{w} & \text{if “0” was transmitted} \end{cases}$$

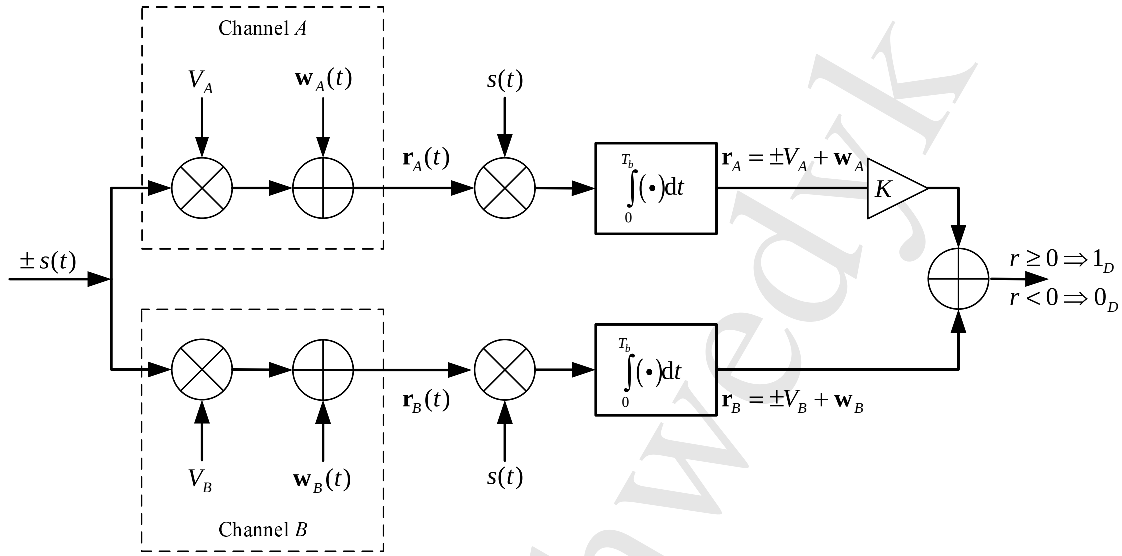


Figure 5.44: A Diversity System.

The probability of error for the decision rule $r \underset{0_D}{\overset{1_D}{\gtrless}} 0$ is

$$\Pr[\text{error}] = Q\left(\frac{V}{\sigma}\right) = Q\left(\frac{KV_A + V_B}{\sqrt{K^2\sigma_A^2 + \sigma_B^2}}\right) \tag{5.56}$$

(b) To minimize Pr[error] one needs to maximize

$$g(K) = \frac{V^2}{\sigma^2} = \frac{(KV_A + V_B)^2}{K^2\sigma_A + \sigma_B} = \frac{V_A^2}{\sigma_A^2}\frac{(K+\alpha)^2}{K^2+\beta}$$

where $\alpha = \dfrac{V_B}{V_A}$, $\beta = \dfrac{\sigma_B^2}{\sigma_A^2}$. Now

$$\begin{aligned}
\frac{\mathrm{d}g(K)}{\mathrm{d}K} &= \frac{V_A^2}{\sigma_A^2}\left[\frac{2(K+\alpha)(K^2+\beta) - (K+\alpha)^2(2K)}{(K^2+\beta)^2}\right] \\
&= \frac{2V_A^2}{\sigma_A^2}\frac{(K+\alpha)[K^2+\beta-(K+\alpha)K]}{(K^2+\beta)^2} \\
&= \frac{2V_A^2}{\sigma_A^2}\frac{(K+\alpha)(\beta-\alpha K)}{(K^2+\beta)^2} = 0
\end{aligned}$$

gives $K = -\alpha$ or $K = \dfrac{\beta}{\alpha} = \dfrac{\sigma_B^2}{\sigma_A^2}\dfrac{V_A}{V_B}$. Since $K > 0$, the optimum value of K to minimize Pr[error] is

$$K^* = \frac{\beta}{\alpha} = \frac{\sigma_B^2}{\sigma_A^2}\frac{V_A}{V_B}$$

(c) When $K = K^* = \frac{\beta}{\alpha}$ we have

$$
\begin{aligned}
g(K^*) &= \frac{(K^*V_A + V_B)^2}{(K^*)^2\sigma_A^2 + \sigma_B^2} = \frac{\left[\frac{\sigma_B^2 V_A}{\sigma_A^2 V_B} V_A + V_B\right]^2}{\left(\frac{\sigma_B^2}{\sigma_A^2}\frac{V_A}{V_B}\right)^2 \sigma_A^2 + \sigma_B^2} \\
&= \frac{(\sigma_B^2 V_A^2 + \sigma_A^2 V_B^2)^2}{\sigma_B^4\sigma_A^2 V_A^2 + \sigma_A^4\sigma_B^2 V_B^2} = \frac{(\sigma_B^2 V_A^2 + \sigma_A^2 V_B^2)^2}{\sigma_A^2\sigma_B^2(\sigma_B^2 V_A^2 + \sigma_A^2 V_B^2)} \\
&= \frac{\sigma_B^2 V_A^2 + \sigma_A^2 V_B^2}{\sigma_A^2\sigma_B^2} = \frac{V_A^2}{\sigma_A^2} + \frac{V_B^2}{\sigma_B^2}
\end{aligned} \tag{5.57}
$$

which means that

$$\frac{V^2}{\sigma^2} = \frac{V_A^2}{\sigma_A^2} + \frac{V_B^2}{\sigma_B^2} \tag{5.58}$$

Note that each ratio in the above is the signal-to-noise ratio (SNR). Thus (5.58) shows that the best possible SNR is the sum of the SNRs in the two different branches. This is also called the *maximal-ratio combining*.

The probability of error for maximal-ratio combining is:

$$\Pr[\text{error}] = Q\left(\sqrt{\frac{V_A^2}{\sigma_A^2} + \frac{V_B^2}{\sigma_B^2}}\right)$$

Remark: The diversity system considered in this assignment is a very popular model in mobile communications when multiple antenna are used at the transmitter/receiver.

(d) If K is set to 1 then the combining of two channels A and B is called *equal-gain combining*:

$$\Pr[\text{error}] = Q\left(\frac{V}{\sigma}\right) = Q\left(\frac{V_A + V_B}{\sqrt{\sigma_A^2 + \sigma_B^2}}\right) \tag{5.59}$$

Of course, as shown above, equal-gain combining (although simple) is inferior to maximal-ratio combining in terms of error performance.

P5.22 Let $s_1(t) \leftrightarrow 0_T$, $s_2(t) \leftrightarrow 1_T$.

(a) $f(r_j|1_T) = \frac{c}{2}\mathrm{e}^{-c|r_j - V|}$; $f(r_j|0_T) = \frac{c}{2}\mathrm{e}^{-c|r_j|}$. Since the samples are statistically independent, the conditional pdfs are:

$$f(r_1, \ldots, r_M|1_T) = \prod_{j=1}^{m} \frac{c}{2}\mathrm{e}^{-c|r_j - V|}; \quad f(r_1, \ldots, r_M|0_T) = \prod_{j=1}^{m} \frac{c}{2}\mathrm{e}^{-c|r_j|}.$$

(b) The log-likelihood ratio becomes:

$$\ln\left\{\frac{\prod_{j=1}^{m}\frac{c}{2}\mathrm{e}^{-c|r_j - V|}}{\prod_{j=1}^{m}\frac{c}{2}\mathrm{e}^{-c|r_j|}}\right\} = c\sum_{j=1}^{m}\left[|r_j| - |r_j - V|\right].$$

If $r_j < 0$ then $|r_j| = -r_j$ and $|r_j - V| = -(r_j - V)$. Therefore the sum becomes $-cm_1V$.

If $r_j < V$ then $|r_j| = r_j$ and $|r_j - V| = -(r_j - V)$. The sum is $2c\sum_{0<r_j<V} r_j - cm_2V$.

If $r_j > V$ then $|r_j| = r_j$ and $|r_j - V| = (r_j - V)$. The sum is cm_3V.

Therefore the sum becomes $2c\sum_{r_j\in(0,V)} r_j + cV(m_3 - m_1 - m_2)$.

But $m = m_3+m_2+m_1$, therefore the sum can be written as $2c\sum_{r_j\in(0,V)} r_j+cV(m_3-m)$.

Note: m is known to us beforehand but m_3 and of course r_j are not. They depend on the transmitted signal and the noise samples.

Further what if $r_j = 0$ or V – where should one put these samples? It should not matter but let us place them with the m_1 and m_3 count, respectively.

(c) The decision rule to maximize the error probability is:

$$\frac{f(r_1,\ldots,r_m|1_T)}{f(r_1,\ldots,r_m|0_T)} \underset{0_D}{\overset{1_D}{\gtrless}} \frac{P_1}{P_2}.$$

In deriving this there were no assumptions made regarding the conditional pdfs. After taking ln, we have

$$\sum_{r_j\in(0,V)} r_j + m_3V \underset{0_D}{\overset{1_D}{\gtrless}} mV + \frac{1}{c}\ln\frac{P_1}{P_2}.$$

Remark: The receiver derived is not necessarily the best there is for the given noise model but it is the optimum on (in terms of minimizing error probability) under the given conditions.

P5.23 (a) The observable is the number of received photons in the time interval T_b seconds. Therefore:

$$P[k \text{ photons emitted in } T_b \text{ seconds}|1_T] = \frac{[(\lambda_s+\lambda_n)T_b]^k \mathrm{e}^{-(\lambda_s+\lambda_n)T_b}}{k!},$$
$$P[k \text{ photons emitted in } T_b \text{ seconds}|0_T] = \frac{[\lambda_n T_b]^k \mathrm{e}^{-\lambda_n T_b}}{k!}.$$

The likelihood ratio test becomes:

$$\frac{[(\lambda_s+\lambda_n)T_b]^k}{[\lambda_n T_b]^k}\mathrm{e}^{-\lambda_s T_b} \underset{0_D}{\overset{1_D}{\gtrless}} \left(\frac{P_1}{P_2}\right)$$

Taking ln gives:

$$k \underset{0_D}{\overset{1_D}{\gtrless}} \ln\left(\frac{\lambda_n}{\lambda_s+\lambda_n}\right)\left[\lambda_s T_b + \ln\frac{P_1}{P_2}\right] \overset{P_1=P_2}{=} \ln\left(\frac{\lambda_n}{\lambda_s+\lambda_n}\right)\lambda_s T_b = T_h.$$

(b) Again assuming the bits are equally probable, one has

$$\begin{aligned}
P[\text{bit error}] &= \frac{1}{2}P[k \geq T_h|0_T] + \frac{1}{2}P[k < T_h|1_T] \\
&= \frac{1}{2}\left[P[k \geq T_h|0_T] + (1 - P[k \geq T_h|1_T])\right] = \frac{1}{2} + \frac{1}{2}P[k \geq T_h|0_T] - \frac{1}{2}P[k \geq T_h|1_T] \\
&= \frac{1}{2} + \frac{1}{2}\Bigg\{ \underbrace{\sum_{k=T_h}^{\infty}}_{\substack{\text{strictly}\\ \text{speaking,}\\ \text{should be}\\ \text{ceiling,}\\ \text{i.e., } \lceil T_h \rceil}} \underbrace{\left[\frac{(\lambda_n T_b)^k \mathrm{e}^{-\lambda_n T_b}}{k!} - \frac{[(\lambda_s+\lambda_n)T_b]^k \mathrm{e}^{-(\lambda_s+\lambda_n)T_b}}{k!}\right]}_{=\left[\frac{(\lambda_n T_b)^k}{k!}\mathrm{e}^{-\lambda_n T_b}\left(1-\left(1+\frac{\lambda_s}{\lambda_n}\right)^k \mathrm{e}^{-\lambda_s T_b}\right)\right]} \Bigg\}.
\end{aligned}$$

The average transmitted energy due to signal photons in one bit interval is $hf\lambda_s T_b$, while that due to background photons is $hf\lambda_n T_b$. Therefore the signal-to-noise ratio is $\text{SNR} = \frac{hf\lambda_s T_b}{hf\lambda_n T_b} = \frac{\lambda_s}{\lambda_n}$. So $P[\text{bit error}]$ depends on the SNR, but in a very complicated manner.

P5.24 (*Signal design for non-uniform sources*)

(a) Simple manipulations give:

$$\begin{aligned}
\sqrt{A} - \frac{B}{\sqrt{A}} &= \sqrt{\frac{(\hat{s}_{22}-\hat{s}_{12})^2}{2N_0}} - 0.5\ln\left(\frac{1-p}{p}\right)\sqrt{\frac{2N_0}{(\hat{s}_{22}-\hat{s}_{12})^2}} \\
&= \frac{\hat{s}_{22}-\hat{s}_{12}}{\sqrt{2N_0}} + \frac{\sqrt{N_0}\ln\left(\frac{p}{1-p}\right)}{\sqrt{2}(\hat{s}_{22}-\hat{s}_{12})}
\end{aligned}$$

$$\begin{aligned}
\frac{T-\hat{s}_{12}}{\sqrt{N_0/2}} &= \frac{0.5(\hat{s}_{12}+\hat{s}_{22}) - \hat{s}_{12} + \frac{1}{2}\frac{N_0}{\hat{s}_{22}-\hat{s}_{12}}\ln\left(\frac{p}{1-p}\right)}{\sqrt{N_0/2}} \\
&= \frac{\hat{s}_{22}-\hat{s}_{12}}{\sqrt{2N_0}} + \frac{\sqrt{N_0}\ln\left(\frac{p}{1-p}\right)}{\sqrt{2}(\hat{s}_{22}-\hat{s}_{12})}
\end{aligned}$$

It then follows that:

$$\sqrt{A} - \frac{B}{\sqrt{A}} = \frac{T-\hat{s}_{12}}{\sqrt{N_0/2}} \Rightarrow Q\left(\sqrt{A} - \frac{B}{\sqrt{A}}\right) = Q\left(\frac{T-\hat{s}_{12}}{\sqrt{N_0/2}}\right)$$

Similarly,

$$\sqrt{A} + \frac{B}{\sqrt{A}} = \frac{\hat{s}_{22}-T}{\sqrt{N_0/2}} = \frac{\hat{s}_{22}-\hat{s}_{12}}{\sqrt{2N_0}} + \frac{\sqrt{N_0}\ln\left(\frac{1-p}{p}\right)}{\sqrt{2}(\hat{s}_{22}-\hat{s}_{12})}$$

Since $Q(x) = 1 - Q(-x)$, then:

$$\Rightarrow Q\left(\frac{\hat{s}_{22}-T}{\sqrt{N_0/2}}\right) = \left[1 - Q\left(\frac{T-\hat{s}_{22}}{\sqrt{N_0/2}}\right)\right] = Q\left(\sqrt{A} + \frac{B}{\sqrt{A}}\right)$$

Thus, the error probability can be written as:

$$\Pr[\text{error}] \;=\; pQ\left(\sqrt{A}-\frac{B}{\sqrt{A}}\right)+(1-p)Q\left(\sqrt{A}+\frac{B}{\sqrt{A}}\right)$$

Since Pr[error] is a decreasing functions of $A \Rightarrow$ Pr[error] is minimized when A is maximized.

Using the energy constraint $\bar{E}_b = E_1 p + E_2(1-p)$ one can write A as the function of a single variable E_1 as follows:

$$A = \frac{E_2 - 2\rho\sqrt{E_1E_2}+E_1}{2N_0} = \frac{E_1 + \frac{\bar{E}_b - E_1 p}{1-p} - 2\rho\sqrt{\frac{\bar{E}_bE_1 - E_1^2 p}{1-p}}}{2N_0} \tag{5.60}$$

Thus, the design problem is to find the optimal value of E_1 in the range $0 \le E_1 \le \bar{E}_b/p$ (E_1 cannot exceed $\bar{E}_b/p$ because $E_2 \ge 0$) so that A is maximum.

(b) For the case of "orthogonal" signalling, $\rho = 0$ and A is simplified to

$$A \;=\; \frac{\bar{E}_b + E_1(1-2p)}{(1-p)2N_0}$$

It is straightforward to see from the above equation that the maximum value of A is $A_{\max} = \bar{E}_b/(2pN_0)$, which occurs when E_1 takes on its maximum value of $E_1 = \bar{E}_b/p$ (and thus $E_2 = 0$). This implies the following optimal signal set:

$$\begin{aligned} s_1(t) &= \sqrt{\bar{E}_b/p}\,\phi_1(t) \\ s_2(t) &= 0 \end{aligned}$$

i.e., we do not use the second basis function. In fact, we have on-off keying (OOK).

If we select $E_1 = E_2 = \bar{E}_b$ as is usually done ("conventional" design), then $A = (\bar{E}_b/N_0)$. Since $p \le 0.5$, $A_{\max}/A = (2p)^{-1} \ge 1$ with equality only when $p = 0.5$.

Note also that, for $p = 0.5$, the two solutions $E_1 = 2\bar{E}_b$, $E_2 = 0$ and $E_1 = E_2 = \bar{E}_b$, are special cases of the general solution (for orthogonal signal set) $E_1 = 2\bar{E}_b\cos^2\phi$, $E_2 = 2\bar{E}_b\sin^2\phi$, where ϕ is an arbitrary angle.

(c) Fig. 5.45 plots the error performance of the optimum design and the conventional design when $p = 0.1$. Observe that, at BER$=10^{-5}$, the gain in SNR by the optimum design is about 7.1 dB.

(d) *Nonorthogonal Signals.* The first and second derivatives of A in (5.60) with respect to E_1 are, respectively:

$$\frac{\mathrm{d}A}{\mathrm{d}E_1} \;=\; \frac{1}{2N_0}\left[\frac{1-2p}{1-p} - \frac{\rho\bar{E}_b(\bar{E}_b - 2E_1p)}{(1-p)^{0.5}(\bar{E}_bE_1 - E_1^2p)}\right] = 0 \tag{5.61}$$

$$\frac{\mathrm{d}^2A}{\mathrm{d}E_1^2} \;=\; \rho\frac{2p(\bar{E}_bE_1 - E_1^2p) + (\bar{E}_b - 2E_1p)^2}{2(1-p)^{0.5}(\bar{E}_bE_1 - E_1^2p)^{1.5}N_0}$$

It can be seen that the sign of the second derivative is determined by the sign of ρ. Thus, for $\rho > 0$, A is a convex function of E_1 (with a minimum), while for $\rho < 0$, A is a concave function of E_1 (with a maximum). Figure 5.46 plots the normalized quantity

$$A_n \;=\; A\frac{2N_0}{\bar{E}_b} = \frac{E_{1n}(1-2p)+1}{1-p} - 2\rho\left(\frac{E_{1n} - E_{1n}^2 p}{1-p}\right)^{0.5}$$

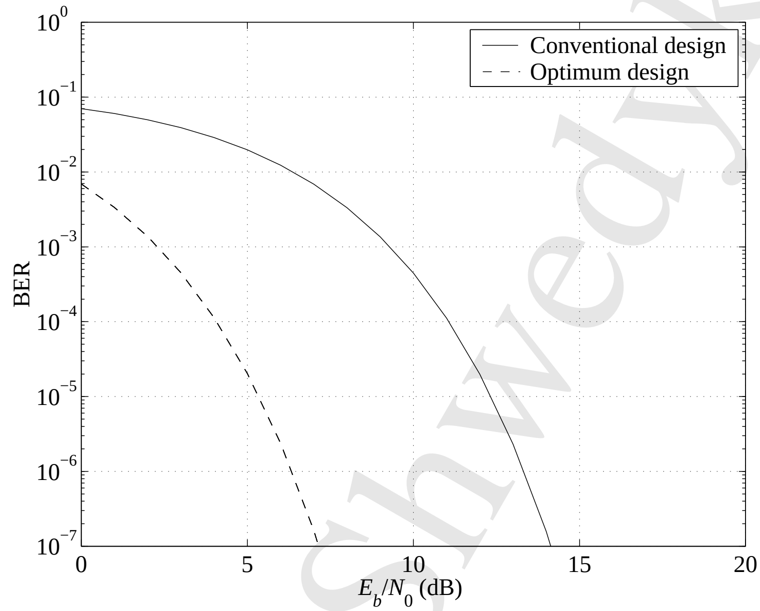


Figure 5.45: Error performance of the optimum and conventional designs.

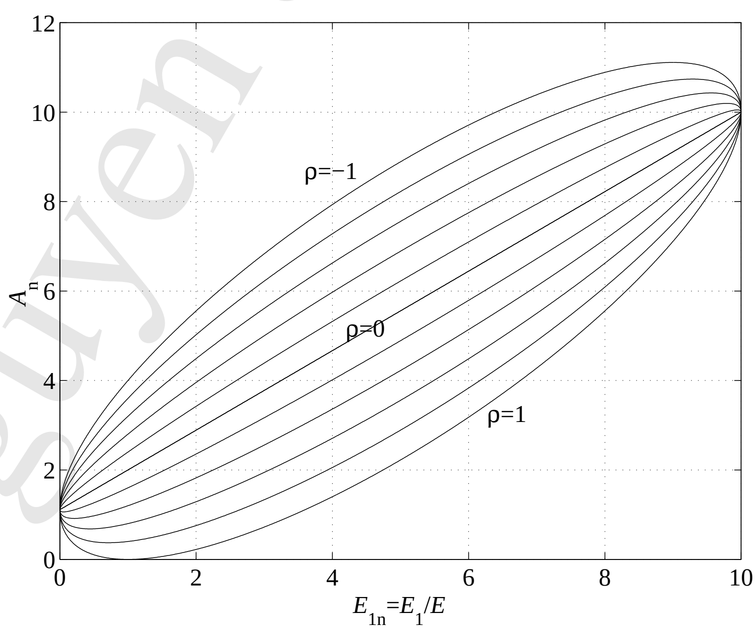


Figure 5.46: Variation of A_n with E_{1n} when ρ varies between -1 and $+1$ in the step of 0.1 ($p = 0.1$).

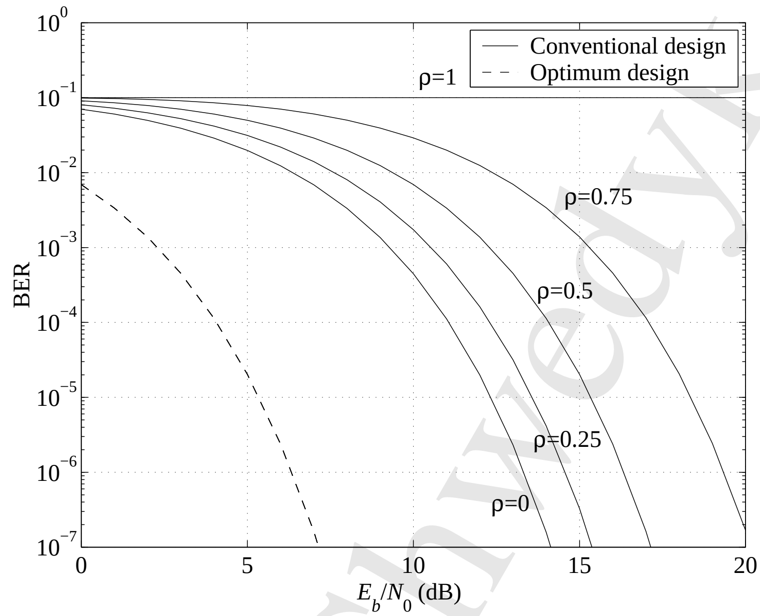


Figure 5.47: Variation of BER of the optimal and conventional systems for $\rho \geq 0$ ($p = 0.1$).

as a function of normalized E_1, $E_{1n} = E_1/\bar{E}_b$, for several values of ρ and $p = 0.1$. We can observe the convexity for $\rho > 0$ and concavity for $\rho < 0$.

Equating (5.61) to zero, the following two roots which give a minimum for $\rho > 0$ and a maximum for $\rho < 0$ are obtained:

$$
\begin{aligned}
E_1 &= \frac{\bar{E}_b}{2p}\left[1 - \frac{1-2p}{[1-4p(1-p)(1-\rho^2)]^{0.5}}\right], & \rho > 0 \\
E_{1,\text{opt}} &= \frac{\bar{E}_b}{2p}\left[1 + \frac{1-2p}{[1-4p(1-p)(1-\rho^2)]^{0.5}}\right], & \rho < 0
\end{aligned}
\tag{5.62}
$$

Thus for $\rho > 0$, the maximum of A_n occurs at the edge with $E_{1,\text{opt}} = \bar{E}_b/p$, $E_{2,\text{opt}} = 0$, $A_{\max} = \bar{E}_b/(2pN_0)$, which is OOK again.

Fig. 5.47 plots the BER as a function of $\bar{E}_b/N_0$ for optimal E_1 and E_2, and also for conventional system (with $E_1 = E_2 = \bar{E}_b$, hence, $A = \bar{E}_b/(1-\rho)/N_0$) for several values of nonnegative ρ and $p = 0.1$. We can see from this figure that for nonnegative ρ, the BER of the optimal system (OOK) is much better than that of the conventional system. For example, when the BER is 10^{-7}, the improvement in $\bar{E}_b/N_0$ is about 7.1 dB for $\rho = 0$ and is even greater for higher values of ρ.

For $\rho < 0$, the maximum of A occurs when $E_1 = E_{1,\text{opt}}$ of (5.62). From (5.62) and (5.60) one has:

$$
E_{2,\text{opt}} = \frac{\bar{E}_b}{2(1-p)}\left[1 + \frac{1-2p}{[1-4p(1-p)(1-\rho^2)]^{0.5}}\right] \tag{5.63}
$$

Substituting (5.62) and (5.63) in (5.60), we obtain the maximum value

$$
A_{\max} = \frac{\bar{E}_b}{2N_0}\left[\frac{1}{2p(1-p)}\left\{1 + \frac{(1-2p)^2}{[1-4p(1-p)(1-\rho^2)]^{0.5}}\right\} - \frac{2\rho|\rho|}{[1-4p(1-p)(1-\rho^2)]^{0.5}}\right]
$$

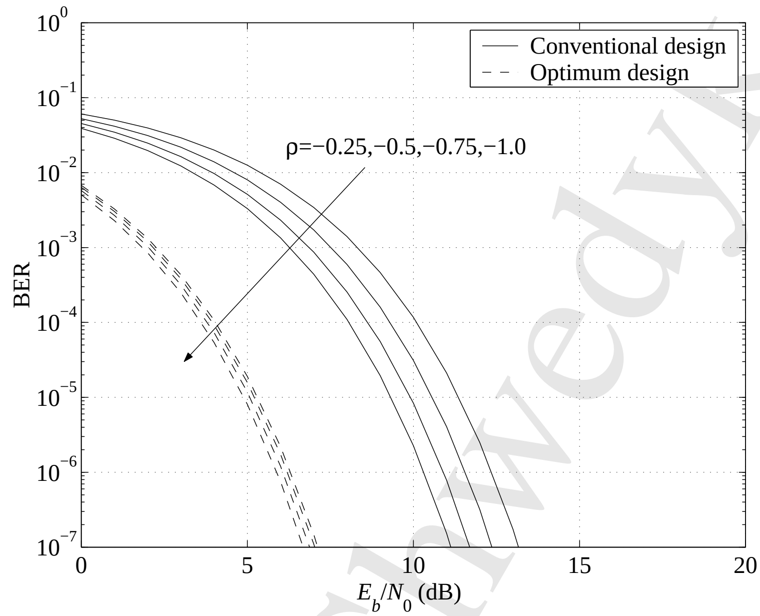


Figure 5.48: Variation of BER of the optimal and conventional systems for $\rho < 0$ ($p = 0.1$).

Fig. 5.48 plots the BER as a function of E_b/N_0 for optimal E_1 and E_2, and for the conventional system, for several values of negative ρ when $p = 0.1$. It can be seen from this figure that for negative values of ρ, there is also a significant improvement in performance relative to the conventional system. For example, when BER is 10^{-7}, the improvement in is about 4.5 dB for $\rho = -1$ and about 6.0 dB for $\rho = -0.25$.

Finally, if there is no constraint on the specific value of ρ (which should be the case when $\rho \neq 0$), it is not hard to see that the optimum signal set corresponds to $\rho = -1$.

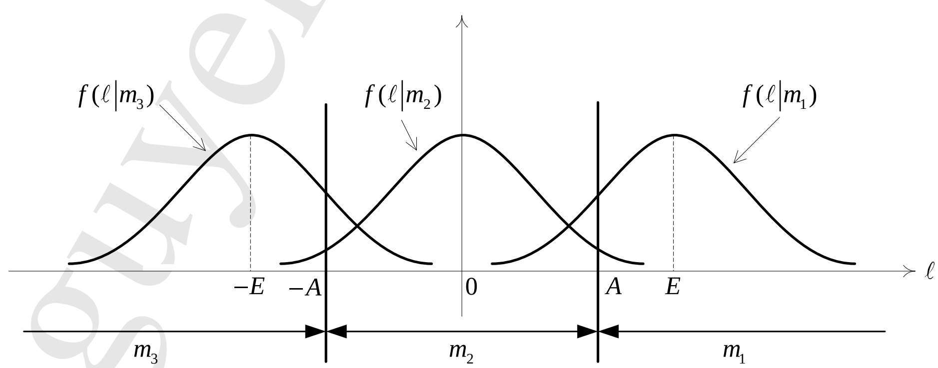


Figure 5.49

P5.25 (a) The decision variable can be expressed as:

$$\ell = \int_0^T \mathbf{r}(t)s(t)\mathrm{d}t = \begin{cases} E + \mathbf{w}, & \text{if } m_1 \text{ was sent} \\ \mathbf{w}, & \text{if } m_2 \text{ was sent} \\ -E + \mathbf{w}, & \text{if } m_3 \text{ was sent} \end{cases}$$

where E is the energy of $\pm s(t)$ and $\mathbf{w} = \int_0^T \mathbf{w}(t)s(t)\mathrm{d}t$ is zero-mean Gaussian random

variable with variance $\sigma^2 = \frac{N_0}{2}E$.
Thus, given m_1 was sent:

$$P[\text{error}|m_1] = P(\boldsymbol{\ell} < A) = 1 - Q\left(\frac{A-E}{\sigma}\right) = Q\left(\frac{E-A}{\sqrt{EN_0/2}}\right)$$

Given m_3 was sent:

$$P[\text{error}|m_3] = P(\boldsymbol{\ell} > -A) == Q\left(\frac{-A+E}{\sqrt{EN_0/2}}\right)$$

Given m_2 was sent:

$$P[\text{error}|m_2] = \Pr(\boldsymbol{\ell} < -A \text{ or } \boldsymbol{\ell} > A) = 2Q\left(\frac{A}{\sqrt{EN_0/2}}\right)$$

(b) The average probability of error when the a priori probabilities of the three signals are $P[m_1] = P[m_3] = \frac{1}{4}$ and $P[m_2] = \frac{1}{2}$ is given by:

$$\begin{aligned} P[\text{error}] &= \frac{1}{4}P[\text{error}|m_1] + \frac{1}{4}P[\text{error}|m_3] + \frac{1}{2}P[\text{error}|m_2] \\ &= \frac{1}{2}Q\left(\frac{-A+E}{\sqrt{EN_0/2}}\right) + Q\left(\frac{A}{\sqrt{EN_0/2}}\right) \end{aligned}$$

(c) Since $P[\text{error}] = f(A)$, to minimize $f(A)$ one needs to find A from $\frac{\mathrm{d}f(A)}{\mathrm{d}A} = 0$. Using the hint provided, one obtains:

$$\begin{aligned} \frac{\mathrm{d}Q\left(\frac{-A+E}{\sqrt{EN_0/2}}\right)}{\mathrm{d}A} &= -\frac{1}{\sqrt{EN_0/2}}\left(-\frac{1}{\sqrt{2\pi}}\right)\exp\left[-\frac{(E-A)^2}{EN_0}\right] \\ &= \frac{1}{\sqrt{\pi EN_0}}\exp\left[-\frac{(E-A)^2}{EN_0}\right] \\ \frac{\mathrm{d}Q\left(\frac{A}{\sqrt{EN_0/2}}\right)}{\mathrm{d}A} &= \frac{1}{\sqrt{EN_0/2}}\left(-\frac{1}{\sqrt{2\pi}}\right)\exp\left[-\frac{A^2}{EN_0}\right] \end{aligned}$$

Thus, setting $\frac{\mathrm{d}f(A)}{\mathrm{d}A} = 0$ gives:

$$\frac{1}{2}\exp\left[-\frac{(E-A)^2}{EN_0}\right] = \exp\left[-\frac{A^2}{EN_0}\right]$$

$$\Leftrightarrow \ln\left(\frac{1}{2}\right) - \frac{(E-A)^2}{EN_0} = -\frac{A^2}{EN_0} \Leftrightarrow \boxed{A = \frac{1}{2}\left[E - N_0\ln\left(\frac{1}{2}\right)\right]}$$

Finally, when $P[m_1] = P[m_3] = P[m_2] = \frac{1}{3}$, then setting $\frac{\mathrm{d}f(A)}{\mathrm{d}A} = 0$ yields $\boxed{A = \frac{E}{2}}$, which is independent of the noise level N_0!

P5.26

$$\begin{aligned} v(t) &= \sum_{k=-\infty}^{\infty} [P_1 s_1(t-kT_b) + P_2 s_2(t-kT_b)] \\ v(t+T_b) &= \sum_{k=-\infty}^{\infty} \Big[P_1 s_1(\underbrace{t+T_b-kT_b}_{t-(k-1)T_b}) + P_2 s_2(\underbrace{t+T_b-kT_b}_{t-(k-1)T_b})\Big]. \end{aligned}$$

Now change (dummy) index variable to $l = k-1$. Then

$$v(t+T_b) = \sum_{l=-\infty}^{\infty} \Big[P_1 s_1(t-lT_b) + P_2 s_2(t-lT_b)\Big] = v(t).$$

Therefore $v(t)$ is periodic with period T_b.

P5.27 (*Regenerative Repeaters*) The number of repeaters needed: $K = 2000\text{km}/20\text{km} = 100$.

(i) If analog repeaters are used:

$$\begin{aligned} &P_b = Q\left(\sqrt{\frac{2E_b}{KN_0}}\right) = 10^{-6} \\ &\Rightarrow \frac{E_b}{N_0} = \frac{K}{2}\left[Q^{-1}(P_b)\right]^2 = \frac{100}{2}\left[Q^{-1}\left(10^{-6}\right)\right]^2 = 1130 = 30.53 \text{ dB} \end{aligned}$$

(ii) If regenerative repeaters are used:

$$\begin{aligned} &P_b = KQ\left(\sqrt{\frac{2E_b}{N_0}}\right) \\ &\Rightarrow \frac{E_b}{N_0} = \frac{1}{2}\left[Q^{-1}\left(\frac{P_b}{K}\right)\right]^2 = \frac{1}{2}\left[Q^{-1}\left(\frac{10^{-6}}{100}\right)\right]^2 = 15.74 = 11.97 \text{ dB} \end{aligned}$$

P5.28 (*Simulation of a binary communication system using antipodal signalling*) The error probability of BPSK signalling is $P[\text{error}] = Q\left(\sqrt{2E_b/N_0}\right)$. Since $E_b = V^2 T_b$, $T_b = 1$ second and $N_0/2 = 1$ watts/Hz, one has $P[\text{error}] = Q(V)$, or $V = Q^{-1}(P[\text{error}])$. The following MATLAB script can be used to perform the simulation.

```
Pe_vec=[10^(-1) 10^(-2) 10^(-3) 10^(-4) 10^(-5)];
L=[10^5,10^5,10^6,10^6,10^7];
% Numbers of information bits to run, corresponding to each Pe
for i=1:length(Pe_vec)
   Pe=Pe_vec(i);
   V(i)=Qinv(Pe);
   % This is the sequence of information bits (0,1):
   b=round(rand(1,L(i)));
   % Convert bits to amplitudes levels of +V and -V volts
   % and add to it AWGN with variance of 1:
   r=V(i)*(2*b-1)+randn(1,length(b));
   % Receiver:
   temp=sign(r);
```

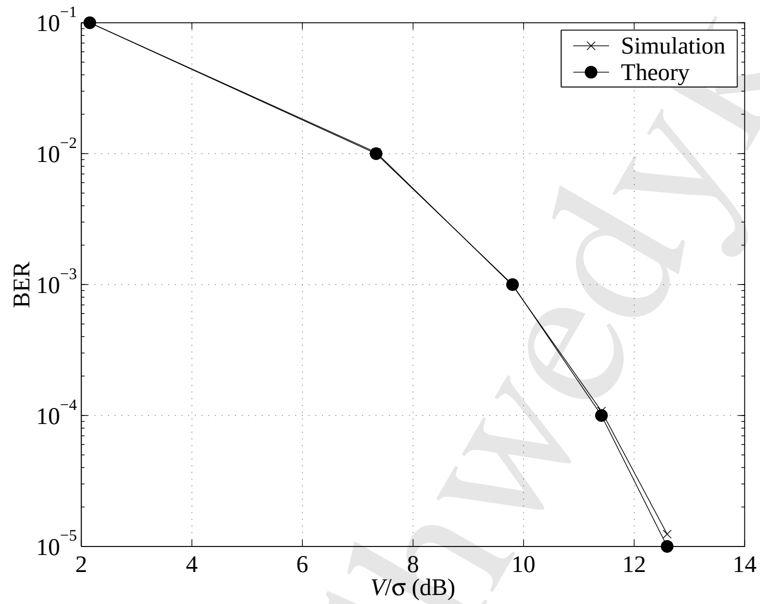


Figure 5.50: BER performance of BPSK signalling.

```
   % if r=0 then arbitrarily decide that bit 1 was transmitted.
   temp(find(temp==0))=1;
   % This is the sequence of demodulated bits (0,1):
   b_hat=(temp+1)/2;
   % Estimation of the error probability:
   BER(i)=length(find(b_hat-b))/length(b);
end
Fig. % Plot the experimental and theoretical probabilities of error (BER)
sim=semilogy(20*log10(V),BER,'marker','x','markersize',8,'color','k');
hold on;
theo=semilogy(20*log10(V),Q(V),'marker','o','markersize',8,'color','k');
legend([sim theo],'Simulation','Theory',+1);
xlabel('{\itV}/{\it\sigma} (dB)','FontName','Times New
Roman','FontSize',16); ylabel('BER','FontName','Times New
Roman','FontSize',16); h=gca;
set(h,'FontSize',16,'XGrid','on','YGrid','on','GridLineStyle',':',...
    'MinorGridLineStyle','none','FontName','Times New Roman');
```

Fig. 5.50 plots both the theoretical and experimental BER curves. As can be seen, the two curves are almost identical, except for $P[\text{error}] = 10^{-5}$. This difference is merely due to the insufficient data for simulation of this BER value (only 10^7 input bits were generated). The difference would disappear if more bits were simulated.

The numbers of bits in error that I got after running the above script are {9992 1021 986 108 124}. Of course these numbers will differ slightly for each execution of the script.

Chapter 6

Baseband Data Transmission

P6.1 (a) See Fig. 6.1-(a).

(b) $d_k = b_k \oplus d_{k-1}$; $d_{-1} = 0$. See Fig. 6.1-(b).

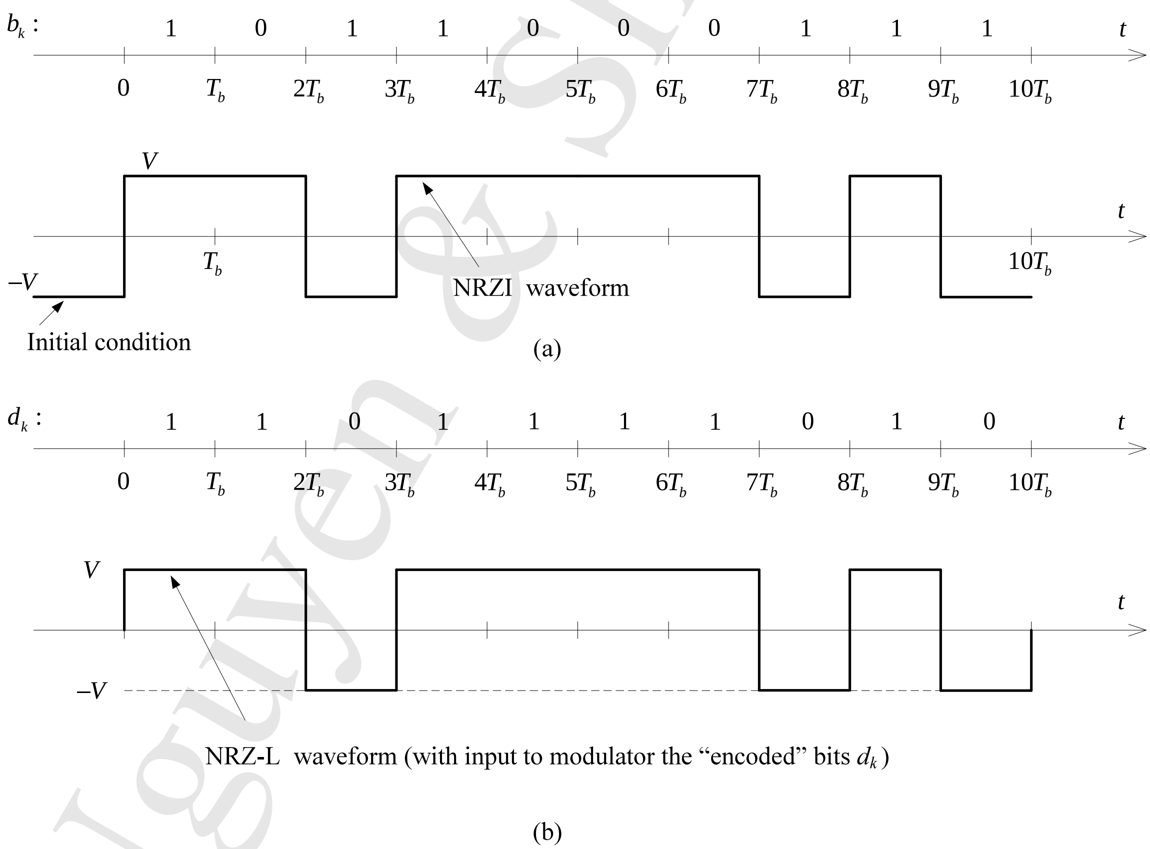


Figure 6.1

Note that the waveforms of (a), (b) are the same, i.e., one can implement the modulator either as in (a) or as in (b), end result is identical.

(c) With polarity reversal at $4T_b$, the transmitted waveform is as shown in Fig. 6.2.

(d) The question is ambiguous a bit since it may refer to either no polarity reversal or a polarity reversal at at $t = 4T_b$. So do it for both situations.

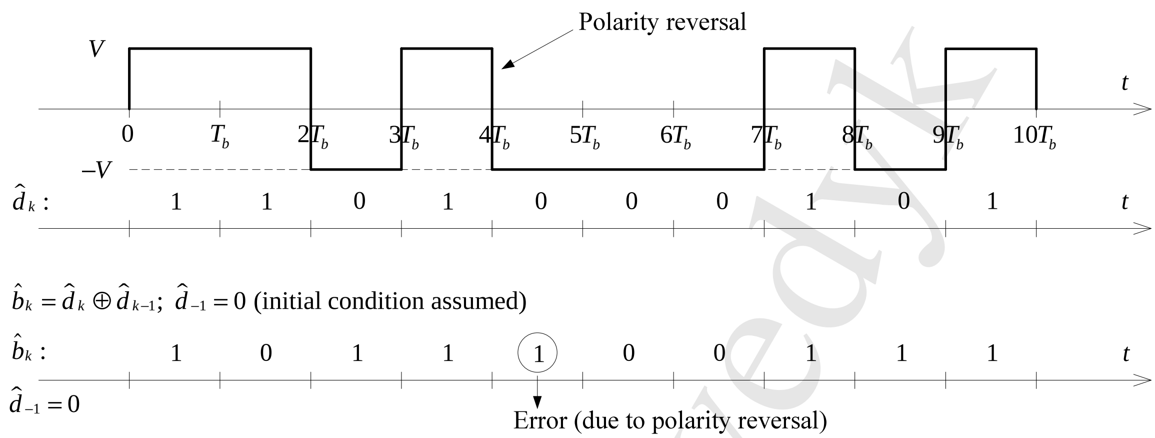


Figure 6.2

(i) With polarity reversal at $t = 4T_b$ and $\hat{d}_{-1} = 1$ (assumed). The picture looks as in Fig. 6.3

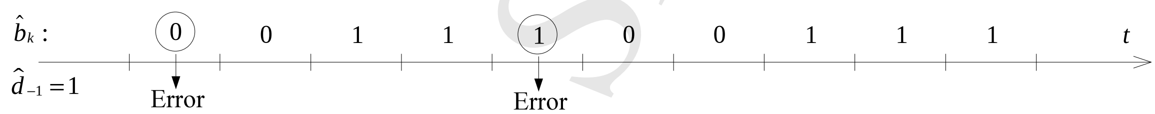


Figure 6.3

(ii) With no polarity reversal at $t = 4T_b$, see Fig. 6.4.

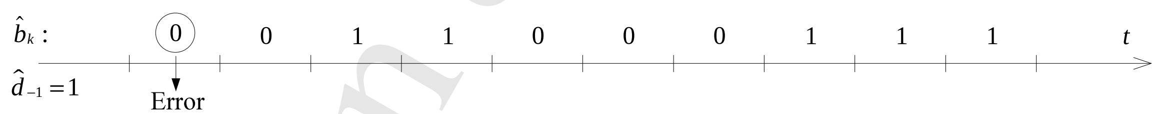


Figure 6.4

Remarks: d_k sequence of (b) was used since with no polarity reversal and no noise, $\hat{d}_k = d_k$. Also from (c) we might expect that only 1 error would occur since assuming $\hat{d}_{-1} = 1$ is equivalent to a polarity reversal. The conclusion is that a polarity reversal causes one bit to be in error.

(e) See Fig. 6.5.

(f) – A polarity reversal leads to a bit error in the interval in which the polarity reversal takes place. The decoder, in the absence of random noise, recovers after this interval. In the presence of random noise, an error due to the random noise leads to a bit error in the next interval as shall be seen in the next problem.
– Note that if the polarity reversal takes place not at $t = kT_b$ then, at least in the absence of random noise this is easily detected and can be corrected.

P6.2 Look at the modulator as first a mapping (encoding) from $\mathbf{b}_k$ to $\mathbf{d}_k$ and then an NRZ-L modulator.

(a) The signals of the NRZ-L modulator are antipodal, in general with energy $= E_b$. The signal space looks as in Fig. 6.6

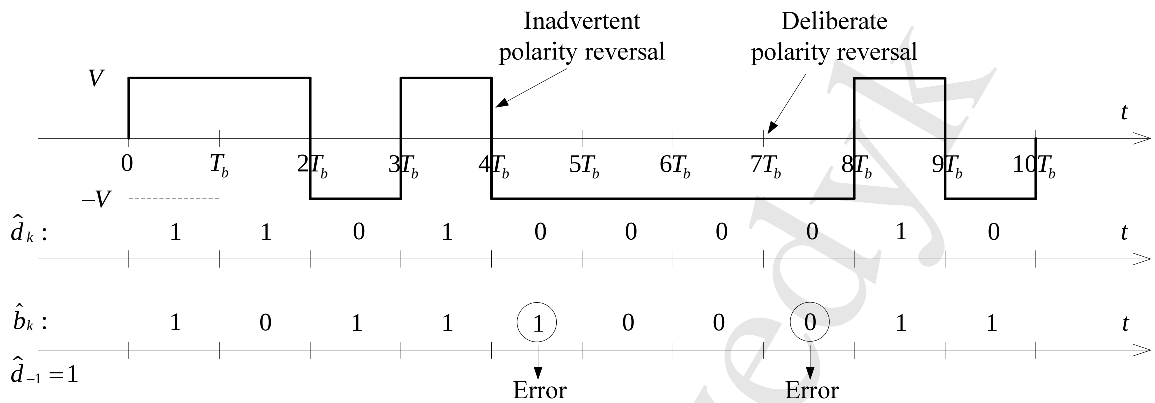


Figure 6.5

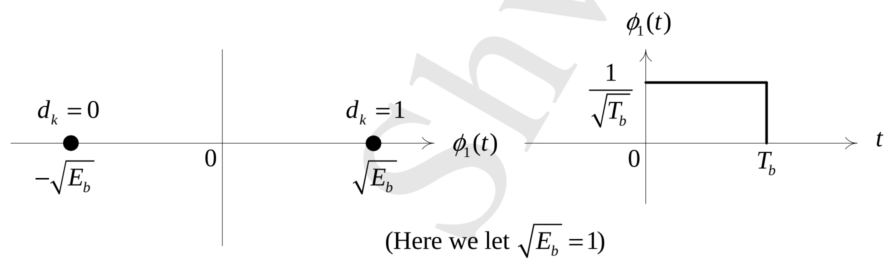


Figure 6.6

(b) With the energy level set to 1 joule the output $\mathbf{s}_k$ is: $\mathbf{d}_k$ = '1' $\rightarrow$ $\mathbf{s}_k = +1$ volt; $\mathbf{d}_k$ = '0' $\rightarrow$ $\mathbf{s}_k = -1$ volt. Using $\mathbf{d}_k$ sequence determined in P6.1(b), the output sequence is shown in Fig. 6.7.

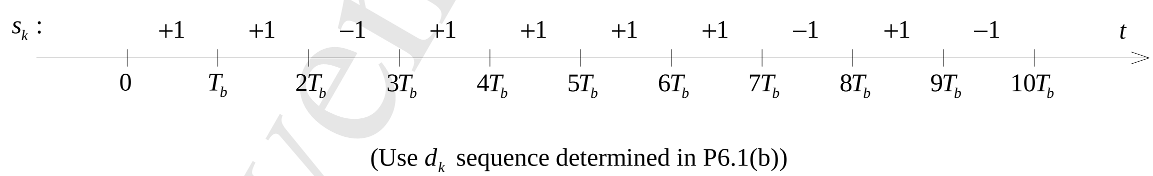


Figure 6.7

(c) $\mathbf{r}_k = \mathbf{s}_k + \mathbf{w}_k$ (volts). See Fig. 6.8

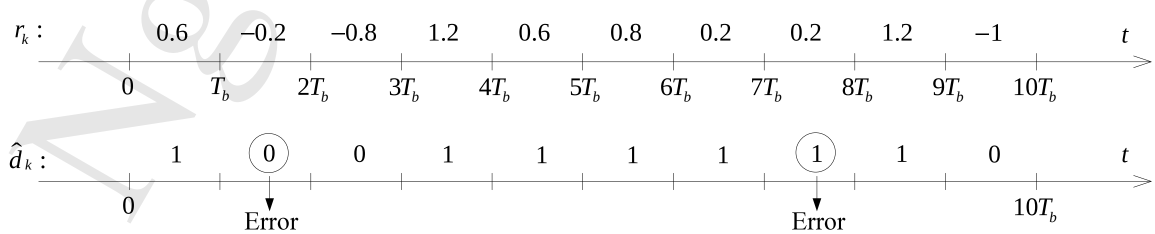


Figure 6.8

Demodulator is a bit-by-bit demodulator, and the decision rule is $\mathbf{r}_k \underset{\hat{\mathbf{d}}_k=1}{\overset{\hat{\mathbf{d}}_k=0}{\gtrless}} 0$. Note the $\mathbf{d}_k$ bit errors at $t = T_b$ and $t = 7T_b$. See Fig. 6.9

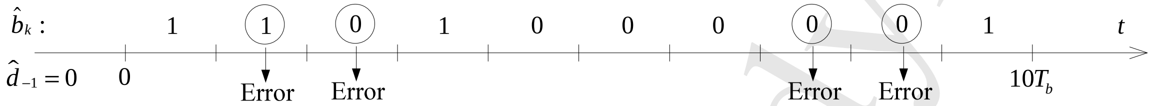


Figure 6.9

The observation is that a bit error in $\hat{\mathbf{d}}_k$ due to random noise results in 2 bit errors in the decoded information bits $\hat{\mathbf{b}}_k$. This is the influence of the memory in the system. If $\mathbf{d}_k$ is in error then since it participates in the determination of both $\mathbf{b}_k$ and $\mathbf{b}_{k+1}$, i.e., $\hat{\mathbf{b}}_k = \hat{\mathbf{d}}_k \oplus \hat{\mathbf{d}}_{k-1}$ and $\hat{\mathbf{b}}_{k+1} = \hat{\mathbf{d}}_{k+1} \oplus \hat{\mathbf{d}}_k$, then both $\hat{\mathbf{b}}_k$, $\hat{\mathbf{b}}_{k+1}$ are both affected by $\hat{\mathbf{d}}_k$. In other words, the error propagates but not that severely.

(d) See Fig. 6.10.

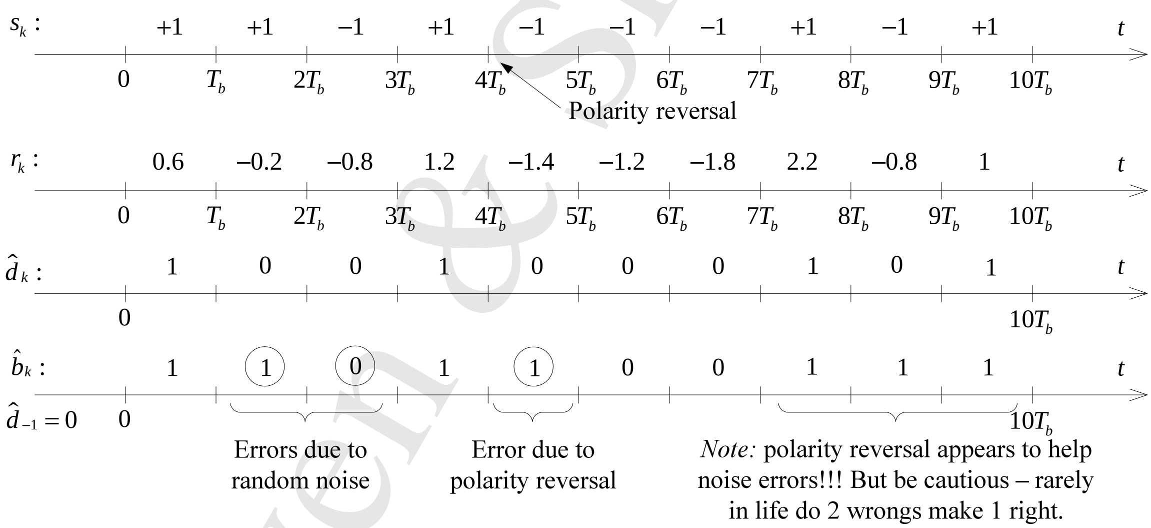


Figure 6.10

P6.3 We assume, as we almost invariably do throughout, that the information bits are equally likely, i.e., $P[\mathbf{b}_k = 0] = P_0 = 1/2$; $P[\mathbf{b}_k = 1] = P_1 = 1/2$. In P6.6, we will discuss the more general case.

(a) Now $\mathbf{d}_k = \mathbf{b}_k \oplus \mathbf{d}_{k-1}$. Note that $\mathbf{d}_k = 0$ if ($\mathbf{b}_k = 0$ and $\mathbf{d}_{k-1} = 0$) or ($\mathbf{b}_k = 1$ and $\mathbf{d}_{k-1} = 1$), which are 2 mutually exclusive random events. Therefore,

$$P[\mathbf{d}_k = 0] = P\left[(\mathbf{b}_k = 0 \text{ and } \mathbf{d}_{k-1} = 0\right] + P\left[(\mathbf{b}_k = 1 \text{ and } \mathbf{d}_{k-1} = 1)\right].$$

Using Bayes' rule:

$$P[\mathbf{d}_k = 0] = P\left[\mathbf{b}_k = 0|\mathbf{d}_{k-1} = 0\right] P\left[\mathbf{d}_{k-1} = 0\right] + P\left[\mathbf{b}_k = 1|\mathbf{d}_{k-1} = 1)\right] P\left[\mathbf{d}_{k-1} = 1\right].$$

But bits $\mathbf{b}_k$ are statistically independent of the $\mathbf{d}_k$ bits, i.e.,

$$P[\mathbf{b}_k = 0|\mathbf{d}_{k-1} = 0] = P[\mathbf{b}_k = 0] = \frac{1}{2} = P[\mathbf{b}_k = 1|\mathbf{d}_{k-1} = 1]$$

$$\therefore \quad P[\mathbf{d}_k = 0] = \frac{1}{2}\Big\{\underbrace{P[\mathbf{d}_{k-1} = 0] + P[\mathbf{d}_{k-1} = 1]}_{\text{surely, this equals 1}}\Big\} = \frac{1}{2} = P[\mathbf{d}_k = 1]$$

(b)

$$P[\mathbf{d}_k \text{ bit in error}] = Q\left(\frac{2E_b}{N_0}\right).$$

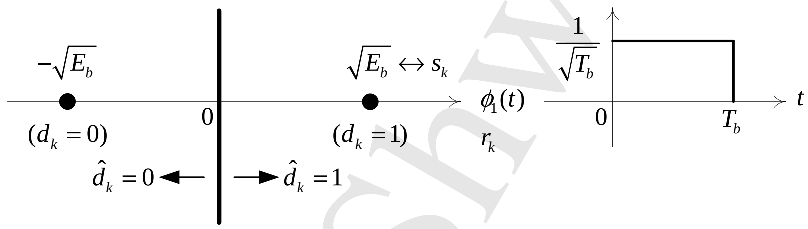


Figure 6.11

(c) (i) No.

(ii) No. 2 wrongs do make a right as we observed in P6.2(d).

(iii) Yes.

(iv) Yes.

$$P\left[\hat{\mathbf{b}}_k \text{ error}\right]$$
$$= P\Big[\underbrace{(\hat{\mathbf{d}}_k \text{ correct and } \hat{\mathbf{d}}_{k-1} \text{ incorrect) or } (\hat{\mathbf{d}}_k \text{ incorrect and } \hat{\mathbf{d}}_{k-1} \text{ correct})}_{\text{mutually exclusive events}}\Big].$$

Also decisions $\hat{\mathbf{d}}_k$, $\hat{\mathbf{d}}_{k-1}$ are statistically independent, since noise terms $\mathbf{w}_k$, $\mathbf{w}_{k-1}$ are statistically independent. Therefore,

$$\begin{aligned} P\left[\hat{\mathbf{b}}_k \text{ error}\right] &= 2P\left[\hat{\mathbf{d}}_k \text{ correct}\right] P\left[\hat{\mathbf{d}}_{k-1} \text{ incorrect}\right] \\ &= 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\left[1 - Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right]. \end{aligned}$$

With NRZ-L, we have $P\left[\hat{\mathbf{b}}_k \text{ error}\right] = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$. Assume that $Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \ll 1$, which is the practical case. Then $P\left[\hat{\mathbf{b}}_k \text{ error for NRZI}\right] \cong 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$, i.e., 2 times that of NRZ-L. The factor 2 reflects the error propagation, i.e., whenever there is a bit error due to random noise, the succeeding bit is also in error.

P6.4 (a) First thing is to define the state set, i.e., what is needed from the past that along with the present input bit, $\mathbf{b}_k$, allows one to determine the output.

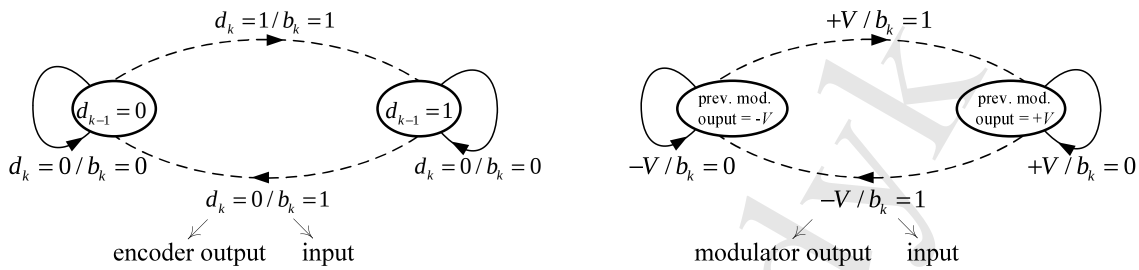


Figure 6.12

- Mapping from $\mathbf{b}_k$ to $\mathbf{d}_k$: state is value of $\mathbf{d}_{k-1}$.
- Mapping from $\mathbf{b}_k$ to modulator level: state is previous value of modulator output.

The state diagrams then look as in Fig. 6.12

(b) The trellis looks as in Fig. 6.13

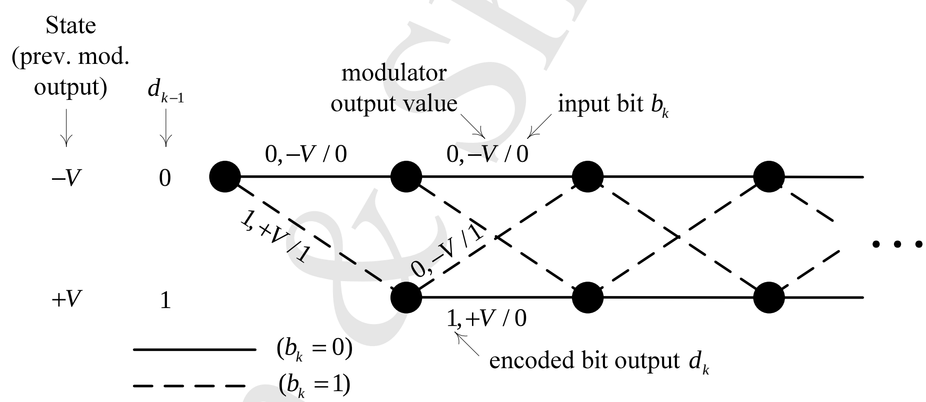


Figure 6.13

(c) The branch "metrics" are $(r_k - \sqrt{E_b})^2$ and $(r_k + \sqrt{E_b})^2$ respectively, $-\sqrt{E_b}$ if the branch corresponds to the $+V$ being transmitted and $+\sqrt{E_b}$ if the level transmitted is $-V$ (see Fig. 6.14).

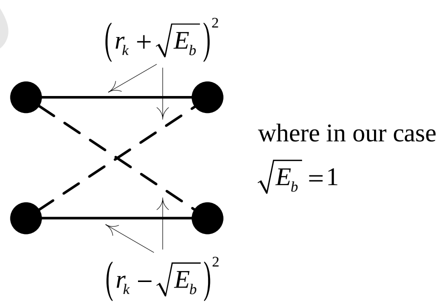


Figure 6.14

Therefore the best path can be found as illustrated in Fig. 6.15.

Note that there are 4 bit errors, which is the same as in P6.2(c). Thus, in this case, the VA does not perform better. But 1 swallow does not make a summer and more will be

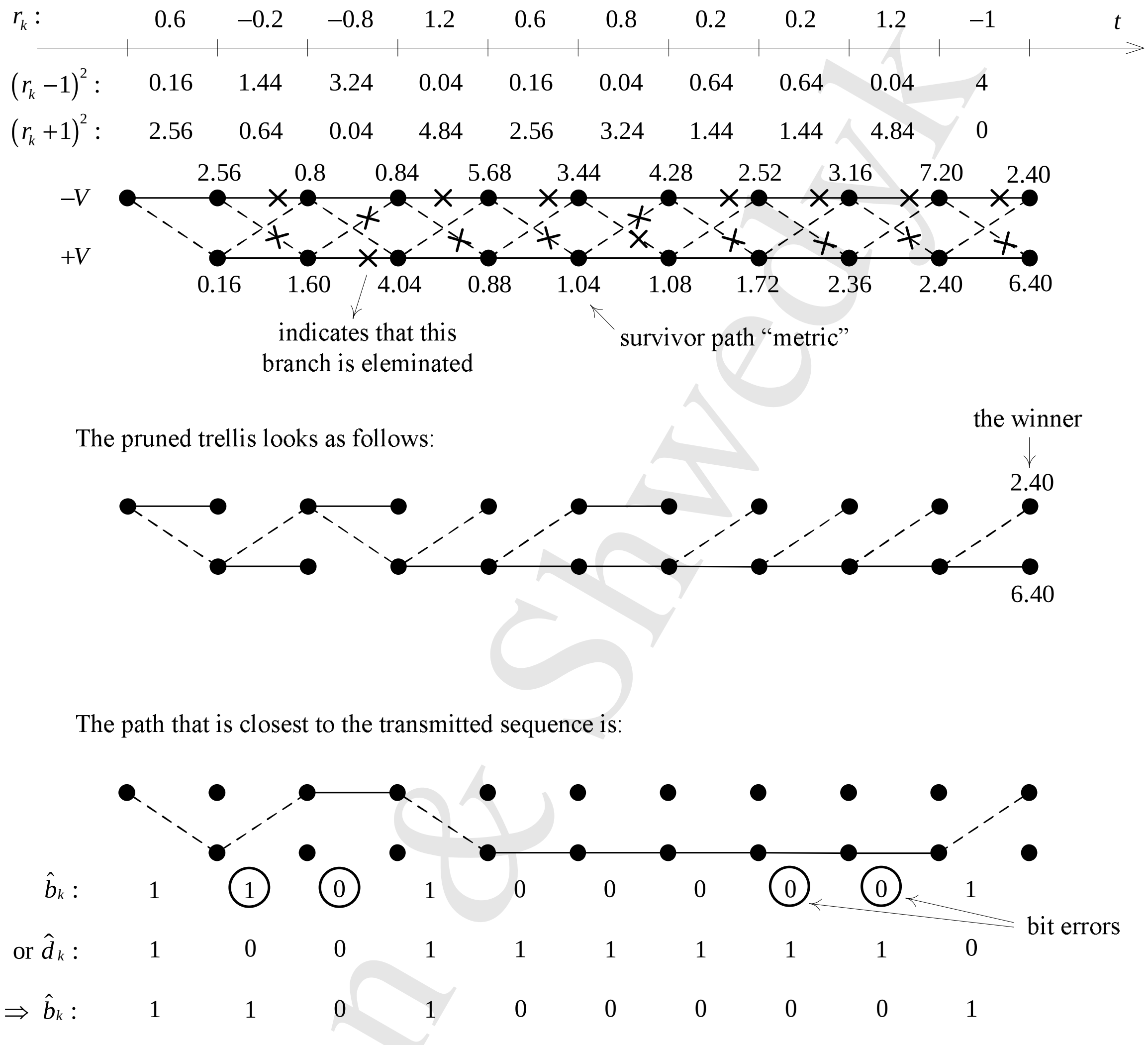


Figure 6.15

said at the end of the problem. But next, we look at P6.4(d).

(d) Refer to Fig. 6.16 for the solution.

Note that there are only 3 bit errors making it appear that the polarity reversal actually helps. But again one should not jump to conclusions. For a proper evaluation of the comparison between bit-by-bit demodulation and sequence demodulation one should resort to simulation, say in Matlab.

P6.5 (a) Consider 2 sections of the trellis as in Fig. 6.17.

Bit pattern $\mathbf{b}_{k-1} = 0$, $\mathbf{b}_k = 0$ corresponds the possible paths as shown in Fig. 6.18.

Bit pattern $\mathbf{b}_{k-1} = 1$, $\mathbf{b}_k = 0$ corresponds to the possible paths shown in Fig. 6.19.

Similarly, bit patterns $\mathbf{b}_{k-1} = 0$, $\mathbf{b}_k = 1$ and $\mathbf{b}_{k-1} = 1$, $\mathbf{b}_k = 1$ correspond to paths shown in Fig. 6.20.

So the bit $\mathbf{b}_k$ is represented by the signals shown in Fig. 6.21.

(b) Choose basis functions as in Fig. 6.22 to represent bit $\mathbf{b}_k$. The signal space plot looks as in Fig. 6.23.

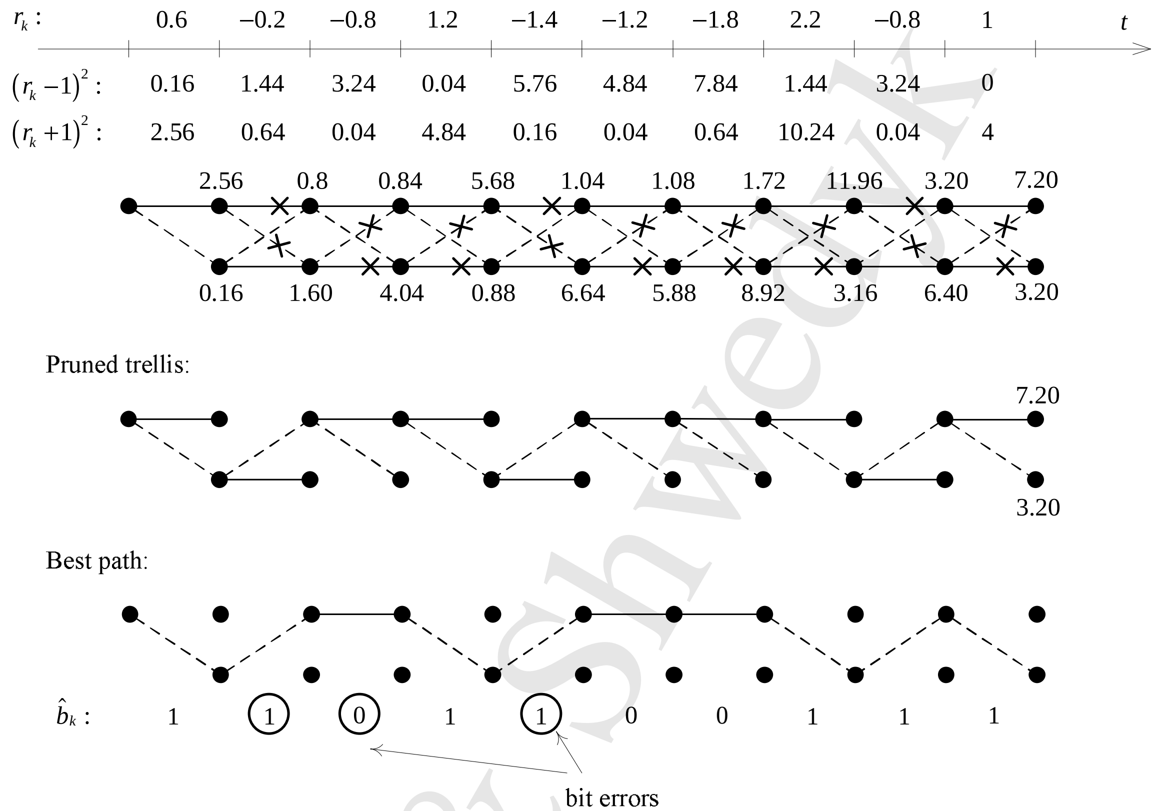


Figure 6.16

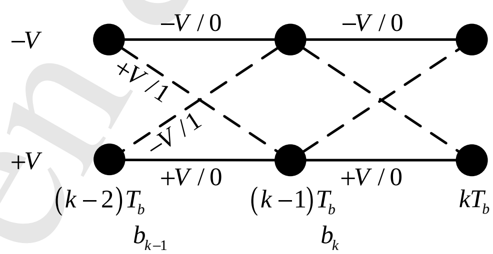


Figure 6.17

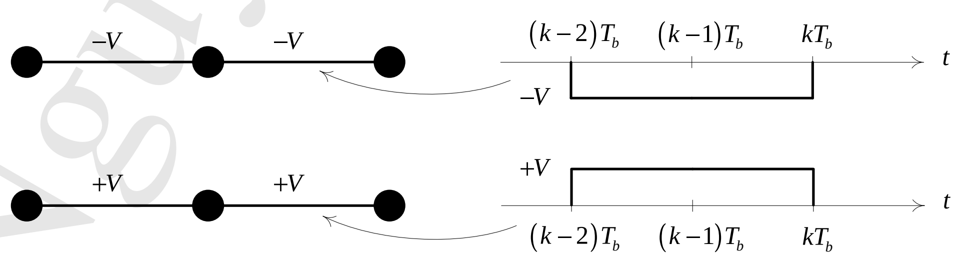


Figure 6.18

(c) The demodulator is to project $\mathbf{r}(t)$ onto $\phi_1(t)$, $\phi_2(t)$ to obtain sufficient statistics $\mathbf{r}_1$, $\mathbf{r}_2$ and choose the signal point they are closest to. The decision space is shown in Fig. 6.23.

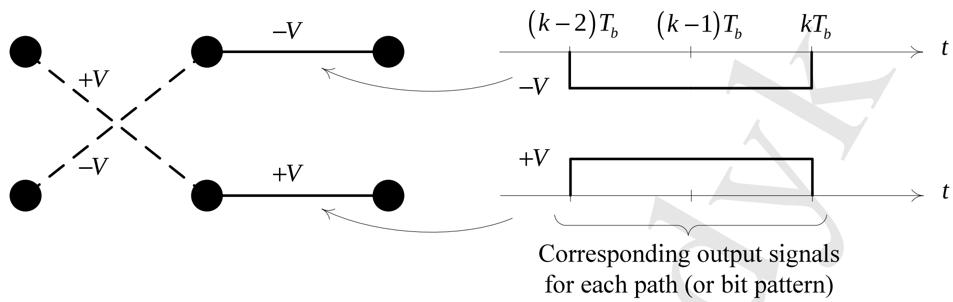


Figure 6.19

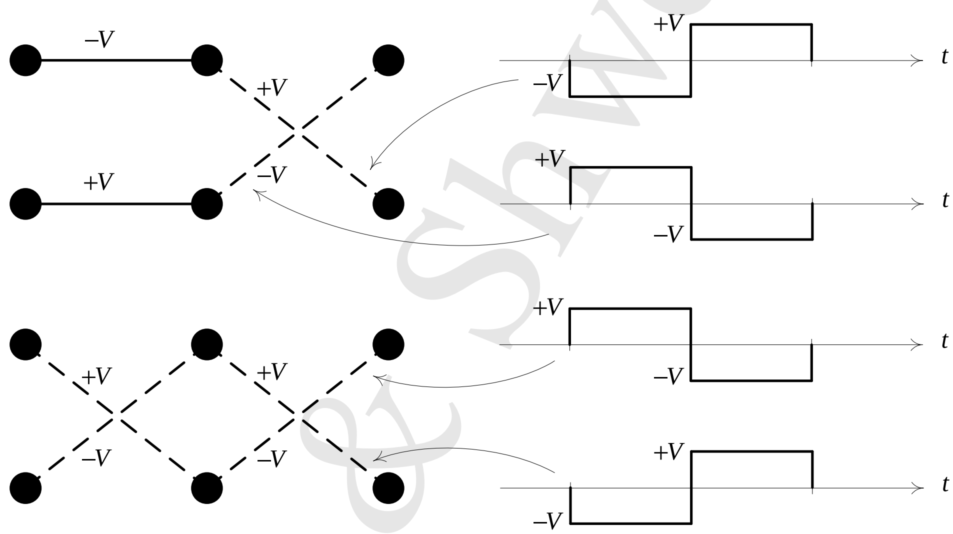


Figure 6.20

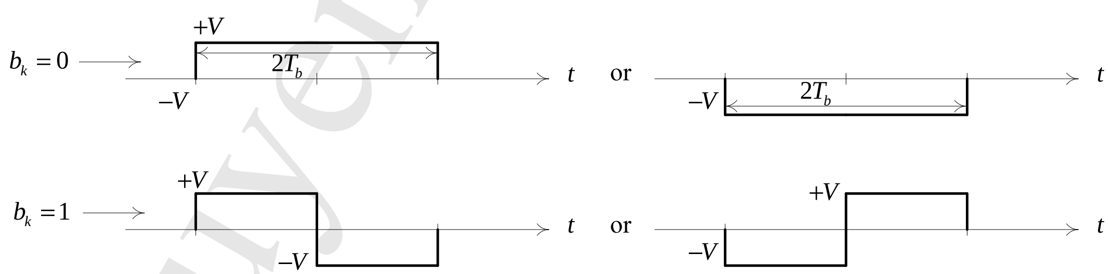


Figure 6.21

(d) As for Miller scheme, one has

$$P[\mathbf{b}_k \text{ is in error}] = 2Q\left(\frac{\sqrt{E}\cos 45^\circ}{\sqrt{N_0/2}}\right)\left[1 - Q\left(\frac{\sqrt{E}\cos 45^\circ}{\sqrt{N_0/2}}\right)\right] = 2Q\left(\sqrt{\frac{E}{N_0}}\right)\left[1 - Q\left(\sqrt{\frac{E}{N_0}}\right)\right].$$

To compare with the expression in P6.3(c) note that $E = 2E_b$ (because we are looking at the signal over 2 bit intervals). Therefore, the error probability is the same.

Remark: We shall use the ideas here when the fading channel and differential phase shift keying (DPSK) is discussed - Chapter 10.

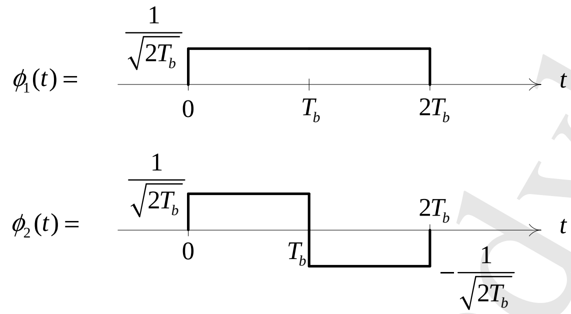


Figure 6.22

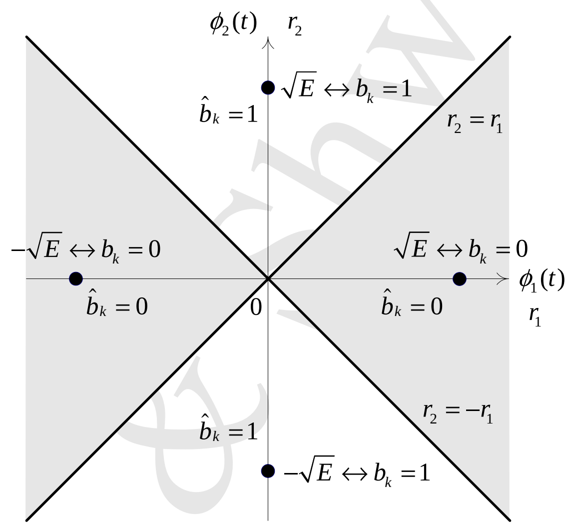


Figure 6.23

P6.6 Consider Table 6.1.

Table 6.1

$\mathbf{d}_{k-1} \to \mathbf{i}$	$\mathbf{d}_k \to \mathbf{j}$	Product $\mathbf{ij}$
$0 \to -1$	$0 \to -1$	$+1$
$0 \to -1$	$1 \to -1$	-1
$1 \to +1$	$0 \to -1$	-1
$1 \to +1$	$1 \to +1$	$+1$

The average or expected value of the product **ij** is given by a weighted sum of the 4 possible values of **ij** where the weight are given by the probability that that value occurs. Therefore,

$$\begin{aligned}
E\{\mathbf{d}_{k-1}\mathbf{d}_k\} &= E\{\mathbf{ij}\} = \sum_{\text{values of } ij} ij\, P[\mathbf{ij}=ij] = \sum_{\text{values of } ij} ij\, P[\mathbf{d}_{k-1}\mathbf{d}_k = ij] \\
&= \sum_{\text{values of } ij} \mathbf{ij}\, P[\mathbf{d}_k = i|\mathbf{d}_{k-1}=j]\, P[\mathbf{d}_{k-1}=j] \\
&= P[\mathbf{d}_k=0|\mathbf{d}_{k-1}=0]\, P[\mathbf{d}_{k-1}=0] - P[\mathbf{d}_k=1|\mathbf{d}_{k-1}=0]\, P[\mathbf{d}_{k-1}=0] \\
&\quad -P[\mathbf{d}_k=0|\mathbf{d}_{k-1}=1]\, P[\mathbf{d}_{k-1}=1] + P[\mathbf{d}_k=1|\mathbf{d}_{k-1}=1]\, P[\mathbf{d}_{k-1}=1]\,.
\end{aligned}$$

Now $\mathbf{d}_k = \mathbf{d}_{k-1} \oplus \mathbf{b}_k$, which means that

$$\begin{aligned}
&P[\mathbf{d}_k = 0|\mathbf{d}_{k-1} = 0] = P[\mathbf{b}_k = 0] = 1/2, \text{ since when } \mathbf{d}_{k-1} = 0, \mathbf{d}_k = 0 \text{ iff } \mathbf{b}_k = 0.\\
&P[\mathbf{d}_k = 1|\mathbf{d}_{k-1} = 0] = P[\mathbf{b}_k = 1] = 1/2\\
&P[\mathbf{d}_k = 0|\mathbf{d}_{k-1} = 1] = P[\mathbf{b}_k = 1] = 1/2\\
&P[\mathbf{d}_k = 1|\mathbf{d}_{k-1} = 1] = P[\mathbf{b}_k = 0] = 1/2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
E\{\mathbf{d}_{k-1}\mathbf{d}_k\} &= \frac{1}{2}P[\mathbf{d}_{k-1} = 0] - \frac{1}{2}P[\mathbf{d}_{k-1} = 0] - \frac{1}{2}P[\mathbf{d}_{k-1} = 1] + \frac{1}{2}P[\mathbf{d}_{k-1} = 1]\\
&= 0, \text{ i.e., uncorrelated.}
\end{aligned}$$

Since the PSD depends only on second order statistics, we conclude that the modulator sees an input of uncorrelated bits (strictly speaking uncorrelated impulses) just as in the NRZ-L case. Therefore the PSD of NRZI is the same as that of NRZ-L.

The above assumes that the input bits $\mathbf{b}_k$ are equally probable. More generally, $P[\mathbf{b}_k = 0] = P_0$ and $P[\mathbf{b}_k = 1] = P_1 = 1 - P_0$. In this case the correlation becomes:

$$\begin{aligned}
E\{\mathbf{d}_{k-1}\mathbf{d}_k\} &= P_0 P[\mathbf{d}_{k-1} = 0] - P_1 P[\mathbf{d}_{k-1} = 1] - P_1 P[\mathbf{d}_{k-1} = 0] + P_0 P[\mathbf{d}_{k-1} = 0]\\
&= (P_0 - P_1)\big\{P[\mathbf{d}_{k-1} = 0] - P[\mathbf{d}_{k-1} = 1]\big\} \neq 0, \text{ if } P_0 \neq P_1.
\end{aligned}$$

Consider now the issue of statistical independence. Intuitively, one feels that because of the functional relationship between $\mathbf{d}_k$ and $\mathbf{d}_{k-1}$, they would be statistically dependent. Consider the case of equally likely $\{\mathbf{b}_k\}$ bits. In P6.3 we showed that in this case the differential bits $\{\mathbf{d}_k\}$ are equally likely, i.e., $P[\mathbf{d}_k] = P[\mathbf{d}_{k-1}] = 1/2$.

Now consider $P[\mathbf{d}_k|\mathbf{d}_{k-1}]$. Then

$$\begin{aligned}
&P[\mathbf{d}_k = 0|\mathbf{d}_{k-1} = 0] = P[\mathbf{b}_k = 0] = 1/2; P[\mathbf{d}_k = 1|\mathbf{d}_{k-1} = 0] = P[\mathbf{b}_k = 1] = 1/2\\
&P[\mathbf{d}_k = 0|\mathbf{d}_{k-1} = 1] = P[\mathbf{b}_k = 1] = 1/2; P[\mathbf{d}_k = 1|\mathbf{d}_{k-1} = 1] = P[\mathbf{b}_k = 0] = 1/2
\end{aligned}$$

which means that $P[\mathbf{d}_k|\mathbf{d}_{k-1}] = P[\mathbf{d}_k]$, i.e., they are statistically independent.

But this is the case only when the input bits are equally probable. Otherwise as shown above the differential bits are correlated and therefore definitely not statistically independent.

Finally, if $P_0 \neq P_1$, i.e., the input bits are not equally probable, it can be shown that as k becomes large, the differential bits $\{\mathbf{d}_k\}$ tend to be equally probable, i.e., $P[\mathbf{d}_k = 1] \to 1/2, P[\mathbf{d}_k = 0] \to 1/2$ which means that the differential bits are uncorrelated. However, in this case they are statistically dependent since $P[\mathbf{d}_k|\mathbf{d}_{k-1}] \neq P[\mathbf{d}_k]$.

P6.7 (a) See Figs. 6.24 and 6.25.

(b) Yes, it is. As in P6.1(c) assume a polarity reversal at $t = 4T_b$. Then as in P6.1(c) the demodulated $\hat{d}_k$ would be reversed (or complemented) from this point on. The output $\hat{b}_k$ would be identical to that in P6.1(c), an error for the bit in interval $4T_b$ to $5T_b$ and then correct decisions.

(c) Nothing much would change. The error performance would be the same, the demodulation/decoding procedure would be the same. What would change is the basis function for the signal space diagram, the signal would now be a normalized biphase signal instead of an normalized NRZ-L signal. The signal space diagram would look the same.

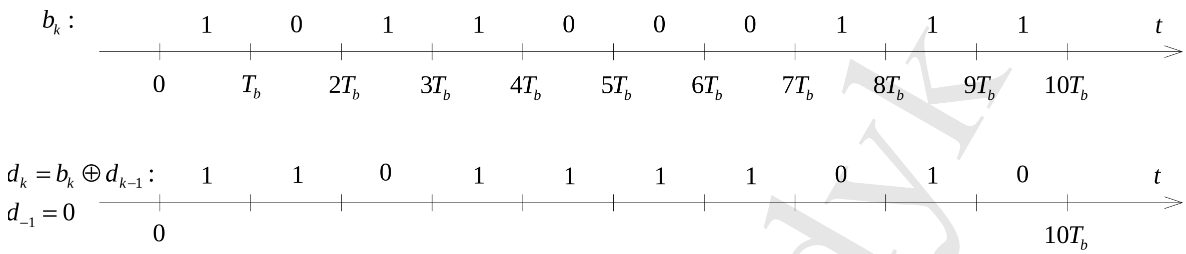


Figure 6.24

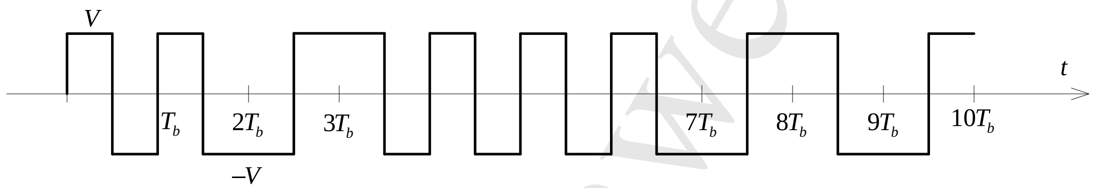


Figure 6.25

P6.8 Error performance is independent of the antipodal signal set. Assuming, of course, the same energy levels and an AWGN channel. The basis function for the signal space would be a normalized version of the signal and the demodulator would project the received signal onto this basis function to generate the sufficient statistic. After this the sufficient statistic is processed the same, regardless of the antipodal signal set.

The PSD would depend on the actual signals used and would be $\frac{1}{T_b}|S(f)|^2$ where $S(f) = \mathcal{F}\{s(t)\}$.

P6.9 (a) See Fig. 6.26.

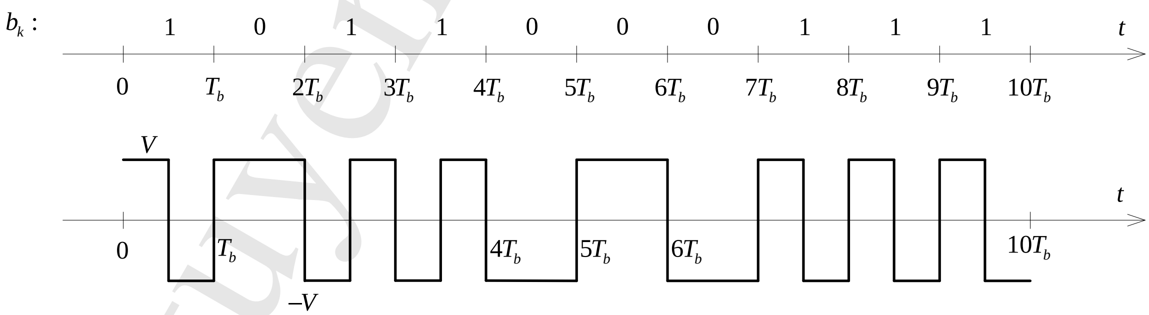


Figure 6.26

(b) Yes and no. Depends on how one demodulates. See (c) as to how one demodulates so that the modulation is immune to polarity reversals.

(c) Note that a 0_T results in one of 2 signals namely $\pm s_0(t)$ and a 1_T results in one of $\pm s_1(t)$ (see Fig. 6.27).

Therefore the signal space looks as shown in Fig. 6.28.

Demodulate by projecting the received signal onto $\phi_0(t)$, $\phi_1(t)$ to generate sufficient statistics $\mathbf{r}_0$, $\mathbf{r}_1$ and decide according to the decision boundaries shown in the signal space diagram.

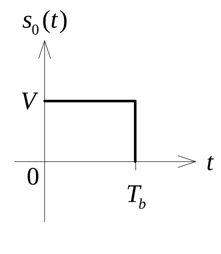


Figure 6.27

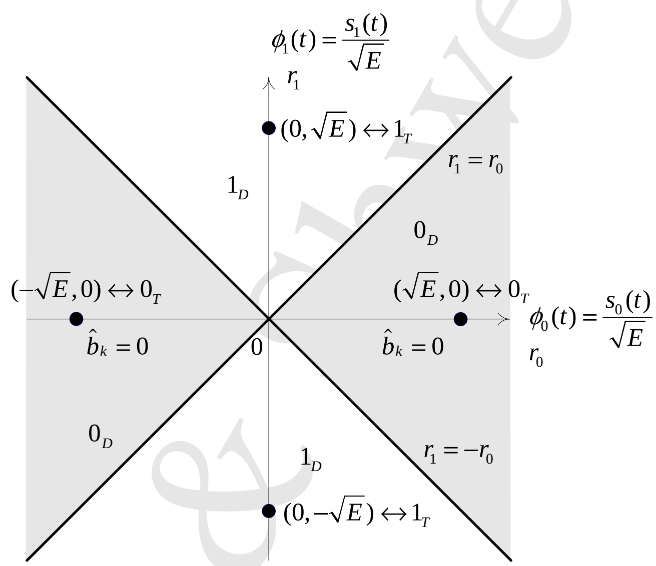


Figure 6.28

(d) By symmetry

$$\begin{aligned}
P[\text{error}] &= P[\mathbf{r}_0, \mathbf{r}_1 \text{ fall in } 1_D \text{ region} \mid 0_T] \\
&= 2\left[Q\left(\frac{\sqrt{E}\cos 45^o}{\sqrt{N_0/2}}\right)\right]\left[1 - Q\left(\frac{\sqrt{E}\sin 45^o}{\sqrt{N_0/2}}\right)\right] = 2Q\left(\sqrt{\frac{E}{N_0}}\right)\left[1 - Q\left(\sqrt{\frac{E}{N_0}}\right)\right] \\
&\cong 2Q\left(\sqrt{\frac{E}{N_0}}\right)
\end{aligned}$$

Remark: Signal space is identical to that of Miller modulation and hence the error performance is the same. The actual transmitted sequence and resultant PSD is however different from Miller.

P6.10 Let $0_T \rightarrow s_0(t)$ and $1_T \rightarrow s_1(t)$ where $s_0(t) \perp s_1(t)$, each of energy E. Polarity reversals mean that $0_T \rightarrow \pm s_0(t)$ and $1_T \rightarrow \pm s_1(t)$. The modification to the demodulator to obtain polarity reversal immunity is to modify the decision regions in the signal space from Fig. 6.29 to Fig. 6.30. Note that the price you pay is a factor of 2 in the error probability expression. However, it only applies if there no polarity reversals. If there are, then you gain a lot.

P6.11 (a) See Fig. 6.31. Yes, immune to polarity reversal.

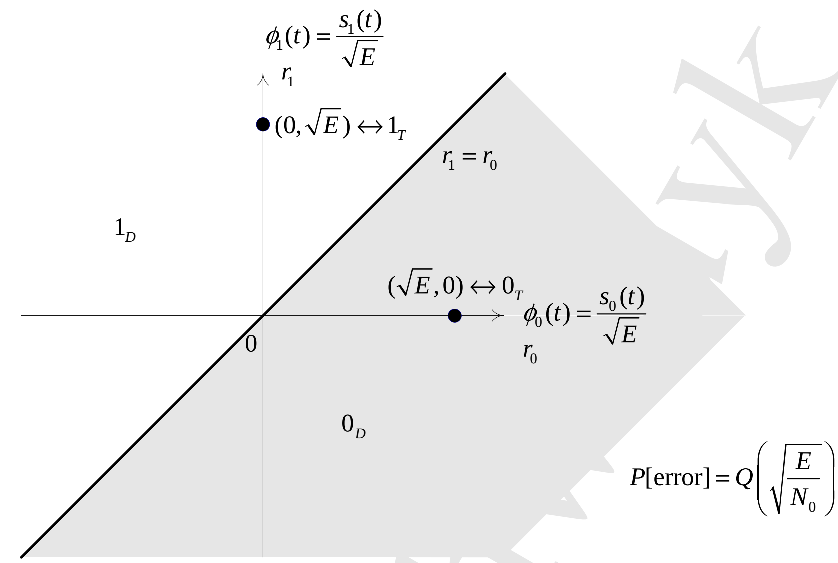


Figure 6.29

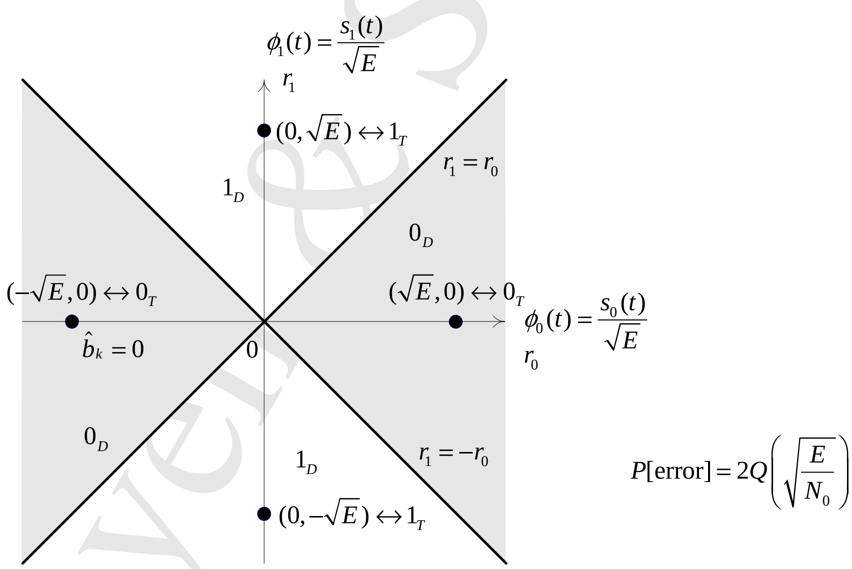


Figure 6.30

(b) See Fig. 6.32. The sufficient statistic is $\mathbf{r} = \int_{t \in T_b} \mathbf{r}(t)\phi(t)\mathrm{d}t$.

(c) $f(r|0_T) \sim \mathcal{N}(0, N_0/2)$, $f(r|1_T, -V) \sim \mathcal{N}(-\sqrt{E}, N_0/2)$; $f(r|1_T, +V) \sim \mathcal{N}(\sqrt{E}, N_0/2)$. Therefore,

$$f(r|1_T) \sim \frac{1}{2}\mathcal{N}(-\sqrt{E}, N_0/2) + \frac{1}{2}\mathcal{N}(\sqrt{E}, N_0/2).$$

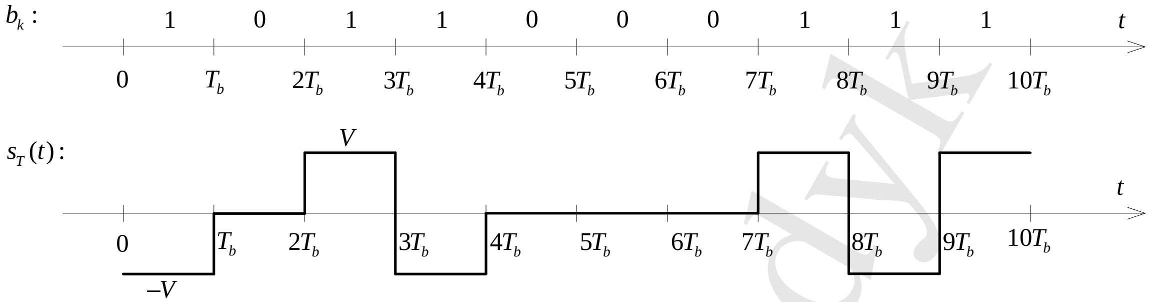


Figure 6.31

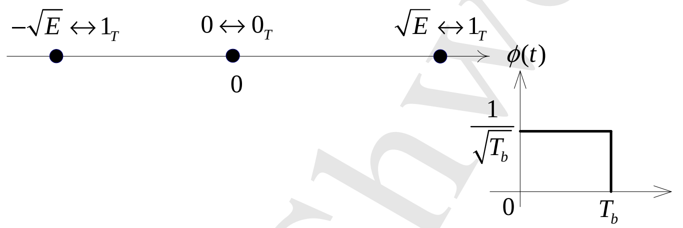


Figure 6.32

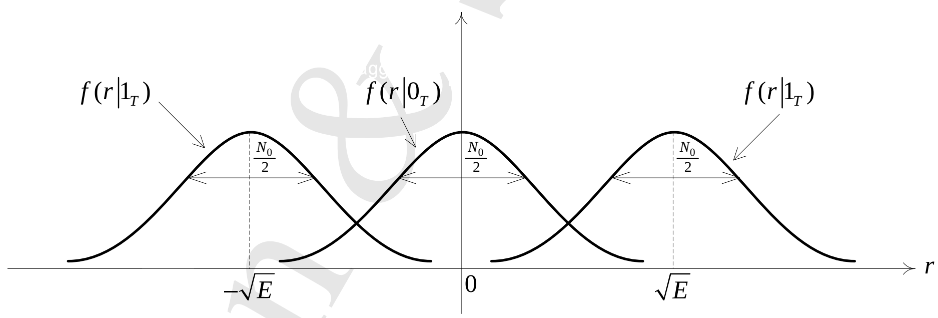


Figure 6.33

(d) The LRT is:

$$\frac{f(r|1_T)}{f(r|0_T)} = \frac{\frac{1}{2}\left[\frac{1}{\sqrt{2\pi}\sqrt{N_0/2}}\mathrm{e}^{\frac{-(r-\sqrt{E})^2}{2(N_0/2)}} + \frac{1}{\sqrt{2\pi}\sqrt{N_0/2}}\mathrm{e}^{\frac{-(r+\sqrt{E})^2}{2(N_0/2)}}\right]}{\frac{1}{\sqrt{2\pi}\sqrt{N_0/2}}\mathrm{e}^{\frac{-r^2}{2(N_0/2)}}} \underset{0_D}{\overset{1_D}{\gtrless}} 1$$

$$\Leftrightarrow \quad \frac{\mathrm{e}^{\frac{2\sqrt{E}}{N_0}r} + \mathrm{e}^{-\frac{2\sqrt{E}}{N_0}r}}{2} \underset{0_D}{\overset{1_D}{\gtrless}} \mathrm{e}^{\frac{E}{N_0}}$$

$$\Leftrightarrow \quad \cosh\left(\frac{2\sqrt{E}}{N_0}r\right) \underset{0_D}{\overset{1_D}{\gtrless}} \mathrm{e}^{\frac{E}{N_0}}.$$

(e) $f(\ell|0) \sim \mathcal{N}(0, 4E/N_0^2)$; $f(\ell|-V) \sim \mathcal{N}(-2E/N_0, 4E/N_0^2)$; $f(\ell|+V) \sim \mathcal{N}(2E/N_0, 4E/N_0^2)$.

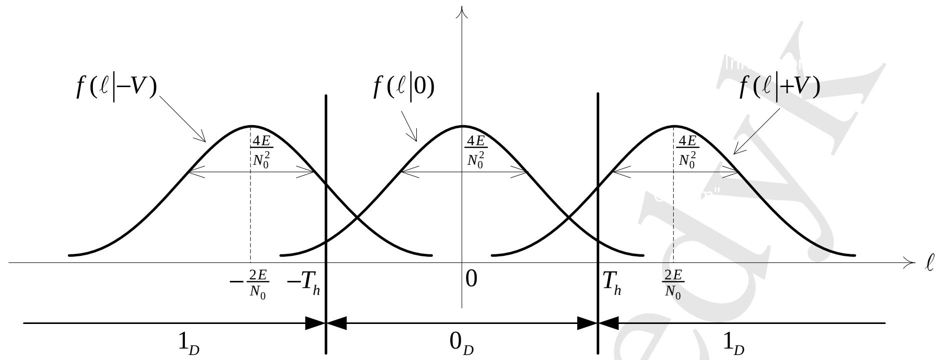


Figure 6.34

$$
\begin{aligned}
P[\text{error} \mid 0_T] &= 2Q\left(\frac{T_h}{2\sqrt{E}/N_0}\right) \\
P[\text{error} \mid 1_T] &= \frac{1}{2}P[\text{error} \mid 1_T, +V] + \frac{1}{2}P[\text{error} \mid 1_T, -V] = P[\text{error} \mid 1_T, +V] \\
&= P[-T_h \le \boldsymbol{\ell} \le T_h \vert\, 1_T, +V] = P[\boldsymbol{\ell} \le T_h \vert\, 1_T, +V] - P[\boldsymbol{\ell} \le -T_h \vert\, 1_T, +V] \\
&= Q\left(\frac{\frac{2E}{N_0} - T_h}{2\sqrt{E}/N_0}\right) - Q\left(\frac{\frac{2E}{N_0} + T_h}{2\sqrt{E}/N_0}\right).
\end{aligned}
$$

P6.12 (a) Since we are only interested in what bit was transmitted, not the signal levels, $\pm V$, 0 we determine the region(s) of r where either $P_1 f(r|-V)$ or $P_3 f(r|+V)$ is $\ge P_2 f(r|0)$. In this region(s) we decide that a 1 was transmitted, in the remaining region(s) we decide a 0 was transmitted.
Let $\sigma^2 \equiv N_0/2$. Then

$$
\begin{aligned}
f(r|-V) &= \frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\frac{(r+\sqrt{E})^2}{2\sigma^2}} \\
f(r|0) &= \frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\frac{r^2}{2\sigma^2}} \\
f(r|+V) &= \frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\frac{r^2}{2\sigma^2}}.
\end{aligned}
$$

Now

$$
\begin{aligned}
P_3 f(r|+V) > P_2 f(r|0) \quad &\Leftrightarrow \quad \frac{1}{4}\frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\frac{(r-\sqrt{E})^2}{2\sigma^2}} > \frac{1}{2}\frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\frac{r^2}{2\sigma^2}} \\
&\underset{\text{algebra)}}{\overset{\text{(after some}}{\Leftrightarrow}} \quad r > \frac{\sigma^2}{\sqrt{E}}\ln 2 + \frac{\sqrt{E}}{2}.
\end{aligned}
$$

In the above region of r choose a 1 transmitted, i.e., 1_D.
Similarly,

$$
P_1 f(r|-V) > P_2 f(r|0) \Rightarrow \frac{1}{4}\frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\frac{(r+\sqrt{E})^2}{2\sigma^2}} > \frac{1}{2}\frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\frac{r^2}{2\sigma^2}} \Rightarrow r < -\left(\frac{\sigma^2}{\sqrt{E}}\ln 2 + \frac{\sqrt{E}}{2}\right).
$$

In this region of r we decide on a 1 being transmitted, i.e., 1_D.
Note: The 2nd result we would expect from the symmetry of the problem. Also the factor $\frac{\sigma^2}{\sqrt{E}}\ln 2$ arises from the fact that $P_2 \neq P_1$ (or P_3).
Therefore $-\left(\frac{N_0}{2\sqrt{E}}\ln 2+\frac{\sqrt{E}}{2}\right) \leq r \leq \left(\frac{N_0}{2\sqrt{E}}\ln 2+\frac{\sqrt{E}}{2}\right)$ choose 0, otherwise choose 1.

(b) Multiply the decision rule of (P6.8) by $\frac{2\sqrt{E}}{N_0}$, i.e., scale the decision space:

$$\begin{cases} -\left(\ln 2+\frac{E}{N_0}\right) \leq \frac{2\sqrt{E}}{N_0}r = \ell \leq \left(\ln 2+\frac{E}{N_0}\right) & \text{choose 0 transmitted} \\ \text{otherwise} & \text{choose 1 transmitted} \end{cases}$$

The new decision variable $\boldsymbol{\ell}$ is Gaussian with a mean value equal to

$$\left(\frac{2\sqrt{E}}{N_0}\right) \times \underbrace{(\text{mean value of } \mathbf{r})}_{\text{depends on the signal transmitted}}$$

and a variance equal to $\left(\frac{4E}{N_0^2}\right) \times \underbrace{(\text{variance of } \mathbf{r})}_{=\frac{N_0}{2}}$. Therefore,

$$f(\ell|-V) \sim \mathcal{N}\left(\frac{-2E}{N_0}, \frac{2E}{N_0}\right);\ f(\ell|0) \sim \mathcal{N}\left(0, \frac{2E}{N_0}\right);\ f(l|+V) \sim \mathcal{N}\left(\frac{2E}{N_0}, \frac{2E}{N_0}\right).$$

To aid in deriving the error probabilities, let us sketch the decision regions(s) as in Fig. 6.35.

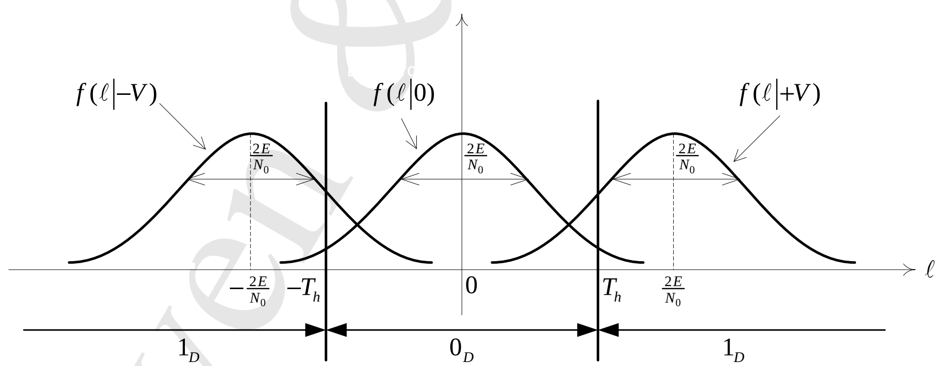


Figure 6.35

$$\begin{aligned}
P[\text{error } |0_T] &= P[(\boldsymbol{\ell} > T_1)|0_T \text{ or } (\boldsymbol{\ell} < -T_1)] = 2P[\boldsymbol{\ell} > T_1|0_T] = 2Q\left(\frac{\ln 2+\frac{2E_b}{N_0}}{\sqrt{\frac{4E_b}{N_0}}}\right) \\
P[\text{error}|1_T] &= P[\{-T_1 < \boldsymbol{\ell} < T_1| -V \text{ xmitted}\} \text{ or } \{-T_1 < \boldsymbol{\ell} < T_1| +V \text{ xmitted}\}] \\
&= 2P[\{-T_1 < \boldsymbol{\ell} < T_1| +V \text{ xmitted}\}] \\
&= 2\left[Q\left(\frac{\frac{2E_b}{N_0}-T_1}{\sqrt{\text{var}}}\right) - Q\left(\frac{\frac{2E_b}{N_0}+T_1}{\sqrt{\text{var}}}\right)\right] \\
&= 2\left[Q\left(\frac{\frac{2E_b}{N_0}-\ln 2}{\sqrt{\frac{4E_b}{N_0}}}\right) - Q\left(\frac{\frac{6E_b}{N_0}+\ln 2}{\sqrt{\frac{4E_b}{N_0}}}\right)\right]
\end{aligned}$$

$$\begin{aligned} P[\text{bit error}] &= P[\text{bit 0 in error}]P[0_T] + P[\text{bit 1 in error}]P[1_T] \\ &= \frac{1}{2}P[\text{error } |0_T] + \frac{1}{2}P[\text{error } |1_T] \end{aligned}$$

P6.13 (a) The sketch(es) looks as follows in Fig. 6.36.

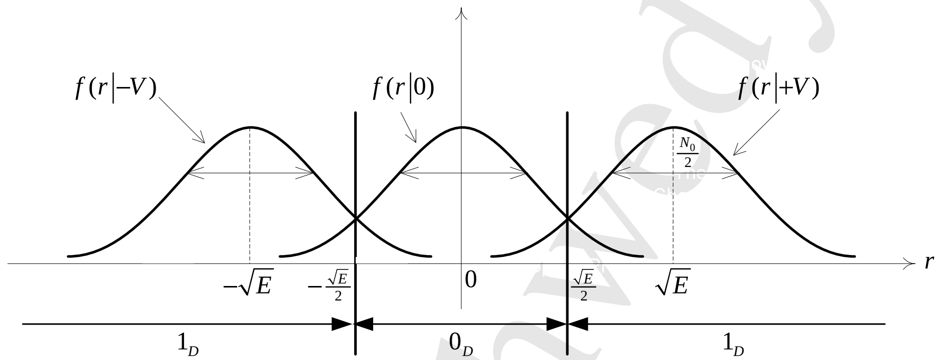


Figure 6.36

The decision rule is

$$\begin{cases} -\frac{\sqrt{E}}{2} \le r \le \frac{\sqrt{E}}{2}, & \text{choose } 0 \\ \text{otherwise,} & \text{choose } 1. \end{cases}$$

(b)

$$\begin{aligned} P[\text{error } |0_T] &= 2Q\left(\frac{\sqrt{E}/2}{\sqrt{N_0/2}}\right) = 2Q\left(\sqrt{\frac{E_b}{N_0}}\right) \\ P[\text{error } |1_T] &= 2\left[Q\left(\sqrt{\frac{E_b}{N_0}}\right) - Q\left(3\sqrt{\frac{E_b}{N_0}}\right)\right] \\ P[\text{bit error}] &= \frac{1}{2}P[\text{error } |0_T] + \frac{1}{2}P[\text{error } |1_T] \\ &= 2Q\left(\sqrt{\frac{E_b}{N_0}}\right) - Q\left(3\sqrt{\frac{E_b}{N_0}}\right) \end{aligned}$$

P6.14 (a) Yes, there should be. The first one, that of P6.11.

(b) Matlab works to be added.

P6.15 (a) What we need to know to from the past is whether the previous level to represent a 1_T is $+V$ or $-V$. Choose these as the states and the state diagram is shown in Fig. 6.37.

(b) Fig. 6.38 shows a trellis diagram.

P6.16 Change the mapping rule as follows (See Fig. 6.39:

- A 0 is always mapped to a level of 0 volts.
- A 1 is mapped to a pulse of level $+V$, if previous it was mapped to a pulse of level $-V$ and vice versa.

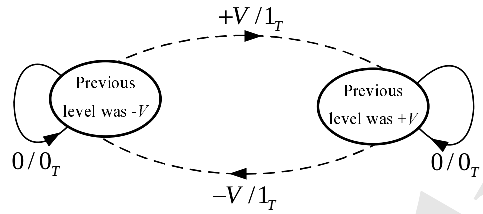


Figure 6.37

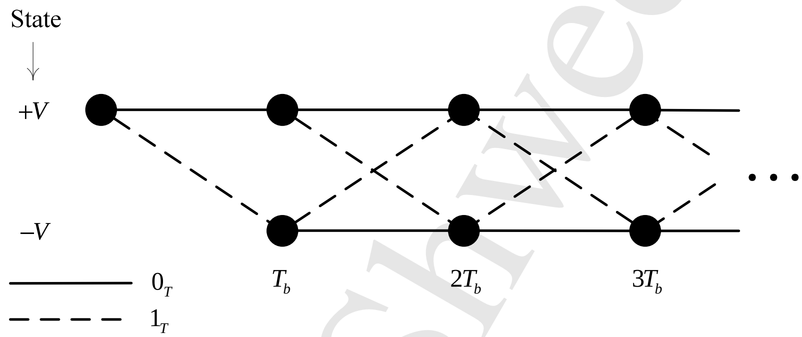


Figure 6.38

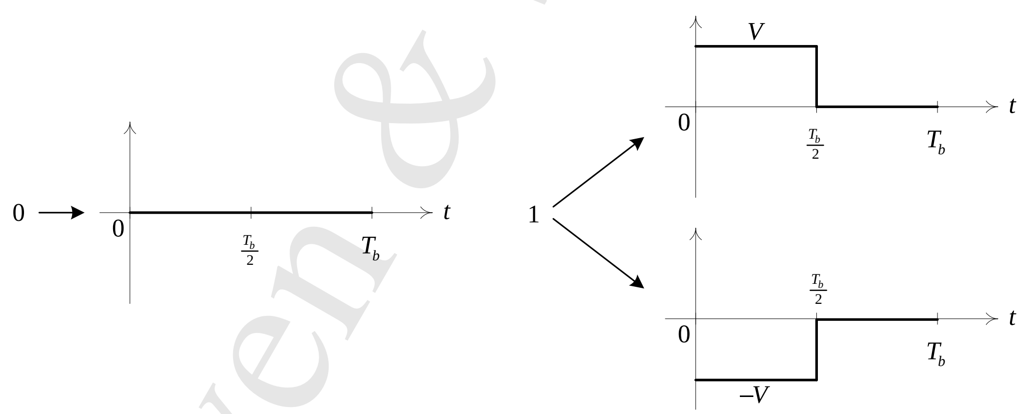


Figure 6.39

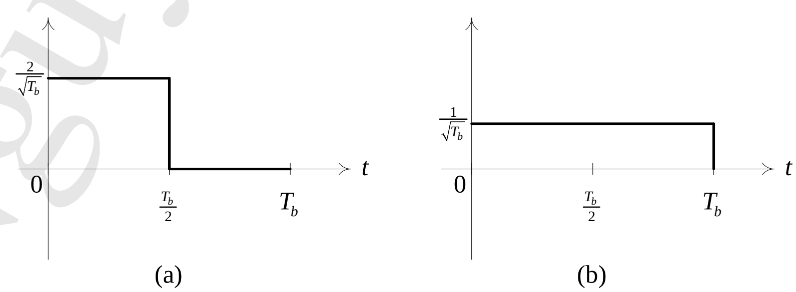


Figure 6.40

In essence nothing much changes. The main difference is the basis function used to represent the signals transmitted. Here it would be the function in Fig. 6.40(a) instead of the one in

Fig. 6.40(b).

If we adjust V here so that the energy level for AMI-RZ is the same as for AMI-NRZ then all error expressions and error performances for the 3 (including that of P6.13) demodulation methods are identical. If the voltage levels are kept the same then the energy E (and hence E_b) is halved for AMI-RZ.

Another difference would be in the PSD.

P6.17 (a) See Fig. 6.41.

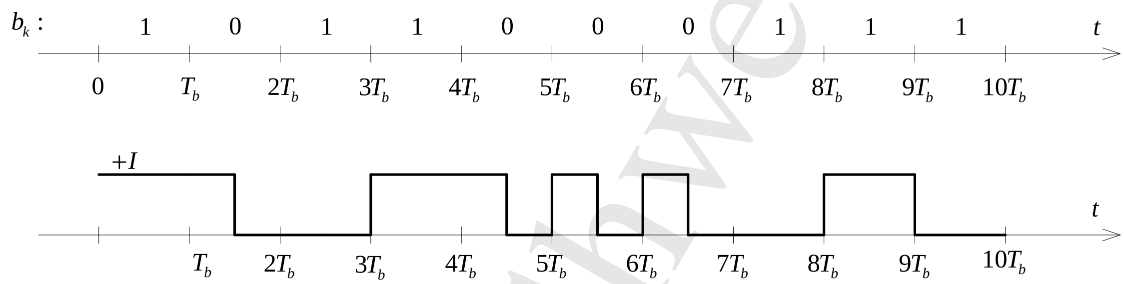


Figure 6.41

(b) See Fig. 6.42.

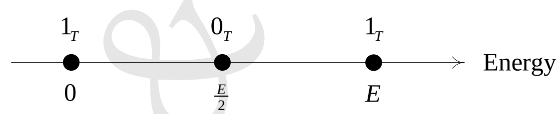


Figure 6.42

(c) Let the sufficient statistics be

$$\begin{aligned} I_1 &= \text{intensity, or some measure of it, in the 1st half} \\ I_2 &= \text{intensity, or some measure of it, in the 2nd half.} \end{aligned}$$

Then a possible decision rule would be as illustrated in Fig. 6.43.

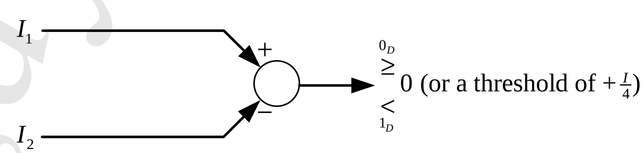


Figure 6.43

Take the polarity inversion to mean that the laser is off when it is on and vice versa. Then the above decision rule is not insensitive to polarity inversion. Indeed the decision would always be 1_D. However it can be modified to be polarity insensitive by the following decision rule:

$$|I_1 - I_2| \overset{0_D}{\underset{1_D}{\gtrless}} + \frac{I}{4}.$$

P6.18 (a) So k, the number of counted photons in time interval T_b, is our sufficient statistic.

$P[k$ photons emitted in a T_b interval $|1_T, 1_T$ represented by laser off$] = \frac{(\lambda_n T_b)^k \mathrm{e}^{-\lambda_n T_b}}{k!}$

$P[k$ photons emitted in a T_b interval $|1_T, 1_T$ represented by laser on$] = \frac{[(\lambda_s+\lambda_n)T_b]^k \mathrm{e}^{-(\lambda_s+\lambda_n)T_b}}{k!}$

$P[k$ photons emitted in a T_b interval $|1_T] = \frac{1}{2}\left[\frac{[(\lambda_s+\lambda_n)T_b]^k \mathrm{e}^{-(\lambda_s+\lambda_n)T_b}}{k!} + \frac{(\lambda_n T_b)^k \mathrm{e}^{-\lambda_n T_b}}{k!}\right]$

$P[k$ photons emitted in a T_b interval $|0_T] = \frac{1}{2}\left[\frac{[(\lambda_s+\lambda_n)\frac{T_b}{2}]^k \mathrm{e}^{-(\lambda_s+\lambda_n)\frac{T_b}{2}}}{k!} + \frac{(\lambda_n \frac{T_b}{2})^k \mathrm{e}^{-\lambda_n \frac{T_b}{2}}}{k!}\right]$

The likelihood ratio test (LRT) is:

$$\frac{1}{2}\,\frac{(\lambda_s+\lambda_n)^k T_b^k \mathrm{e}^{-(\lambda_s+\lambda_n)T_b} + \lambda_n^k T_b^k \mathrm{e}^{-\lambda_n T_b}}{(\lambda_s+\lambda_n)^k \frac{T_b^k}{2^k} \mathrm{e}^{-(\lambda_s+\lambda_n)\frac{T_b}{2}} + \lambda_n^k \frac{T_b^k}{2^k} \mathrm{e}^{-\lambda_n \frac{T_b}{2}}} \underset{0_D}{\overset{1_D}{\gtrless}} 1$$

$$\Leftrightarrow \frac{\left[\left(1+\frac{\lambda_s}{\lambda_n}\right)^k \mathrm{e}^{-\lambda_s T_b} + 1\right] \mathrm{e}^{-\lambda_n T_b}}{\left[\left(1+\frac{\lambda_s}{\lambda_n}\right)^k \mathrm{e}^{-\lambda_s T_b/2} + 1\right] \mathrm{e}^{-\lambda_n T_b/2}} \underset{0_D}{\overset{1_D}{\gtrless}} \frac{2}{2^k} = 2^{-k+1}$$

$$\Leftrightarrow \frac{\left[\left(1+\frac{\lambda_s}{\lambda_n}\right)^k \mathrm{e}^{-\lambda_s T_b} + 1\right]}{\left[\left(1+\frac{\lambda_s}{\lambda_n}\right)^k \mathrm{e}^{-\lambda_s T_b/2} + 1\right]} 2^{(k-1)} \underset{0_D}{\overset{1_D}{\gtrless}} \mathrm{e}^{\lambda_n T_b/2}.$$

(b) Unfortunately, an analytical expression for the error performance of the receiver in (a) appears to be intractable. Nonetheless one can resort to simulation.

P6.19 (a) See Fig. 6.44.

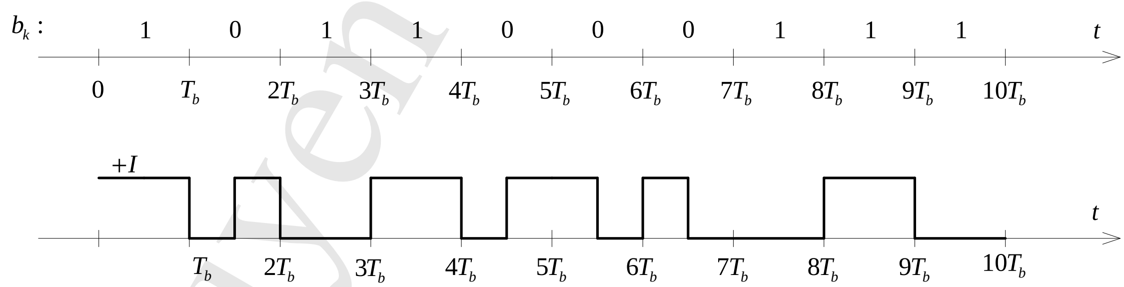


Figure 6.44

(b) Yes. But again depends on how decision rule is implemented. But basically if the demodulator decides on whether the laser is on or off for the whole bit interval OR on for only 1/2 the bit interval then the modulation is immune to polarity reversals.

(c) Let the axis be the energy of the transmitted pulse. The signals then plots as in Fig. 6.45.

(d) Let the states be

(i) previous 1 represented by $+I$, i.e., laser on for the whole bit interval,

(ii) previous 1 represented by 0, i.e., laser off for the whole bit interval.

The state diagram is shown in Fig. 6.46.

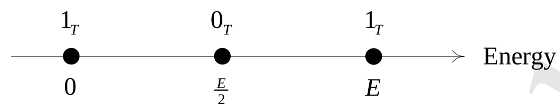


Figure 6.45

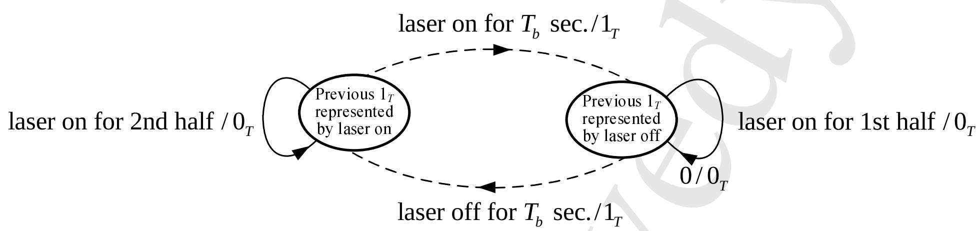


Figure 6.46

P6.20 (a) $P[\mathbf{c}_k = 1] = P[\mathbf{c}_k = -1] = \frac{1}{4}$; $P[\mathbf{c}_k = 0] = \frac{1}{2}$.

(b) $E\{\mathbf{c}_k^2\} = 0P[\mathbf{c}_k = 0] + 1P[\mathbf{c}_k = 1] + 1P[\mathbf{c}_k = -1] = \frac{1}{2}$.

Consider now $E\{\mathbf{c}_k\mathbf{c}_{k+1}\}$. Set up a table (see Table 6.2) for the possibilities for $\mathbf{c}_k$, $\mathbf{c}_{k+1}$ and the product. $\therefore E\{\mathbf{c}_k\mathbf{c}_{k+1}\} = -1(1/4) + 0(1/4) + 0(1/4) + 0(1/4) = -1/4$

Table 6.2

$\mathbf{b}_k$	$\mathbf{b}_{k+1}$	$P[\mathbf{b}_k, \mathbf{b}_{k+1}]$	$\mathbf{c}_k$	$\mathbf{c}_{k+1}$		$\mathbf{c}_k\mathbf{c}_{k+1}$	$P[\mathbf{c}_k\mathbf{c}_{k+1}] = P[\mathbf{b}_k\mathbf{b}_{k+1}]$
1	1	1/4	1 -1	-1 1	$\rightarrow$	-1	1/4
0	1	1/4	0 0	1 -1	$\rightarrow$	0	1/4
1	0	1/4	1 -1	0 0	$\rightarrow$	0	1/4
0	0	1/4	0	0	$\rightarrow$	0	1/4

Now consider $\mathcal{E}\{\mathbf{c}_k\mathbf{c}_{k+n}\}$, $n > 1$ and the Table 6.3. If the bits $\mathbf{b}_k$ are statistically independent and equally probable then each of combinations of $\mathbf{c}_k$, $\mathbf{c}_{k+n}$ is equally probable, i.e., $P[\mathbf{c}_k\mathbf{c}_{k+n}] = 1/9$.

$\therefore E\{\mathbf{c}_k\mathbf{c}_{k+n}\} = 1(1/9) - 1(1/9) - 1(1/9) + 1(1/9) = 0$. Therefore they are uncorrelated.

Now Equation (5.127) in the textbook states that:

$$S(f) = \frac{|P(f)|^2}{T_b} \sum_{m=-\infty}^{\infty} R_{\mathbf{c}}(m) \mathrm{e}^{-j2\pi m f T_b}$$

Table 6.3

$\mathbf{c}_k$	$\mathbf{c}_{k+n}$	$\mathbf{c}_k\mathbf{c}_{k+n}$
1	1	1
1	−1	−1
1	0	0
−1	1	−1
−1	−1	1
−1	0	0
0	1	0
0	−1	0
0	0	0

where $R_{\mathbf{c}}(m)$ is the autocorrelation we have just determined and

$$\begin{aligned} P(f) &= \mathcal{F}\{\text{NRZ signal}\} = \int_0^{T_b} V\mathrm{e}^{-j2\pi ft}\mathrm{d}t \\ &= \frac{V}{\pi f}\mathrm{e}^{-j\pi fT_b}\underbrace{\left[\frac{\mathrm{e}^{j\pi fT_b}-\mathrm{e}^{-j\pi fT_b}}{2j}\right]}_{\sin(\pi fT_b)} \\ &\Rightarrow\ |P(f)|^2 = V^2T_b^2\frac{\sin^2(\pi fT_b)}{(\pi fT_b)^2} \end{aligned}$$

Finally,

$$S(f) = V^2T_b\frac{\sin^2(\pi fT_b)}{(\pi fT_b)^2}\underbrace{\left[R_{\mathbf{c}}(-1)\mathrm{e}^{j2\pi fT_b}+R_{\mathbf{c}}(0)+R_{\mathbf{c}}(1)\mathrm{e}^{-j2\pi fT_b}\right]}_{=\underbrace{-\frac{1}{4}[2\cos(2\pi fT_b)]+\frac{1}{2}}_{=\sin^2(\pi fT_b)}}.$$

P6.21 (a) The decision rule of (6.14) is compute $d_i = \sqrt{\int_0^{nT_b}[r(t)-S_i(t)]^2\mathrm{d}t}$ for each possible transmitted sequence and choose the smallest. Since d_i is obviously ≥ 0 this decision rule can be restated as compute d_i^2 for each possible transmitted sequence and choose the smallest, i.e.,

compute

$$d_i^2 = \int_0^{nT_b} r^2(t)\mathrm{d}t - 2\int_0^{nT_b} r(t)S_i(t)\mathrm{d}t + \int_0^{nT_b} S_i^2(t)\mathrm{d}t,\ i=1,\ldots,M=2^n,$$

and choose smallest.

Now the term $\int_0^{nT_b} r^2(t)\mathrm{d}t$ is the same for each i and can be discarded. Similarly $\int_0^{nT_b} S_i^2(t)\mathrm{d}t$ is the same regardless of i, i.e., for Miller modulation each transmitted signal (sequence) has the same energy. It may also be discarded. Therefore the decision rule can be stated as:

$$\begin{array}{l}\text{Compute}\\ -2\displaystyle\int_0^{nT_b} r(t)S_i(t)\mathrm{d}t,\ i=1,\ldots,m=2^n,\\ \text{and choose smallest.}\end{array}$$

The factor 2 does not affect which i yields the smallest relative value. Multiplying though by -1 changes the smallest into largest. Therefore the decision rule is:

$$\begin{array}{l}\text{Compute}\\ \displaystyle\int_0^{nT_b} r(t)S_i(t)\mathrm{d}t,\ i=1,\ldots,m=2^n,\\ \text{and choose smallest.}\end{array}$$

Writing this as $\displaystyle\sum_{j=1}^{n}\int_{(j-1)T_b}^{jT_b} r_j(t)S_{ij}(t)\mathrm{d}t = \sum_{j=1}^{n}[r_1^{(j)}, r_2^{(j)}]\begin{bmatrix} s_{i1}^{(j)} \\ s_{i2}^{(j)} \end{bmatrix}$ we get the desired result.

(b) The steps are essentially the same as before, except now one computes the crosscorrelation between the signal projections and the signal components along a branch, adds to the surviving value of the state from which the branch emanates and compares with all the incoming path values at the state on which the branch terminates. Choose the largest and discard all others. Do this for each state.

Table 6.4: Cross correlation table: $r_1s_1 + r_2s_2$

(s_1, s_2)	$0 \to T_b$	$T_b \to 2T_b$	$2T_b \to 3T_b$	$3T_b \to 4T_b$
$s_1(t) \to (1,0)$	-0.2	0.2	-0.61	-1.1
$s_2(t) \to (0,1)$	-0.4	-0.8	0.5	0.1
$s_3(t) \to (-1,0)$	0.2	-0.2	0.61	1.1
$s_4(t) \to (0,-1)$	0.4	0.8	-0.5	-0.1

(c) Trellis showing the branch correlations, i.e., $r_1^{(j)}s_{i1}^{(j)} + r_2^{(j)}s_{i2}^{(j)}$, is in Fig. 6.47.
The pruned trellis showing the *survivors* & partial path correlations is in Fig. 6.48.
To compare with the minimum distance approach remember that the minimum distance corresponds to maximum correlation, i.e., the relationship is an inverse one.

P6.22 (a) By error probability, what is meant is *bit error probability.* We are not all that interested in symbol error probability. From the symmetry, we can say that

$$P[\text{bit error}] = P[\text{bit error }|0_T] = P[\text{bit error }|s_1(t)] \text{ (or } P[\text{bit error }|s_2(t)]).$$

A bit error occurs when $s_1(t)$ is transmitted whenever the sufficient statistics (r_1, r_2) are closer to either $s_3(t)$, $s_4(t)$ then to $s_1(t)$ (or $s_2(t)$). The geometrical picture looks as in Fig. 6.49.
Clearly,

$$P[\text{bit error}] = Q\left(\frac{\sqrt{E}}{\sqrt{2}\sqrt{N_0/2}}\right) = Q\left(\sqrt{\frac{E}{N_0}}\right).$$

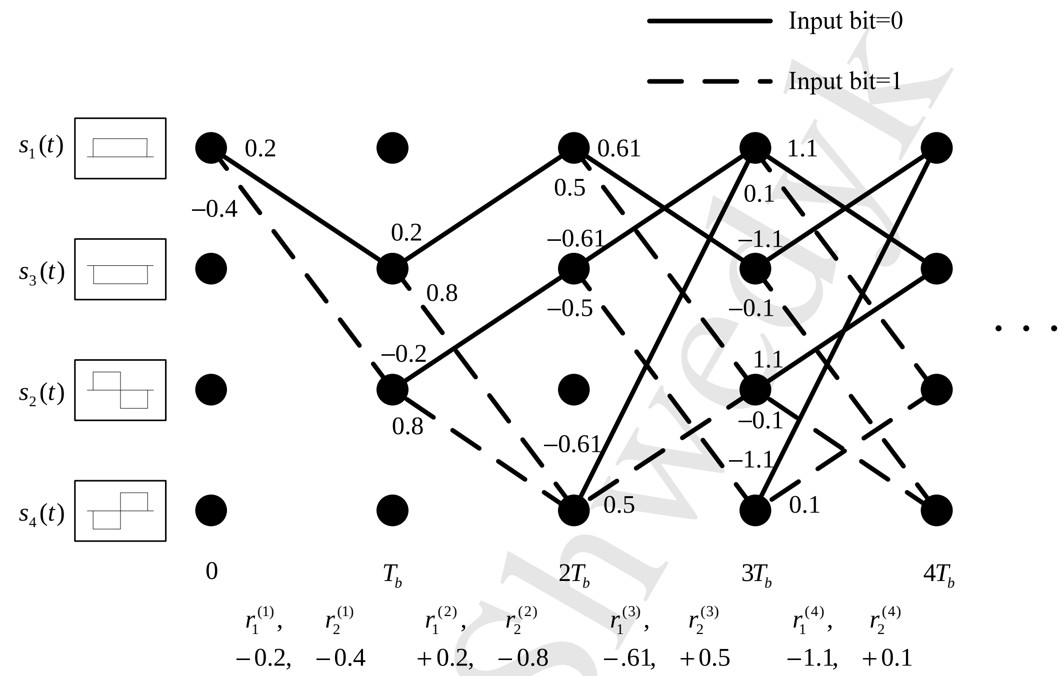


Figure 6.47

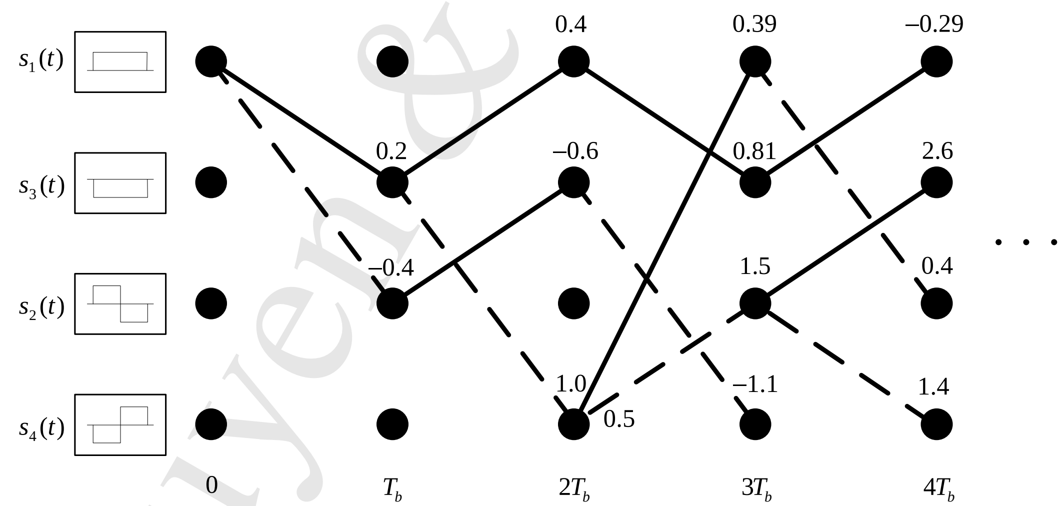
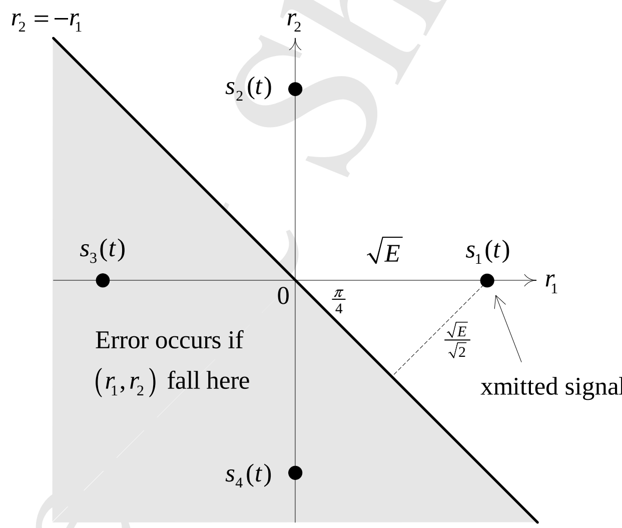


Figure 6.48

(b) Matlab works to be added.

P6.23 Matlab works to be added.



$r_2 = -r_1$
r_2
$s_2(t)$
$s_3(t)$
$\sqrt{E}$
$s_1(t)$
r_1
0
$\frac{\pi}{4}$
$\frac{\sqrt{E}}{\sqrt{2}}$
Error occurs if
(r_1, r_2) fall here
xmitted signal
$s_4(t)$

Figure 6.49

Chapter 7

Basic Digital Passband Modulation

P7.1

$$\left.\begin{array}{l}\text{Antenna diameter: } d = \dfrac{\lambda}{4} \\ \text{wave length: } \lambda = \dfrac{c}{f} = \dfrac{3\times 10^8}{f}\end{array}\right\} \Rightarrow\ d = \frac{c}{4f} = \frac{3\times 10^8}{4f}$$

(a) Direct coupling:

$$f = 4\text{kHz} \Rightarrow\ d = \frac{3\times 10^8}{4\times 4\times 10^3} = 18.75\times 10^3(\text{m}) = 18.75(\text{km})$$

(b) Via modulation with carrier frequency $f_c = 1.2$GHz

$$\Rightarrow d = \frac{3\times 10^8}{4f_c} = \frac{3\times 10^8}{4\times 1.2\times 10^9} = 62.5\times 10^{-3}(\text{m}) = 6.25(\text{cm})$$

The above results clearly show that carrier-wave or passband modulation is an essential step for all systems involving radio transmission.

P7.2 Express NRZ signal in terms of NRZ-L signal as follows:

$$\mathbf{s}_{\text{NRZ}}(t) \quad = \quad 1/2\left[\mathbf{s}_{\text{NRZ-L}}(t) + 1\right] \tag{7.1}$$

Note that the factor 1/2 scales the autocorrelation by $(1/2)^2 = 1/4$, the constant 1 shifts (vertically) the autocorrelation by 1 and $\mathbf{s}_{\text{NRZ-L}}(t)$ has an autocorrelation of $R_{\mathbf{s}_{\text{NRZ-L}}}(\tau)$. Putting all this together we have

$$R_{\mathbf{s}_{\text{NRZ}}}(\tau) = \frac{1}{4}\left[R_{\mathbf{s}_{\text{NRZ-L}}}(\tau) + 1\right] = \begin{cases}\frac{1}{4} + \frac{1}{4}\left(1 - \frac{|\tau|}{T_b}\right), & |\tau| \le T_b \\ \frac{1}{4}, & |\tau| > T_b\end{cases}$$

Try to generalize this: Let $\mathbf{y}(t) = K[\mathbf{x}(t) + A]$ where K, A are constants. $\mathbf{x}(t)$ has mean value $m_{\mathbf{x}}$ and autocorrelation, $R_{\mathbf{x}}(\tau)$. What then is $R_{\mathbf{y}}(\tau)$?

P7.3 Consider (7.41), $P[m_2|m_1]$. This probability is the volume under $f(\hat{r}_2, \hat{r}_1|m_1)$ in the 1st quadrant as shown below. Note that given m_1, $\hat{\mathbf{r}}_2$ and $\hat{\mathbf{r}}_1$ are statistically independent Gaussian random variables, variance $= \frac{N_0}{2}$, and means $-\sqrt{\frac{E_s}{2}}$, $\sqrt{\frac{E_s}{2}}$, respectively. The volume is therefore

$$\int_{\hat{r}_2=0}^{\infty}\int_{\hat{r}_1=0}^{\infty} f(\hat{r}_2|m_1)f(\hat{r}_1|m_1)\mathrm{d}\hat{r}_2\mathrm{d}\hat{r}_1 = \left[\int_{\hat{r}_2=0}^{\infty} f(\hat{r}_2|m_1)\mathrm{d}\hat{r}_2\right]\left[\int_{\hat{r}_1=0}^{\infty} f(\hat{r}_1|m_1)\mathrm{d}\hat{r}_1\right]$$

$$
\begin{aligned}
P[m_2|m_1] &= \left[\frac{1}{\sqrt{2\pi}\sqrt{\frac{N_0}{2}}}\int_{\hat{r}_2=0}^{\infty} \mathrm{e}^{-\frac{\left(\hat{r}_2+\sqrt{\frac{E_s}{2}}\right)^2}{2\left(\frac{N_0}{2}\right)}}\mathrm{d}\hat{r}_2\right]\left[\frac{1}{\sqrt{2\pi}\sqrt{\frac{N_0}{2}}}\int_{\hat{r}_1=0}^{\infty} \mathrm{e}^{-\frac{\left(\hat{r}_1+\sqrt{\frac{E_s}{2}}\right)^2}{2\left(\frac{N_0}{2}\right)}}\mathrm{d}\hat{r}_1\right] \\
&= \left[1-Q\left(\frac{\sqrt{E_s/2}}{N_0/2}\right)\right]Q\left(\frac{\sqrt{E_s/2}}{N_0/2}\right) = \left[1-Q\left(\frac{E_s}{N_0}\right)\right]Q\left(\frac{E_s}{N_0}\right)
\end{aligned}
$$

Graphically the 2 integrals above compute the areas illustrated in Fig. 7.1

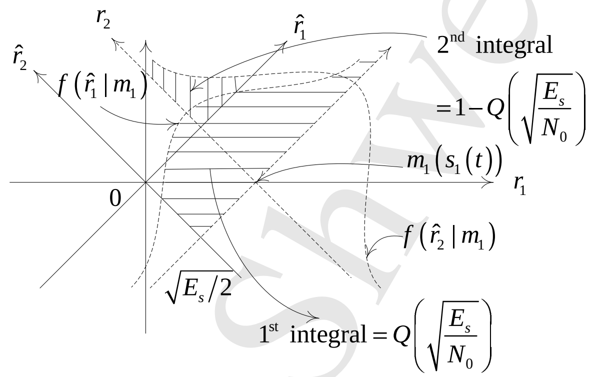


Figure 7.1: Areas to compute $P[m_2|m_1]$.

Similarly $P[m_3|m_1]$ is given by the volume under $f(\hat{r}_2, \hat{r}_1|m_1)$ in the 2nd quadrant, i.e., by the product of the areas in Fig. 7.2.

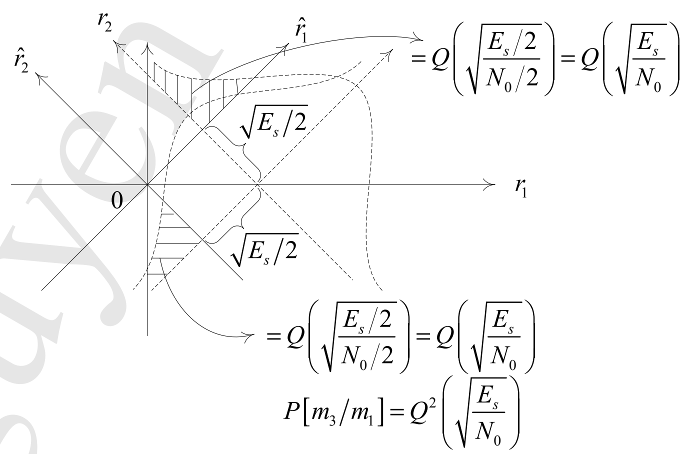
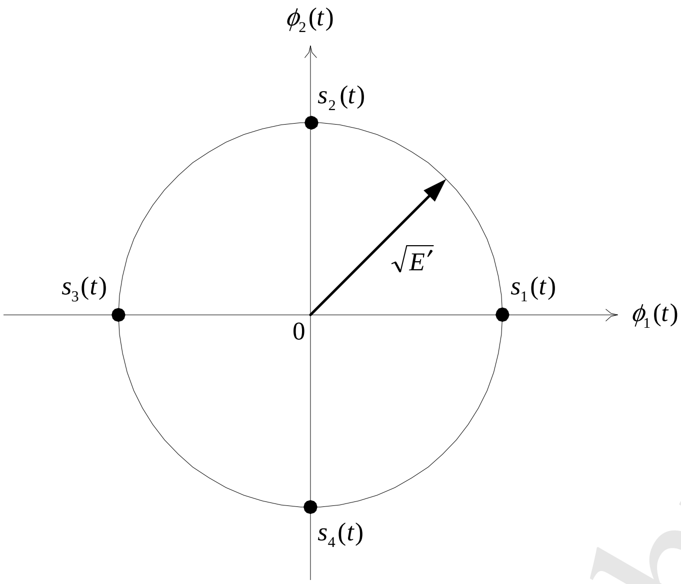


Figure 7.2: Areas to compute $P[m_3|m_1]$.

Finally $P[m_4|m_1] = P[m_2|m_1]$ – as can be seen graphically.

P7.4 (a) As in the case of QPSK with Gray mapping, to determine the bit error probability for the above mapping it is necessary to distinguish between the different message errors (or signal errors). Again because of symmetry it is sufficient to consider only a specific



Bit Pattern	Signal Transmitted
00	$s_1(t)$
11	$s_2(t)$
10	$s_3(t)$
01	$s_4(t)$

Figure 7.3: QPSK modulation.

signal, say $s_1(t)$. Then the different error probabilities are

$$P[s_2(t)|s_1(t)] = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\left[1-Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right] \tag{7.2}$$

$$P[s_3(t)|s_1(t)] = Q^2\left(\sqrt{\frac{2E_b}{N_0}}\right) \tag{7.3}$$

$$P[s_4(t)|s_1(t)] = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\left[1-Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right] \tag{7.4}$$

Note that the above result does not depends on the specific mapping. Then the bit error probability is given by

$$\begin{aligned}P[\text{bit error}] &= \\ &= 1.0P[s_2(t)|s_1(t)] + 0.5P[s_4(t)|s_1(t)] + 0.5P[s_3(t)|s_1(t)] = \\ &= 1.5Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - Q^2\left(\sqrt{\frac{2E_b}{N_0}}\right) \approx 1.5Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\end{aligned} \tag{7.5}$$

(b) Recall that the bit error probability of QPSK employing Gray mapping is $Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$. Compared to (7.5) we see that with anti-Gray mapping the bit error probability is increased by a factor of 3/2. The plots of both error probabilities are shown in Figure 7.4.

P7.5 (a) Since the symbols are equiprobable and the channel is AWGN the demodulator is a minimum distance one. The decision boundary and decision regions are shown in Fig. 7.5.

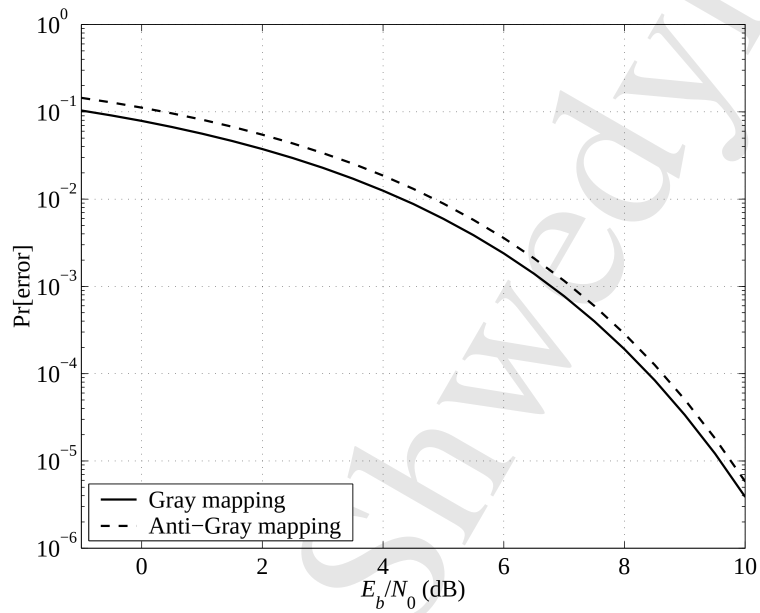


Figure 7.4: Error performance of QPSK with Gray and Anti-Gray mapping.

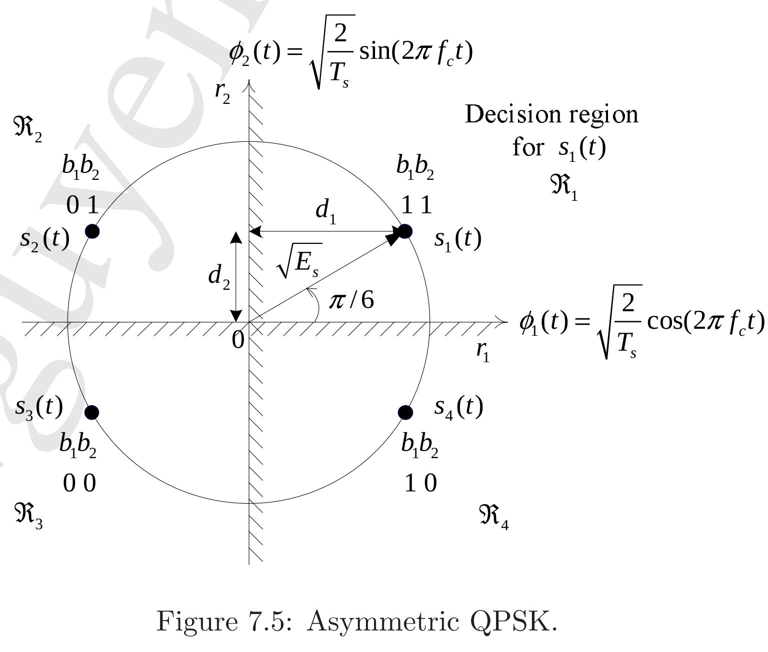


Figure 7.5: Asymmetric QPSK.

(b) In Fig. 7.5, $d_1 = \sqrt{E_s}\cos\theta$ and $d_2 = \sqrt{E_s}\sin\theta$, where $\theta = \pi/6$.

$$\begin{aligned}
P[\text{symbol error}] &= 1 - \Pr[\text{symbol correct}] \\
&= 1 - \Pr[\text{symbol correct}|s_1(t)] \quad \text{(due to symmetry)} \\
&= 1 - P[(\mathbf{r}_1,\mathbf{r}_2) \in \Re_1|s_1(t)] = 1 - P[\mathbf{r}_1 \geq 0|s_1(t)]P[\mathbf{r}_2 \geq 0|s_1(t)] \\
&= 1 - \left[1 - Q\left(\frac{d_1}{\sqrt{N_0/2}}\right)\right]\left[1 - Q\left(\frac{d_2}{\sqrt{N_0/2}}\right)\right] \\
&= Q\left(\frac{d_1}{\sqrt{N_0/2}}\right) + Q\left(\frac{d_2}{\sqrt{N_0/2}}\right) - Q\left(\frac{d_1}{\sigma}\right)Q\left(\frac{d_2}{\sqrt{N_0/2}}\right) \\
&= Q\left(\sqrt{\frac{2E_s}{N_0}}\cos\theta\right) + Q\left(\sqrt{\frac{2E_s}{N_0}}\sin\theta\right) - Q\left(\sqrt{\frac{2E_s}{N_0}}\cos\theta\right)Q\left(\sqrt{\frac{2E_s}{N_0}}\sin\theta\right).
\end{aligned}$$

(c)

$$\begin{aligned}
P[b_1 \text{ in error}] &= P[b_1 \text{ in error}|s_1(t)] = P[(\mathbf{r}_1,\mathbf{r}_2) \in \Re_2 \text{ or } \Re_3|s_1(t)] \\
&= P[\mathbf{r}_1 < 0|s_1(t)] = Q\left(\frac{d_1}{\sqrt{N_0/2}}\right) = Q\left(\sqrt{\frac{2E_s}{N_0}}\cos\theta\right). \qquad (7.6)
\end{aligned}$$

Similarly,

$$P[b_2 \text{ in error}] = Q\left(\frac{d_2}{\sqrt{N_0/2}}\right) = Q\left(\sqrt{\frac{2E_s}{N_0}}\sin\theta\right). \qquad (7.7)$$

With $\theta = \pi/6$, $\sin\theta = \frac{1}{2} < \cos\theta = \frac{\sqrt{3}}{2}$. Since the Q-function is a monotonically decreasing function of its argument, $P[b_1 \text{ in error}] < P[b_2 \text{ in error}]$ and thus bit b_1 is more protected than bit b_2.

Note that in general, if $0 < \theta < \pi/4$, then bit b_1 is more protected. If $\pi/4 < \theta < \pi/2$, then b_2 is more protected. And if $\theta = \pi/4$?

(d)

$$\begin{aligned}
P[b_1 \text{ in error}] &= Q\left(\sqrt{1.5\frac{E_s}{N_0}}\right) \leq 10^{-3} \Rightarrow 1.5\frac{E_s}{N_0} \geq \left[Q^{-1}(10^{-3})\right]^2 = 3.09^2 \\
&\Rightarrow E_s \geq \frac{3.09^2}{1.5}N_0 = \frac{3.09^2}{1.5} \times 10^{-6} = 6.37 \times 10^{-6} \text{ (joules)}. \qquad (7.8)
\end{aligned}$$

Choose equality to keep E_s as small as possible.

P7.6

$$\begin{aligned}
\text{Sending:} \quad s(t) &= \cos(2\pi f_c t);\ f_c = 1.2\text{GHz} \\
\text{Receiving:} \quad r(t) &= \cos[2\pi f_c(t + T_d)] = \cos(2\pi f_c t + 2\pi f_c T_d)
\end{aligned}$$

where T_d is the propagation delay corresponding to the distance d. Assuming ideal electromagnetic propagation at the speed of light, then $T_d = \frac{d}{c}$, where $c = 3 \times 10^8$m/s.

(a) If the mobile user moves in-line away from the base station (to point B), or in-line toward the base station (to point C), then the difference in propagation delay is:

$$T_d' = \pm\frac{d'}{c}$$

$\Rightarrow$ The difference in phase shift is:

$$\begin{aligned}\Delta\theta &= 2\pi f_c T_d' = \pm\frac{2\pi f_c d'}{c} = \pm 2\pi \\ \Rightarrow d' &= \frac{2\pi c}{2\pi f_c} = \frac{c}{f_c} = \frac{3\times 10^8}{1.2\times 10^9} = 0.25(\text{m}) = 25 \text{ cm}\end{aligned}$$

Thus a movement of only 25cm causes a maximum phase rotation of 2π radian.

(b) Of course we do not care about a 2π phase rotation because it does not change the transmitted signal. Typically, a phase rotation of $\pi/2$ or π is a more serious problem (as will be seen in Problem 7.7). The minimum distances of user's movement to cause a $\frac{\pi}{2}$ and π phase rotations are as follows:

$$\begin{aligned}\Delta\theta &= \pi/2 \Rightarrow d" = d'/4 = 25/4 = 6.25 \text{ cm} \\ \Delta\theta &= \pi \Rightarrow d" = d'/2 = 25/2 = 12.5 \text{ cm}\end{aligned}$$

If BPSK is used then it would appear that moving your cell phone from one ear to another would (depending on the size of your head) have a severe effect on the receiver performance. One should either track the incoming phase, say by a phase-locked loop which is the subject of Chapter 12 or consider modulation/demodulation techniques that are present in the face of phase uncertainty – this is the subject of Chapter 10.

P7.7

$$\mathbf{r}(t) = \begin{cases} s_1^R(t) + \mathbf{w}(t) = 0 + \mathbf{w}(t) & : \text{“}0_T\text{”} \\ s_2^R(t) + \mathbf{w}(t) = \sqrt{E}\sqrt{\dfrac{2}{T_b}}\cos(2\pi f_c t + \theta) + \mathbf{w}(t) & : \text{“}1_T\text{”}\end{cases}$$

Without the noise, the receiver sees one of the two signals

$$\begin{aligned} &s_1^R(t) = 0 && : \text{“}0_T\text{”} \\ &s_2^R(t) = \sqrt{E}\sqrt{\tfrac{2}{T_b}}\cos(2\pi f_c t + \theta) && : \text{“}1_T\text{”}\end{aligned}$$

Write $s_2^R(t)$ as follows:

$$s_2^R(t) = \left(\sqrt{E}\cos\theta\right)\underbrace{\sqrt{\frac{2}{T_b}}\cos(2\pi f_c t)}_{\phi_1(t)} + \left(\sqrt{E}\sin\theta\right)\underbrace{\sqrt{\frac{2}{T_b}}\sin(2\pi f_c t)}_{\phi_2(t)}.$$

Thus, for an arbitrary θ two basis functions are required to represent $s_1^R(t)$ and $s_2^R(t)$ as shown in Fig. 7.6(a).

(a) If the receiver assumes that $\theta = 0$, then the optimum decision boundary is a line perpendicular to $\phi_1(t)$ and at $\sqrt{E}/2$ distance from $s_1^R(t)$ (see Figure 7.6-(a)). With this receiver, the decision is based on $\mathbf{r}_1$:

$$\mathbf{r}_1 = \int_0^{T_b}\mathbf{r}(t)\phi_1(t)\mathrm{d}t = \begin{cases} 0 + \mathbf{w}_1 & : \text{“}0_T\text{”} \\ \sqrt{E}\cos\theta + \mathbf{w}_1 & : \text{“}1_T\text{”}\end{cases}$$

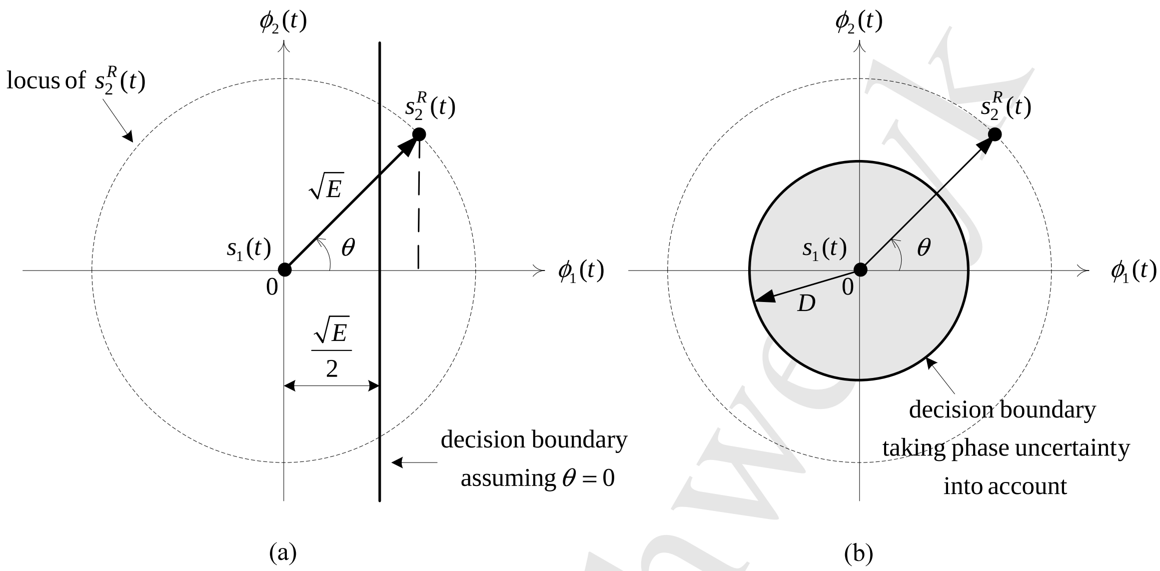


Figure 7.6: Signal space diagram for ASK with phase uncertainty.

where, as usual, $\mathbf{w}_1$ is a zero-mean Gaussian random variable with variance $N_0/2$. The decision rule can be expressed as

$$\mathbf{r}_1 \underset{0_T}{\overset{1_T}{\gtrless}} \sqrt{E}/2 \tag{7.9}$$

The error performance of the receiver that implements the above decision rule can be calculated as follows:

$$\begin{aligned} P[\text{error}] &= P[0_T]P[\mathbf{r}_1 \geq \sqrt{E}/2|s_1(t)] + P[1_T]P[\mathbf{r}_1 \leq \sqrt{E}/2|s_2(t)] \\ &= \frac{1}{2}Q\left(\frac{\sqrt{E}/2}{\sqrt{N_0/2}}\right) + \frac{1}{2}Q\left(\frac{\sqrt{E}\cos\theta - \sqrt{E}/2}{\sqrt{N_0/2}}\right) \\ &= 0.5Q\left(\sqrt{\frac{E}{2N_0}}\right) + 0.5Q\left(\sqrt{\frac{2E}{N_0}}[\cos\theta - 0.5]\right) \end{aligned} \tag{7.10}$$

Equation (7.10) shows that $P[\text{error}]$ increases as θ increases from 0 to π. This implies that the phase uncertainty degrades the error performance of the optimum receiver designed for ASK assuming no phase shift.

– For $\theta = 30^o \Rightarrow \cos\theta = \frac{\sqrt{3}}{2}$. Then

$$P[\text{error}] = 0.5Q\left(\sqrt{\frac{E}{2N_0}}\right) + 0.5Q\left(\sqrt{\frac{E}{2N_0}}(\sqrt{3}-1)\right)$$

– For $\theta = 60^o \Rightarrow \cos\theta = \frac{1}{2}$. Then

$$P[\text{error}] = 0.5Q\left(\sqrt{\frac{E}{2N_0}}\right) + \frac{1}{4}$$

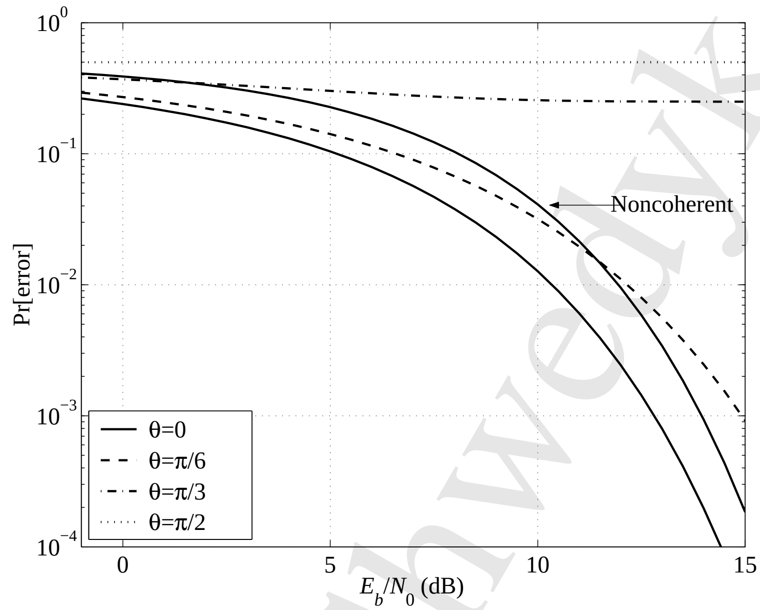


Figure 7.7: Error performance of ASK under phase uncertainty.

– For $\theta = 90^o \Rightarrow \cos\theta = 0$. One has

$$\begin{aligned} P[\text{error}] &= 0.5Q\left(\sqrt{\frac{E}{2N_0}}\right) + 0.5Q\left(-\sqrt{\frac{E}{2N_0}}\right) \\ &= 0.5Q\left(\sqrt{\frac{E}{2N_0}}\right) + 0.5\left[1 - Q\left(\sqrt{\frac{E}{2N_0}}\right)\right] = 0.5 \end{aligned}$$

The result for $\theta = 90^o$ is expected. Since in this case the 2 signals have the same projection along $\phi_1(t)$, i.e., they are indistinguishable. We would expect that the best we can do is to guess, since $\mathbf{r}_1$ gives no information about which signal was transmitted and the error probability should be 1/2.
Fig. 7.7 plots the error performance of the conventional optimum receiver for different values of θ as calculated above. It is clear that the knowledge of the phase θ is crucial for the implementation of the "conventional" optimum receiver. All the receivers discussed in the text so far require this knowledge and they are called the "COHERENT" receivers. For coherent receivers, a circuit known as phase-locked loop (PLL) needs to be built to estimate the phase of the incoming signals (see Chapter 12).

(b) Without PLL, the receiver needs to partition the signal space into two regions that are robust to the phase uncertainty. A rather obvious choice for the decision boundary is a circle centered at $s_1^R(t)$ and with some diameter D as shown in Fig. 7.6-(b). In essence, this receiver makes the decision based on the output energy, not the voltage values.

An interesting question is "what is the optimum value of D to minimize the probability of error for the above receiver?". Finding the answer to this question involves Rician and Rayleigh probability density functions (see Chapter 10). It can be shown that the optimum value of D is:

$$D = \frac{\sqrt{E}}{\sqrt{2}}\left(1 + \frac{4N_0}{E}\right). \tag{7.11}$$

The above shows that the optimum "threshold" depends on the actual noise level (channel condition). Typically $E/N_0 \gg 1$ or $N_0/E \ll 1$ and D is well approximated by

$$D \approx \sqrt{\frac{E}{2}}. \tag{7.12}$$

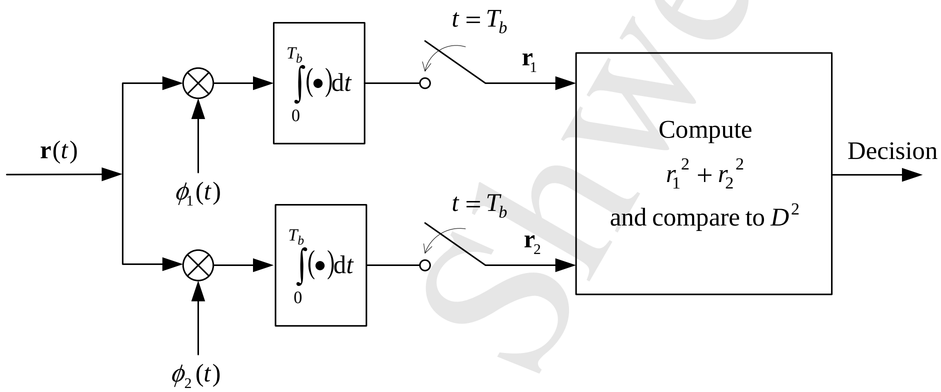


Figure 7.8: Implementation of the noncoherent receiver for ASK.

The receiver in Fig. 7.8 does not require the phase information and it is called the "noncoherent" receiver. With D given by (7.12), it can be shown that the error probability of the noncoherent receiver is approximately:

$$P[\text{error}] \approx \frac{1}{2}\mathrm{e}^{-\frac{E}{4N_0}}$$

The above performance is also plotted in Fig. 7.7 for comparison. Observe that there is a loss of about 1 to 2 dB in SNR compared to coherent receiver. However, the noncoherent receiver is much simpler to implement since it does not require a PLL.

A final remark is that the notation E in this question is the energy of signal $s_2(t)$; not the average energy per bit. The average energy per bit for this case is $E_b = E/2$.

P7.8 (a) See Fig. 7.9.

(b) The receiver in this case projects onto the $\phi_1(t)$ axis to get r_1 and compares r_1 to a threshold of 0, i.e., $r_1 \underset{0_D}{\overset{1_D}{\gtrless}} 0$ where it is assumed that $1_T \leftrightarrow s_1^T(t)$, $0_T \leftrightarrow s_2^T(t)$.

The signal projection is either $\pm k\sqrt{E_b}\cos\theta$ and the noise projection has a variance of $N_0/2$. Therefore

$$P[\text{error}] = Q\left(\frac{k\sqrt{E_b}\cos\theta}{\sqrt{N_0/2}}\right) = Q\left(k\sqrt{\frac{2E_b}{N_0}}\cos\theta\right).$$

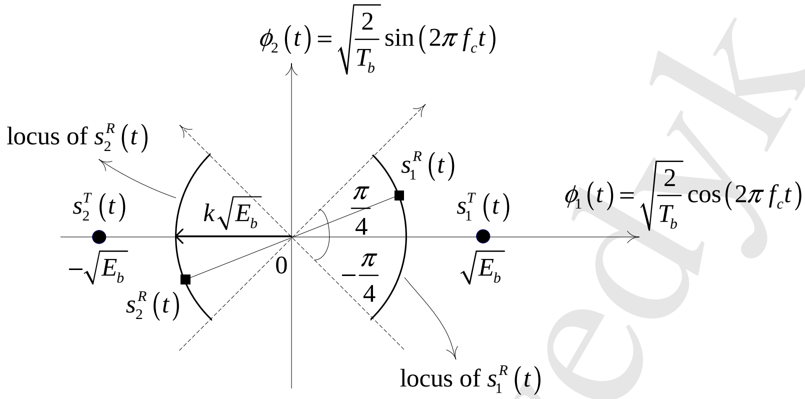


Figure 7.9

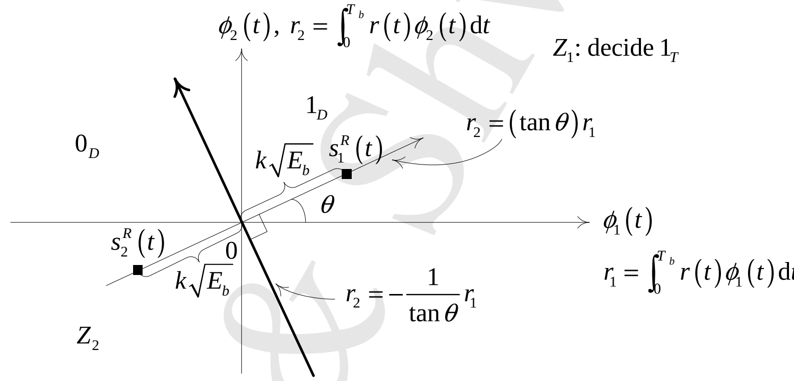


Figure 7.10

(c) See Fig. 7.10.

$$P[\text{error}] = Q\left(k\sqrt{\frac{2E_b}{N_0}}\right)$$

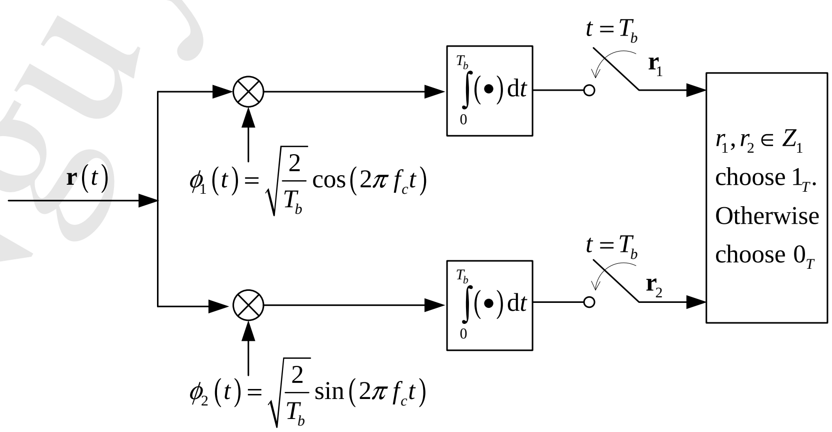


Figure 7.11

Remark: Another receiver, and a simpler one, is one that needs only ONE projection. What is it?

P7.9 Signal space diagrams at the transmitter and receiver are shown below:

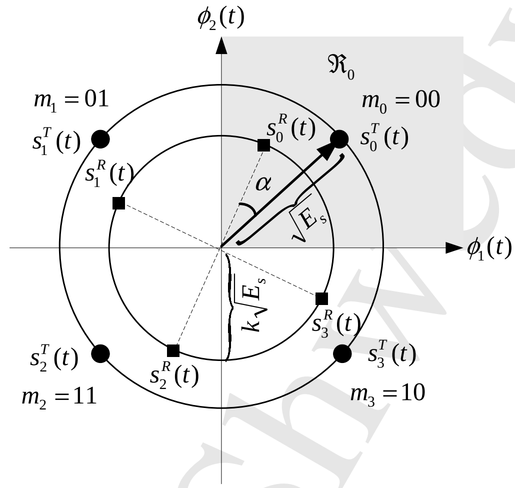


Figure 7.12

(b) The symbol error probability can be determined as follows:

$$P[\text{error}] = \sum_{i=0}^{3} P[\text{error}|s_i^T(t)]P[s_i^T(t)] = P[\text{error}|s_i^T(t)] = P[\text{error}|s_0^T(t)]$$

The above is obvious given the symmetry of both the transmit and receive signals and the fact that the four transmit signals are equally likely. Given $s_0^T(t)$ was transmitted, one has

$$\mathbf{r}(t) = s_0^R(t) + \mathbf{w}(t); \quad 0 \le t \le T_s$$

Therefore,

$$\mathbf{r}_1 = s_{0,1}^R + \mathbf{w}_1 = k\sqrt{E_s}\cos\left(\alpha + \frac{\pi}{4}\right) + \mathbf{w}_1$$
$$\mathbf{r}_2 = s_{0,2}^R + \mathbf{w}_2 = k\sqrt{E_s}\sin\left(\alpha + \frac{\pi}{4}\right) + \mathbf{w}_2$$

where $\mathbf{w}_1$ and $\mathbf{w}_2$ are i.i.d Gaussian random variables with zero mean and variance $\frac{N_0}{2}$. Thus:

$$\begin{aligned}
P[\text{error}|s_0^T(t)] &= 1 - P[\text{correct}|s_0^T(t)] = 1 - P[(\mathbf{r}_1, \mathbf{r}_2) \in \mathcal{R}_0|s_0^T(t)] \\
&= 1 - P[\mathbf{r}_1 \ge 0|s_0^T(t)]P[\mathbf{r}_2 \ge 0|s_0^T(t)] \\
&= 1 - \left[1 - Q\left(\frac{k\sqrt{E_s}\sin(\alpha + \pi/4)}{\sqrt{N_0/2}}\right)\right]\left[1 - Q\left(\frac{k\sqrt{E_s}\cos(\alpha + \pi/4)}{\sqrt{N_0/2}}\right)\right]
\end{aligned}$$

$$= Q\left(k\sin(\alpha + \pi/4)\sqrt{\frac{2E_s}{N_0}}\right) + Q\left(k\cos(\alpha + \pi/4)\sqrt{\frac{2E_s}{N_0}}\right)$$
$$-Q\left(k\sin(\alpha + \pi/4)\sqrt{\frac{2E_s}{N_0}}\right) Q\left(k\cos(\alpha + \pi/4)\sqrt{\frac{2E_s}{N_0}}\right)$$

Note that the third term in the above expression can be safely ignored.

(c) To compute the bit error probability, need to compute the following message error probabilities given that $s_0^T(t)$ (or message $m_0 = 00$) was transmitted:

$$\begin{aligned}
P[m_1|m_0] &= P[\mathbf{r}_1 \leq 0|m_0]P[\mathbf{r}_2 \geq 0|m_0] \\
&= Q\left(k\cos(\alpha+\pi/4)\sqrt{\frac{2E_s}{N_0}}\right)\left[1 - Q\left(k\sin(\alpha+\pi/4)\sqrt{\frac{2E_s}{N_0}}\right)\right] \\
P[m_2|m_0] &= P[\mathbf{r}_1 \leq 0|m_0]P[\mathbf{r}_2 \leq 0|m_0] \\
&= Q\left(k\cos(\alpha+\pi/4)\sqrt{\frac{2E_s}{N_0}}\right)Q\left(k\sin(\alpha+\pi/4)\sqrt{\frac{2E_s}{N_0}}\right) \\
P[m_3|m_0] &= P[\mathbf{r}_1 \geq 0|m_0]P[\mathbf{r}_2 \leq 0|m_0] \\
&= Q\left(k\sin(\alpha+\pi/4)\sqrt{\frac{2E_s}{N_0}}\right)\left[1 - Q\left(k\cos(\alpha+\pi/4)\sqrt{\frac{2E_s}{N_0}}\right)\right]
\end{aligned}$$

Finally:

$$\begin{aligned}
P[\text{bit error}] &= 0.5P[m_1|m_0] + 1.0P[m_2|m_0] + 0.5P[m_3|m_0] \\
&= 0.5\left[Q\left(k\cos(\alpha+\pi/4)\sqrt{\frac{2E_s}{N_0}}\right) + Q\left(k\sin(\alpha+\pi/4)\sqrt{\frac{2E_s}{N_0}}\right)\right]
\end{aligned} \tag{7.13}$$

The bit error error probability of QPSK with no phase and attenuation uncertainty is

$$P[\text{bit error}] = Q\left(\sqrt{\frac{E_s}{N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \tag{7.14}$$

Thus, as long as $\alpha \neq 0$ and/or $k \neq 1$, one of the arguments of Q functions in (7.13) is smaller than $\sqrt{\frac{E_s}{N_0}}$. Therefore, it makes the bit error probability in (7.13) worse than that in (7.14).

P7.10 (a) See Fig. 7.13.

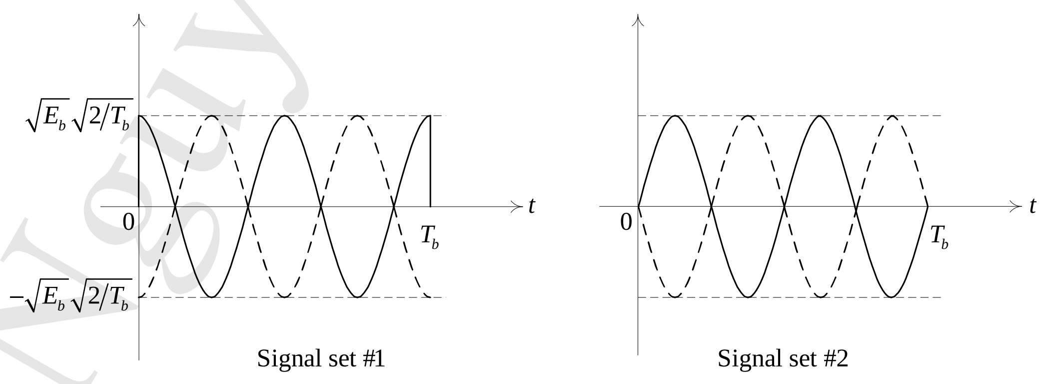


Figure 7.13

Signal #1 has a discontinuity $\Rightarrow$ PSD decays asymptotically as $\frac{1}{f^2}$.
Signal #2 has a discontinuity after 1 differentiation $\Rightarrow$ PSD decays asymptotically as $\frac{1}{f^4}$.

(b) To determine the PSDs, shift the signal set by $\frac{T_b}{2}$ (to the left) and use the modulation model (for antipodal signalling) in Fig. 7.14, where $h(t)=\sqrt{E_b}\sqrt{\frac{2}{T_b}}\cos(2\pi f_c t)$ or $h(t)=\sqrt{E_b}\sqrt{\frac{2}{T_b}}\sin(2\pi f_c t)$, $|t|\le\frac{T_b}{2}$.

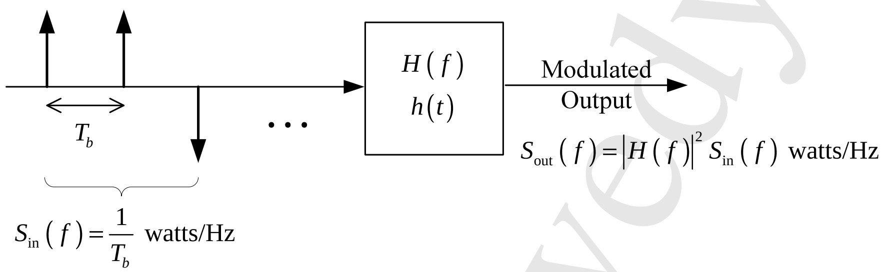


Figure 7.14

To find $H(f)$ express $h(t)$ as $K\left[e^{j2\pi f_c t}\pm e^{-j2\pi f_c t}\right]$ where $K=\sqrt{E_b}\sqrt{\frac{2}{T_b}}\left(\frac{1}{2}\right)$, + sign for signal set #1; $K=\sqrt{E_b}\sqrt{\frac{2}{T_b}}\left(\frac{1}{2j}\right)$, − sign for signal set #2.
Then $|H(f)|^2=|K|^2|\mathcal{F}\left\{e^{j2\pi f_c t}\pm e^{-j2\pi f_c t}\right\}|^2$. Note $|K|^2=\frac{E_b}{2T_b}$ is the same for either signal set.

$$\begin{aligned}
\mathcal{F}\left\{e^{j2\pi f_c t}\pm e^{-j2\pi f_c t}\right\} &= \int_{-\frac{T_b}{2}}^{\frac{T_b}{2}}\left\{e^{j2\pi f_c t}\pm e^{-j2\pi f_c t}\right\}e^{-j2\pi f t}\mathrm{d}t\\
&= \int_{-\frac{T_b}{2}}^{\frac{T_b}{2}}\left\{e^{j2\pi(f-f_c)t}\pm e^{-j2\pi(f+f_c)t}\right\}\mathrm{d}t\\
&= \underbrace{\frac{e^{j2\pi(f-f_c)T_b}-e^{-j2\pi(f-f_c)T_b}}{j2}}_{\sin\pi(f-f_c)T_b}\frac{1}{\pi(f-f_c)}\pm\underbrace{\frac{e^{j2\pi(f+f_c)T_b}-e^{-j2\pi(f+f_c)T_b}}{j2}}_{\sin\pi(f+f_c)T_b}\frac{1}{\pi(f+f_c)}
\end{aligned}\tag{7.15}$$

Now $f_c=\frac{k}{T_b}$ (usual assumption). Therefore $\sin\pi(f-f_c)T_b=\sin(\pi f T_b-k\pi)=\sin(\pi f T_b)\cos(k\pi)$ and $\sin\pi(f+f_c)T_b=\sin(\pi f T_b+k\pi)=\sin(\pi f T_b)\cos(k\pi)$. Therefore the Fourier transform becomes

$$\sin(\pi f T_b)\cos(k\pi)\left[\frac{1}{f-f_c}\pm\frac{1}{f+f_c}\right]=\sin(\pi f T_b)\cos(k\pi)\left[\frac{(f+f_c)\pm(f-f_c)}{f^2-f_c^2}\right].$$

So for signal set #1, it is $(\sin\pi f T_b\cos(k\pi))\left[\frac{2f}{f^2-f_c^2}\right]$, while for signal set #2, it is = $(\sin\pi f T_b\cos(k\pi))\left[\frac{2f_c}{f^2-f_c^2}\right]$. Therefore the PSD of signal set #1 is

$$|K|^2\sin^2(\pi f T_b)\left[\frac{4f^2}{(f^2-f_c^2)^2}\right]=\frac{E_b}{2T_b}\sin^2(\pi f T_b)\left[\frac{4f^2}{(f^2-f_c^2)^2}\right],$$

and that of signal set #2 is:

$$|K|^2\sin^2(\pi f T_b)\left[\frac{4f_c^2}{(f^2-f_c^2)^2}\right]=\frac{E_b}{2T_b}\sin^2(\pi f T_b)\left[\frac{4f_c^2}{(f^2-f_c^2)^2}\right],$$

where $\cos^2(k\pi) = 1$.

Observe that the PSD of signal set #1 $\propto \frac{1}{f^2}$ and that of signal set #2 $\propto \frac{1}{f^4}$ for large f, as expected.

(c) Need to set up equations for plotting. Note that

$$\frac{4f^2}{(f^2-f_c^2)^2} = \frac{4f^2}{(f^2-\frac{k^2}{T_b^2})^2} = \frac{4f^2T_b^4}{(f^2T_b^2-k^2)^2} = 4T_b^2\frac{(fT_b)^2}{[(fT_b)^2-(f_cT_b)^2]^2},$$

and

$$\frac{4f_c^2}{(f^2-f_c^2)^2} = \frac{4f_c^2T_b^4}{(f^2T_b^2-k^2)^2} = 4T_b^2\frac{(f_cT_b)^2}{[(fT_b)^2-(f_cT_b)^2]^2}.$$

Therefore the two PSDs are:

$$(2E_bT_b)\sin^2(\pi fT_b)\left[\frac{(fT_b)^2}{[(fT_b)^2-(f_cT_b)^2]^2}\right] \quad \text{(signal set \#1)}$$
$$(2E_bT_b)\sin^2(\pi fT_b)\left[\frac{(f_cT_b)^2}{[(fT_b)^2-(f_cT_b)^2]^2}\right] \quad \text{(signal set \#2)}$$

The normalized PSDs, normalized by $(2E_bT_b)$ are plotted in Fig. 7.15. The Plots are versus fT_b with f_cT_b as a parameter. Observe that as more and more cycles (i.e., f_cT_b is larger and larger) the difference between the two PSDs becomes negligible. The reason is that the maximum slope is larger and larger (ignoring the discontinuity for signal set #1).

(d) So how do we reconcile this? I.e., how does one reconcile that the PSD ignores phase information (a statement made in Chapter 3) and that phase appears to play a role in the slope of the PSD, at least in this case.

P7.11 (a)

$$\begin{aligned} \text{P[error]} &= pQ\left(\sqrt{\frac{2E_b}{N_0}}\right) + (1-p)Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \\ &= Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\sqrt{2\text{SNR}}\right) \end{aligned} \tag{7.16}$$

which is the same as in the case of that p is actually $1/2$. Here SNR $= E_b/N_0$.

(b) Now the receiver knows that $p \neq 1/2$. It needs to adjust the threshold accordingly to minimize P[error]. The optimal threshold is found as follows:

$$\frac{f(\mathbf{r}|1_T)}{f(\mathbf{r}|0_T)} \underset{0_D}{\overset{1_D}{\gtrless}} \frac{1-p}{p}$$

$$\frac{\frac{1}{\sqrt{2\pi\sqrt{N_0/2}}}\mathrm{e}^{-(\mathbf{r}-\sqrt{E_b})^2/N_0}}{\frac{1}{\sqrt{2\pi\sqrt{N_0/2}}}\mathrm{e}^{-(\mathbf{r}+\sqrt{E_b})^2/N_0}} \underset{0_D}{\overset{1_D}{\gtrless}} \frac{1-p}{p} \tag{7.17}$$

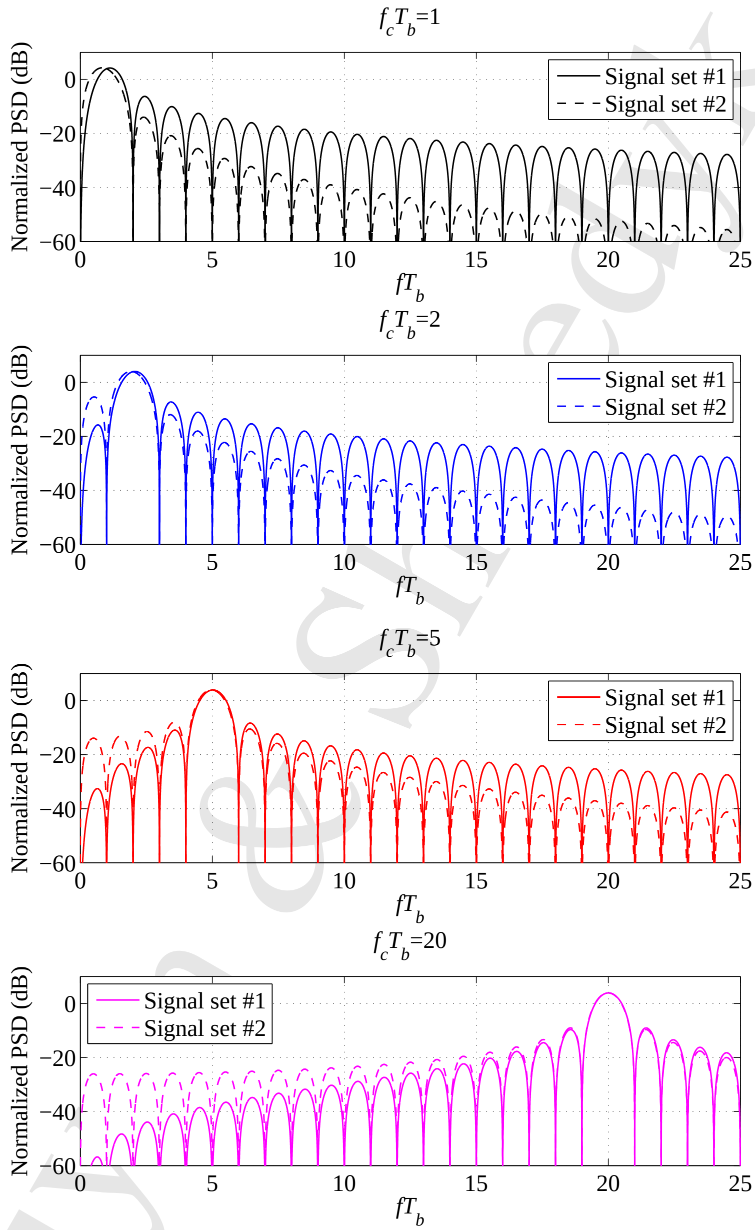


Figure 7.15: Normalized PSDs of two signal sets.

$$\mathbf{r} \underset{0_D}{\overset{1_D}{\gtrless}} \frac{N_0}{4\sqrt{E_b}} \ln\left(\frac{1-p}{p}\right) = \underbrace{\frac{\sqrt{E_b}}{4\mathrm{SNR}} \ln\left(\frac{1-p}{p}\right)}_{T_h}.$$

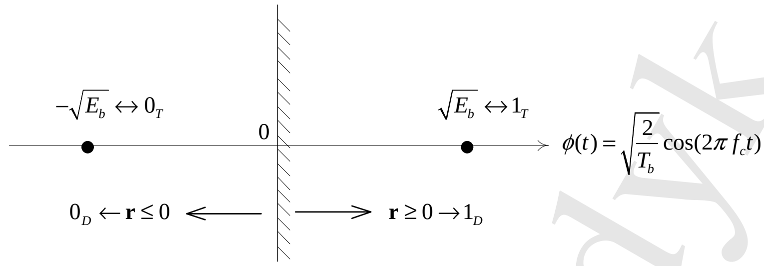


Figure 7.16: Decision regions of the BPSK demodulator with the assumption of $p = 1/2$.

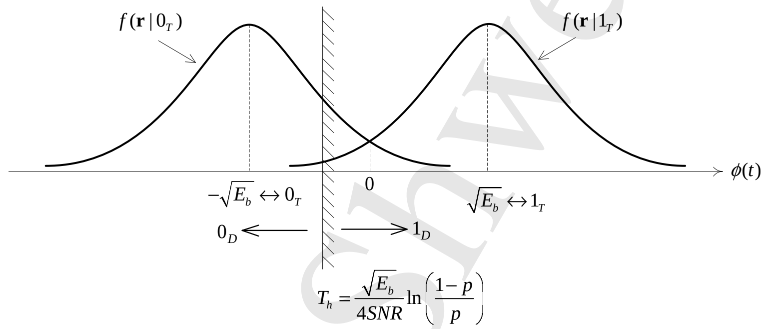


Figure 7.17: Decision regions of the optimum BPSK demodulator when p is known at the receiver.

$$
\begin{aligned}
\text{P[error]} &= (1-p)Q\left(\frac{\sqrt{E_b}+T_h}{\sqrt{N_0/2}}\right) + pQ\left(\frac{\sqrt{E_b}-T_h}{\sqrt{N_0/2}}\right) \\
&= (1-p)Q\left(\frac{1+\frac{1}{4\text{SNR}}\ln\left(\frac{1-p}{p}\right)}{\frac{1}{\sqrt{2}\sqrt{\text{SNR}}}}\right) + pQ\left(\frac{1-\frac{1}{4\text{SNR}}\ln\left(\frac{1-p}{p}\right)}{\frac{1}{\sqrt{2}\sqrt{\text{SNR}}}}\right) \\
&= (1-p)Q\left(\sqrt{2}\left[\sqrt{\text{SNR}}+\frac{1}{4\sqrt{\text{SNR}}}\ln\left(\frac{1-p}{p}\right)\right]\right) \\
&\quad +pQ\left(\sqrt{2}\left[\sqrt{\text{SNR}}-\frac{1}{4\sqrt{\text{SNR}}}\ln\left(\frac{1-p}{p}\right)\right]\right) \qquad (7.18)
\end{aligned}
$$

As a small check, when $p = 1/2$ one has $\text{P[error]} = Q\left(\sqrt{2}\sqrt{\text{SNR}}\right)$, as expected.

(c) The error performance curves based on the demodulator of (a) and that of (b) for $p = 0.6, 0.7, 0.8, 0.9$ are shown in Fig. 7.18. Observe that the error performance curves are quite insensitive to the a priori probability value p.

P7.12
```
close all;
clear all;
[y,fs,nbits]=wavread('Filename');
%Normalize the values in y
y=2^(nbits-1)+2^(nbits-1)*y;
% Change each floating point value to an unsigned integer
if nbits==8
    y=uint8(y);
elseif nbits==16
```

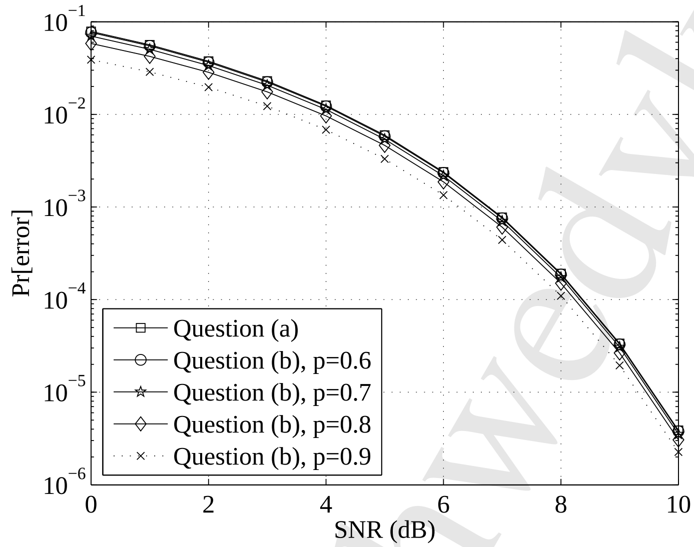


Figure 7.18: Error performance of BPSK demodulators with different values of the *a priori* probability p.

```
    y=uint16(y);
elseif nbits==32
    y=uint32(y);
elseif nbits==64
    y=uint64(y);
else
    fprintf('The number of bits per sample used in...
                this wav file is out of the range !');
    exit
end
% Count the number of bit 1s and 0s
count_bit1=0;
count_bit0=0;
L=numel(y);
for l=1:L
    sample=bitget(y(l),1:nbits);
    count_bit1=count_bit1 + sum(sample);
    count_bit0=count_bit0 + nbits - sum(sample);
end
fprintf('\n Number of bits per sample: $\%d$',nbits);
fprintf('\n Number of bit 1s: $\%d$',count_bit1);
fprintf('\n Number of bit 0s: $\%d$',count_bit0);
fprintf('\n Percentage of bit 1s: $\%3.2f$',100*count_bit1/(L*nbits));
fprintf('\n Percentage of bit 0s: $\%3.2f$',100*count_bit0/(L*nbits));
```

The following are results for the wave file tested:

```
Number of bits per sample: 16
Number of bit 1s: 2832449
Number of bit 0s: 2132159
Percentage of bit 1s: 57.05
```

```
Percentage of bit 0s: 42.95
```

P7.13
```
mark_vec=['^';'o';'s';'d']; color_vec=['r';'b';'m';'g'];
L=1000; % number of QPSK symbol;
Pevec=[10^(-1),10^(-2),10^(-3),10^(-4)]; for h=1:4
    Pe=Pevec(h);
    sigma=(Qinv(Pe)*sqrt(2))^(-1);
    text_title=['{\itL}=',num2str(L),';Pr[bit error]=',num2str(Pe)];
    b=round(rand(1,2*L)); % This is the binary information sequence
    for i=1:L
        BP=b((i-1)*2+1:i*2); % Take two bits at a time to map to a QPSK symbol
        if BP==[0 0]
            phi_1(i)=1;phi_2(i)=0;
        elseif BP==[0 1]
            phi_1(i)=0;phi_2(i)=1;
        elseif BP==[1 1]
            phi_1(i)=-1;phi_2(i)=0;
        elseif BP==[1 0]
            phi_1(i)=0;phi_2(i)=-1;
        end
    r1=phi_1(i)+sigma*randn(1,1);
    r2=phi_2(i)+sigma*randn(1,1);
    figure(h); % Plot in a seperate figure for each value of Pr[error]
    plot(r1,r2,'marker',mark_vec(bi2de(BP)+1,:),'markersize',6,'markerfacecolor',...
        color_vec(bi2de(BP)+1,:),'color',color_vec(bi2de(BP)+1,:));
    hold on;
    figure(5); % Plot in the same figure for all four values of Pr[error]
    subplot(2,2,h);
    plot(r1,r2,'marker',mark_vec(bi2de(BP)+1,:),'markersize',6,'markerfacecolor',...
        color_vec(bi2de(BP)+1,:),'color',color_vec(bi2de(BP)+1,:));
    hold on;
    end
    x=[-3:0.01:3];
    figure(h);
    plot(x,x,x,-x,'marker','none','color','k','linewidth',1.5);
    title(text_title,'FontName','Times New Roman','FontSize',16)
    xlabel('{\it\phi}_1({\itt})','FontName','Times New Roman','FontSize',16);
    ylabel('{\it\phi}_2({\itt})','FontName','Times New Roman','FontSize',16);
    h1=gca;
    set(h1,'FontSize',16,'XGrid','on','YGrid','on','GridLineStyle',':',...
        'MinorGridLineStyle','none','FontName','Times New Roman');
    axis([-3 3 -3 3]);axis equal;
    figure(5);
    subplot(2,2,h);
    plot(x,x,x,-x,'marker','none','color','k','linewidth',1.5);
    title(text_title,'FontName','Times New Roman','FontSize',16)
    xlabel('{\it\phi}_1({\itt})','FontName','Times New Roman','FontSize',16);
    ylabel('{\it\phi}_2({\itt})','FontName','Times New Roman','FontSize',16);
    h1=gca;
    set(h1,'FontSize',16,'XGrid','on','YGrid','on','GridLineStyle',':',...
        'MinorGridLineStyle','none','FontName','Times New Roman');
    axis([-3 3 -3 3]);axis equal;
end
```

Note that the program is by no means "optimal". Your program may be different, even though they accomplish the same task.

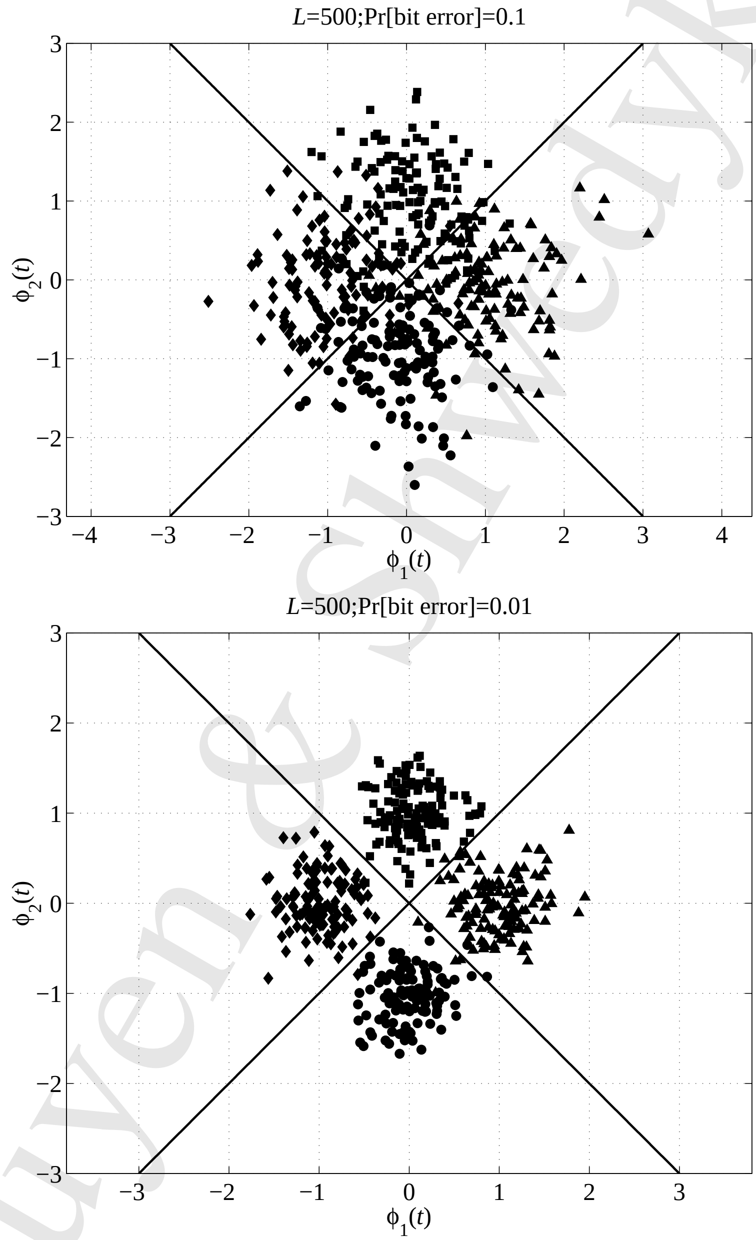


Figure 7.19

Fig. 7.20 plots the received QPSK signals in the signal space diagram. It is clear that with a higher signal-to-noise ratio the received signals stay closer to the corresponding transmitted signals. This makes it easier for the demodulation, hence a smaller bit or symbol error probability.

P7.14 (a) BASK signal set is:

$$\begin{cases} 0_T: & 0 \\ 1_T: & \sqrt{E}\sqrt{\frac{2}{T_b}}[u(t)-u(t-T_b)]\cos 2\pi f_c t \end{cases} \tag{7.19}$$

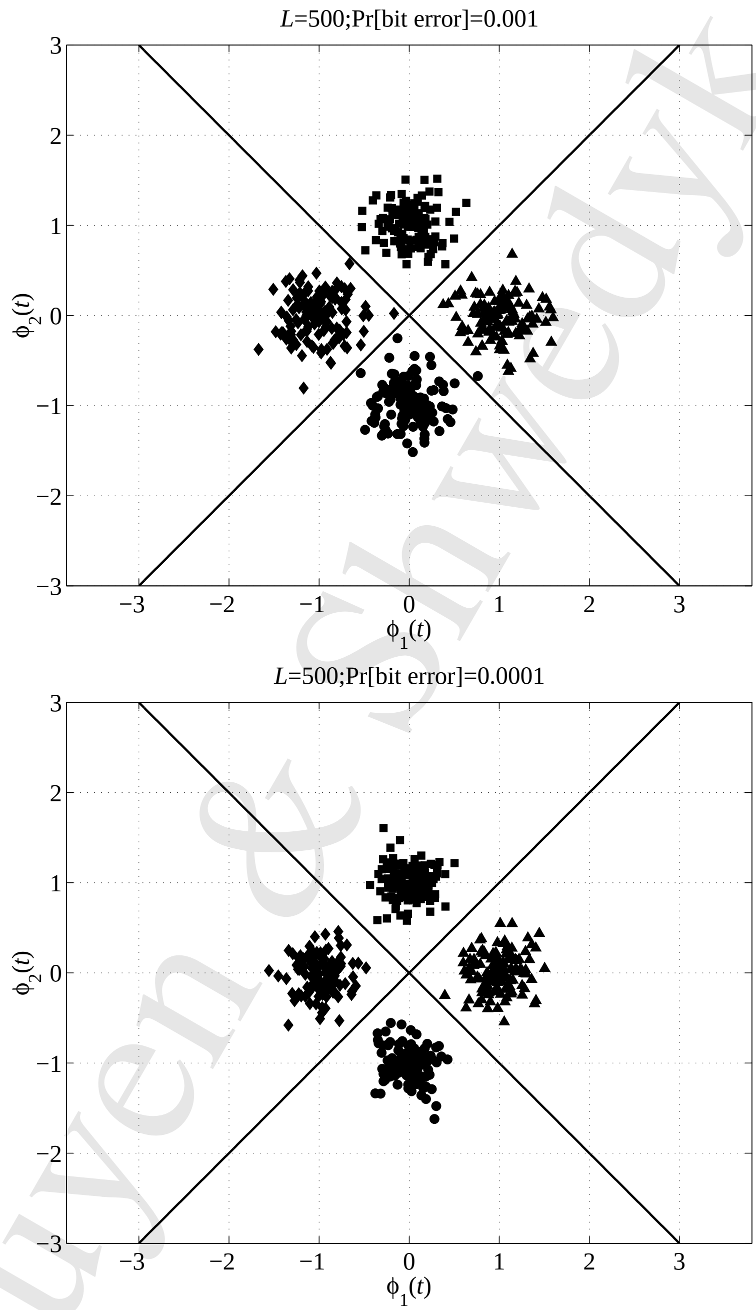


Figure 7.20

which maybe rewritten as:

$$\begin{cases} 0_T: & \mathcal{R}\left\{0\mathrm{e}^{j2\pi f_c t}\right\} & \Rightarrow s_{\mathrm{BB}}(t)=0 \\ 1_T: & \mathcal{R}\left\{\sqrt{E}\sqrt{\frac{2}{T_b}}[u(t)-u(t-T_b)]\mathrm{e}^{j2\pi f_c t}\right\} & \Rightarrow s_{\mathrm{BB}}(t)=\sqrt{E}\sqrt{\frac{2}{T_b}}[u(t)-u(t-T_b)] \end{cases} \tag{7.20}$$

They are plotted in Fig. 7.21.

In baseband the signal is NRZ or BASK is NRZ modulation shifted up by the carrier frequency, f_c Hz.

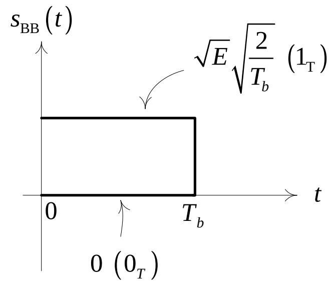


Figure 7.21

(b) BPSK signal set is:

$$\begin{cases} 0_T: & -\sqrt{E_b}\sqrt{\frac{2}{T_b}}[u(t)-u(t-T_b)]\cos(2\pi f_c t) \\ 1_T: & \sqrt{E_b}\sqrt{\frac{2}{T_b}}[u(t)-u(t-T_b)]\cos(2\pi f_c t) \end{cases}$$

which maybe written as:

$$\begin{cases} 0_T: & \mathcal{R}\left\{-\sqrt{E_b}\sqrt{\frac{2}{T_b}}[u(t)-u(t-T_b)]\mathrm{e}^{j2\pi f_c t}\right\} & \Rightarrow s_{\mathrm{BB}}(t) = -\sqrt{E_b}\sqrt{\frac{2}{T_b}}[u(t)-u(t-T_b)] \\ 1_T: & \mathcal{R}\left\{\sqrt{E_b}\sqrt{\frac{2}{T_b}}[u(t)-u(t-T_b)]\mathrm{e}^{j2\pi f_c t}\right\} & \Rightarrow s_{\mathrm{BB}}(t) = \sqrt{E_b}\sqrt{\frac{2}{T_b}}[u(t)-u(t-T_b)] \end{cases}$$

The baseband signals are shown in Fig. 7.22.

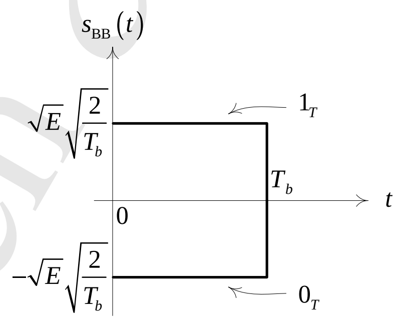


Figure 7.22

In baseband the signal set is that of NRZ-L or BPSK is NRZ-L modulation shifted up by the carrier frequency, f_c Hz.

(c) BFSK signal set is: $\begin{cases} 0_T: & \sqrt{E_b}\sqrt{\frac{2}{T_b}}\cos(2\pi f_1 t) \\ 1_T: & \sqrt{E_b}\sqrt{\frac{2}{T_b}}\cos(2\pi f_2 t) \end{cases}$ $0 \le t \le T_b$ (assume $f_1 < f_2$).

The signal set can be rewritten as:

$$0_T \quad : \quad \mathcal{R}\left\{\sqrt{E_b}\sqrt{\frac{2}{T_b}}[u(t)-u(t-T_b)]\mathrm{e}^{j2\pi f_1 t}\right\},$$

$$1_T \quad : \quad \mathcal{R}\left\{\sqrt{E_b}\sqrt{\frac{2}{T_b}}[u(t)-u(t-T_b)]\mathrm{e}^{j2\pi f_2 t}\right\}$$

(i) Let $f_s = f_1$. Then immediately, when for 0_T : $s_{\text{BB}}(t) = \sqrt{E_b}\sqrt{\frac{2}{T_b}}[u(t) - u(t-T_b)]$.

For 1_T, write the passband signal as: $\mathcal{R}\Big\{ \underbrace{\sqrt{E_b}\sqrt{\frac{2}{T_b}}[u(t) - u(t-T_b)]\mathrm{e}^{j2\pi f_2 t}\mathrm{e}^{-j2\pi f_1 t}}_{\text{This is } s_{\text{BB}}(t).} \mathrm{e}^{j2\pi f_1 t}\Big\}$

$$0_T \quad : \quad s_{\text{BB}}(t) = \sqrt{E_b}\sqrt{\frac{2}{T_b}}[u(t) - u(t-T_b)] \tag{7.21}$$

$$1_T \quad : \quad s_{\text{BB}}(t) = \sqrt{E_b}\sqrt{\frac{2}{T_b}}[u(t) - u(t-T_b)]\mathrm{e}^{j2\pi(f_2-f_1)t} \tag{7.22}$$

$$= \sqrt{E_b}\sqrt{\frac{2}{T_b}}[u(t) - u(t-T_b)]\{\cos 2\pi(f_2-f_1)t + j\sin 2\pi(f_2-f_1)t\} \tag{7.23}$$

(ii) Let $f_s = f_2$. Basically the roles of f_1 and f_2 are interchanged. Therefore from (i):

$$0_T \quad : \quad s_{\text{BB}}(t) = \sqrt{E_b}\sqrt{\frac{2}{T_b}}[u(t) - u(t-T_b)]\{\cos 2\pi(f_2-f_1)t - j\sin 2\pi(f_2-f_1)t\}$$

$$1_T \quad : \quad s_{\text{BB}}(t) = \sqrt{E_b}\sqrt{\frac{2}{T_b}}[u(t) - u(t-T_b)]$$

(iii) Let $f_s = \frac{f_1+f_2}{2}$. Then

$$0_T : s_{\text{PB}}(t) = \mathcal{R}\Big\{ \underbrace{\sqrt{E_b}\sqrt{\frac{2}{T_b}}[u(t) - u(t-T_b)]\mathrm{e}^{j2\pi f_1 t}\mathrm{e}^{-j2\pi\left(\frac{f_1}{2}\right)t}\mathrm{e}^{-j2\pi\left(\frac{f_2}{2}\right)t}}_{=s_{\text{BB}}(t)=\sqrt{E_b}\sqrt{\frac{2}{T_b}}[u(t)-u(t-T_b)]\mathrm{e}^{-j2\pi\left(\frac{f_2-f_1}{2}\right)t}} \mathrm{e}^{-j2\pi\left(\frac{f_2+f_1}{2}\right)t}\Big\}$$

$$1_T : s_{\text{PB}}(t) = \mathcal{R}\Big\{ \underbrace{\sqrt{E_b}\sqrt{\frac{2}{T_b}}[u(t) - u(t-T_b)]\mathrm{e}^{j2\pi f_2 t}\mathrm{e}^{-j2\pi\left(\frac{f_1}{2}\right)t}\mathrm{e}^{-j2\pi\left(\frac{f_2}{2}\right)t}}_{=s_{\text{BB}}(t)=\sqrt{E_b}\sqrt{\frac{2}{T_b}}[u(t)-u(t-T_b)]\mathrm{e}^{j2\pi\left(\frac{f_2-f_1}{2}\right)t}} \mathrm{e}^{-j2\pi\left(\frac{f_2+f_1}{2}\right)t}\Big\}$$

Or if we let $\Delta f \equiv f_2 - f_1$ be the frequency separation then:

$$0_T \quad : \quad s_{\text{BB}}(t) = \sqrt{E_b}\sqrt{\frac{2}{T_b}}[u(t) - u(t-T_b)]\mathrm{e}^{-j2\pi\left(\frac{\Delta f}{2}\right)t}$$

$$1_T \quad : \quad s_{\text{BB}}(t) = \sqrt{E_b}\sqrt{\frac{2}{T_b}}[u(t) - u(t-T_b)]\mathrm{e}^{j2\pi\left(\frac{\Delta f}{2}\right)t}$$

To plot the signals in baseband, let $f_2 - f_1 = \Delta f = \frac{1}{T_b}$, the minimum frequency separation for "noncoherent" orthogonality. The various signals are plotted in Figs. 7.23, 7.24 and 7.25.

Remarks:

(i) In passband the transmitted energy is 0 or E (for BASK) or E_b (for BPSK or BFSK). What is the energy in the equivalent baseband signal(s)? Note – for a complex signal, $x(t)$, the energy is given by $\int_{\infty}^{\infty}|x(t)|^2\mathrm{d}t = \int_{\infty}^{\infty}x(t)x^*(t)\mathrm{d}t$. How would you explain the factor of 2?

(ii) BPSK, BFSK are antipodal and orthogonal (with the proper choice of $f_2 - f_1$). Are these characteristics preserved in baseband?

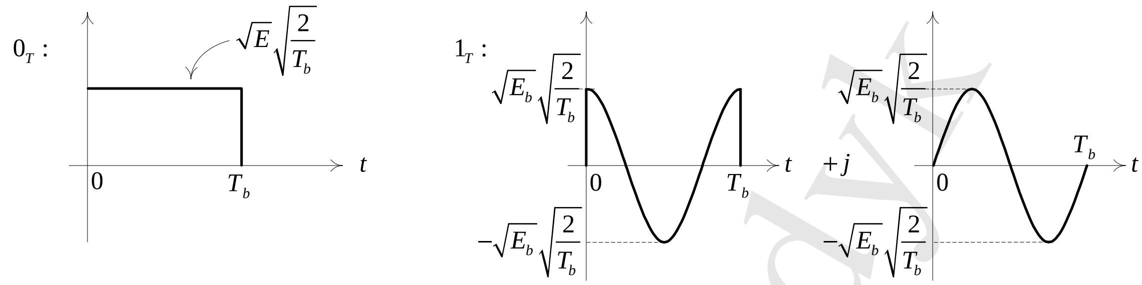


Figure 7.23: Case (i): $f_s = f_1$.

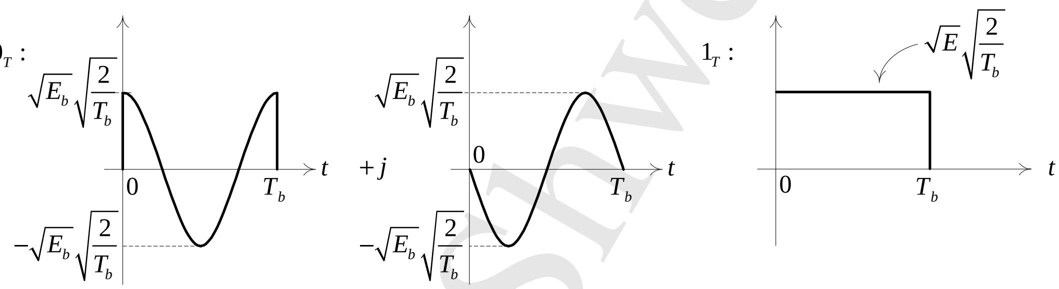


Figure 7.24: Case (ii): $f_s = f_2$.

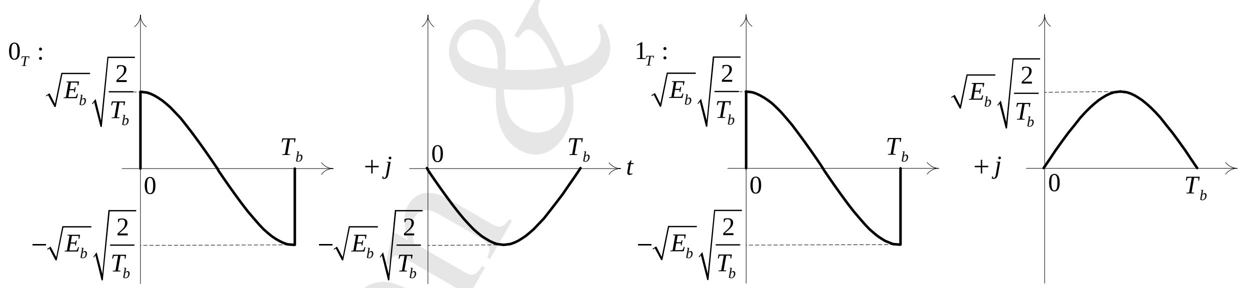


Figure 7.25: Case (iii): $f_s = \frac{f_1+f_2}{2}$.

(iii) What would happen in (b) to the $s_{\text{BB}}(t)$'s if instead of $f_s = f_c$, the shift frequency, also commonly called representation frequency, is chosen to be $f_s = f_c + f_{\text{offset}}$? Try to draw some general conclusions or go to P7.15.

P7.15 One gets a real equivalent baseband signal if the shift or representation frequency, f_s, is chosen so that passband signals' Fourier transform has the following properties: its magnitude function, $|S_{PB}(f)|$, is <u>even</u> about f_s and its phase function, $\angle S(f)$, is <u>odd</u> about f_s or $|S(f - f_s)| = |S(-f - f_s)|$ and $\angle S(f - f_s) = -\angle S(-f - f_s)$.

Otherwise the equivalent baseband signal is complex.

So given the spectrum of a passband signal as in Fig. 7.26, how would you choose f_s to have a real equivalent baseband signal?

TO CONCLUDE – THE CHOICE OF THE REPRESENTATION FREQUENCY IS ARBITRARY AND IS YOUR DECISION. BUT THERE ARE GOOD, BAD & UGLY CHOICES.

P7.16 Let the QPSK signals be represented by the following signal set:

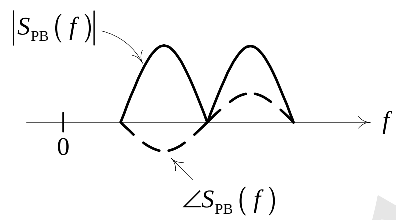


Figure 7.26

$$
\begin{array}{lcl}
00 \rightarrow \sqrt{E_s}\sqrt{\frac{2}{T_s}}\cos(2\pi f_c t) & \overset{\text{choosing } f_s=f_c}{\Longrightarrow} & s_{\text{BB}}(t) = \sqrt{E_s}\sqrt{\frac{2}{T_s}}[u(t)-u(t-T_s)] \\
01 \rightarrow \sqrt{E_s}\sqrt{\frac{2}{T_s}}\cos(2\pi f_c t + \frac{\pi}{2}) & & s_{\text{BB}}(t) = \sqrt{E_s}\sqrt{\frac{2}{T_s}}[u(t)-u(t-T_s)]\mathrm{e}^{j\frac{\pi}{2}} \\
10 \rightarrow \sqrt{E_s}\sqrt{\frac{2}{T_s}}\cos(2\pi f_c t + \pi) & & s_{\text{BB}}(t) = \sqrt{E_s}\sqrt{\frac{2}{T_s}}[u(t)-u(t-T_s)]\mathrm{e}^{j\pi} \\
11 \rightarrow \sqrt{E_s}\sqrt{\frac{2}{T_s}}\cos(2\pi f_c t + \frac{3\pi}{2}) & & s_{\text{BB}}(t) = \sqrt{E_s}\sqrt{\frac{2}{T_s}}[u(t)-u(t-T_s)]\mathrm{e}^{-j\frac{\pi}{2}}
\end{array}
$$

What do the plots of the baseband signals look like?

What happens to the baseband signals if the transmitted signal set is shifted by say θ radians?

Before going to Problems 7.17 to 7.20 we first become comfortable with Equation (7.4). The term $S_B(f-f_s)$ represents $S_{\text{B}}(f)$ shifted to the right by f_s Hz, while the term $S_B(-f-f_s)$ represents $S_{\text{B}}(f)$ flipped around the vertical axis, i.e., $S_B(-f)$ and then (because of the flip or $-f$ argument) shifted to the left by f_s Hz. Summing the two gives $S_{\text{n}}(f)$. Here we worked from $S_{\text{B}}(f)$ to $S_{\text{n}}(f)$ but one can work the other way. As another graphical example consider $S_{\text{n}}(f)$ as shown in Fig. 7.27.

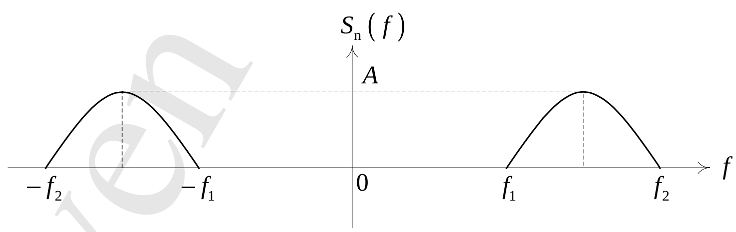


Figure 7.27

Whatever it is, $S_{\text{n}}(f)$, is real positive and even and therefore a valid PSD. To determine the PSD of the equivalent baseband process the representation frequency, f_s, must be chosen. Let us choose it to be $\frac{f_2+f_1}{2}$. Then the PSD of the equivalent baseband process is shown in Fig. 7.28.

A word about the auto-correlation function. It is as usual the inverse Fourier transform of the PSD, i.e., $R_{\text{B}}(\tau) = \int_{-\infty}^{\infty} S_{\text{B}}(f)\mathrm{e}^{j2\pi f\tau}\mathrm{d}f$.

Note that the PSD of Fig. 7.31 results in a complex auto-correlation function while the PSD above results in a real auto-correlation function.

Real or complex depends on the shape of the passband PSD and also on the choice of f_s. A complex auto-correlation function means that the equivalent baseband process is complex. Keep in mind that it is the passband process that is the physical (real) process, the baseband process is first an equivalent one, i.e., a mathematical construct.

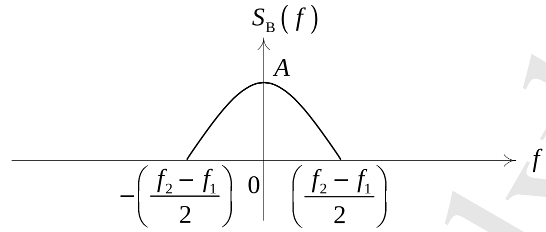


Figure 7.28

Some questions you can ask yourself are: What is the equivalent baseband process PSD when f_s above is chosen to be $= f_1$?, $= f_2$? When would the auto-correlation function be real?

P7.17 (a) Let $S_n(f)$ be as in Fig. 7.29. Let $H(f)$ be as in Fig. 7.30. Certainly $|H(f)|^2 = S_n(f)$. Choose $f_s = f_1$ which implies that $H_B(f)$ is as in Fig. 7.31

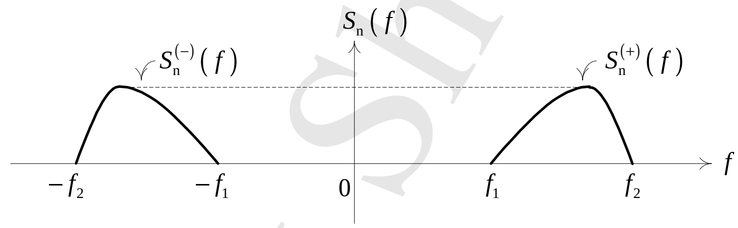


Figure 7.29

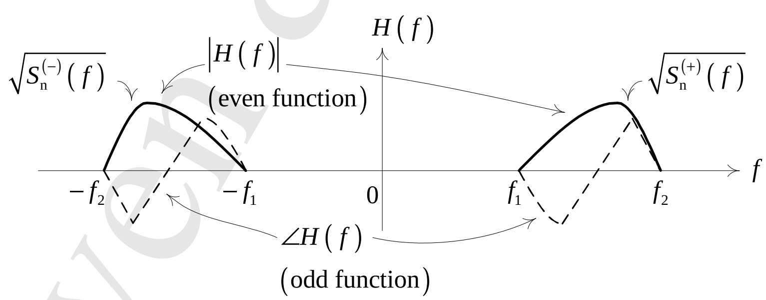


Figure 7.30

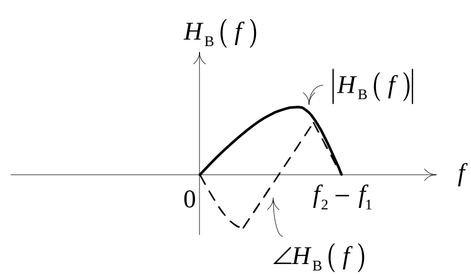


Figure 7.31

Now show that indeed $H(f) = H_B(f - f_s) + H_B(-f - f_s)^*$. Term $H_B(f - f_s)$ of course gives the spectrum in Fig. 7.32

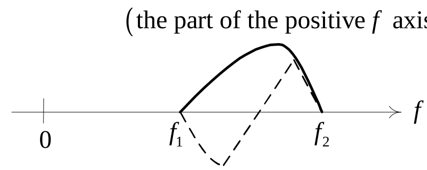


Figure 7.32

Now $H_{\mathrm{B}}(-f-f_s)$ is again a flip of $H_{\mathrm{B}}(f)$ about the vertical axis and a shift to the left by $f_s = f_1$ Hz. The result is shown on the left of Fig. 7.33

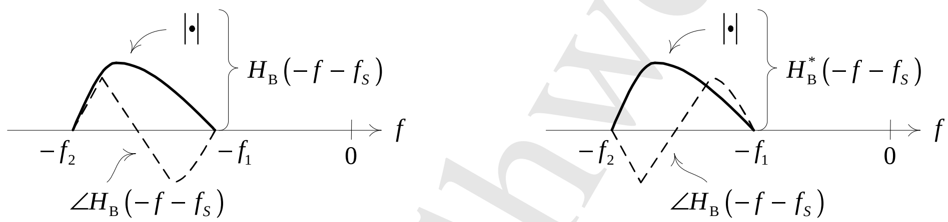


Figure 7.33

But we want the overall phase function $\angle H(f)$ to be an odd function ($h(t)$ is real). Therefore conjugate (not as a French verb but as complex conjugate) $H_{\mathrm{B}}(-f-f_s) \rightarrow H_{\mathrm{B}}(-f-f_s)^*$. This flips the phase over since if $x = |x|\mathrm{e}^{j\angle x}, x^* = |x|\mathrm{e}^{j(-\angle x)} \Rightarrow \angle x^* = -\angle x$. Therefore the result is as shown on the right of Fig. 7.33.

Remarks:

- The reason that there was no complex conjugate in Eqn. (P7.4) is because $S_{\mathrm{n}}(f)$ is a PSD and hence the phase is zero (always).
- Different f_s can be chosen. You may wish to see what is the effect?

(e) $\mathbf{n}_{\mathrm{B}}(t) = \mathrm{e}^{-j2\pi f_s t} \int_{-\infty}^{\infty} \mathbf{w}(t-\lambda) h_{\mathrm{B}}(\lambda) \mathrm{e}^{j2\pi f_s \lambda} \mathrm{d}\lambda$. Should be a baseband process. Facetious answer would be because of the subscript B in $\mathbf{n}_{\mathrm{B}}(t)$.

More seriously $-\mathbf{n}(t)$ is a passband process with its PSD lying "around" f_s Hz. Therefore $\mathbf{n}_{\mathrm{B}}(t)$ should be a baseband process with its PSD "around" 0 Hz and shifted up and down by $\mathrm{e}^{j2\pi f_s t}$ to lie around f_s Hz.

(f) $\mathbf{n}_I(t) = \mathcal{R}\{\mathbf{n}_{\mathrm{B}}(t)\} = \mathcal{R}\left\{\left[\int_{-\infty}^{\infty} \mathbf{w}(t-\lambda) h_{\mathrm{B}}(\lambda) \mathrm{e}^{j2\pi f_s \lambda} \mathrm{d}\lambda\right] \mathrm{e}^{-j2\pi f_s t}\right\}$

$$\mathbf{n}_Q(t) = -\mathcal{I}\{\mathbf{n}_{\mathrm{B}}(t)\} = -\mathcal{I}\left\{\left[\int_{-\infty}^{\infty} \mathbf{w}(t-\lambda) h_{\mathrm{B}}(\lambda) \mathrm{e}^{j2\pi f_s \lambda} \mathrm{d}\lambda\right] \mathrm{e}^{-j2\pi f_s t}\right\}.$$

P7.18 (a) They are Gaussian because they are obtained by linear operations.

$$E\left\{\mathbf{n}_{\mathrm{B}}(t)\mathbf{n}_{\mathrm{B}}(t+\tau)\right\} = E\Bigg\{\left[\left\{\int_{\lambda=-\infty}^{\infty}\mathbf{w}(t-\lambda)h_{\mathrm{B}}(\lambda)\mathrm{e}^{j2\pi f_s\lambda}\mathrm{d}\lambda\right\}\mathrm{e}^{-j2\pi f_s t}\right]$$
$$\left\{\int_{u=-\infty}^{\infty}\mathbf{w}(t+\tau-u)h_{\mathrm{B}}(u)\mathrm{e}^{j2\pi f_s u}\mathrm{d}u\right\}\mathrm{e}^{-j2\pi f_s(t+\tau)}\Bigg\}$$
$$= \mathrm{e}^{-j2\pi f_s t}\mathrm{e}^{-j2\pi f_s(t+\tau)}\times$$
$$\int_{\lambda=-\infty}^{\infty}\underbrace{\int_{u=-\infty}^{\infty}\underbrace{E\left\{\mathbf{w}(t-\lambda)\mathbf{w}(t+\tau-u)\right\}}_{=\delta(\tau-u+\lambda)\quad(\mathbf{w}(t)\text{ is white noise})}h_{\mathrm{B}}(u)\mathrm{e}^{j2\pi f_s(\lambda+u)}\mathrm{d}u}_{h_{\mathrm{B}}(\tau+\lambda)\mathrm{e}^{j2\pi f_s(t+2\lambda)}\quad(\text{Impulse sifts out value at }\tau-\lambda=0\Rightarrow u=\tau+\lambda)}\,h_{\mathrm{B}}(\lambda)\mathrm{d}\lambda$$
$$= \mathrm{e}^{-j2\pi f_s(2t+\tau)}\int_{\lambda=-\infty}^{\infty}h_{\mathrm{B}}(\lambda)h_{\mathrm{B}}(\tau+\lambda)\mathrm{e}^{j2\pi f_s\tau}\mathrm{e}^{j2\pi(2f_s)\lambda}\mathrm{d}\lambda$$
$$= \mathrm{e}^{-j4\pi f_s t}\int_{\lambda=-\infty}^{\infty}h_{\mathrm{B}}(\lambda)h_{\mathrm{B}}(\tau+\lambda)\mathrm{e}^{j2\pi(2f_s)\lambda}\mathrm{d}\lambda$$

Now Fourier transform with respect to τ:

$$\begin{aligned}\mathcal{F}_{\tau}\left\{\mathcal{R}_{\mathbf{n}_{\mathrm{B}}}(t,\tau)\right\} &= \mathrm{e}^{-j4\pi f_s t}\int_{\tau=-\infty}^{\infty}\left[\int_{\lambda=-\infty}^{\infty}h_{\mathrm{B}}(\lambda)h_{\mathrm{B}}(\tau+\lambda)\mathrm{e}^{j2\pi(2f_s)\lambda}\mathrm{d}\lambda\right]\mathrm{e}^{-j2\pi f\tau}\mathrm{d}\tau\\ &= \mathrm{e}^{-j4\pi f_s t}\int_{\lambda=-\infty}^{\infty}h_{\mathrm{B}}(\lambda)\mathrm{e}^{j2\pi(2f_s)\lambda}\left[\int_{\tau=-\infty}^{\infty}h_{\mathrm{B}}(\tau+\lambda)\mathrm{e}^{-j2\pi 2f\tau}\mathrm{d}\tau\right]\mathrm{d}\lambda\end{aligned}$$

(b)

$$\begin{aligned}S_{\mathrm{n}}(f) &= S_{\mathrm{B}}(f-f_s)+S_{\mathrm{B}}(-f-f_s) = |H(f)|^2 = |H_{\mathrm{B}}(f-f_s)+H_{\mathrm{B}}^*(-f-f_s)|^2\\ &= [H_{\mathrm{B}}(f-f_s)+H_{\mathrm{B}}^*(-f-f_s)][H_{\mathrm{B}}^*(f-f_s)+H_{\mathrm{B}}(-f-f_s)]\\ &= \underbrace{H_{\mathrm{B}}(f-f_s)H_{\mathrm{B}}^*(-f-f_s)}_{=|H_{\mathrm{B}}(f-f_s)|^2=S_{\mathrm{B}}(f-f_s)}+\underbrace{H_{\mathrm{B}}(f-f_s)H_{\mathrm{B}}(-f-f_s)}_{0}+\underbrace{H_{\mathrm{B}}^*(-f-f_s)H_{\mathrm{B}}^*(f-f_s)}_{0}\\ &\quad+\underbrace{H_{\mathrm{B}}^*(f-f_s)H_{\mathrm{B}}(-f-f_s)}_{=|H_{\mathrm{B}}(-f-f_s)|^2=S_{\mathrm{B}}(-f-f_s)}\end{aligned}$$

$$\therefore S_{\mathrm{B}}(f) = |H_{\mathrm{B}}(f)|^2$$

(c) We know that $H(f) = H_{\mathrm{B}}(f-f_s)+H_{\mathrm{B}}^*(-f-f_s)$. Therefore we show that

$$\mathcal{F}\left\{2\mathcal{R}\left\{h_{\mathrm{B}}(t)\mathrm{e}^{j2\pi f_s t}\right\}\right\} = H(f)$$

which means $h(t) = 2\mathcal{R}\left\{h_{\mathrm{B}}(t)\mathrm{e}^{j2\pi f_s t}\right\}$. To show this express the real operation as follows; $\mathcal{R}\{x\} = \dfrac{x+x^*}{2}$.

$$\therefore \quad \mathcal{F}\{h(t)\} = H(f) = \mathcal{F}\left\{h_{\mathrm{B}}(t)\mathrm{e}^{j2\pi f_s t} + h_{\mathrm{B}}^*(t)\mathrm{e}^{-j2\pi f_s t}\right\}$$
$$= \underbrace{\int_{-\infty}^{\infty} h_{\mathrm{B}}(t)\underbrace{\mathrm{e}^{j2\pi f_s t}\mathrm{e}^{-j2\pi f t}}_{\mathrm{e}^{-j2\pi(f-f_s t)t}}\mathrm{d}t}_{H_{\mathrm{B}}(f-f_s)} + \underbrace{\underbrace{\int_{-\infty}^{\infty} h_{\mathrm{B}}^*(t)\mathrm{e}^{-j2\pi f_s t}\mathrm{e}^{-j2\pi f t}\mathrm{d}t}_{=\left[\underbrace{\int_{-\infty}^{\infty} h_{\mathrm{B}}(t)\mathrm{e}^{-j2\pi(-f-f_s)t}\mathrm{d}t}_{H_{\mathrm{B}}(-f-f_s)}\right]^*}}_{H_{\mathrm{B}}^*(-f-f_s)}$$

or $H(f) = H_{\mathrm{B}}(f-f_s) + H_{\mathrm{B}}^*(-f-f_s)$ as we wished to show.

(d) $\mathbf{n}(t) = \int_{-\infty}^{\infty}\mathbf{w}(t-\lambda)h(\lambda)\mathrm{d}\lambda = \int_{-\infty}^{\infty}\mathbf{w}(t-\lambda)\left[\mathcal{R}\left\{h_{\mathrm{B}}(\lambda)\mathrm{e}^{j2\pi f_s t}\mathrm{d}\lambda\right\}\right]\mathrm{d}\lambda$
Interchange the integral and $\mathcal{R}\{\}$ operation. Note that $\mathbf{w}(\cdot)$ is real and can be "pulled" inside the $\mathcal{R}\{\}$ operation. Therefore

$$\mathbf{n}(t) = \mathcal{R}\left\{\int_{\lambda=-\infty}^{\infty}\mathbf{w}(t-\lambda)h_{\mathrm{B}}(\lambda)\mathrm{e}^{j2\pi f_s\lambda}\mathrm{d}\lambda\right\}.$$

Change the variables in the inner bracket to $u = \tau+\lambda$. It becomes

$$\mathrm{e}^{j2\pi f\lambda}\int_{u=-\infty}^{\infty} h_{\mathrm{B}}(u)\mathrm{e}^{-j2\pi f u}\mathrm{d}u = \mathrm{e}^{j2\pi f\lambda}H_{\mathrm{B}}(f)$$
$$\mathcal{F}\{\mathcal{R}_{\mathbf{n}_{\mathrm{B}}}(t,\tau)\} = \mathrm{e}^{-j4\pi f_s t}H_{\mathrm{B}}(f)\underbrace{\int_{\lambda=-\infty}^{\infty}\mathrm{d}\lambda h_{\mathrm{B}}(\lambda)\mathrm{e}^{-j2\pi(-f-2f_s)\lambda}}_{H_{\mathrm{B}}(-f-2f_s)}$$

But $H_{\mathrm{B}}(f)$ and $H_{\mathrm{B}}(-f-2f_s)$ do not overlap $\Rightarrow$ product is zero. Therefore $\mathcal{R}_{\mathbf{n}_{\mathrm{B}}}(t,\tau) = 0$
But, on the other hand, $\mathbf{n}_{\mathrm{B}}(t) = \mathbf{n}_I(t) - j\mathbf{n}_Q(t)$. Therefore

$$\begin{aligned}
E\{\mathbf{n}_{\mathrm{B}}(t)\mathbf{n}_{\mathrm{B}}(t+\tau)\} &= E\{[\mathbf{n}_I(t)-j\mathbf{n}_Q(t)][\mathbf{n}_I(t+\tau)-j\mathbf{n}_Q(+\tau)]\}\\
&= [E\{\mathbf{n}_I(t)\mathbf{n}_I(t+\tau)\} - E\{\mathbf{n}_Q(t)\mathbf{n}_Q(t+\tau)\}]\\
&\quad -j[E\{\mathbf{n}_I(t)\mathbf{n}_I(t+\tau)\} + E\{\mathbf{n}_Q(t)\mathbf{n}_Q(t+\tau)\}]\\
&= [\mathcal{R}_I(\tau)-\mathcal{R}_Q(\tau)] - j[\mathcal{R}_I(\tau)+\mathcal{R}_Q(\tau)]
\end{aligned}$$

But we just showed that this expectation is equal to zero. Therefore $\mathcal{R}_I(\tau) = \mathcal{R}_Q(\tau)$ and $\mathcal{R}_{I,Q}(\tau) = -\mathcal{R}_{Q,I}(\tau)$
Consider now:

$$E\{\mathbf{n}_{\mathrm{B}}^*(t)\mathbf{n}_{\mathrm{B}}(t+\tau)\} = E\left\{\left[\int_{\lambda=-\infty}^{\infty}\mathbf{w}(t-\tau)h_{\mathrm{B}}^*(\lambda)\mathrm{e}^{-j2\pi f_s\lambda}\mathrm{d}\lambda\right]\mathrm{e}^{-j2\pi f_s t}\right.$$
$$\left.\left[\int_{u=-\infty}^{\infty}\mathbf{w}(t+\tau-u)h_{\mathrm{B}}(u)\mathrm{e}^{j2\pi f_s u}\mathrm{d}u\right]\mathrm{e}^{-j2\pi f_s(t+\tau)}\right\}$$
$$= \mathrm{e}^{-j2\pi f_s\tau}\left[\int_{\lambda=-\infty}^{\infty}\int_{u=-\infty}^{\infty}\underbrace{E\{\mathbf{w}(t-\lambda)\mathbf{w}(t+\lambda-u)\}}_{=\delta(\tau-u+\lambda)\text{ as impulse occurs at }u=\tau+\lambda} h_{\mathrm{B}}^*(\lambda)h_{\mathrm{B}}(u)\mathrm{e}^{-j2\pi f_s(\lambda-u)}\mathrm{d}\lambda\mathrm{d}u\right]$$

Considering the integral w.r.t. u, the impulse sifts out the value of the integral at $u=\tau+\lambda$. Therefore

$$\begin{aligned} E\left\{\mathbf{n}_{\mathrm{B}}^{*}(t)\mathbf{n}_{\mathrm{B}}(t+\tau)\right\} &= \mathrm{e}^{-j2\pi f_s\tau}\int_{\lambda=-\infty}^{\infty} h_{\mathrm{B}}^{*}(\lambda)h_{\mathrm{B}}(\tau+\lambda)\mathrm{e}^{-j2\pi f_s(\lambda-\tau-\lambda)}\mathrm{d}\lambda \\ &= \int_{\lambda=-\infty}^{\infty} h_{\mathrm{B}}^{*}(\lambda)h_{\mathrm{B}}(\tau+\lambda)\mathrm{d}\lambda. \end{aligned}$$

Now Fourier transform the above w.r.t τ, call it $S_{\mathrm{B}}(f)$:

$$\begin{aligned} S_{\mathrm{B}}(f) &= \int_{\tau=-\infty}^{\infty}\left[\int_{\lambda=-\infty}^{\infty} h_{\mathrm{B}}^{*}(\lambda)h_{\mathrm{B}}(\tau+\lambda)\mathrm{d}\lambda\right]\mathrm{e}^{-j2\pi f\tau} \\ &= \int_{\lambda=-\infty}^{\infty}\left[\int_{\tau=-\infty}^{\infty} h_{\mathrm{B}}(\tau+\lambda)\mathrm{e}^{-j2\pi f\tau}\mathrm{d}\tau\right]h_{\mathrm{B}}^{*}(\lambda)\mathrm{d}\lambda \end{aligned}$$

Change variable in the τ integral to $u=\tau+\lambda$. It becomes

$$\int_{u=-\infty}^{\infty} h_{\mathrm{B}}(u)\mathrm{e}^{-j2\pi f(u-\lambda)}\mathrm{d}u = \mathrm{e}^{j2\pi f\lambda}\int_{u=-\infty}^{\infty} h_{\mathrm{B}}(u)\mathrm{e}^{-j2\pi fu}\mathrm{d}u = \mathrm{e}^{j2\pi f\lambda}H_{\mathrm{B}}(f)$$

$$\therefore S_{\mathrm{B}}(f) = H_{\mathrm{B}}(f)\left[\int_{\lambda=-\infty}^{\infty} h_{\mathrm{B}}^{*}(\lambda)\mathrm{e}^{j2\pi f\lambda}\mathrm{d}\lambda\right] = H_{\mathrm{B}}(f)\Bigg[\underbrace{\int_{\lambda=-\infty}^{\infty} h_{\mathrm{B}}(\lambda)\mathrm{e}^{-j2\pi f\lambda}\mathrm{d}\lambda}_{H_{\mathrm{B}}(f)}\Bigg]^{*}$$

$$\therefore S_{\mathrm{B}}(f) = H_{\mathrm{B}}(f)H_{\mathrm{B}}^{*}(f) = |H_{\mathrm{B}}(f)|^2 \text{ watts/Hz}$$

(e) From (a) we have that $R_I(\tau)=R_Q(\tau)$ and $R_{I,Q}(\tau)=-R_{Q,I}(\tau)$. In (b) we find out that $E\left\{\mathbf{n}_{\mathrm{B}}^{*}(t)\mathbf{n}_{\mathrm{B}}(t+\tau)\right\}$ is the inverse Fourier transform of $S_{\mathrm{B}}(f)$, i.e., $=\int_{-\infty}^{\infty} S_{\mathrm{B}}(f)\mathrm{e}^{-j2\pi f\tau}\mathrm{d}f$. But it also equals
$E\left\{[\mathbf{n}_I(t)+j\mathbf{n}_Q(t)][\mathbf{n}_I(t+\tau)-j\mathbf{n}_Q(t+\tau)]\right\} = [R_I(\tau)+R_Q(\tau)]+j[-R_{I,Q}(\tau)+R_{Q,I}(\tau)]$
or

$$\begin{aligned} \underbrace{2R_I(\tau)-j2R_{I,Q}(\tau)}_{\text{or } (2R_Q(\tau)+j2R_{Q,I}(\tau))} &= \int_{-\infty}^{\infty} S_{\mathrm{B}}(f)\mathrm{e}^{-j2\pi f\tau}\mathrm{d}f \\ &= \int_{-\infty}^{\infty} S_{\mathrm{B}}(f)\cos(2\pi f\tau)\mathrm{d}f + j\int_{-\infty}^{\infty} S_{\mathrm{B}}(f)\sin(2\pi f\tau)\mathrm{d}f \end{aligned}$$

$$\begin{aligned} \Rightarrow R_I(\tau) &= R_Q(\tau) = \frac{1}{2}\int_{-\infty}^{\infty} S_{\mathrm{B}}(f)\cos(2\pi f\tau)\mathrm{d}f \\ R_{I,Q}(\tau) &= -R_{Q,I}(\tau) = -\frac{1}{2}\int_{-\infty}^{\infty} S_{\mathrm{B}}(f)\sin(2\pi f\tau)\mathrm{d}f \end{aligned}$$

(f) It is even. The crosscorrelation function are 0, since $S_{\mathrm{B}}(t)\sin(2\pi f\tau)$ is an odd function and the area under it is zero. Therefore the I & Q processes are statically independent.

P7.19

$$\begin{aligned} Y(f) &= X(f)H(f) = [X_{\mathrm{B}}(f-f_s)+X_{\mathrm{B}}^{*}(-f-f_s)][H_{\mathrm{B}}(f-f_s)+H_{\mathrm{B}}^{*}(-f-f_s)] \\ &= X_{\mathrm{B}}(f-f_s)H_{\mathrm{B}}(f-f_s)+X_{\mathrm{B}}^{*}(-f-f_s)H_{\mathrm{B}}^{*}(-f-f_s) \\ &\quad +\underbrace{X_{\mathrm{B}}(f-f_s)H_{\mathrm{B}}^{*}(-f-f_s)}_{=0}+\underbrace{X_{\mathrm{B}}^{*}(-f-f_s)H_{\mathrm{B}}(f-f_s)}_{=0} \end{aligned}$$

But $Y(f)$ can be also written as $Y(f) = Y_{\text{B}}(f - f_s) + Y^*_{\text{B}}(-f - f_s)$, i.e.,

$$\begin{aligned}
& Y_{\text{B}}(f - f_s) + Y^*_{\text{B}}(-f - f_s) = X_{\text{B}}(f - f_s)H_{\text{B}}(f - f_s) + X^*_{\text{B}}(-f - f_s)H^*_{\text{B}}(-f - f_s) \\
\Rightarrow \quad & Y_{\text{B}}(f - f_s) = X_{\text{B}}(f - f_s)H_{\text{B}}(f - f_s) \Rightarrow Y_{\text{B}}(f) = X_{\text{B}}(f)H_{\text{B}}(f) \\
\Rightarrow \quad & y_{\text{B}}(t) = x_{\text{B}}(t) \otimes h_{\text{B}}(t)
\end{aligned}$$

P7.20 $h_{\text{B}}(t) = s_{\text{B}}(T_b - t)$.

Chapter 8

M-ary Signaling Techniques

P8.1 (a) Basically the procedure is to take the previous Gray code, flip it over to create a bottom half. Insert a leading zero in the top half and the leading one in the bottom half. So for 3-bit sequences, one obtains:

$$
\begin{array}{rccl}
\text{leading zeros} & \left\{\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{array}\right. & & \text{top half} \\
\hline
\text{leading ones} & \left\{\begin{array}{ccc} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{array}\right. & & \text{bottom half}
\end{array}
$$

(b)

a_1	a_2	a_3	a_4	a_5	b_1	b_2	b_3	b_4	b_5
0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	1
0	0	0	1	0	0	0	0	1	1
0	0	0	1	1	0	0	0	1	0
0	0	1	0	0	0	0	1	1	0
0	0	1	0	1	0	0	1	1	1
0	0	1	1	0	0	0	1	0	1
0	0	1	1	1	0	0	1	0	0
0	1	0	0	0	0	1	1	0	0
0	1	0	0	1	0	1	1	0	1
0	1	0	1	0	0	1	1	1	1
0	1	0	1	1	0	1	1	1	0
0	1	1	0	0	0	1	0	1	0
0	1	1	0	1	0	1	0	1	1
0	1	1	1	0	0	1	0	0	1
0	1	1	1	1	0	1	0	0	0

For the bottom half: flip the top half over and set b_1=1.

(c) The direct method appears to be more straightforward since it appears that you do not need to generate the previous Gray bit codes to get the final one. But in reality there is no difference. One can adapt the inductive method to produce a Gray code by considering the pattern of zeros and ones in successive columns - Left as an exercise.

P8.2 (*4-ASK modulation*)

(a)+(b) With $M = 4$,

$$\text{P[symbol error]} = \frac{3}{2}Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) \Rightarrow \frac{\Delta}{\sqrt{2N_0}} = Q^{-1}\left(\frac{2}{3}\text{P[symbol error]}\right). \tag{8.1}$$

The results of $\frac{\Delta}{\sqrt{2N_0}}, Q\left(\frac{\Delta}{\sqrt{2N_0}}\right), Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right), Q\left(\frac{5\Delta}{\sqrt{2N_0}}\right)$ corresponding to different P[symbol error] are given in the Table 8.1.

Table 8.1

P[symbol error]	$\frac{\Delta}{\sqrt{2N_0}}$	$Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)$	$Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right)$	$Q\left(\frac{5\Delta}{\sqrt{2N_0}}\right)$
10^{-1}	1.5011	6.67×10^{-2}	3.35×10^{-6}	3.06×10^{-14}
10^{-2}	2.4747	6.67×10^{-3}	5.67×10^{-14}	1.81×10^{-35}
10^{-3}	3.2087	6.67×10^{-4}	3.10×10^{-22}	3.17×10^{-58}
10^{-4}	3.8202	6.67×10^{-5}	1.04×10^{-30}	1.28×10^{-81}

(c) It is clear that

$$\begin{aligned} \text{P}[\{01\}_D|\{00\}_T] &= Q\left(\frac{\frac{\Delta}{2}}{\sqrt{N_0/2}}\right) - Q\left(\frac{\frac{3\Delta}{2}}{\sqrt{N_0/2}}\right) \\ &= Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) - Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right) \approx Q\left(\frac{\Delta}{\sqrt{2N_0}}\right). \end{aligned} \tag{8.2}$$

Similarly, one can find that

$$\text{P}[\{11\}_D|\{00\}_T] = Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right) - Q\left(\frac{5\Delta}{\sqrt{2N_0}}\right) \approx Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right), \tag{8.3}$$

$$\text{P}[\{10\}_D|\{00\}_T] = Q\left(\frac{5\Delta}{\sqrt{2N_0}}\right). \tag{8.4}$$

(d) It can be seen from the signal space that the same set of conditional symbol error probabilities applies for the case that symbol 10 is transmitted. The set will be different when either $\{01\}$ or $\{11\}$ is transmitted.

Compared with $Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)$ the term $Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right)$ and $Q\left(\frac{5\Delta}{\sqrt{2N_0}}\right)$ are negligible at any level of P[symbol error] and can be ignored. It is the error that occurs when a signal is mistaken with a nearest neighbor that dominates. Therefore, in terms of neighbors the most important ones are the nearest ones.

P8.3 (a)

$$
\begin{aligned}
P[\text{bit error}] &= P[\text{bit error}|\ \{00\}_T]P[\{00\}_T] + P[\text{bit error}|\ \{01\}_T]P[\{01\}_T] \\
&\quad + P[\text{bit error}|\ \{11\}_T]P[\{11\}_T] + P[\text{bit error}|\ \{10\}_T]P[\{10\}_T] \\
P[\{00\}_T] &= P[\{01\}_T] = P[\{11\}_T] = P[\{10\}_T] = \frac{1}{4} \\
P[\text{bit error}|\ \{00\}_T] &= P[\text{bit error}|\ \{10\}_T] \\
P[\text{bit error}|\ \{01\}_T] &= P[\text{bit error}|\ \{11\}_T] \\
\therefore \qquad P[\text{bit error}] &= \frac{1}{2}P[\text{bit error}|\ \{00\}_T] + \frac{1}{2}P[\text{bit error}|\ \{01\}_T] \\
P[\text{bit error}|\ \{00\}_T] &= \frac{1}{2}\left[Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) - Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right)\right] \\
&\quad + \left[Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right) - Q\left(\frac{5\Delta}{\sqrt{2N_0}}\right)\right] + \frac{1}{2}Q\left(\frac{5\Delta}{\sqrt{2N_0}}\right) \\
P[\text{bit error}|\ \{01\}_T] &= \frac{1}{2}Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) + \frac{1}{2}\left[Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) - Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right)\right] \\
&\quad + \frac{1}{2}Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right) \\
\Rightarrow \qquad P[\text{bit error}] &= \frac{3}{4}Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) + \frac{1}{2}Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right) - \frac{1}{4}Q\left(\frac{5\Delta}{\sqrt{2N_0}}\right).
\end{aligned}
$$

(b) Due to symmetry $P[b_1 \text{ in error}] = P[b_1 \text{ in error}|b_1 = 0]$

$$
\begin{aligned}
P[b_1 \text{ in error}|b_1 = 0] &= P\left[\mathbf{r}_I > 0 \middle|\ s_T = -\frac{\Delta}{2}, b_1 = 0\right] P\left[s_T = -\frac{\Delta}{2}\middle| b_1 = 0\right] \\
&\quad + P\left[\mathbf{r}_I > 0 \middle|\ s_T = -\frac{3\Delta}{2}, b_1 = 0\right] P\left[s_T = -\frac{3\Delta}{2}\middle| b_1 = 0\right] \\
\Rightarrow P[b_1 \text{ in error}] &= \frac{1}{2}\left[Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) + Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right)\right].
\end{aligned}
$$

$$
P[b_2 \text{ in error}] = P[b_2 \text{ in error}|b_2 = 0]P[b_2 = 0] + P[b_2 \text{ in error}|b_2 = 1]P[b_2 = 1]
$$

$$
\begin{aligned}
P[b_2 \text{ in error}|b_2 = 0] &= 2P\left[-\Delta < \mathbf{r}_I < \Delta \middle|\ s_T = -\frac{3\Delta}{2}, b_2 = 0\right] P\left[s_T = -\frac{3\Delta}{2}\middle| b_2 = 0\right] \\
&= 2\left[Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) - Q\left(\frac{5\Delta}{\sqrt{2N_0}}\right)\right]\left(\frac{1}{2}\right) = Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) - Q\left(\frac{5\Delta}{\sqrt{2N_0}}\right)
\end{aligned}
$$

$$
\begin{aligned}
P[b_2 \text{ in error}|b_2 = 1] &= 2P\left[\mathbf{r}_I < -\Delta \text{ or } \mathbf{r}_I > \Delta \middle|\ s_T = -\frac{\Delta}{2}, b_2 = 1\right] P\left[s_T = -\frac{\Delta}{2}\middle| b_2 = 1\right] \\
&= 2\left[Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) + Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right)\right]\left(\frac{1}{2}\right) = Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) + Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right) \\
\Rightarrow P[b_2 \text{ in error}] &= \frac{1}{2}\left[2Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) + Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right) - Q\left(\frac{5\Delta}{\sqrt{2N_0}}\right)\right] \\
&= Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) + \frac{1}{2}Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right) - \frac{1}{2}Q\left(\frac{5\Delta}{\sqrt{2N_0}}\right).
\end{aligned}
$$

The arithmetic averaging of the two individual bit error probabilities gives:

$$P[\text{bit error}] = \frac{3}{4}Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) + Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right) - \frac{1}{4}Q\left(\frac{5\Delta}{\sqrt{2N_0}}\right) \quad \text{(same as in } (a)\text{)}.$$

The first observation is that the individual bit errors are not the same, the probability of b_2 being in error is higher by approximately a factor of 2. The overall bit error probability is somewhere between the two individual bit error probabilities. If one makes the reasonable approximations based on the result of P8.2(b) then

$$\begin{aligned} P[\text{bit error}] &= \frac{3}{4}Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) \\ P[b_1 \text{ in error}] &= \frac{1}{2}Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) \\ P[b_2 \text{ in error}] &= Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) \end{aligned}$$

Using the approximate expression we have $P[\text{bit error}] = \frac{1}{2}Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)$, which is close to the exact expression.

P8.4 (a) From Fig. 8.27 in the textbook, one can obtain the decision rule for bit b_2 as follows:

$$|r_I| \underset{b_2=1}{\overset{b_2=0}{\gtrless}} \Delta. \tag{8.5}$$

(b) Denote $s_1(t) = \{00\}, s_2(t) = \{01\}, s_3(t) = \{11\}, s_4(t) = \{10\}$. The error probabilities can be calculated as:

$$\begin{aligned} \text{P}[b_1 \text{ in error}] &= \frac{1}{4}\sum_{i=1}^{4} \text{P}[b_1 \text{ in error}|s_i(t)] \\ &= \frac{1}{4}\left[Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right) + Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) + Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) + Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right)\right] \\ &= \frac{1}{2}\left[Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) + Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right)\right]. \end{aligned} \tag{8.6}$$

$$\begin{aligned} \text{P}[b_2 \text{ in error}] &= \frac{1}{4}\sum_{i=1}^{4} \text{P}[b_2 \text{ in error}|s_i(t)] = \frac{1}{2}\left(\text{P}[b_2 \text{ in error}|s_1(t)] + \text{P}[b_2 \text{ in error}|s_2(t)]\right) \\ &= \frac{1}{2}\left[Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) - Q\left(\frac{5\Delta}{\sqrt{2N_0}}\right) + Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) + Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right)\right] \\ &= Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) + \frac{1}{2}Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right) - \frac{1}{2}Q\left(\frac{5\Delta}{\sqrt{2N_0}}\right). \end{aligned} \tag{8.7}$$

Then the average bit error probability can be calculated as

$$\begin{aligned} \text{P}[\text{bit error}] &= \frac{\text{P}[b_1 \text{ in error}] + \text{P}[b_2 \text{ in error}]}{2} \\ &= \frac{3}{4}Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) + \frac{1}{2}Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right) - \frac{1}{4}Q\left(\frac{5\Delta}{\sqrt{2N_0}}\right). \end{aligned} \tag{8.8}$$

The error probabilities are identical to those of P8.3.

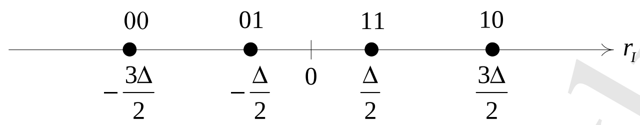


Figure 8.1

P8.5 (a) With symbol demodulation, $P[b_1 \text{ in error}]$ remains unchanged from P8.3(a) since the pattern for b_1 is identical. On the other hand,

$$\begin{aligned} P[b_2 \text{ in error}] &= P[b_2 \text{ in error}|b_2=0]P[b_2=0]+P[b_2 \text{ in error}|b_2=1]P[b_2=1] \\ &= \frac{1}{2}P[b_2 \text{ in error}|b_2=0]+\frac{1}{2}P[b_2 \text{ in error}|b_2=1] \end{aligned}$$

$$\begin{aligned} P[b_2 \text{ in error}|b_2=0] &= P\left[-\Delta<\mathbf{r}_I<0\middle|\, s_T=-\frac{3\Delta}{2},b_2=0\right]P\left[s_T=-\frac{3\Delta}{2}\middle|b_2=0\right] \\ &\quad +P\left[\mathbf{r}_I>\Delta\middle|\, s_T=-\frac{3\Delta}{2},b_2=0\right]P\left[s_T=-\frac{3\Delta}{2}\middle|b_2=0\right] \\ &\quad +P\left[-\Delta<\mathbf{r}_I<0\middle|\, s_T=\frac{\Delta}{2},b_2=0\right]P\left[s_T=\frac{\Delta}{2}\middle|b_2=0\right] \\ &\quad +P\left[\mathbf{r}_I>\Delta\middle|\, s_T=\frac{\Delta}{2},b_2=0\right]P\left[s_T=\frac{\Delta}{2}\middle|b_2=0\right] \\ &= \frac{1}{2}\left[Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)-Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right)+Q\left(\frac{5\Delta}{\sqrt{2N_0}}\right)\right. \\ &\qquad \left.+Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)-Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right)+Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)\right] \\ &= \frac{3}{2}Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)-Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right)+\frac{1}{2}Q\left(\frac{5\Delta}{\sqrt{2N_0}}\right) \\ \Rightarrow P[\text{bit error}] &= Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)-\frac{1}{4}Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right)+\frac{1}{4}Q\left(\frac{5\Delta}{\sqrt{2N_0}}\right). \end{aligned}$$

Compared with Gray coding the bit error probability is somewhat larger, basically $Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)$ as compared to $\frac{3}{4}Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)$, but not dramatically so.

(b) Now consider demodulation directly to the bits. The decision rules are:

$$r_I \underset{b_1=0}{\overset{b_1=1}{\gtrless}} 0$$

$$\begin{cases} (r_I<-\Delta) \text{ or } (0<r_I<\Delta) \Rightarrow b_2=0 \\ (-\Delta<r_I<0) \text{ or } (r_I>0) \Rightarrow b_2=1 \end{cases}$$

In terms of error probabilities, $P[b_1 \text{ in error}]$ should be the same as in P8.4(b). $P[b_2 \text{ in error}]$

is determined as follows:

$$\begin{aligned}
&P[b_2 \text{ in error}|b_2=0] \\
&= P\left[-\Delta < \mathbf{r}_I < 0\middle|\, s_T=-\frac{3\Delta}{2}, b_2=0\right] P\left[s_T=-\frac{3\Delta}{2}\middle|b_2=0\right] \\
&\quad + P\left[\mathbf{r}_I > \Delta\middle|\, s_T=-\frac{3\Delta}{2}, b_2=0\right] P\left[s_T=-\frac{3\Delta}{2}\middle|b_2=0\right] \\
&\quad + P\left[-\Delta < \mathbf{r}_I < 0\middle|\, s_T=\frac{\Delta}{2}, b_2=0\right] P\left[s_T=\frac{\Delta}{2}\middle|b_2=0\right] \\
&\quad + P\left[\mathbf{r}_I > \Delta\middle|\, s_T=\frac{\Delta}{2}, b_2=0\right] P\left[s_T=\frac{\Delta}{2}\middle|b_2=0\right] \\
&= \frac{1}{2}\left\{\left[Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)-Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right)\right]+Q\left(\frac{5\Delta}{\sqrt{2N_0}}\right)\right. \\
&\quad \left.+\left[Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)-Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right)\right]+Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)\right\} \\
&= \frac{3}{2}Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)-Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right)+\frac{1}{2}Q\left(\frac{5\Delta}{\sqrt{2N_0}}\right)
\end{aligned}$$

$$\begin{aligned}
P[b_2 \text{ in error}|b_2=1] &= \frac{1}{2}P\left[\mathbf{r}_I < -\Delta\middle|\, s_T=-\frac{\Delta}{2}, b_2=1\right] \\
&\quad + \frac{1}{2}P\left[0<\Delta<\mathbf{r}_I\middle|\, s_T=-\frac{\Delta}{2}, b_2=1\right] \\
&\quad + \frac{1}{2}P\left[\mathbf{r}_I < -\Delta\middle|\, s_T=\frac{3\Delta}{2}, b_2=1\right] \\
&\quad + \frac{1}{2}P\left[0<\mathbf{r}_I<\Delta\middle|\, s_T=\frac{3\Delta}{2}, b_2=1\right] \\
&= \frac{1}{2}\left\{Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)+\left[Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)-Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right)\right]\right. \\
&\quad \left.+Q\left(\frac{5\Delta}{\sqrt{2N_0}}\right)+\left[Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)-Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right)\right]\right\} \\
&= \frac{3}{2}Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)-Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right)+\frac{1}{2}Q\left(\frac{5\Delta}{\sqrt{2N_0}}\right) \\
\Rightarrow P[b_2 \text{ in error}] &= \frac{1}{2}P[b_2 \text{ in error}|b_2=0]+\frac{1}{2}P[b_2 \text{ in error}|b_2=1] \\
&= \frac{3}{2}Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)-Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right)+\frac{1}{2}Q\left(\frac{5\Delta}{\sqrt{2N_0}}\right).
\end{aligned}$$

$$\begin{aligned}
P[\text{bit error}] &= \frac{1}{2}P[b_2 \text{ in error}]+\frac{1}{2}P[b_1 \text{ in error}] \\
&= \frac{1}{2}\left[2Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)-\frac{1}{2}Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right)+\frac{1}{2}Q\left(\frac{5\Delta}{\sqrt{2N_0}}\right)\right] \\
&= Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)-\frac{1}{4}Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right)+\frac{1}{4}Q\left(\frac{5\Delta}{\sqrt{2N_0}}\right).
\end{aligned}$$

Again, the result is the same as demodulating to the symbol.

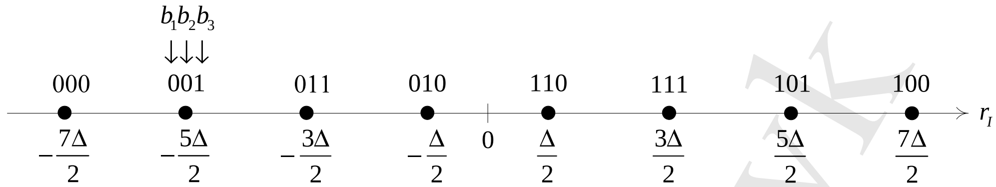


Figure 8.2

P8.6 (a)

$$r_I \underset{b_1=1}{\overset{b_1=0}{\gtrless}} 0; \qquad |r_I| \underset{b_2=0}{\overset{b_2=1}{\gtrless}} 2\Delta; \qquad \text{If } \{|r_I| > 3\Delta \text{ or } |r_I| < \Delta\} \Rightarrow b_3 = 0, \text{ else } b_3 = 1.$$

Graphically, the decision regions are illustrated in Fig. 8.3.

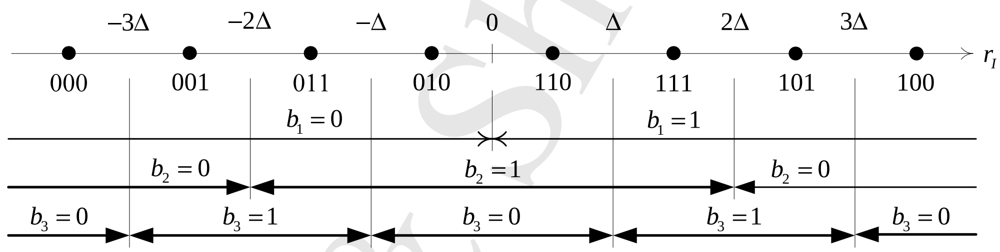


Figure 8.3

(b) Consider first bit b_1:

$$\begin{aligned}
P[b_1 \text{ in error}] &= P[b_1 \text{ in error}|b_1 = 0]P[b_1 = 0] \\
&\quad + P[b_1 \text{ in error}|b_1 = 1]P[b_1 = 1] \\
&= P[b_1 \text{ in error}|b_1 = 0] \quad \text{(by symmetry)} \\
&= P\left[r_I > 0 \middle| s_T \text{ is one of } -\frac{7\Delta}{2}, -\frac{5\Delta}{2}, -\frac{3\Delta}{2}, -\frac{\Delta}{2}, b_1 = 0\right] \\
&\quad \times \underbrace{P\left[s_T \text{ is one of } -\frac{7\Delta}{2}, -\frac{5\Delta}{2}, -\frac{3\Delta}{2}, -\frac{\Delta}{2} \middle| b_1 = 0\right]}_{=1/4}
\end{aligned}$$

$$\Rightarrow P[b_1 \text{ in error}] = \frac{1}{4}\left[Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) + Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right) + Q\left(\frac{5\Delta}{\sqrt{2N_0}}\right) + Q\left(\frac{7\Delta}{\sqrt{2N_0}}\right)\right].$$

Now consider bit b_2:

$$\begin{aligned}
P[b_2 \text{ in error}] &= P[b_2 \text{ in error}|b_2 = 0]P[b_2 = 0] \\
&\quad + P[b_2 \text{ in error}|b_2 = 1]P[b_2 = 1]
\end{aligned}$$

$$\begin{aligned}
&P[b_2 \text{ in error}|b_2=0] = \\
&\quad P\left[-2\Delta < r_I < 2\Delta \middle| \, s_T \text{ is one of } -\frac{7\Delta}{2}, -\frac{5\Delta}{2}, \frac{5\Delta}{2}, \frac{7\Delta}{2}, b_2=0\right] \\
&\quad \times \underbrace{P\left[s_T \text{ is one of } -\frac{7\Delta}{2}, -\frac{5\Delta}{2}, \frac{5\Delta}{2}, \frac{7\Delta}{2} \middle| b_2=0\right]}_{=1/4} \\
&= 2\left[Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right) - Q\left(\frac{11\Delta}{\sqrt{2N_0}}\right) + Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) - Q\left(\frac{9\Delta}{\sqrt{2N_0}}\right)\right]\left(\frac{1}{4}\right).
\end{aligned}$$

$$\begin{aligned}
&P[b_2 \text{ in error}|b_2=1] = \\
&\quad P\left[\mathbf{r}_I < -2\Delta \text{ or } \mathbf{r}_I > 2\Delta \middle| \, s_T \text{ is one of } -\frac{3\Delta}{2}, -\frac{\Delta}{2}, \frac{\Delta}{2}, \frac{3\Delta}{2}, b_2=1\right] \\
&\quad \times \underbrace{P\left[s_T \text{ is one of } -\frac{3\Delta}{2}, -\frac{\Delta}{2}, \frac{\Delta}{2}, \frac{3\Delta}{2} \middle| b_2=1\right]}_{=1/4} \\
&= 2\left[Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) + Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right) + Q\left(\frac{5\Delta}{\sqrt{2N_0}}\right) + Q\left(\frac{7\Delta}{\sqrt{2N_0}}\right)\right]\left(\frac{1}{4}\right).
\end{aligned}$$

$$\begin{aligned}
\Rightarrow P[b_2 \text{ in error}] &= \frac{1}{2}\left[2Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) + 2Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right) + Q\left(\frac{5\Delta}{\sqrt{2N_0}}\right)\right. \\
&\qquad \left. + Q\left(\frac{7\Delta}{\sqrt{2N_0}}\right) - Q\left(\frac{9\Delta}{\sqrt{2N_0}}\right) - Q\left(\frac{11\Delta}{\sqrt{2N_0}}\right)\right].
\end{aligned}$$

Lastly consider bit b_3:

$$\begin{aligned}
&P[b_3 \text{ in error}|b_3=0] = \\
&\quad P\left[-3\Delta < \mathbf{r}_I < -\Delta \text{ or } \Delta < \mathbf{r}_I < 3\Delta \middle| \, s_T \text{ is one of} -\frac{7\Delta}{2}, -\frac{\Delta}{2}, \frac{\Delta}{2}, \frac{7\Delta}{2}, b_3=0\right] \\
&\quad \times \underbrace{P\left[s_T \text{ is one of } -\frac{7\Delta}{2}, -\frac{\Delta}{2}, \frac{\Delta}{2}, \frac{7\Delta}{2} \middle| b_3=0\right]}_{=1/4} \\
&= \frac{1}{4}\left\{2\left[Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) - Q\left(\frac{5\Delta}{\sqrt{2N_0}}\right) + Q\left(\frac{9\Delta}{\sqrt{2N_0}}\right) - Q\left(\frac{13\Delta}{\sqrt{2N_0}}\right)\right]\right. \\
&\qquad \left. +2\left[Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) - Q\left(\frac{5\Delta}{\sqrt{2N_0}}\right) + Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right) - Q\left(\frac{7\Delta}{\sqrt{2N_0}}\right)\right]\right\} \\
&= \frac{1}{2}\left[2Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) + Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right) - 2Q\left(\frac{5\Delta}{\sqrt{2N_0}}\right)\right. \\
&\qquad \left. -Q\left(\frac{7\Delta}{\sqrt{2N_0}}\right) + Q\left(\frac{9\Delta}{\sqrt{2N_0}}\right) - Q\left(\frac{13\Delta}{\sqrt{2N_0}}\right)\right].
\end{aligned}$$

$$P[b_3 \text{ in error}|b_3=1] =$$

$$\frac{1}{4}P\left[(\mathbf{r}_I<-3\Delta)\text{ or }(-\Delta<\mathbf{r}_I<\Delta)\text{ or }(\mathbf{r}_I>3\Delta)\,\Big|\, s_T \text{ is one of } -\frac{5\Delta}{2},-\frac{3\Delta}{2},\frac{3\Delta}{2},\frac{5\Delta}{2},b_3=1\right]$$

$$\begin{aligned}
&= \frac{1}{4}\cdot 2\cdot\left[Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)+Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right)-Q\left(\frac{7\Delta}{\sqrt{2N_0}}\right)+Q\left(\frac{11\Delta}{\sqrt{2N_0}}\right)\right.\\
&\qquad\left.+Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right)+Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)-Q\left(\frac{5\Delta}{\sqrt{2N_0}}\right)+Q\left(\frac{9\Delta}{\sqrt{2N_0}}\right)\right]\\
&= \frac{1}{2}\left[2Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)+2Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right)-Q\left(\frac{5\Delta}{\sqrt{2N_0}}\right)\right.\\
&\qquad\left.-Q\left(\frac{7\Delta}{\sqrt{2N_0}}\right)+Q\left(\frac{9\Delta}{\sqrt{2N_0}}\right)+Q\left(\frac{11\Delta}{\sqrt{2N_0}}\right)\right].
\end{aligned}$$

$$\begin{aligned}
\Rightarrow P[b_3 \text{ in error}] = \frac{1}{4}\left[4Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)+3Q\left(\frac{3\Delta}{\sqrt{2N_0}}\right)-3Q\left(\frac{5\Delta}{\sqrt{2N_0}}\right)\right.\\
\left.-2Q\left(\frac{7\Delta}{\sqrt{2N_0}}\right)+2Q\left(\frac{9\Delta}{\sqrt{2N_0}}\right)+Q\left(\frac{11\Delta}{\sqrt{2N_0}}\right)-Q\left(\frac{13\Delta}{\sqrt{2N_0}}\right)\right].
\end{aligned}$$

Using the result of P8.2 we ignore, with confidence, all terms except $Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)$. Therefore:

$$\begin{aligned}
P[b_1 \text{ in error}] &= \frac{1}{4}Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)\\
P[b_2 \text{ in error}] &= \frac{1}{2}Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)\\
P[b_3 \text{ in error}] &= Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)\\
\therefore\ P[\text{bit error}] &= \frac{1}{3}\cdot\frac{7}{4}\cdot Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)=\frac{7}{12}Q\left(\frac{\Delta}{\sqrt{2N_0}}\right).
\end{aligned}$$

P8.7 Consider rectangular QAM of λ bits with λ_I bits assigned to the I axis, λ_Q bits to the Q axis ($\lambda=\lambda_I+\lambda_Q$). Gray code the λ_I, λ_Q bit patterns separately, concatenate them and assign the corresponding bit pattern to the I,Q signal.

Example: $\lambda_I=\lambda_Q=2$. Gray code for either axis is $00, 01, 11, 10$. Therefore the result is as shown in Fig. 8.4.

P8.8 (a) See Fig. 8.4 for the bit pattern $b_1b_2b_3b_4$. Then the decision rules are:

$$\mathbf{r}_Q \underset{b_1=0}{\overset{b_1=1}{\gtrless}} 0; \qquad |\mathbf{r}_Q| \underset{b_2=1}{\overset{b_2=0}{\gtrless}} \frac{\Delta}{2}; \qquad \mathbf{r}_I \underset{b_3=0}{\overset{b_3=1}{\gtrless}} 0; \qquad |\mathbf{r}_I| \underset{b_4=1}{\overset{b_4=0}{\gtrless}} \frac{\Delta}{2}.$$

Note: $\mathbf{r}_I, \mathbf{r}_Q$ are statistically independent Gaussian random variables, variance $N_0/2$ and a mean value which depends on the transmitted signal.

(b) Since rectangular 16-QAM is basically 2 independent 4-ASK modulations the results of P8.2 apply directly.

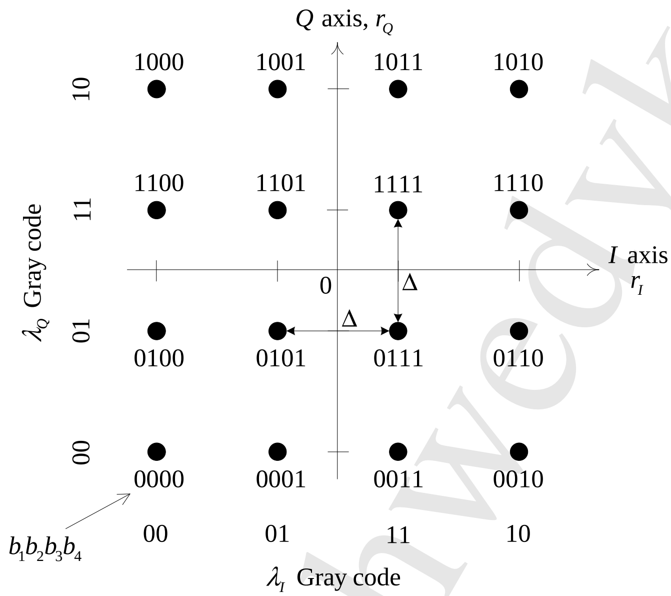


Figure 8.4

(c) Make this bit a b_1 or b_3 bit.

P8.9 (a) Constellation (a): The average symbol energy is equal to the squared distance of any signal from the origin. In terms of Δ this is $E_s = \left(\frac{\Delta}{2\sin 22.5^\circ}\right)^2 = 1.71\Delta^2$ joules/symbol.

Constellation (b): The sum of the squared distances of each signal from the origin is $= 4\left(\frac{\Delta^2}{4}+\frac{\Delta^2}{4}\right) + 4\left(\frac{9\Delta^2}{4}+\frac{\Delta^2}{4}\right) = 12\Delta^2$. The average energy is $E_s = \frac{12\Delta^2}{8} = 1.5\Delta^2$ joules/symbol.

Constellation (c): Again sum of the squared distances is $= 4\left(\frac{\Delta^2}{4}+\frac{\Delta^2}{4}\right) + 4\left(\Delta + \frac{\sqrt{2}}{2}\Delta\right)^2 = (8+4\sqrt{2})\Delta^2$. Therefore $E_s = 1 + \frac{\sqrt{2}}{2}\Delta^2 = 1.71\Delta^2$ joules/symbol.

Constellation (d): Sum of the squared distances is $2\left(\frac{\Delta^2}{4}\right) + 2\left(\frac{9\Delta^2}{4}\right) + 4\left(\frac{\Delta^2}{4}+\Delta^2\right) = 10\Delta^2$. Therefore $E_s = \frac{10\Delta^2}{8} = 1.25\Delta^2$ joules/symbol.

So from most efficient to least efficient the ranking is (d), (b), (c), (a).
Note: Since each symbol (in each constellation) represents 3 bits, the average bit energy, $E_b = E_s/3$ joules/bit.

(b) See Fig. 8.5.

(c) See Fig. 8.6.

(d) The triangular constellation is most efficient at $1.125\Delta^2$ joules/symbol, which is

$$10\log_{10}\left(\frac{1.25}{1.125}\right) = 0.46 \text{ dB}$$

better than the best constellation in Fig. 8.28.

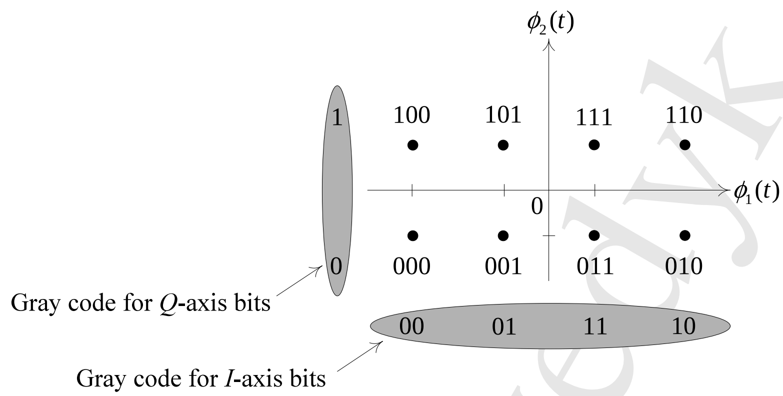


Figure 8.5

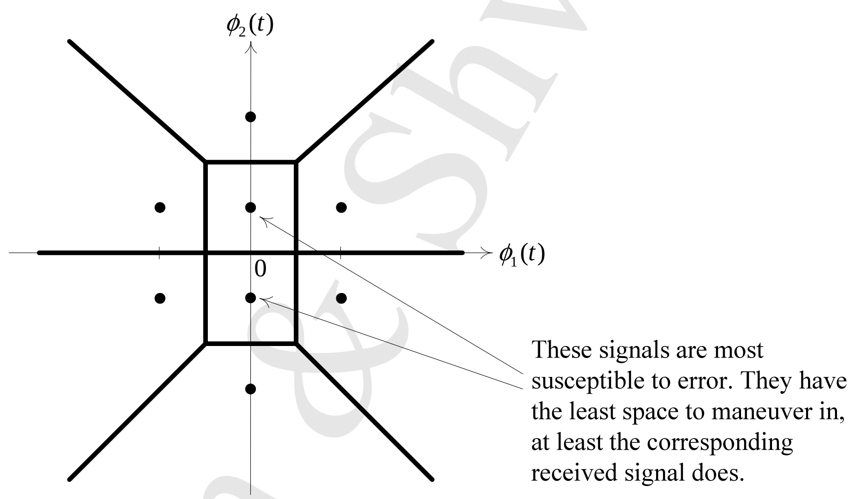
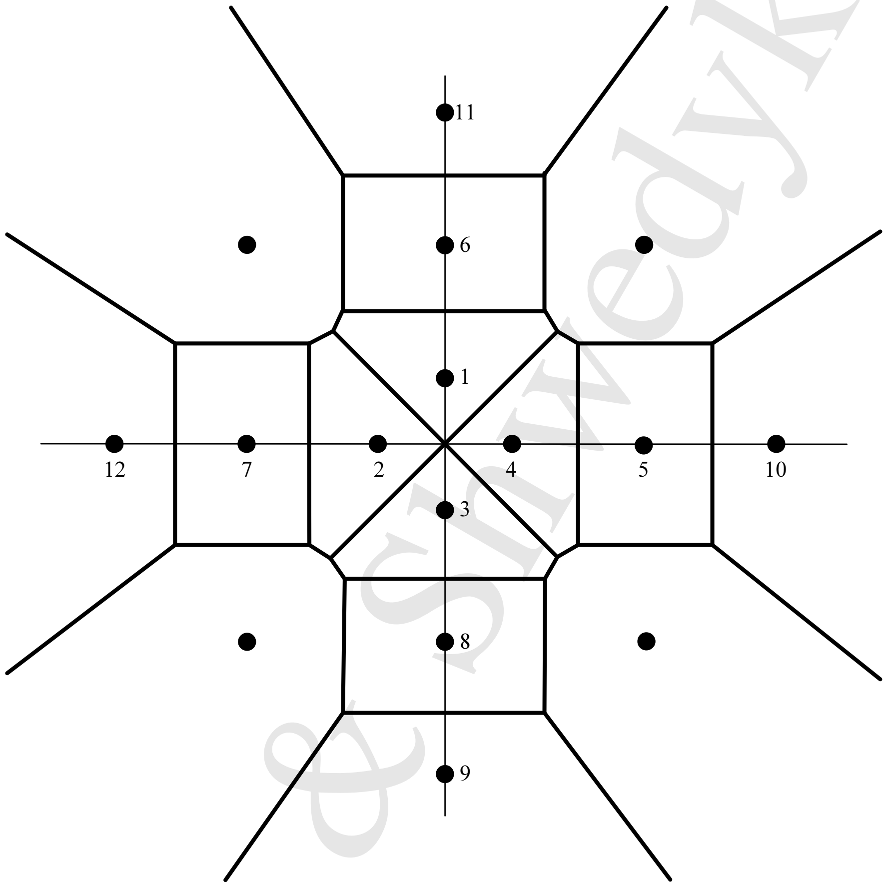


Figure 8.6

P8.10 (a) Fig. 8.7 shows the signal constellation with the signals to be Gray coded numbered. A solution is also given and explained on the figure.

(b) Assume that all the 16 signal points are equally likely. The optimum decision boundaries of the minimum distance receiver are also sketched in Fig. 8.7.



A graph of the signals to be Gray coded and their nearest neighbours looks as follows:

$1 \rightarrow 6 \rightarrow 11$
$\downarrow$
$2 \rightarrow 7 \rightarrow 12$
$\downarrow$
$3 \rightarrow 8 \rightarrow 9$
$\downarrow$
$4 \rightarrow 5 \rightarrow 10$

The 4-bit Gray code is:

$0000 \rightarrow 1$
$0001 \rightarrow 2$
$0011 \rightarrow 3$
$0010 \rightarrow 4$
0100
$0110 \rightarrow 5$
$0111 \rightarrow 10$
$0101 \rightarrow 7$
$1101 \rightarrow 12$
1100
1111
1110
$1010 \rightarrow 11$
$1001 \rightarrow 9$
$1011 \rightarrow 8$
$1000 \rightarrow 6$

Somewhat heuristically assign the bit patterns to signals as shown. The remaining 4 bit patterns can be arbitrarily assigned to the 4 remaining signals.

Figure 8.7: V.29 constellation and “Gray mapping”.

P8.11 (a) The two constellations have roughly the same symbol error performance since the minimum distance for both constellations is 2.

To see which one is more power-efficient, one needs to compute the average (or total) energies for the two constellations.

$$\bar{E}_{(a)} = \frac{1}{16}\left[4\{(3^2+3^2)+(1^2+1^2)+2(3^2+1^2)\}\right] = 10.0 \text{ (joules)}$$

$$\bar{E}_{(b)} = \frac{1}{16}\left\{4[(\sqrt{2})^2+(\sqrt{2}+2)^2+(\sqrt{2}+4)^2+(\sqrt{2}+6)^2]\right\} = 24.49 \text{ (joules)}$$

From the above results it can be seen that constellation (a) is much more power-efficient than constellation (b).

(b) See Fig. 8.8. Gray code by trial and error - but intelligent trial and error.

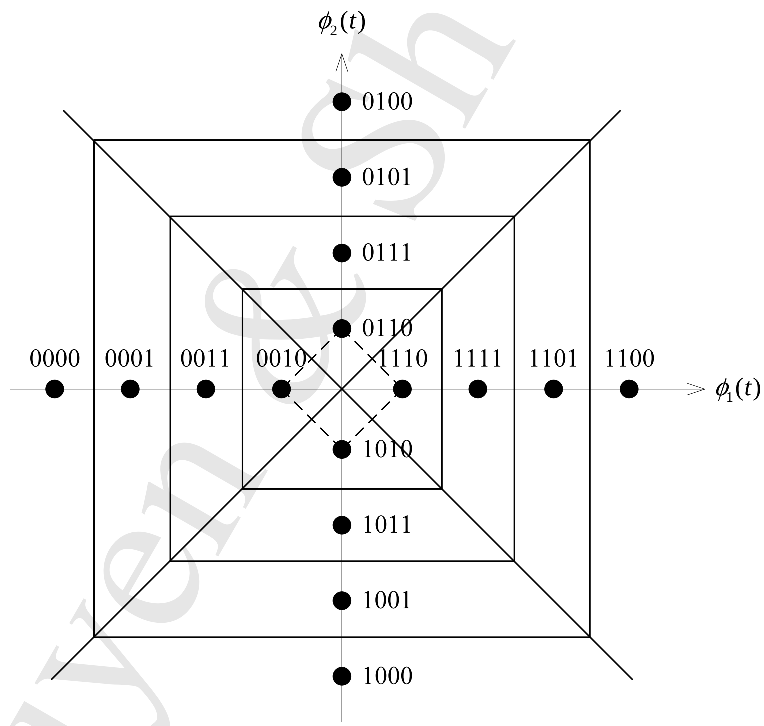


Figure 8.8

(c) See Fig. 8.8. The four outer (corner) signals are least susceptible to error since they have the least nearest neighbors, hence and the most space for the received signal to roam in.

P8.12 (a) By observation, $d_{\min}$ is maximized when $\Delta_h = \Delta_v = \Delta$ $(= d_{\min})$, which means that $\tan\theta = \frac{(\Delta/2)}{(3\Delta/2)} = \frac{1}{3} \Rightarrow \theta = 18.4°$.

(b) Gray code "vertical bits" and "horizontal bits" separately and then concatenate them. The result is shown in Fig. 8.9.

(c) For the present constellation: $E_s = \left(\frac{3}{2}d_{\min}\right)^2 + \left(\frac{1}{2}d_{\min}\right)^2 = \frac{5}{2}\left(d_{\min}\right)^2 \Rightarrow d_{\min} = 0.63\sqrt{E_s}$. For 8-PSK the geometry looks like in Fig. 8.10 and $d_{\min} = 0.765\sqrt{E_s}$, which is better.

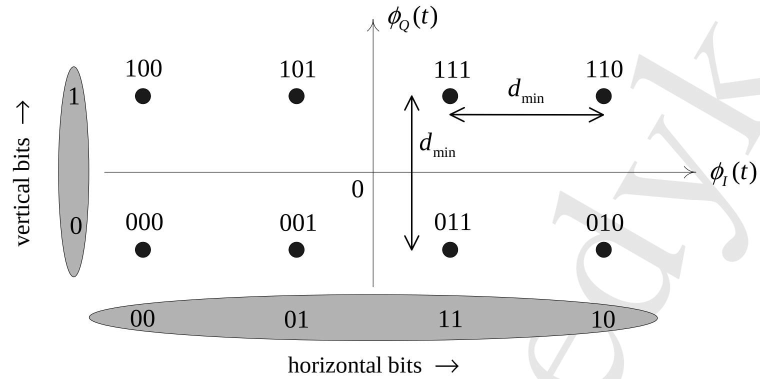


Figure 8.9

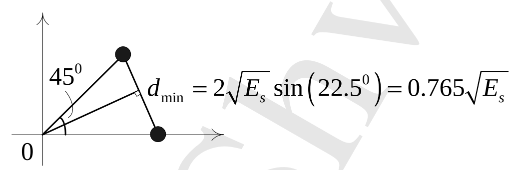


Figure 8.10

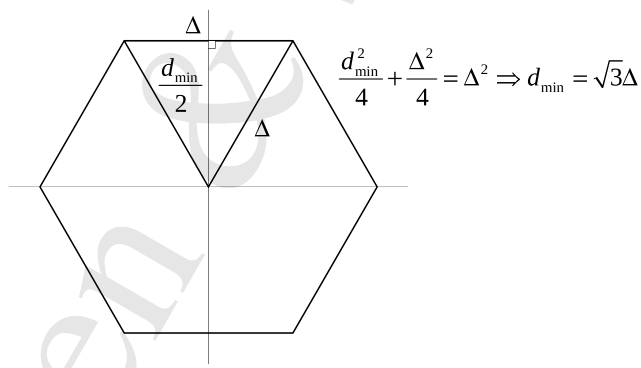


Figure 8.11

P8.13 (a) See Fig. 8.11.

(b)
- Have 6 signals with energy $= d_{\min}^2 = 3\Delta^2$ joules.
- 1 signal with zero energy.
- 1 signal with $(d_{\min}\cos 30^\circ)^2 + \left(\frac{3d_{\min}}{2}\right)^2 = \left(\frac{3}{4} + \frac{9}{4}\right) 3\Delta^2 = 9\Delta^2$. Therefore

$$E_s = \frac{[6(3) + 9]\,\Delta^2}{8} = \frac{27}{8}\Delta^2 \quad \text{joules/signal.}$$
$$E_b = \frac{E_s(\text{joules/signal})}{3(\text{ bits/signal})} = \frac{9}{8}\Delta^2 \quad \text{joules/bit.}$$

(c) See Fig. 8.12.

(d) No, not possible. The signal at the origin has 6 nearest neighbors. Not possible to find six 3-bit patterns that differ from its 3-bit pattern by only 1 bit. At most there are three 3-bit patterns.

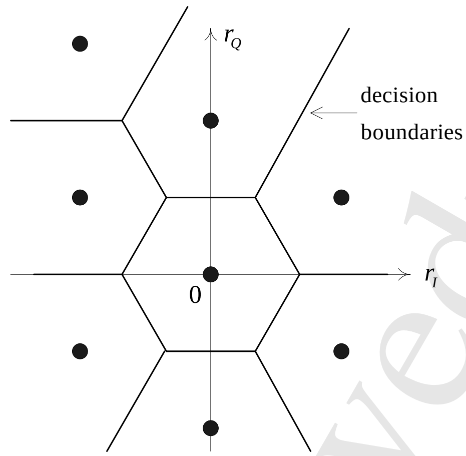


Figure 8.12

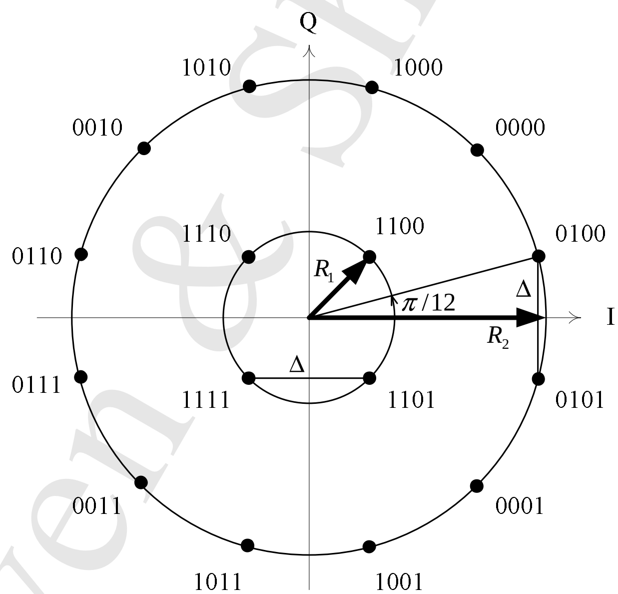


Figure 8.13: The 16APSK constellation used in DVB-S2.

P8.14 (a) From Fig. 8.13, one can calculate R_1, R_2 in terms of Δ as follows:

$$\frac{\Delta}{2} = R_2 \sin\frac{\pi}{12} \Rightarrow R_2 = \frac{\Delta}{2\sin\frac{\pi}{12}} = 1.932\Delta. \tag{8.9}$$

$$\Delta = R_1\sqrt{2} \Rightarrow R_1 = \frac{\Delta}{\sqrt{2}} = 0.707\Delta. \tag{8.10}$$

The average symbol energy is:

$$E_s^{(1)} = \frac{1}{16}\left(12R_2^2 + 4R_1^2\right) = \frac{1}{16}\left(44.79\Delta^2 + 2\Delta^2\right) = 2.9245\Delta^2. \tag{8.11}$$

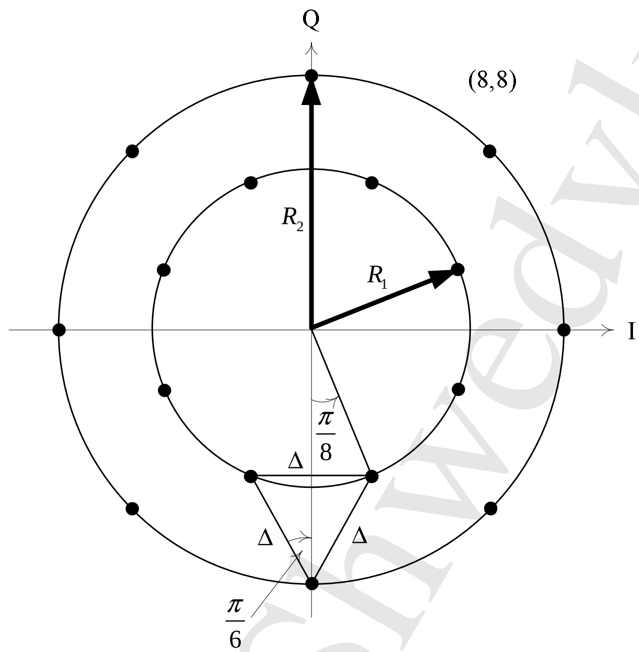


Figure 8.14: The modified 16APSK constellation.

(b) Similarly, from Fig. 8.14, R_1, R_2 can be calculated as:

$$R_1 = \frac{\Delta}{2\sin\frac{\pi}{8}} = 1.307\Delta. \tag{8.12}$$

$$R_2 = \frac{\Delta}{2\tan\frac{\pi}{8}} + \Delta\cos\frac{\pi}{6} = 2.073\Delta. \tag{8.13}$$

The average symbol energy is:

$$\begin{aligned} E_s^{(2)} &= \frac{1}{16}\left(8R_1^2 + 8R_2^2\right) = \frac{1}{2}\left(R_1^2 + R_2^2\right) \\ &= \frac{1}{2}\left(1.708\Delta^2 + 4.297\Delta^2\right) = 3.003\Delta^2. \end{aligned} \tag{8.14}$$

Since the minimum distances of both constellations are the same, the two constellations perform approximately identically in AWGN. The modified constellation has a larger average energy ($E_s^{(2)} > E_s^{(1)}$). Therefore it is less power-efficient compared to DVB-S2 16APSK.

Furthermore, the demodulator for the (8,8) signal constellation appears to be more complicated than that of the (4,12) constellation. But to back this up you should propose block diagrams of the demodulators.

(c) The (8,8) constellation cannot be Gray coded. This is based on the observation that 3 signals which are at equal distance ($d_{\min}$) from each other, i.e., lie on the vertices of an equilateral triangle, cannot be Gray coded. This is true regardless of the length of the bit sequence.

Proof:

1) Let a, b, c be 3 unique n-bit sequences.

2) Let a and b differ in one bit, say in the jth position.
3) Now suppose c differs from a by only one bit. This cannot be in the jth position because then it would be identical to b. So let it differ in the ith position.
4) But this means that c differs from b in the jth and ith positions, hence violating the Gray mapping rule. Remember Gray code means that 2 binary sequences that are at minimum distance differ by only one bit.

The (4,12) constellation, however, can be Gray coded. See a Gray code table for 4-bit sequences in Fig. 8.15.

0000	0000	For the signals on the inner circle
0001	0100	
0011	1100	
0010	1000	
0110	0001	For the signals on the outer circle
0111	0011	
0101	0010	
0100	0110	
1100	0111	
1101	0101	
1111	1101	
1110	1111	
1010	1110	
1011	1010	
1001	1011	
1000	1001	

Figure 8.15

P8.15 (a) The probability of symbol error can be written as

$$\begin{aligned} P[\text{symbol error}] &= P[(\text{error on I axis}) \text{ or } (\text{error on Q axis})] \\ &= P[(\text{error on I axis})] + P[(\text{error on Q axis})] \\ &\quad -P[(\text{error on I axis}) \text{ and } (\text{error on Q axis})] \end{aligned}$$

Now $P[(\text{error on I axis})]$ is that of M-ASK with $M = M_I = 2^{\lambda_I}$ (number of signal components along I axis).
Similarly $P[(\text{error on Q axis})]$ is that of M-ASK with $M = M_Q = 2^{\lambda_Q}$.
Finally $P[(\text{error on I axis}) \text{ and } (\text{error on Q axis})] = P[(\text{error on I axis})]P[(\text{error on Q axis})]$ because the 2 events are statistically independent. This is because the noise components $\mathbf{w}_I, \mathbf{w}_Q$ along the respective I and Q axes are statistically independent.
Using Eqn. (8.18) on page 307 we obtain

$$\begin{aligned} P[\text{symbol error}] &= \frac{2(M_I-1)}{M_I} Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) + \frac{2(M_Q-1)}{M_Q} Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) \\ &\quad -\frac{4(M_I-1)(M_Q-1)}{M_I M_Q} Q^2\left(\frac{\Delta}{\sqrt{2N_0}}\right) \end{aligned}$$

Since it is more meaningful to express the error probability in terms of E_b, the average transmitted energy per bit, we find the relationship between E_b and Δ. First we determine the average transmitted energy per symbol (or signal). Now the transmitted signal can be written as

$$\begin{aligned} s_{ij}(t) &= (2i-1-M_I)\frac{\Delta}{2}\sqrt{\frac{2}{T_s}}\cos(2\pi f_c t) + (2i-1-M_Q)\frac{\Delta}{2}\sqrt{\frac{2}{T_s}}\cos(2\pi f_c t) \\ & \quad i=1,2,\ldots,M_I=2^{\lambda_I}; \quad j=1,2,\ldots,M_Q=2^{\lambda_Q} \end{aligned}$$

The energy of the (i,j)th signal is $E_{ij}=(2i-1-M_I)^2\frac{\Delta^2}{4}+(2j-1-M_Q)^2\frac{\Delta^2}{4}$. Therefore

$$E_s = \frac{1}{M_I\cdot M_Q}\sum_{i=1}^{M_I}\sum_{j=1}^{M_Q}E_{ij} = \frac{\Delta^2}{4m}\sum_{i=1}^{M_I}\sum_{j=1}^{M_Q}\left[(2i-1-M_I)^2+(2j-1-M_Q)^2\right]$$

Consider the sum $\sum_{i=1}^{M_I}\sum_{j=1}^{M_Q}(2i-1-M_I)^2$ (other sum is the same except M_I, M_Q are interchanged). It equals $\left\{\sum_{i=1}^{M_I}\left[4i^2-4(1+M_I)i+(1+M_I)^2\right]\sum_{j=1}^{M_Q}1\right\}$. Using the identities $\sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}$; $\sum_{k=1}^{n}k=\frac{n(n+1)}{2}$ and straightforward algebra, the sum becomes $\frac{(M_I^2-1)M}{3}$, where $M=M_I\cdot M_Q=2^{\lambda}$.

More algebra yields $E_s=\frac{\Delta^2}{12}\left\{M_I^2+M_Q^2-2\right\}$ joules/symbol.

Now $E_b=\frac{E_s}{\lambda}=\frac{E_s}{\log_2 M}$ joules/bit which means that $\Delta=\frac{2\sqrt{3}\sqrt{\log_2 ME_b}}{\sqrt{M_I^2+M_Q^2-2}}$. Therefore

$$\begin{aligned} P[\text{symbol error}] &= \frac{2(M_I-1)}{M_I}Q\left(\sqrt{\frac{6\log_2 M}{M_I^2+M_Q^2-2}\cdot\frac{E_b}{N_0}}\right) \\ &\quad +\frac{2(M_Q-1)}{M_Q}Q\left(\sqrt{\frac{6\log_2 M}{M_I^2+M_Q^2-2}\cdot\frac{E_b}{N_0}}\right) \\ &\quad -\frac{4(M_I-1)(M_Q-1)}{M}Q^2\left(\sqrt{\frac{6\log_2 M}{M_I^2+M_Q^2-2}\cdot\frac{E_b}{N_0}}\right) \end{aligned}$$

It is worthwhile, because of all the algebra, to see if the above error probability expression reduces to (8.47), page 320 of the text for square QAM, i.e., $M_I=M_Q=\sqrt{M}$. Also plots of $P[\text{symbol error}]$ versus $\frac{E_b}{N_0}$ for non-square QAM are also of interest. To reduce the possibilities consider the special case of $\lambda_I=\lambda_Q+1 \Rightarrow \lambda_I=\frac{\lambda+1}{2}$, $\lambda_Q=\frac{\lambda-1}{2}$, λ odd. Special case but the most practical one. Left as a further exercise.

It is also of interest to see how the average energies divide between the inphase and quadrature axes.

Along the quadrature axis we have

$$E_Q=\frac{1}{M_Q}\sum_{j=1}^{M_Q}(2i-1-M_I)^2\frac{\Delta^2}{4}=\frac{\Delta^2}{12}(M_Q^2-1) \quad \text{joules/“quadrature symbol”}$$

Similarly $E_I=\frac{\Delta^2}{12}(M_I^2-1)$ joules/“inphase symbol”.

Note $E_Q + E_I = E_s$ (as would be expected); further for square QAM, $E_I = E_Q = \frac{E_s}{2}$, i.e., the energy is divided evenly.

Now consider non-square QAM

$$\frac{E_I}{E_Q} = \frac{M_I^2 - 1}{M_Q^2 - 1} = \frac{2^{2\lambda_I} - 1}{2^{2\lambda_Q} - 1}$$

Now let $\lambda_I = \lambda_Q + 1$. Then the ratio becomes

$$\frac{E_I}{E_Q} = \frac{2^{2\lambda} - 1}{2^{2\lambda} - 1}, \quad \lambda \text{ odd.}$$

As λ becomes large, $\frac{E_I}{E_Q} \to 4$, indeed it does this quite rapidly.

(b) The union upper bound is obtained by simply ignoring the overlapping regions, i.e., P[(error on I axis) and (error on Q axis)] or the $Q^2(\cdot)$ term.

P8.16 In general,

$$\begin{aligned} P[\text{symbol error}] &= \sum_{i=1}^{M} P\left[\text{error}|s_i(t)\right] P[s_i(t)] \\ &= \frac{1}{M}\sum_{i=1}^{M} P\left[\text{error}|s_i(t)\right] \quad \text{(since signals are equally probable)} \\ &\leq P[\text{worst-case conditional error}] \end{aligned}$$

From the geometry the worst-case conditional error occurs when an "inner" signal is transmitted and equals the volume under the 2-dimensional pdf in the region outside the square or 1 − (volume under pdf inside the square). See Fig. 8.16.

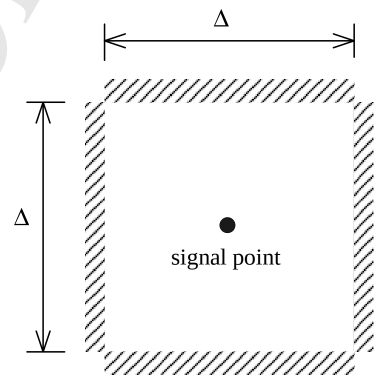


Figure 8.16

Volume under pdf inside the square $= \left[1 - 2Q\left(\frac{\Delta}{2\sqrt{N_0/2}}\right)\right]^2$. Therefore,

$$P[\text{symbol error}] \leq 1 - \left[1 - 2Q\left(\frac{\Delta}{2\sqrt{N_0/2}}\right)\right]^2.$$

Now from P8.15, $\Delta = \frac{2\sqrt{3}\sqrt{E_s}}{\sqrt{M_I^2+M_Q^2-2}}$. It then follows that

$$P[\text{symbol error}] \le 1 - \left[1 - 2Q\left(\sqrt{\frac{6E_s}{(M_I^2+M_Q^2-2)N_0}}\right)\right]^2.$$

A somewhat looser upper bound can be obtained by noting that $2M \le M_I^2 + M_Q^2 < 2M^2$. For square QAM, $M_I^2 + M_Q^2 = 2M$ and the result, Eqn. (8.48), follows immediately. For non-square QAM where $M_I^2 + M_Q^2 < 2M^2$ the reasoning is as follows:

Replacing $M_I^2 + M_Q^2$ by $2M^2$ $\Rightarrow$ the argument of the $Q(\cdot)$ function is smaller $\Rightarrow$ that $Q(\cdot)$ is larger $\Rightarrow$ $1-2Q(\cdot)$ is smaller $\Rightarrow$ $[1-2Q(\cdot)]^2$ is smaller $\Rightarrow$ $1-[1-2Q(\cdot)]^2$ is larger $\Rightarrow$

$$P[\text{symbol error}] \le 1 - \left[1 - 2Q\left(\sqrt{\frac{6E_s}{(2M^2-2)N_0}}\right)\right]^2. \tag{8.15}$$

Note that $\frac{6E_s}{(2M^2-2)N_0} = \frac{3E_s}{(M^2-1)N_0} = \frac{3\lambda E_b}{(M^2-1)N_0}$.

Remark: We show that indeed $M_I^2 + M_Q^2 \ge 2M$ or $2^{2\lambda_I} + 2^{2\lambda_Q} \ge 2\cdot 2^{\lambda}$. Let $\lambda_I = \lambda_Q + k, k \ge 1 \Rightarrow \lambda_I = \frac{\lambda+k}{2}, \lambda_Q = \frac{\lambda-k}{2}$. Then $2^{2\lambda_I} + 2^{2\lambda_Q} = 2^{\lambda}2^{k} + 2^{\lambda}2^{-k} = 2^{\lambda}(2^k + 2^{-k})$ and since $2^k + 2^{-k} > 2$ for any $k \ge 1$, then indeed $M_I^2 + M_Q^2 > 2M$. The inequality $M_I^2 + M_Q^2 < 2M^2$ is quite obvious.

P8.17 The verification involves 2 changes of variable. First in the integral w.r.t. λ, let $x = \sqrt{\frac{2}{N_0}}\lambda$. The integral becomes

$$\int_{x=-\infty}^{\sqrt{\frac{2}{N_0}}r_1} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}}\,\mathrm{d}x$$

Now let $y = \sqrt{\frac{2}{N_0}}r_1$. Then

$$P[\text{correct}] = \frac{1}{\sqrt{2\pi}}\int_{y=-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{x=-\infty}^{y}\mathrm{e}^{-\frac{x^2}{2}}\,\mathrm{d}x\right]^{M-1}\mathrm{e}^{-\frac{\left(y-\sqrt{\frac{2E_s}{N_0}}\right)^2}{2}}\,\mathrm{d}y$$

Finally observe that $E_s = \lambda E_b = \log_2 M E_b$ and that $P[\text{error}] = 1 - P[\text{correct}]$.

P8.18 The transmitted signal can be written as:

$$s_i(t) = (\pm\sqrt{E_b})\phi_1(t) + (\pm\sqrt{E_b})\phi_2(t) + \cdots + (\pm\sqrt{E_b})\phi_j(t) + \cdots + (\pm\sqrt{E_b})\phi_\lambda(t), \quad i = 1, 2, \ldots, M,$$

where the jth component is $+\sqrt{E_b}$ if the jth bit of the transmitted bit sequence is 1 and $-\sqrt{E_b}$ if the jth bit is 0; $j = 1, 2, \ldots, \lambda$.

The demodulator consists of projecting the received signal $r(t) = s_i(t) + w(t)$ onto $\phi_1(t)$, $\phi_2(t)$, $\ldots$, $\phi_\lambda(t)$ to generate the sufficient statistics $r_1, r_2, \ldots, r_\lambda$, and using minimum distance to carve up the decision space. Geometrically this consists of choosing the signal point that lies in the quadrant that $r_1, r_2, \ldots, r_\lambda$ fall in (convince yourself of this for $\lambda = 2$ and $\lambda = 3$; perhaps $\lambda = 4$ if you are really ambitious).

Now because of symmetry (and the fact that the signals are equally probable), one has

$$P[\text{error}] = P[\text{error}|\text{a specific signal or bit sequence}] = 1 - P[\text{correct}|\text{a specific signal}]$$

Choose for the specific signal the bit sequence $(111\cdots 11)$, or $(\sqrt{E_b}, \sqrt{E_b}, \sqrt{E_b}, \cdots, \sqrt{E_b}, \sqrt{E_b})$, the signal that lies in the first quadrant. Now,

$$P[\text{correct}|\text{this specific signal}] = P[\mathbf{r}_1 > 0, \mathbf{r}_2 > 0, \cdots, \mathbf{r}_\lambda > 0]$$

where the $\mathbf{r}_j$'s are Gaussian, mean value $\sqrt{E_b}$, variance $N_0/2$ and statistically independent. Therefore,

$$P[\text{correct}|\text{this specific signal}] = \prod_{j=1}^{\lambda} P[\mathbf{r}_j > 0] = \left[1 - Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right]^\lambda.$$

Finally,

$$P[\text{error}] = 1 - \left[1 - Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right]^\lambda.$$

Remark: The signal points also lie on the surface of a hypersphere of radius $\sqrt{\lambda E_b}$.

P8.19 (a) Simply put, the transmission bandwidth is halved. It is directly proportional to the dimensionality of the signal space.

(b) The sufficient statistics are generated by projecting the received signal onto the $\frac{M}{2}$ orthonormal bases (which are $s_i(t)$ normalized to unit energy) to obtain $r_1, r_2, \ldots, r_j, \ldots, r_{M/2}$. Statistically $\mathbf{r}_j$ is Gaussian with variance of $\frac{N_0}{2}$ and a mean value of 0, $+\sqrt{E_b}$, or $-\sqrt{E_b}$. Zero if the transmitted is not $\pm s_j(t)$, $+\sqrt{E_b}$ if the transmitted signal is $+s_j(t)$ and $-\sqrt{E_b}$ is the transmitted signal is $-s_j(t)$. Note that we assume, for convenience, that the signal $s_j(t)$ has energy E_b. To obtain the decision rule consider just the special case of $M = 4$ and generalize from there.

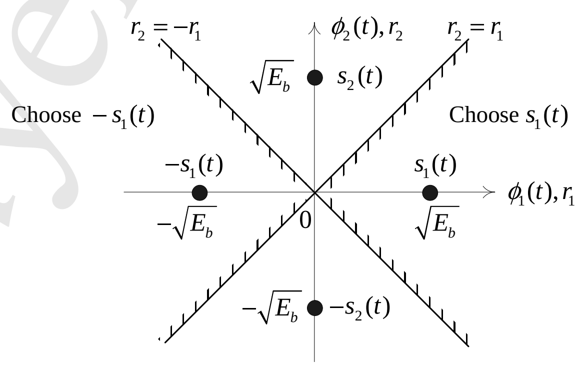


Figure 8.17

The decision rule is: If $|r_1| > |r_2|$ and $\begin{cases} r_1 > 0 \text{ choose } s_1(t) \\ r_1 < 0 \text{ choose } -s_1(t) \end{cases}$.

<u>To generalize:</u>

Compute $|r_j|, j = 1, 2, \ldots, \frac{M}{2}$. Determine maximum $|r_j|, j = 1, 2, \ldots, \frac{M}{2}$. Then the decision is $s_i(t)$ if $r_j > 0$ and $-s_i(t)$ if $r_j < 0$.

(c) Due to the symmetry of the decision space we have

$$\begin{aligned}P[\text{symbol error}] &= P[\text{symbol error}|\text{any specific transmitted signal}]\\ &= 1 - P[\text{correct decision}|\text{any specific transmitted signal}].\end{aligned}$$

Choose the specific transmitted signal to be $s_1(t)$. Then $s_1(t)$ is chosen if

$$(\mathbf{r}_1 > 0) \text{ and } (-\mathbf{r}_1 \leq \mathbf{r}_2 \leq \mathbf{r}_1) \text{ and } (-\mathbf{r}_1 \leq \mathbf{r}_3 \leq \mathbf{r}_1) \text{ and } \cdots \text{ and } (-\mathbf{r}_1 \leq \mathbf{r}_{\frac{M}{2}} \leq \mathbf{r}_1).$$

To determine the probability of the above happening, fix the random variable r_1 at a specific value, say $\mathbf{r}_1 = r_1 \geq 0$. Note that the probability of this is $f_{\mathbf{r}_1}(r_1|s_1(t))\mathrm{d}r_1 = \mathcal{N}\left(\sqrt{E_b}, \frac{N_0}{2}\right)\mathrm{d}r_1$. Then the probability of the event

$$\left[(-r_1 \leq \mathbf{r}_2 \leq r_1) \text{ and } (-r_1 \leq \mathbf{r}_3 \leq r_1) \text{ and } \cdots \text{ and } (-r_1 \leq \mathbf{r}_{\frac{M}{2}} \leq r_1)|s_1(t)\right]$$

is given by $\prod_{i=2}^{\frac{M}{2}} P\left[-r_1 \leq \mathbf{r}_i \leq r_1|s_1(t)\right]$ where the random variables are Gaussian, zero-mean (given that the transmitted signal is $s_1(t)$), variance $\frac{N_0}{2}$ and statistically independent.

Therefore the probability becomes

$$\left[\frac{1}{\sqrt{2\pi}\sqrt{\frac{N_0}{2}}}\int_{-r_1}^{r_1} \mathrm{e}^{-\frac{r_i^2}{2(N_0/2)}}\mathrm{d}r_i\right]^{\frac{M}{2}-1} = \left[1 - 2Q\left(r_1\sqrt{\frac{2}{N_0}}\right)\right]^{\frac{M}{2}-1}$$

Now $0 \leq r_1 \leq \infty$, i.e., we find the weighted sum of the above, weighted by the probability of $\mathbf{r}_1 = r_1$:

$$P[\text{symbol error}] = 1 - \left[\frac{1}{\sqrt{\pi N_0}}\int_{r_1=0}^{\infty}\left[1 - 2Q\left(r_1\sqrt{\frac{2}{N_0}}\right)\right]^{\frac{M}{2}-1}\mathrm{e}^{-\frac{(r_1-\sqrt{E_b})^2}{N_0}}\mathrm{d}r_1\right].$$

It would be useful to plot and compare the above error probability with that of orthogonal modulation (such as M-FSK), either for the same number of signal points, or the same dimensionality of the signal space.

P8.20 (a) See Fig. 8.18.

(b) It can be seen from Fig. 8.18 that the USSB waveform does not have the desirable property of constant envelope of BPSK signal. This is the price one has to pay for a reduced (half) bandwidth of the USSB signal.

P8.21 One has to establish a bandwidth definition and also the error probability at which the signalling techniques are compared. In the text (Section 8.7 and Fig. 8.26) the bandwidth W is defined as the reciprocal of the symbol duration. Other possible definition that could have been used are null-to-null bandwidth, 95% power bandwidth and so on. The error probability used to determine the SNR is 10^{-5}.

(a) Using Eqn. (8.33), $\left(\frac{r_b}{W}\right)_{\text{FSK}} = \frac{2\log_2 M}{M}$, we have

- 2-FSK: $\left(\frac{r_b}{W}\right) = \frac{2\log_2 2^1}{2} = 1$.

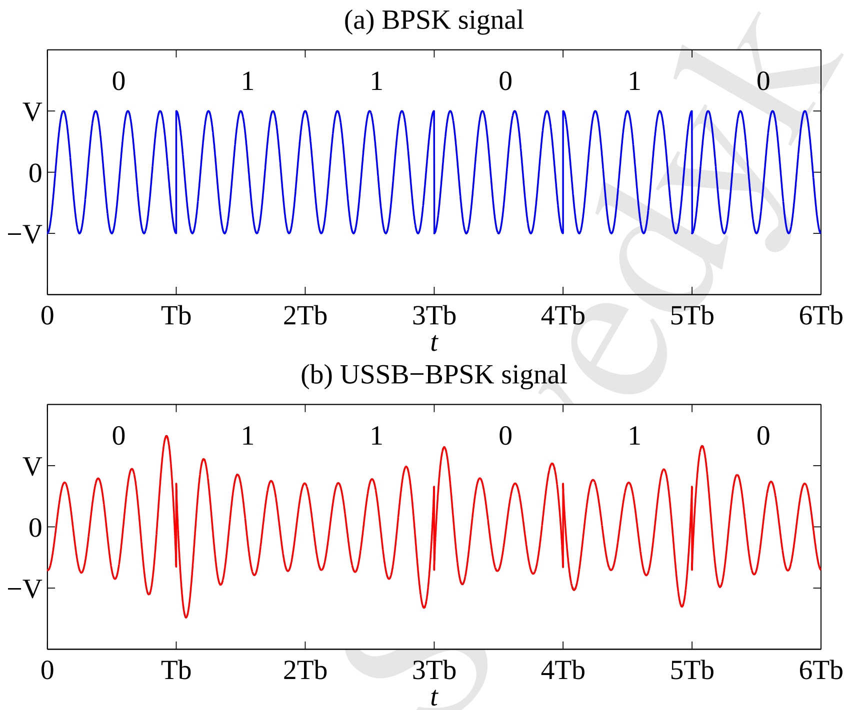


Figure 8.18

- 4-FSK: $\left(\frac{r_b}{W}\right) = \frac{2\log_2 2^2}{4} = 1.$

And using Fig. 8.23 we have at $P[\text{symbol error}] = 10^{-5}$:

- 2-FSK: $\frac{E_b}{N_0} = 12.6$ dB.
- 4-FSK: $\frac{E_b}{N_0} = 10.2$ dB.

(b) The coordinates for the plot are given below.

(i) For FSK, use Fig. 8.83 to determine $\frac{E_b}{N_0}$ and Eqn. (8.83) for $\frac{r_b}{W}$. The results are:

M	=	8	16	32	64	
$\frac{E_b}{N_0}$ @ 10^{-2}	=	5.1	4.5	4.2	3.5	(dB)
$\frac{E_b}{N_0}$ @ 10^{-7}	=	10.4	9.3	8.5	7.8	(dB)
$\left(\frac{r_b}{W}\right)$	=	0.75	0.5	0.3125	0.1875	(bits/sec/Hz)

(ii) For PSK, use Fig. 8.15 and Eqn. (8.39) for $M = 64$ to determine $\frac{E_b}{N_0}$. Use Eqn. (8.81) for $\frac{r_b}{W}$. The results are:

M	=	2	4	8	16	32	64	
$\frac{E_b}{N_0}$ @ 10^{-2}	=	4.4	5.3	8.4	13	18	22.7	(dB)
$\frac{E_b}{N_0}$ @ 10^{-7}	=	11.25	11.6	14.8	19.5	24.5	29.7	(dB)
$\left(\frac{r_b}{W}\right)$	=	1	2	3	4	5	6	(bits/sec/Hz)

(iii) For QAM, use Fig. 8.20 to determine $\frac{E_b}{N_0}$ and use Eqn. (8.81) for $\frac{r_b}{W}$. The results are:

M	=	4	16	64	
$\frac{E_b}{N_0}$ @ 10^{-2}	=	5.3	9.7	14.4	(dB)
$\frac{E_b}{N_0}$ @ 10^{-7}	=	11.4	15.6	20	(dB)
$\left(\frac{r_b}{W}\right)$	=	2	4	6	(bits/sec/Hz)

Note for QAM: $W = \frac{1}{T_s} = \frac{1}{\lambda T_b} = \frac{r_b}{\log_2 M} \Rightarrow \frac{r_b}{W} = \log_2 M$.

The plots on power-bandwidth plane are shown in Fig. 8.19. Note that the points shift toward Shannon's capacity curve for a larger value of SER. This does not mean that it is beneficial to increase the SER requirement. It merely says that one can approach the Shannon's curve easier by lowering the transmission reliability requirement. Shannon's work, however, promises to approach the curve with an arbitrarily low SER!

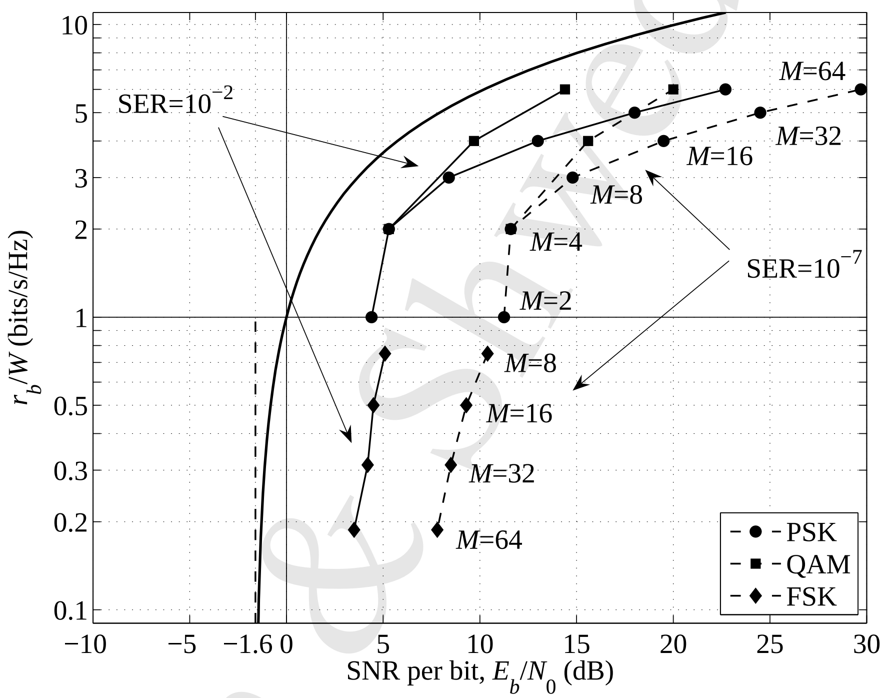


Figure 8.19

P8.22 (a) Invoking the sampling theorem we have $2W$ samples/sec.

(b) The answer in (a) means that in T seconds there are $2WT$ independent time samples; independent in the sense that they are all needed to represent the time signal.

(c) Any set of orthogonal functions are linearly independent in the usual mathematical sense that there is no set of nonzero coefficients $\{c_1, c_2, \ldots, c_N\}$, N being the number of orthogonal functions such that $\sum_{k=1}^{N} c_k \phi_k(t) = 0$. Each orthogonal function can provide an independent sample. Since we need $2WT$ independent samples then it follows that one needs $2WT$ orthogonal functions.

(d)

$$S_{\text{RECT}}(f) = \int_{-\frac{T}{2}}^{\frac{T}{2}} \mathrm{e}^{-j2\pi T}\mathrm{d}t = VT\frac{\sin(\pi f T)}{\pi f T}$$

and in passband this becomes

$$\frac{VT}{2}\left[\frac{\sin(\pi(f-f_c)T)}{\pi(f-f_c)T} + \frac{\sin(\pi(f+f_c)T)}{\pi(f+f_c)T}\right].$$

From the sketch of the above function it is (or should be) readily seen that the null-to-null bandwidth is $W = \frac{2}{T}$ Hz. This means that $WT = 2$ and that $2WT = 4$, i.e., we can fit 4 orthogonal time functions in this bandwidth.

Remark: The above derivation is very heuristic, it however agrees well with the sophisticated mathematical analysis given by Landau, Pollack & Slepian. Their definition of bandwidth is different; remember no time-limited signal can be bandlimited - the Gordian knot of signal analysis. The achieved result however agrees very well with their conclusions. Perhaps the moral is: one should not let sophisticated mathematics stand in the way of good engineering. Always keep in mind math for engineers is a tool, albeit a very important one.

P8.23 (a) To show that two time functions are orthogonal over an interval of T seconds, one needs to show that $\int_{t\in T} s_1(t)s_2(t)\mathrm{d}t = 0$. Since here we are dealing with sinusoids we can dispense with the integration by recalling that an integer number of cycles of a sinusoid in the time interval T means the area (or the integral of it) under it is zero. Recall further the following trig identities:

$$\begin{aligned}
\cos x \cos y &= \frac{1}{2}\left[\cos(x+y) + \cos(x-y)\right] \\
\sin x \sin y &= \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right] \\
\cos x \sin y &= \frac{1}{2}\left[\sin(x+y) - \sin(x-y)\right]
\end{aligned}$$

<u>Show $\phi_1'(t) \perp \phi_2'(t)$</u>

$$\begin{aligned}
\phi_1'(t)\phi_2'(t) &= \cos\left(\frac{\pi t}{T}\right)\sin\left(\frac{\pi t}{T}\right)\cos^2(2\pi f_c t) = \frac{1}{2}\sin\left(\frac{2\pi t}{T}\right)\frac{1}{2}\left[1+\cos\left(2\pi\frac{k}{T}t\right)\right] \\
&= \frac{1}{4}\left\{\sin\left(\frac{2\pi t}{T}\right) + \frac{1}{2}\sin\left(2\pi\frac{k+1}{T}t\right) - \frac{1}{2}\sin\left(2\pi\frac{k-1}{T}t\right)\right\}
\end{aligned}$$

The sinusoids have 1, $(k+1)$, $(k-1)$ cycles, respectively, over T sec and therefore the area under them is 0. Therefore $\phi_1'(t)$ and $\phi_2'(t)$ are orthogonal.

<u>Show $\phi_1'(t) \perp \phi_3'(t)$</u>

$$\begin{aligned}
\phi_1'(t)\phi_3'(t) &= \cos^2\left(\frac{\pi t}{T}\right)\cos(2\pi f_c t)\sin(2\pi f_c t) = \frac{1}{4}\left[1+\cos\left(\frac{2\pi t}{T}\right)\right]\sin\left(2\pi\frac{k}{T}t\right) \\
&= \frac{1}{4}\left\{\sin\left(2\pi\frac{k}{T}t\right) + \frac{1}{2}\sin\left(2\pi\frac{k+1}{T}t\right) + \frac{1}{2}\sin\left(2\pi\frac{k-1}{T}t\right)\right\}
\end{aligned}$$

Here there are k, $(k+1)$, $(k-1)$ cycles respectively over T sec $\Rightarrow$ area $= 0 \Rightarrow \phi_1'(t) \perp \phi_3'(t)$.

<u>Show $\phi_1'(t) \perp \phi_4'(t)$</u>

$$\begin{aligned}
\phi_1'(t)\phi_4'(t) &= \cos\left(\frac{\pi t}{T}\right)\sin\left(\frac{\pi t}{T}\right)\cos(2\pi f_c t)\sin(2\pi f_c t) = \frac{1}{4}\sin\left(\frac{2\pi t}{T}\right)\sin\left(2\pi\frac{k}{T}t\right) \\
&= \frac{1}{8}\left\{\cos\left(2\pi\frac{k-1}{T}t\right) - \cos\left(2\pi\frac{k+1}{T}t\right)\right\}
\end{aligned}$$

$\Rightarrow (k-1), (k+1)$ cycles over $T \Rightarrow$ area $= 0 \Rightarrow \phi_1'(t) \perp \phi_4'(t)$.

Show $\phi_2'(t) \perp \phi_3'(t)$

$$\phi_2'(t)\phi_3'(t) = \sin\left(\frac{\pi t}{T}\right)\cos\left(\frac{\pi t}{T}\right)\cos(2\pi f_c t)\sin(2\pi f_c t)$$

Note this product is the same as $\phi_1'(t)\phi_4'(t)$. Therefore it follows that the area under $\phi_2'(t)\phi_3'(t)$ is zero over T sec and that $\phi_2'(t) \perp \phi_3'(t)$.

Show $\phi_2'(t) \perp \phi_4'(t)$

$$\begin{aligned}
\phi_2'(t)\phi_4'(t) &= \sin^2\left(\frac{\pi t}{T}\right)\cos(2\pi f_c t)\sin(2\pi f_c t) \\
&= \left[1-\cos^2\left(\frac{\pi t}{T}\right)\right]\cos(2\pi f_c t)\sin(2\pi f_c t) \\
&= \cos(2\pi f_c t)\sin(2\pi f_c t) - \cos^2\left(\frac{\pi t}{T}\right)\cos(2\pi f_c t)\sin(2\pi f_c t)
\end{aligned}$$

The area under the first term is 0 over T sec because as noted in the problem the two carriers are orthogonal over T sec. The second term is the same as $\phi_1'(t)\phi_3'(t)$ and we have shown that this area is zero. Therefore $\phi_2'(t) \perp \phi_4'(t)$.

Show $\phi_3'(t) \perp \phi_4'(t)$

$$\begin{aligned}
\phi_3'(t)\phi_4'(t) &= \cos\left(\frac{\pi t}{T}\right)\sin\left(\frac{\pi t}{T}\right)\sin^2(2\pi f_c t) \\
&= \underbrace{\cos\left(\frac{\pi t}{T}\right)\sin\left(\frac{\pi t}{T}\right)}_{\text{area}=0 \text{ because } p_1(t)\perp p_2(t)} \underbrace{-\cos\left(\frac{\pi t}{T}\right)\sin\left(\frac{\pi t}{T}\right)\cos^2(2\pi f_c t)}_{\text{area}=0\text{, from } \phi_1'(t)\perp\phi_2'(t)}
\end{aligned}$$

Therefore $\phi_3'(t) \perp \phi_4'(t)$.

Remark: Given n functions, one needs to check $(n-1)!$ conditions to see that they form an orthogonal set. Here $n=4$ and $(n-1)! = 3! = 6$ conditions.

(b) Though one should find the normalizing factor for each basis function, we'll determine it for the 1st one and trust the gods (or intuition) it is the same for the other three. Now

$$\begin{aligned}
\int_{-\frac{T}{2}}^{\frac{T}{2}} \left(\phi_1'(t)\right)^2 \mathrm{d}t &= \int_{-\frac{T}{2}}^{\frac{T}{2}} \cos^2\left(\frac{\pi t}{T}\right)\cos^2(2\pi f_c t)\,\mathrm{d}t \\
&= \int_{-\frac{T}{2}}^{\frac{T}{2}} \left[\frac{1}{2}+\frac{1}{2}\cos\left(\frac{2\pi t}{T}\right)\right]\left[\frac{1}{2}+\frac{1}{2}\cos\left(\frac{2\pi(2f_c)t}{T}\right)\right]\mathrm{d}t = \frac{T}{4}
\end{aligned}$$

Therefore multiply each basis function by $\sqrt{\frac{4}{T}} = \frac{2}{\sqrt{T}}$ to normalize.

(c) Let $\phi_i(t) = \frac{2}{\sqrt{T}}\phi_i'(t), i = 1,2,3,4, -\frac{T}{2} \le t \le \frac{T}{2}$. See Fig. 8.20.

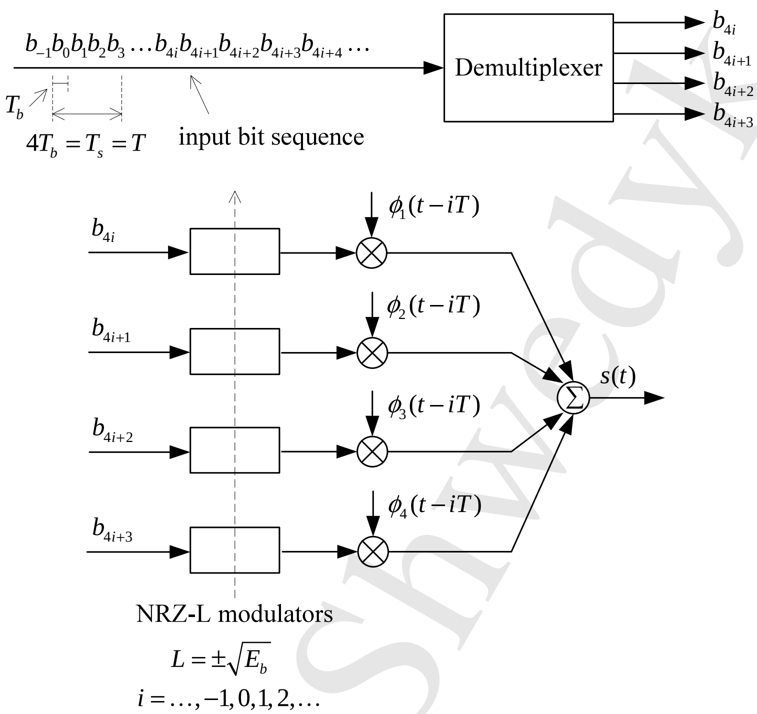


Figure 8.20

(d) The transmitted signal $s(t)$ is given by

$$s(t) = \pm\sqrt{E_b}\phi_1(t) \pm \sqrt{E_b}\phi_2(t) \pm \sqrt{E_b}\phi_3(t) \pm \sqrt{E_b}\phi_4(t)$$

where it is $+\sqrt{E_b}$ if the bit is 1 and $-\sqrt{E_b}$ if the bit is 0 (assumed).

The signal space is 4 dimensional, with basis functions $\phi_i(t)$, $i = 1, 2, 3, 4$. The signals lie on the vertices of a four-dimensional hypercube with coordinates ranging from $(-\sqrt{E_b}, -\sqrt{E_b}, -\sqrt{E_b}, -\sqrt{E_b})$ to $(+\sqrt{E_b}, +\sqrt{E_b}, +\sqrt{E_b}, +\sqrt{E_b})$ corresponding to bit patterns (0000) to (1111) (assuming that $0 \leftrightarrow -\sqrt{E_b}, 1 \leftrightarrow \sqrt{E_b}$).

There are 16 signals, one on each vertex. The minimum distance between two signals occurs when a pair differ in only one component and is equal to $2\sqrt{E_b}$.
Finally the number of nearest neighbors is 4. Note that in an n-dimensional hypercube any vertex has n nearest vertices.

P8.24 (a) The generation of the sufficient statistics is the same for the symbol demodulator and for the bit demodulator(s). See Fig. 8.21. The difference between the two is in the decision rule.

The decision rule for the symbol demodulator is: determine which quadrant r_1, r_2, r_3, r_4 fall in and choose the signal (or bit pattern it represents) that lies in that quadrant.
For example, if $(r_1 > 0, r_2 < 0, r_3 > 0, r_4 > 0)$ choose signal $(+\sqrt{E_b}, -\sqrt{E_b}, +\sqrt{E_b}, +\sqrt{E_b})$ (or bit pattern 1011).

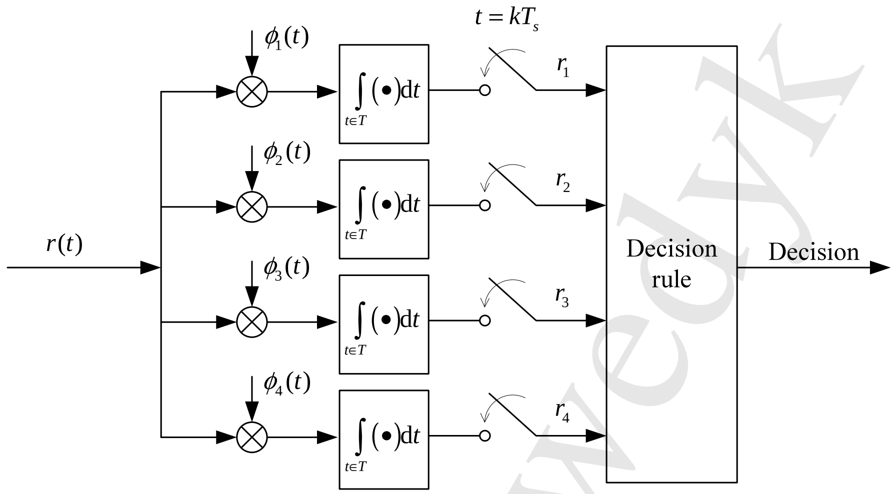


Figure 8.21

For the bit demodulator the decision rule is:

$$r_1 \underset{b_{4i}=0}{\overset{b_{4i}=1}{\gtrless}} 0; \quad r_2 \underset{b_{4i+1}=0}{\overset{b_{4i+1}=1}{\gtrless}} 0; \quad r_3 \underset{b_{4i+2}=0}{\overset{b_{4i+2}=1}{\gtrless}} 0; \quad r_4 \underset{b_{4i+3}=0}{\overset{b_{4i+3}=1}{\gtrless}} 0.$$

(b) Due to symmetry we have

$$\begin{aligned} P[\text{symbol error}] &= P[\text{symbol error}|\text{a specific signal}] \\ &= 1 - P[\text{correct}|\text{a specific signal}] \\ &= 1 - P\left[\text{correct}\middle|\text{specific signal}\left(+\sqrt{E_b}, +\sqrt{E_b}, +\sqrt{E_b}, +\sqrt{E_b}\right)\right] \end{aligned}$$

$$\begin{aligned} & P\left[\text{correct}\middle|\left(+\sqrt{E_b}, +\sqrt{E_b}, +\sqrt{E_b}, +\sqrt{E_b}\right)\right] \\ &\quad = P\left[r_1 \geq 0, r_2 \geq 0, r_3 \geq 0, r_4 \geq 0 \middle| \left(+\sqrt{E_b}, +\sqrt{E_b}, +\sqrt{E_b}, +\sqrt{E_b}\right)\right] \end{aligned}$$

Under the given conditions the random variables $\mathbf{r}_i$ are Gaussian with mean $\sqrt{E_b}$, variance $N_0/2$, and statistically independent. The probability is therefore

$$\left[\frac{1}{\sqrt{2\pi}\sqrt{N_0/2}}\int_0^{\infty} \mathrm{e}^{-\frac{(x-\sqrt{E_b})^2}{2(N_0/2)}}\mathrm{d}x\right]^4 = \left[1 - Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right]^4$$

Therefore

$$P[\text{symbol error}] = 1 - \left[1 - Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right]^4$$

$$P[\text{bit error for the bit demodulator}] = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

Note that the above result is just a special case of the more general result derived in P8.18 for a hypercube constellation.

The bit error probability of the symbol demodulator is the same as that of the bit demodulator. To see this consider bit b_{4i}, i.e., the one that corresponds to sufficient statistic r_i. A symbol error is caused by any one of r_1, r_2, r_3, r_4 or any subset of them being of the wrong sign. However bit b_{4i} would still be correct if r_i is of the right polarity regardless of what r_2, r_3, r_4 are. The probability of this is $Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$.

The bit error probability of Q^{2}PSK is the same as BPSK and QPSK/OQPSK/MSK.

P8.25 (a) Substituting in for the $\phi_i(t)$'s and collecting terms we have:

$$s(t) = \frac{2\sqrt{E_b}}{\sqrt{T}}\left\{\left(b_1\cos\frac{\pi t}{T} + b_2\sin\frac{\pi t}{T}\right)\cos(2\pi f_c t) + \left(b_3\cos\frac{\pi t}{T} + b_4\sin\frac{\pi t}{T}\right)\sin(2\pi f_c t)\right\}$$

The envelope is given by (ignoring the constant $\frac{2\sqrt{E_b}}{\sqrt{T}}$):

$$e(t) = \left[\left(b_1\cos\frac{\pi t}{T} + b_2\sin\frac{\pi t}{T}\right)^2 + \left(b_3\cos\frac{\pi t}{T} + b_4\sin\frac{\pi t}{T}\right)^2\right]^{\frac{1}{2}}$$

Using the fact that $\cos^2 x + \sin^2 x = 1$ (always) and that $b_i^2 = 1$, one has

$$\begin{aligned} e(t) &= \left[2 + 2(b_1b_2 + b_3b_4)\cos\frac{\pi t}{T}\sin\frac{\pi t}{T}\right]^{\frac{1}{2}} = \left[2 + 2(b_1b_2 + b_3b_4)\sin\frac{2\pi t}{T}\right]^{\frac{1}{2}} \\ &= \left[2 + 2(b_1b_2 + b_3b_4)\sin\frac{\pi t}{2T_b}\right]^{\frac{1}{2}}. \end{aligned} \tag{8.16}$$

Plot of $e(t)$ for all three different combinations of $b_1b_2 + b_3b_4$ is shown in Fig. 8.22.

(b) Need $b_1b_2 + b_3b_4 = 0 \Rightarrow b_1 = -\frac{b_3b_4}{b_2}$. Price paid is a reduction in the information rate. Instead of 4 information bits transmitted every T seconds, there are only 3 information bits transmitted now.

The coding scheme is represented by the following table:

Real Number Arithmetic				Bolean Algebra			
b_2	b_3	b_4	b_1	b_2	b_3	b_4	b_1
−1	−1	1	+1	0	0	0	1
−1	−1	1	−1	0	0	1	0
−1	1	−1	−1	0	1	0	0
−1	1	1	+1	0	1	1	1
1	−1	−1	−1	1	0	0	0
1	−1	1	+1	1	0	1	1
1	1	−1	+1	1	1	0	1
1	1	1	−1	1	1	1	0

Note that $b_1 = 1$ if there is an even number of ones, otherwise $b_1 = 0$. The encoder can be implemented by the following logic:

$$b_1 = \overline{b_2 \oplus b_3 \oplus b_4}$$

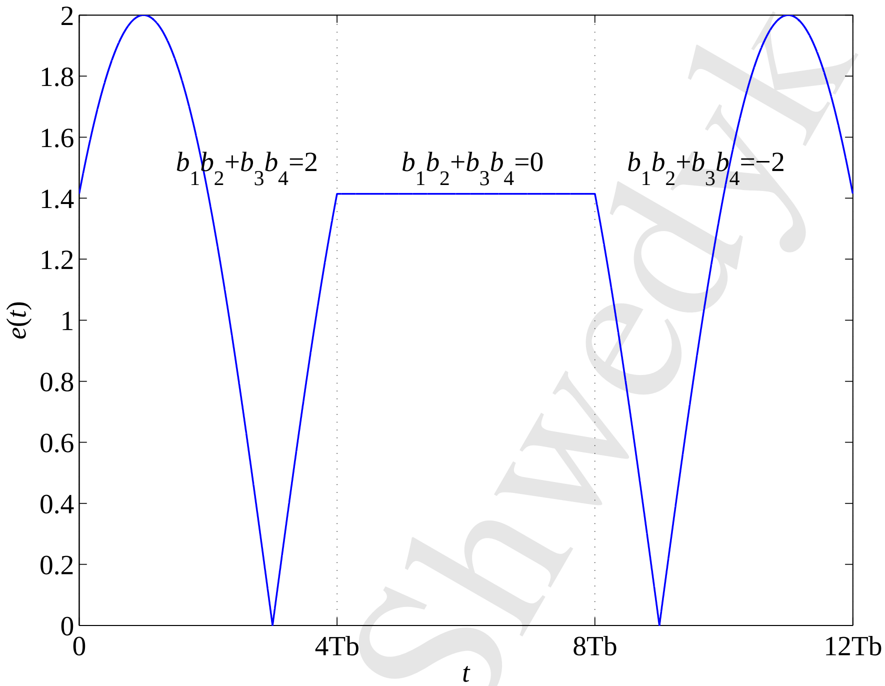


Figure 8.22

(c) Eight – one for each 3-bit allowable bit pattern.

$$d_{\min} = \left[\left(2\sqrt{E_b}\right)^2 + \left(2\sqrt{E_b}\right)^2\right]^{\frac{1}{2}} = 2\sqrt{2}\sqrt{E_b}.$$

(Note that signals closest together differ in 2 components).

(d) The block diagram is the same as that of P8.24(a). One difference could be that the sufficient statistics for b_1 (b_{4i}) does not need to be generated. Only bits b_2, b_3, b_4 represent information. The bit error probability is $Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$.

(e) Yes, it can be used for error detection. After demodulating the bits to $\hat{b}_1, \hat{b}_2, \hat{b}_3, \hat{b}_4$, determine if $\hat{b}_1 = \overline{\hat{b}_2 \oplus \hat{b}_3 \oplus \hat{b}_4}$. If this is not true then an error has been made. Note that the error could be on the "redundant" bit b_1. If the relationship is satisfied then either no errors are made or an undetectable number of errors have been made. You may wish to convince yourself that an odd number errors (i.e., 1 or 3) are detectable while an even number (2 or 4) are not.

P8.26 (a) Consider $E\{\mathbf{s}_i(t)\mathbf{s}_j(t+\tau)\}$, $i \neq j$. This, with a bit of creativity in notation, is:

$$\sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} E\{\mathbf{b}_{ik}\mathbf{b}_{jl}\} \begin{smallmatrix}\cos\\ \text{or}\\ \sin\end{smallmatrix} \left\{\frac{\pi t}{4T_b}\right\} \begin{smallmatrix}\cos\\ \text{or}\\ \sin\end{smallmatrix} \left\{\frac{\pi(t+\tau)}{4T_b}\right\}$$

But $E\{\mathbf{b}_{ik}\mathbf{b}_{jl}\} = 0\ \forall i, j; i \neq j$. Therefore any two signals are uncorrelated.

(b)

$$\begin{aligned}
R_{s_1}(\tau) &= E\left\{\mathbf{s}_1(t)\mathbf{s}_1(t+\tau)\right\} = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} E\left\{\mathbf{b}_{1k}\mathbf{b}_{1l}\right\}\cos\left\{\frac{\pi t}{4T_b}\right\}\cos\left\{\frac{\pi(t+\tau)}{4T_b}\right\} \\
R_{s_3}(\tau) &= \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} E\left\{\mathbf{b}_{3k}\mathbf{b}_{3l}\right\}\cos\left\{\frac{\pi t}{4T_b}\right\}\cos\left\{\frac{\pi(t+\tau)}{4T_b}\right\}
\end{aligned}$$

But $E\left\{\mathbf{b}_{1k}\mathbf{b}_{1l}\right\} = E\left\{\mathbf{b}_{3k}\mathbf{b}_{3l}\right\}$, since both $= 1$ if $k = l$ and $= 0$ if $k \neq l$. Therefore the two autocorrelations are the same.

Use the same argument to show that $R_{s_2}(\tau) = R_{s_4}(\tau)$.

(c) The PSD component due to $s_1(t)$ (or $s_3(t)$) is $\frac{1}{4T_b}|H(f)|^2$.

To find $H(f)$, determine $\mathcal{F}\left\{\cos\left(\frac{\pi t}{4T_b}\right)\right\}$ and $\mathcal{F}\left\{u(t+2T_b) - u(t-2T_b)\right\}$ which are $\frac{1}{2}\left[\delta(f-\frac{1}{8T_b}) + \delta(f+\frac{1}{8T_b})\right]$ and $\frac{4T_b\sin(4\pi fT_b)}{4\pi fT_b}$, respectively.
Now convolve and simplify to obtain:

$$H(f) = -2\pi T_b\frac{\cos(4\pi fT_b)}{(4\pi fT_b)^2 - \frac{\pi^2}{4}}$$

For $s_2(t)$, $h(t) = \sin\left(\frac{\pi t}{4T_b}\right)\left[u(t+2T_b) - u(t-2T_b)\right]$ and we now convolve $\frac{1}{2j}\left[\delta(f-\frac{1}{8T_b}) + \delta(f+\frac{1}{8T_b})\right]$ and $\frac{4T_b\sin(4\pi fT_b)}{4\pi fT_b}$. This gives

$$H(f) = \frac{16\pi T_b}{j}\frac{fT_b\cos(4\pi fT_b)}{(4\pi fT_b)^2 - \frac{\pi^2}{4}}$$

The overall PSD is

$$\pi^2 T_b\cos^2(4\pi fT_b)\frac{1+64(fT_b)^2}{\left[(4\pi fT_b)^2 - \frac{\pi^2}{4}\right]^2} \text{ (watts/Hz).}$$

(d) Plot of the resultant PSD is shown in Fig. 8.23.

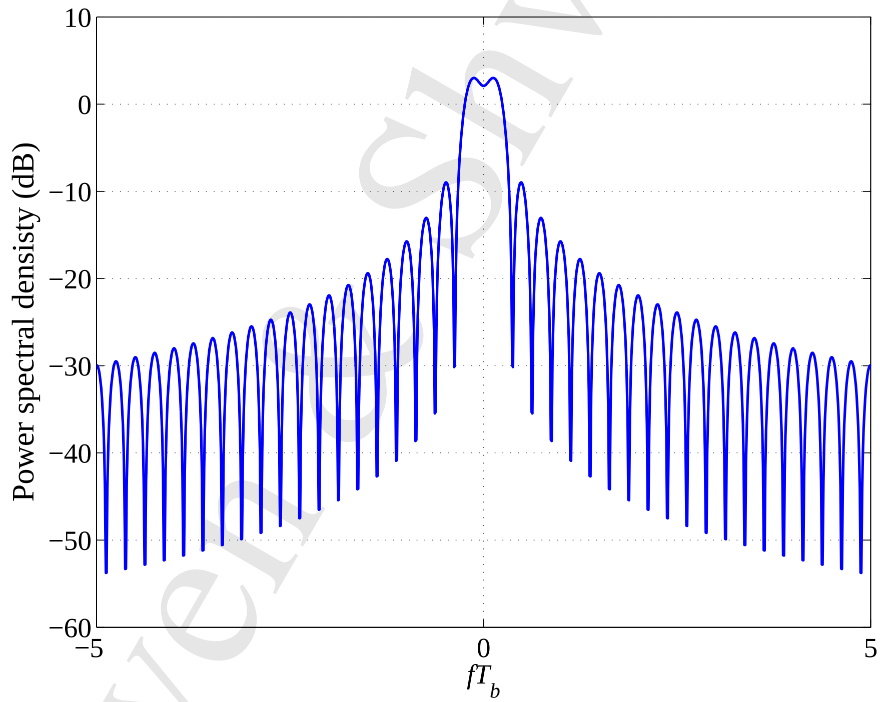


Figure 8.23

Chapter 9

Signaling Over Bandlimited Channels

P9.1 A bandwidth of 4 kHz in the passband (i.e., around f_c) means that in baseband we have 2 kHz of bandwidth. Since a bit rate of 9600 bit/second means that at minimum $\frac{1}{2T_b} = \frac{r_b}{2} = 4800$ Hz of bandwidth is needed to transmit with zero ISI, obviously an M-ary modulation is required to reduce the bandwidth requirement. Since QAM has been specified as the modulation paradigm, determine the number of bits a symbol needs to carry (or represent) to achieve not only zero ISI but also a $\beta \geq 0.5$ (the roll-off factor spec.). Do this by trial and error: start with $\lambda = 3$, i.e., 3 bits/symbol. Then the symbol transmission rate is $r_s = \frac{9600 \text{ bits/second}}{3 \text{ bits/symbol}} =$ 3200 symbols/sec. Using the hint this means that the required bandwidth needed for zero ISI is $\frac{1}{2T_s} = \frac{r_s}{2} = 1.6$ kHz. Since we have 2 kHz this means that we have an excess of bandwidth with the excess being 0.4 kHz. But is this enough to achieve a $\beta \geq 0.5$. Let's check this

$$\frac{1+\beta}{2T_s} = \frac{r_s}{2} + \frac{r_s}{2}\beta = 2 \text{ kHz} \Rightarrow \beta = \frac{0.4}{1.6} = 0.25 < 0.5$$

Back to the drawing board. (An auxiliary question is: When the developer of the drawing board made an error in the drawing board what did she go back to?)

Try $\lambda = 4$. Now $r_s = \frac{9600}{4} = 2400$ symbols/sec. Since $\frac{r_s}{2} = 1.2$ kHz we have 800 Hz (or 0.8 kHz) excess bandwidth $\Rightarrow$ 1.2 kHz + 0.8 kHz = 2 kHz $\Rightarrow \beta = \frac{0.8}{1.2} = \frac{2}{3} > 0.5$ so this spectrum is satisfied, i.e., 16-QAM is the modulation to use.

Finally need to satisfy the bit error probability of 10^{-6}. To do this consider the 16-QAM modulation as 2 4-ASK modulation, i.e., 2 bits on the inphase axis, 2 bits on the quadrature axis. From Eqn. (8.27) we have

$$P[\text{symbol error}] = 2P[\text{bit error}] = 2 \times 10^{-6}$$

And from Fig. 8.8, using the $M = 4$ curve, this means $\frac{E_b}{N_0} \simeq 14.5$ dB or $10^{1.45} = 28.2$ so $E_b = 28.2 \times 10^{-8}$ joules. The average transmitted power is

$$P_{\text{av}} = \frac{E_s}{T_s} = \frac{4E_b}{T_s} = 4 \times 28.2 \times 10^{-8} \times 2400 = 2.707 \times 10^{-3} \text{ (watts)} = 2.707 \text{ (mW)} \qquad (9.1)$$

P9.2 First in "baseband" the channel looks like in Fig. 9.1, where $h_c(t) = h_c'(t) 2\cos(2\pi f_c t)$; $f_c = 1.8$ kHz. We therefore have 1.5 kHz if bandwidth to play with.

(a) Now $T_s = \lambda T_b$ or $\lambda = \frac{r_b}{r_s} = \frac{9600}{2400} = 4$ bits/symbol $\Rightarrow$ 16-QAM.

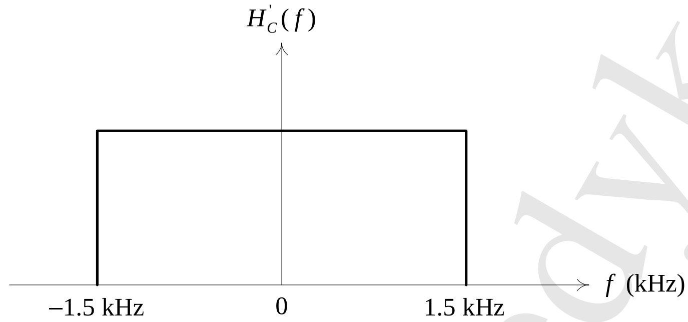


Figure 9.1

This means that $\frac{1}{2T_s} = \frac{r_s}{2} = 1.2$ kHz is the minimum bandwidth needed to achieve zero ISI. But we have 1.5 kHz of bandwidth or an excess of 0.3 kHz. Therefore β can be larger than 0. Utilizing the entire bandwidth, β is determined from:

$$\frac{1+\beta}{2T_s} = \frac{r_s}{2}(1+\beta) = 1.2(1+\beta) = 1.5 \Rightarrow \beta = 0.25$$

(b) In "passband" the spectrum looks as in Fig. 9.2 (showing just the positive frequency portion).

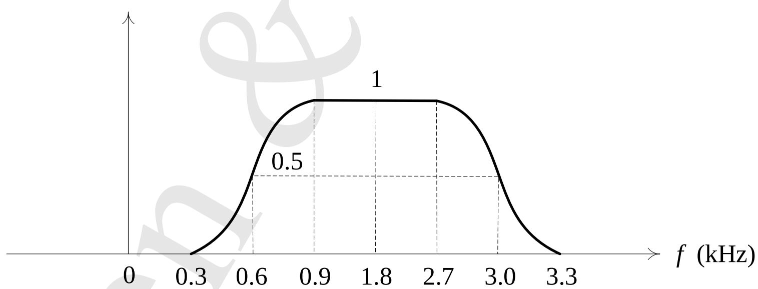


Figure 9.2

P9.3 With 75% excess bandwidth, then $\beta = 0.75$. Now,

$$|H_R(f)| = \frac{|S_R(f)|^{\frac{1}{2}}}{|H_C(f)|^{\frac{1}{2}}}; \quad |H_T(f)| = \frac{K_2}{K_1}\frac{|S_R(f)|^{\frac{1}{2}}}{|H_C(f)|^{\frac{1}{2}}}$$

Assume $K_2 = K_1$ and that $S_R(f)$ is raised-cosine. Therefore

$$H_R(f) = H_T(f) = \begin{cases} \frac{\sqrt{T_s}}{|1+\alpha\cos(2\pi f T_s)|^{\frac{1}{2}}}, & |f| \leq \frac{0.25}{2T_s} \\ \frac{\sqrt{T_s}\cos\left[\frac{\pi T_s}{1.5}\left(|f| - \frac{0.25}{2T_s}\right)\right]}{|1+\alpha\cos(2\pi f T_s)|^{\frac{1}{2}}}, & \frac{0.25}{2T_s} \leq |f| \leq \frac{1.75}{2T_s} \\ 0, & |f| \geq \frac{1.75}{2T_s} \end{cases}$$

Note that $1 + \alpha\cos(2\pi f T_s)$ for $0 < f < \frac{1.75}{2T_s} = \frac{7}{8T_s}$ plots as in Fig. 9.3 (where it is assumed that $0 < \alpha < 1$).

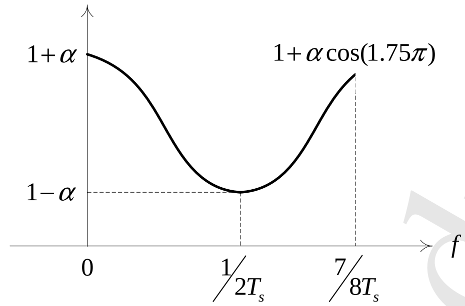


Figure 9.3

P9.4 (a)

$$H_C(f) = 1 + \frac{\alpha}{2}\left[e^{j2\pi f t_0} + e^{-j2\pi f t_0}\right]; \quad s(t) \leftrightarrow S(f)$$

Therefore

$$Y(f) = S(f)H_C(f) = S(f) + \frac{\alpha}{2}S(f)e^{j2\pi f t_0} + \frac{\alpha}{2}S(f)e^{-j2\pi f t_0}.$$

Now, multiplying $S(f)$ by $e^{j2\pi f t_0}$ means a time shift of t_0 seconds in the time domain (see Table 2.3 – Property 2). Therefore

$$y(t) = s(t) + \frac{\alpha}{2}s(t+t_0) + \frac{\alpha}{2}s(t-t_0)$$

Remark: The fact that $s(t)$ is bandlimited to W Hz is necessary. Otherwise we could not make use of the property. Why?

(b) The output of the matched filter, based on the hint is

$$y_0(t) = \int_{-\infty}^{\infty}\left[s(\lambda) + \frac{\alpha}{2}s(\lambda+t_0) + \frac{\alpha}{2}s(\lambda-t_0)\right]s(t-(T-\lambda))\mathrm{d}\lambda.$$

And at $t = kT$:

$$\begin{aligned} y_0(kT) &= \int_{-\infty}^{\infty} s(\lambda)s((k-1)T-\lambda)\mathrm{d}\lambda + \frac{\alpha}{2}\int_{-\infty}^{\infty} s(\lambda-t_0)s((k-1)T-\lambda)\mathrm{d}\lambda \\ &\quad + \frac{\alpha}{2}\int_{-\infty}^{\infty} s(\lambda+t_0)s((k-1)T-\lambda)\mathrm{d}\lambda \end{aligned}$$

(c) If $t_0 = T$, the above becomes

$$\begin{aligned} y_0(kT) &= \int_{-\infty}^{\infty} s(\lambda)s((k-1)T-\lambda)\mathrm{d}\lambda + \frac{\alpha}{2}\int_{-\infty}^{\infty} s(\lambda-T)s((k-1)T-\lambda)\mathrm{d}\lambda \\ &\quad + \frac{\alpha}{2}\int_{-\infty}^{\infty} s(\lambda+T)s((k-1)T-\lambda)\mathrm{d}\lambda \end{aligned}$$

Change variables in the last two integrals to $\lambda - T \to \lambda$ and $\lambda + T \to \lambda$, respectively. Then

$$\begin{aligned} y_0(kT) &= \int_{-\infty}^{\infty} s(\lambda)s((k-1)T-\lambda)\mathrm{d}\lambda + \frac{\alpha}{2}\int_{-\infty}^{\infty} s(\lambda)s((k-2)T-\lambda)\mathrm{d}\lambda \\ &\quad + \frac{\alpha}{2}\int_{-\infty}^{\infty} s(\lambda)s(kT-\lambda)\mathrm{d}\lambda \end{aligned}$$

Let $x(t) = s(t) * s(t-T)$. Then the above can also be written as:

$$y_0(kT) = x(kT) + \frac{\alpha}{2}x((k-1)T) + \frac{\alpha}{2}x((k+1)T)$$

where the first term is the desired signal component, while the last two terms account for ISI.

P9.5 (a) The sufficient statistic is

$$\begin{aligned} 1_T : \mathbf{r}_k &= \sqrt{E_b} \pm 0.25\sqrt{E_b} + \mathbf{w}_k \\ 0_T : \mathbf{r}_k &= -\sqrt{E_b} \pm \underbrace{0.25\sqrt{E_b}}_{\text{due to ISI}} + \underbrace{\mathbf{w}_k}_{\mathcal{N}\left(0,\frac{N_0}{2}\right)} \end{aligned}$$

The decision space looks as shown in Fig. 9.4. By symmetry, we have

$$\begin{aligned} P[\text{bit error}] &= \frac{1}{2}Q\left(\frac{0.75\sqrt{E_b}}{\sqrt{N_0/2}}\right) + \frac{1}{2}Q\left(\frac{1.25\sqrt{E_b}}{\sqrt{N_0/2}}\right) \\ &= \frac{1}{2}Q\left(\sqrt{\frac{9}{8}\frac{E_b}{N_0}}\right) + \frac{1}{2}Q\left(\sqrt{\frac{25}{8}\frac{E_b}{N_0}}\right) \end{aligned}$$

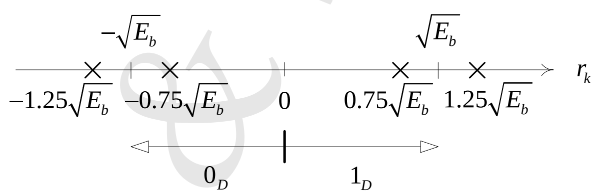


Figure 9.4

(b) With no ISI, one has $P[\text{bit error}] = Q\left(\frac{\sqrt{2E_b}}{\sqrt{N_0}}\right)$. The two error probabilities are plotted in Fig. 9.5. Observe that at the bit error probability of 10^{-5} the difference in SNR is about 2.2 dB.

P9.6 (a) Solve questions of this type graphically (usually), i.e., slide $S_R(f)$ along the f axis until the sum results in a constant level between $-\frac{1}{2T_b}$ to $\frac{1}{2T_b}$ Hz. See Fig. 9.6.

(b) No. Spectrum would not be flat in 0 to $\frac{1}{2T_b}$ Hz.

(c) Excess bandwidth is 1 kHz.

P9.7 (a) To transmit with zero-ISI we need to find a signalling rate $r_b = 1/T_b$ bits/sec such that in the frequency band $-\frac{r_b}{2} = -\frac{1}{2T_b} \le f \le \frac{1}{2T_b} = \frac{r_b}{2}$ (Hz) the spectrum $S_R(f)$ and its aliases $S_R\left(f - \frac{1}{T_b}\right)$ and $S_R\left(f + \frac{1}{T_b}\right)$ add up to a constant value. For the two spectra given, the answers are (see Figs. 9.7 and 9.8):

$$\begin{aligned} r_b &= 2 \times 2000 = 4000 \text{ bits/sec} = 4 \text{ kbps for (a)} \\ r_b &= 2 \times 1500 = 3000 \text{ bits/sec} = 3 \text{ kbps for (b)} \end{aligned}$$

Remark: One could also determine $\frac{r_b}{2}$ from the frequency at which $S_R(f)$ has the required symmetry about $\frac{r_b}{2}$ (which is?).

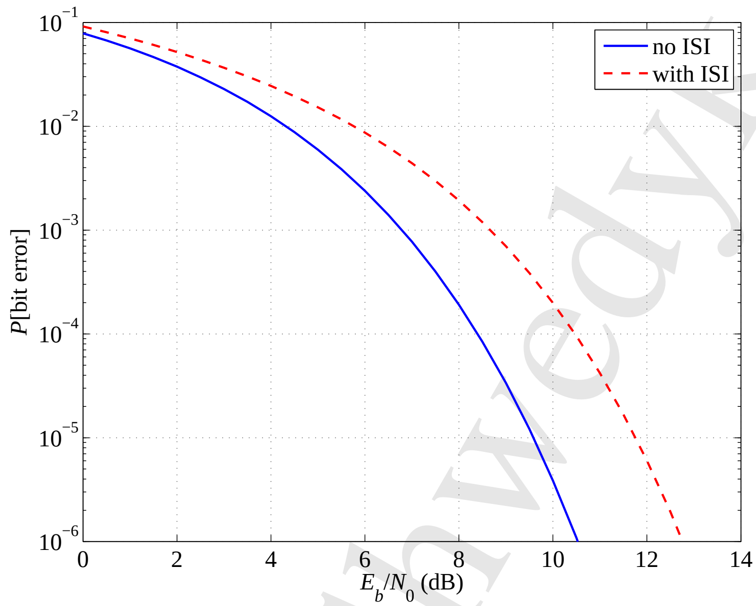


Figure 9.5

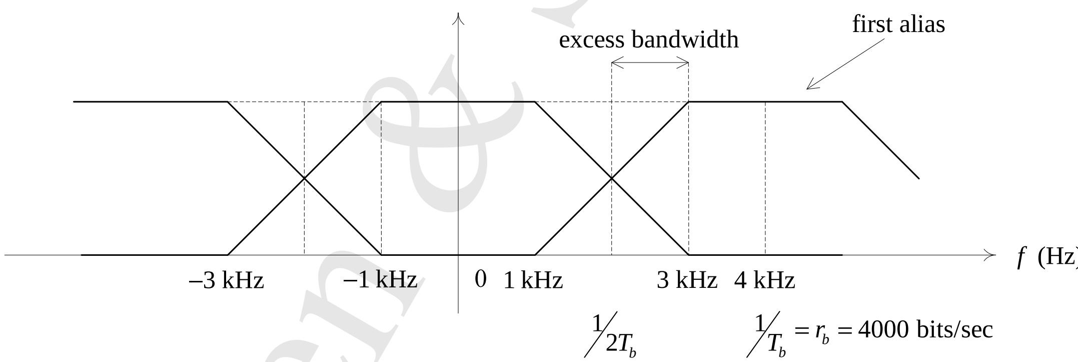


Figure 9.6

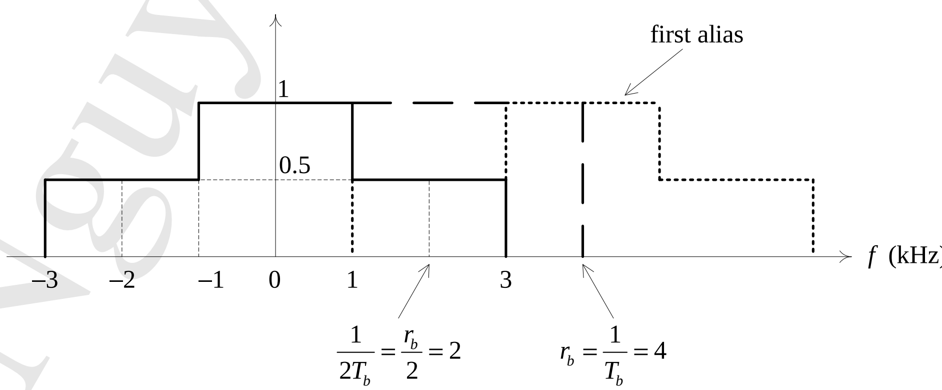


Figure 9.7

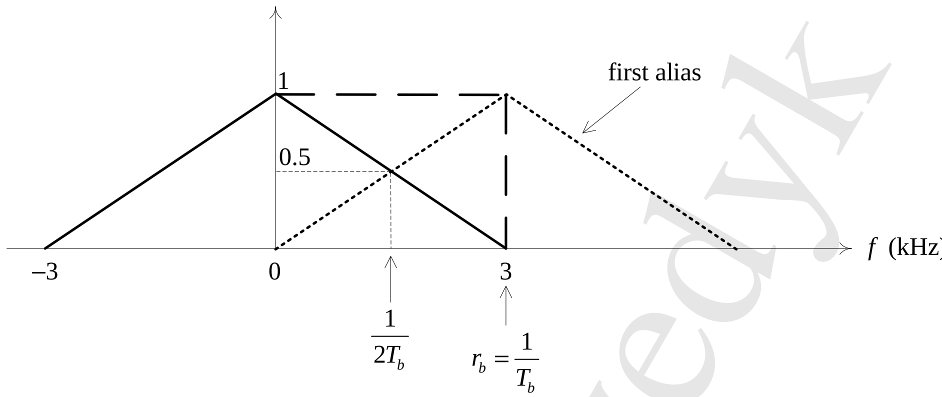


Figure 9.8

(b) Clearly the excess bandwidths are

$$\begin{aligned} 3000 - 2000 &= 1000 \text{ Hz (or 50\%) for (a)} \\ 3000 - 1500 &= 1500 \text{ Hz (or 100\%) for (b)} \end{aligned}$$

(c) See Fig. 9.9.

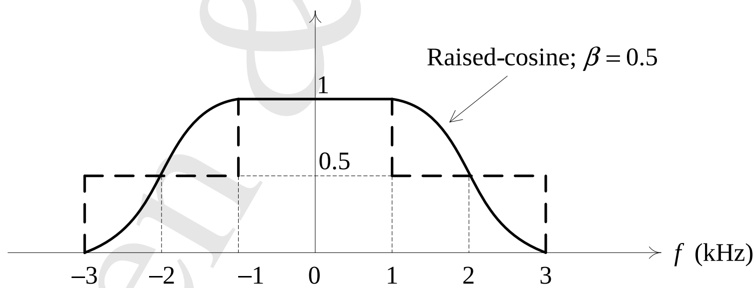


Figure 9.9

(d) The raised-cosine spectrum is preferred because its eye diagram is considerably more open. The wide opening is desirable under synchronization error and additive white Gaussian noise in order to minimize the effect of ISI. Note that the relative shapes of the two eye diagrams are expected since the RC spectrum is smoother than the other spectrum for the same bit rate and excess bandwidth.

P9.8 As mentioned on page 348, $S_R(f)$ should have a certain symmetry about the $\frac{1}{2T_b}$ point. Graphically, at least for $S_R(f)$ functions that have zero phase, this symmetry can be checked by checking to see if $S_R(f)$ is: i) flat from 0 to some $f_1 \leq \frac{1}{2T_b}$; ii) the portion from f_1 to $\frac{1}{T_b} - f_1$ has odd symmetry about the $\left(\frac{1}{2T_b}, \frac{1}{2}\right)$ point; iii) is zero for $f > \frac{1}{T_b} - f_1$. Applying this to the four spectra given:

(i) Yes. Choose $\frac{1}{2T_b}$ to be $\frac{2B}{3}$ Hz $\Rightarrow r_b = \frac{4}{3}B$ bits/second.

(ii) Yes. Choose $\frac{1}{2T_b}$ to be $\frac{B}{2}$ Hz $\Rightarrow r_b = B$ bits/second.

(iii) Yes. Here $\frac{1}{2T_b} = \frac{2B}{3}$ Hz $\Rightarrow r_b = \frac{4}{3}B$ bits/second.

(iv) No. The required symmetry does not exist on the f axis.

P9.9 See Table 9.1.

Table 9.1

$k =$	0	1	2	3	4	5	6	7	8	9	10	11	12	
$b_k =$		1	1	0	0	1	0	1	0	1	1	0	1	
$d_k =$	0	1	0	0	0	1	1	0	0	1	0	0	1	
$V_k =$	$-V$	$+V$	$-V$	$-V$	$-V$	V	V	$-V$	$-V$	V	$-V$	$-V$	V	(ignore noise)
$y_k =$		0	0	$-2V$	$-2V$	0	$2V$	0	$-2V$	0	0	$-2V$	0	
$\hat{b}_k =$		1	1	0	0	1	0	1	0	1	1	0	1	

P9.10 (a)

$$S_R^{(\text{sampled})}(t) = \sum_{k=-\infty}^{\infty} s(kT_b)\delta(t-kT_b) = \sum_{k=0}^{N-1} s_k\delta(t-kT_b)$$

Therefore

$$S_R^{(\text{sampled})}(f) = \sum_{k=0}^{N-1} s_k \int_{-\infty}^{\infty} \delta(t-kT_b)\mathrm{e}^{-j2\pi ft}\mathrm{d}t = \sum_{k=0}^{N-1} s_k \mathrm{e}^{-j2\pi fkT_b}$$

From the sampling theorem, Section 4.11, pages 136-139, Eqn. (4.5) we know that

$$\sum_{k=-\infty}^{\infty} S_R\left(f - \frac{k}{T_b}\right) = T_b S_R^{(\text{sampled})}(f)$$

Ignoring the aliases, i.e., considering only $k=0$ term:

$$S_R(f) = \begin{cases} T_b S_R^{(\text{sampled})}(f) = \sum_{k=0}^{N-1} T_b s_k \mathrm{e}^{-j2\pi fkT_b}, & -\frac{1}{2T_b} \le f \le \frac{1}{2T_b} \\ 0, & \text{otherwise.} \end{cases}$$

(b)

$$\begin{aligned} s_R(t) &= \mathcal{F}^{-1}\{S_R(f)\} = \sum_{k=0}^{N-1} T_b s_k \int_{-\frac{1}{2T_b}}^{\frac{1}{2T_b}} \mathrm{e}^{-j2\pi fkT_b}\mathrm{e}^{j2\pi ft}\mathrm{d}f \\ &= \sum_{k=0}^{N-1} T_b s_k \int_{-\frac{1}{2T_b}}^{\frac{1}{2T_b}} \mathrm{e}^{j2\pi(t-kT_b)f}\mathrm{d}f \\ &= \sum_{k=0}^{N-1} T_b s_k \frac{\mathrm{e}^{j2\pi(t-kT_b)\frac{1}{2T_b}} - \mathrm{e}^{-j2\pi(t-kT_b)f\frac{1}{2T_b}}}{j2\pi(t-kT_b)} \\ &= \sum_{k=0}^{N-1} s_k \frac{\sin\left(\frac{\pi}{T_b}(t-kT_b)\right)}{\frac{\pi}{T_b}(t-kT_b)} \end{aligned}$$

P9.11 (a)

$$s_R(t) = j2T_b \int_{-\frac{1}{2T_b}}^{\frac{1}{2T_b}} \left(\frac{\mathrm{e}^{j2\pi f T_b} - \mathrm{e}^{-j2\pi f T_b}}{2j} \right) \mathrm{e}^{j2\pi f t} \mathrm{d}f$$

After integration this becomes

$$s_R(t) = \frac{T_b}{\pi} \left\{ \frac{\sin\left(\frac{\pi}{T_b}(t+T_b)\right)}{(t+T_b)} - \frac{\sin\left(\frac{\pi}{T_b}(t-T_b)\right)}{(t-T_b)} \right\}$$

Now $\sin\left(\frac{\pi}{T_b}(t+T_b)\right) = \sin\left(\frac{\pi}{T_b}(t-T_b)\right) = -\sin\left(\frac{\pi t}{T_b}\right)$ so

$$s_R(t) = \frac{T_b}{\pi} \sin\left(\frac{\pi t}{T_b}\right) \left\{ \frac{-1}{t+T_b} + \frac{1}{t-T_b} \right\} = \frac{2T_b^2}{\pi} \frac{\sin\left(\frac{\pi t}{T_b}\right)}{t^2 - T_b^2} = \frac{2}{\pi} \frac{\sin\left(\frac{\pi t}{T_b}\right)}{\left(\frac{t}{T_b}\right)^2 - 1}$$

$s_R(t)$ decays as $\frac{1}{t^2}$ which is expected since $S_R(f)$ needs to be differentiated twice to produce an impulse(s). Use the duality concept and the result of P2.41:

$s_R(kT_b) = 0$ at $k = 0, \pm 2, \pm 3, \pm 4, \ldots$. At $k = \pm 1$, i.e., $\pm T_b$, $s_R(\pm T_b)$ becomes $\frac{0}{0}$. Using L'Hospitale's rule we have $s(T_b) = -1$, $s(-T_b) = +1$.

(b) Ignoring any random noise: $y_k = V_{k+1} + V_{k-1}$ Associate value V_{k+1} with bit b_k. V_{k-1} is due to bit b_{k-2} and an ISI term, i.e., the interference comes not from the previous bit but from the one before it. Since $V_k = \pm V$, there are three received levels, $\pm 2V$ and 0.

(c) As a precoder form $d_k = b_k \oplus d_{k-2}$. See Fig. 9.10

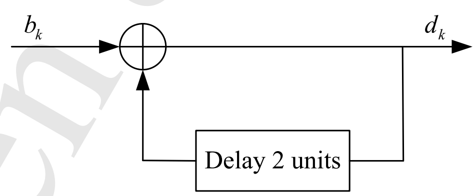


Figure 9.10

(d) 2.1 dB. Note that this is very similar to duobinary.

P9.12 (a) Using the result of P9.10, Eqn. P9.3,

$$S_R(f) = \begin{cases} T_b \left[2 + \frac{2\left(\mathrm{e}^{j2\pi f T_b} + \mathrm{e}^{-j2\pi f T_b}\right)}{2} \right] = 2T_b\left(1 + \cos(2\pi f T_b)\right) & \\ = 2T_b(1 + \cos^2 \pi f T_b - \sin^2 \pi f T_b) = 4T_b \cos^2 \pi f T_b, & -\frac{1}{2T_b} \le f \le \frac{1}{2T_b} \\ 0, & \text{otherwise.} \end{cases}$$

(b) It decays as $\frac{1}{t^3}$. Need to differentiate $S_R(f)$ three times before an impulse(s) appears.

(c) $y_k = V_{k-1} + 2V_k + V_{k+1}$ (ignoring the random noise).

(d) There are five levels, $\pm 4V, \pm 2V, 0$.

(e) To design the encoder, one wants a unique y_k to correspond to b_k. Note further that there are 2 interfering terms which implies that $d_k = f(b_k, d_{k-1}, d_{k-2})$. Given this let us build a truth table of (b_k, d_{k-1}, d_{k-2}) and the corresponding d_k so that a unique y_k is produced. Finally let $b_k = 0$ correspond to the levels $0, \pm 4V$ and $b_k = 1$ to the level $\pm 2V$.

b_k	d_{k-1}	d_{k-2}	$\Rightarrow$	d_k	$y_{k-1} = V_{k-2} + 2V_{k-1} + V_k$
0	0	0		0	$= -2 - 1 - 1 = -4$
0	0	1		1	$= -2 + 1 + 1 = 0$
0	1	0		0	$= +2 - 1 - 1 = 0$
0	1	1		1	$= +2 + 1 + 1 = +4$
1	0	0		1	$= -2 - 1 + 1 = -2$
1	0	1		0	$= -2 + 1 - 1 = -2$
1	1	0		1	$= +2 - 1 + 1 = +2$
1	1	1		0	$= +2 + 1 - 2 = +2$

Note that if $y_{k-1} = \pm 4V$ or 0 then $b_k = 0$ and if $y_{k-1} = \pm 2V$ then $b_k = 0$. Therefore the decision space looks as in Fig. 9.11:

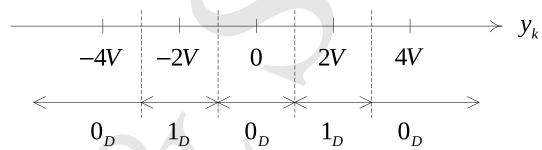


Figure 9.11

Note further that we are assuming, as usual, that the random noise sample $\mathbf{w}_k$ is Gaussian, zero-mean and of variance $\sigma^2_{\mathbf{w}}$ (watts).

(f) To determine the SNR degradation, we need to determine $\left(\frac{V^2}{\sigma^2_{\mathbf{w}}}\right)_{\max}$ from Eqn. (9.37) where $|S_R(f)| = 4T_b \cos^2(\pi f T_b)$. Therefore

$$\left(\frac{V^2}{\sigma^2_{\mathbf{w}}}\right)_{\max} = P_T T_b \left[\sqrt{\frac{N_0}{2}} \int_{-\frac{1}{2T_b}}^{\frac{1}{2T_b}} 4T_b \cos^2(\pi f T_b)\right]^2$$

Note that $\cos^2(\pi f T_b) = \frac{1}{2}[1+\cos(2\pi f T_b)]$ and do the integration to get $\left(\frac{V^2}{\sigma^2_{\mathbf{w}}}\right)_{\max} = \frac{P_T T_b}{2N_0}$. Therefore the probability of error is $Q\left(\sqrt{\frac{P_T T_b}{2N_0}}\right)$ (ignoring constant factors before the $Q(\cdot)$ function(s)).

Compared with the ideal binary case where $P[\text{error}] \sim Q\left(\sqrt{\frac{P_T T_b}{2N_0}}\right)$, we see that P_T needs to be increased by a factor of 4 which implies an SNR degradation $10 \log_{10} 4 = 6$ dB.

P9.13 (a) The Fourier transform of the sampled spectrum

$$\begin{aligned} S_R^{\text{sampled}}(f) &= T_b \mathcal{F}\{-\delta(t+T_b) + 2\delta(t) - \delta(t-T_b)\} \\ &= T_b \left\{-\mathrm{e}^{j2\pi f T_b} + 2 - \mathrm{e}^{-j2\pi f T_b}\right\} = 2T_b[1 - \cos(2\pi f T_b)] \end{aligned}$$

The spectrum of the continuous time signal is

$$S_R(f) = \begin{cases} 2T_b[1-\cos^2(\pi f T_b)] = 4T_b\sin^2(\pi f T_b), & |f| \le \frac{1}{2T_b} \\ 0, & \text{otherwise.} \end{cases}$$

(b) The spectrum of $S_R(f)$ looks as in Fig. 9.12. Since it has a discontinuity, $s_R(t)$ decays as $\frac{1}{t}$.

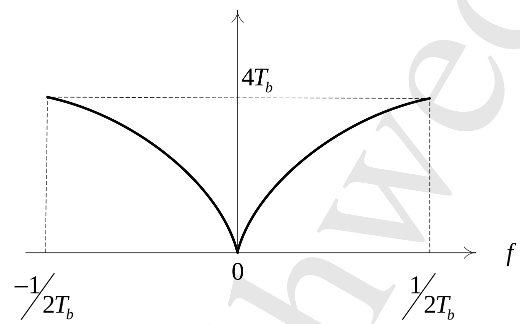


Figure 9.12

(c) Without random noise $y_k = -V_{k-1} + 2V_k - V_{k+1}$ where $V_j = \pm V$.

(d) The sampled output $y_k = -(\pm V) + 2(\pm V) - (\pm V)$. Looking at all combinations $s_k = -4V, -2V, 0, +2V, +4V$, i.e., there are 5 levels.

(e) To establish the coder's table, note that there are 2 interfering terms which implies that a coded bit should depend on 2 previous coded bits as well as the present information bit, i.e., $d_k = f(b_k, d_{k-1}, d_{k-2})$. Further we wish $b_k = 0$ and $b_k = 1$ to be associated with levels that are distinct. Let $b_k = 0$ correspond to the levels $0, \pm 4V$ and $b_k = 1$ correspond to levels $\pm 2V$. Therefore

b_k	d_{k-1}	d_{k-2}	$\Rightarrow$	d_k	$y_{k-1} = V_{k-2} + V_{k-1} + V_k$
0	0	0		0	$= -1 - 2 - 1 = -4$
0	0	1		0	$= -1 + 2 - 1 = 0$
0	1	0		1	$= +1 - 2 + 1 = 0$
0	1	1		1	$= +1 + 2 + 1 = +4$
1	0	0		1	$= -1 - 2 + 1 = -2$
1	0	1		1	$= -1 + 2 + 1 = +2$
1	1	0		0	$= +1 - 2 - 1 = -2$
1	1	1		0	$= +1 + 2 - 1 = +2$

(f) Using Eqn. (9.39) we have

$$\left(\frac{V^2}{\sigma_{\mathbf{w}}^2}\right)_{\max} = P_T T_b \left[\sqrt{\frac{N_0}{2}} \int_{-\frac{1}{2T_b}}^{\frac{1}{2T_b}} 2T_b\left[1-\cos(2\pi f T_b)\right] \mathrm{d}f\right]^2 = \frac{P_T T_b}{2}$$

so $P[\text{error}] \sim Q\left(\sqrt{\frac{P_T T_b}{2N_0}}\right) \Rightarrow$ a 6 dB degradation in SNR.

P9.14 See Fig. 9.13. The eye diagram with the duobinary signalling (left figure) is significantly more open, reflecting the fact that its overall impulse response, $s_R(t)$, decays as $1/t^2$, while that of P9.11 decays as $1/t$.

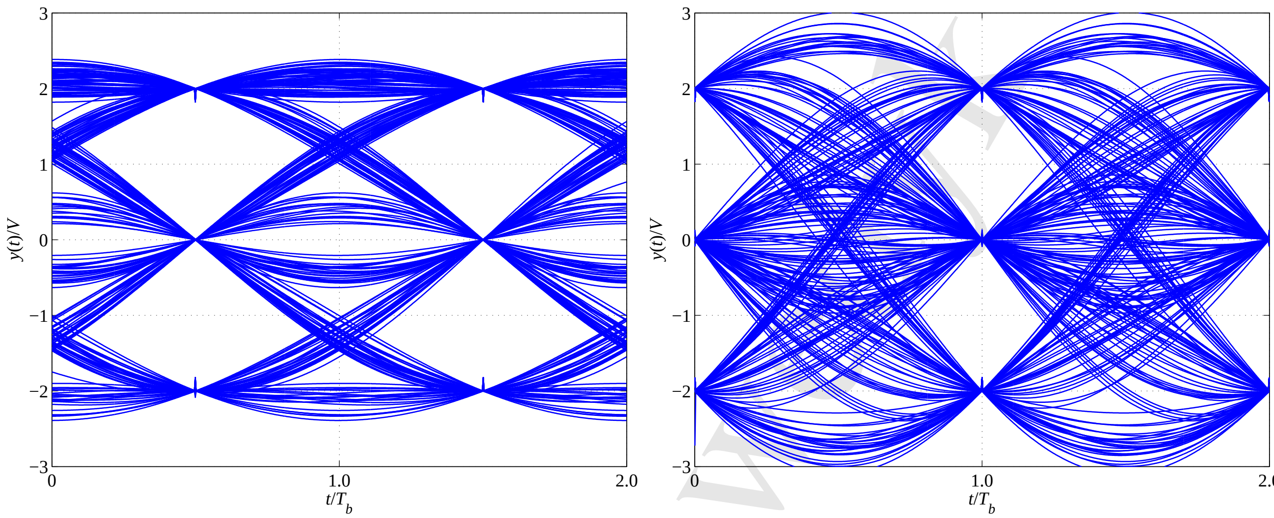


Figure 9.13: Eye diagrams with the duobinary modulation (left) and with the spectrum of P9.11 (right). Note that for duobinary modulation the sampling times need to be delayed by $T_b/2$.

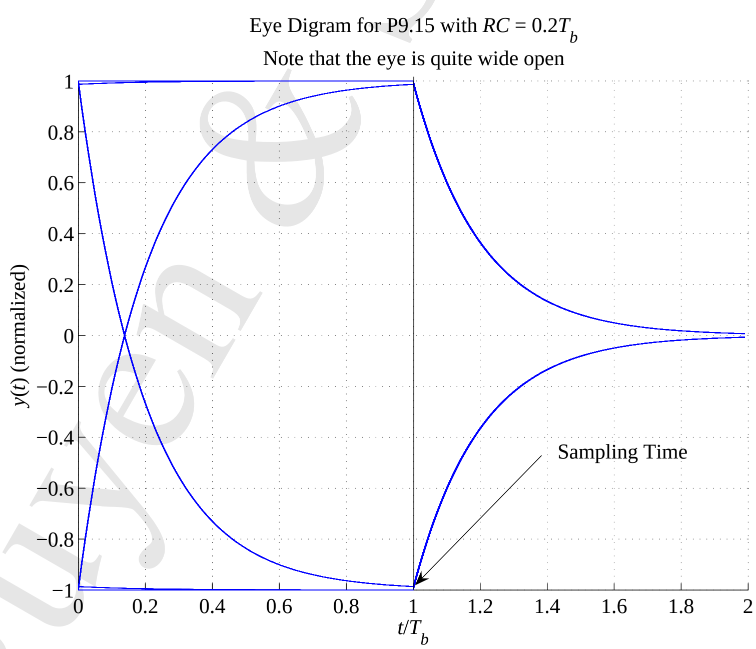


Figure 9.14

P9.15 See Figs. 9.14, 9.15, 9.16.

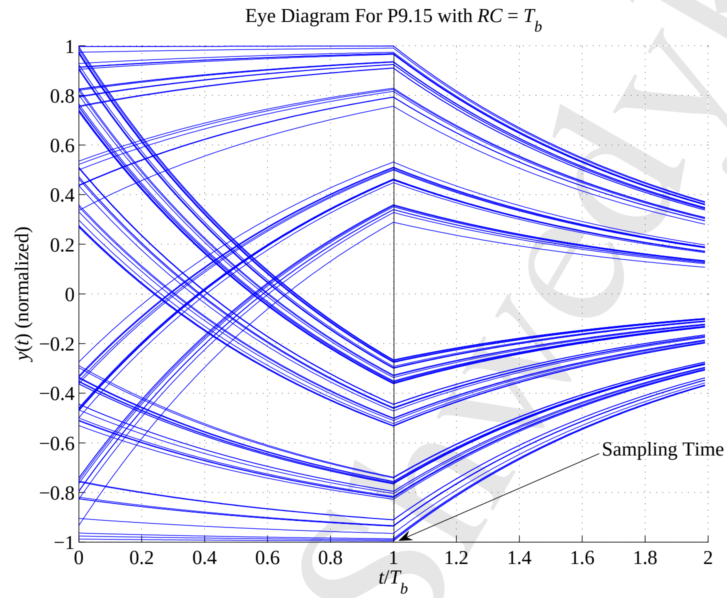


Figure 9.15

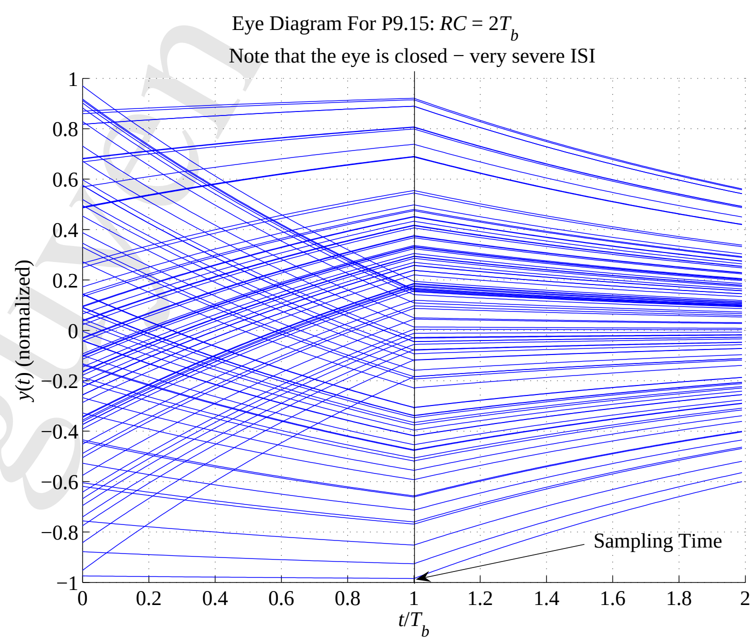


Figure 9.16

Chapter 10

Signaling Over Fading Channels

P10.1 The two equations of interest are

$$P[\text{error}]_{\text{BASK}} = \frac{1}{2}\mathrm{e}^{-T_h^2/N_0} + \frac{1}{2}\left[1 - Q\left(\sqrt{\frac{2E}{N_0}}, \sqrt{\frac{2}{N_0}}T_h\right)\right] \tag{10.1}$$

where T_h is the threshold used in BASK and $E = 2E_b$, and

$$P[\text{error}]_{\text{BFSK}} = \frac{1}{2}\mathrm{e}^{-E_b/2N_0} \tag{10.2}$$

When one chooses $T_h = \frac{\sqrt{E}}{2} = \frac{\sqrt{2E_b}}{2}$ then $P[\text{error}]_{\text{BASK}}$ simplifies to:

$$P[\text{error}]_{\text{BASK}} = \frac{1}{2}\mathrm{e}^{-E_b/2N_0} + \frac{1}{2}\left[1 - Q\left(2\sqrt{\frac{E_b}{N_0}}, \sqrt{\frac{E_b}{N_0}}\right)\right]$$

Since the first term in the above expression is $P[\text{error}]_{\text{BFSK}}$, while the second term is always positive for finite E_b/N_0, it follows immediately that $P[\text{error}]_{\text{BASK}} > P[\text{error}]_{\text{BFSK}}$.

P10.2 The plot of the normalized optimum threshold, namely $T_h^{(\text{normalized})} = \frac{T_h}{\sqrt{\frac{E_b}{2}}}$ is shown in Fig. 10.1. The plot shows that $T_h \geq \frac{\sqrt{E}}{2} = \sqrt{\frac{E_b}{2}}$. This fact can be shown mathematically as follows.

Start with $T_h = \frac{N_0}{2\sqrt{E}} I_0^{-1}\left(\mathrm{e}^{E/N_0}\right)$ where $E = 2E_b$. Therefore

$$T_h = \frac{N_0}{2\sqrt{2E_b}} I_0^{-1}\left(\mathrm{e}^{2E_b/N_0}\right) = \sqrt{\frac{E_b}{2}}\frac{1}{\left(\frac{2E_b}{N_0}\right)} I_0^{-1}\left(\mathrm{e}^{2E_b/N_0}\right)$$

or

$$T_h^{(\text{normalized})} = \frac{T_h}{\sqrt{\frac{E_b}{2}}} = \frac{I_0^{-1}\left(\mathrm{e}^{2E_b/N_0}\right)}{\left(\frac{2E_b}{N_0}\right)}$$

Now $\mathrm{e}^{x\cos\theta} = I_0(x) + 2\sum_{k=1}^{\infty} I_k(x)\cos(k\theta)$ and $I_k(x) > 0$ for all $k > -1$ and $x > 0$. Therefore with $\theta = 0$ we have $\mathrm{e}^x = I_0(x) + 2\sum_{k=1}^{\infty} I_k(x) \geq I_0(x)$, which also means $I_0^{-1}(\mathrm{e}^x) \geq x$, $\forall x > 0$. Therefore $T_h^{(\text{normalized})} \geq 1$.

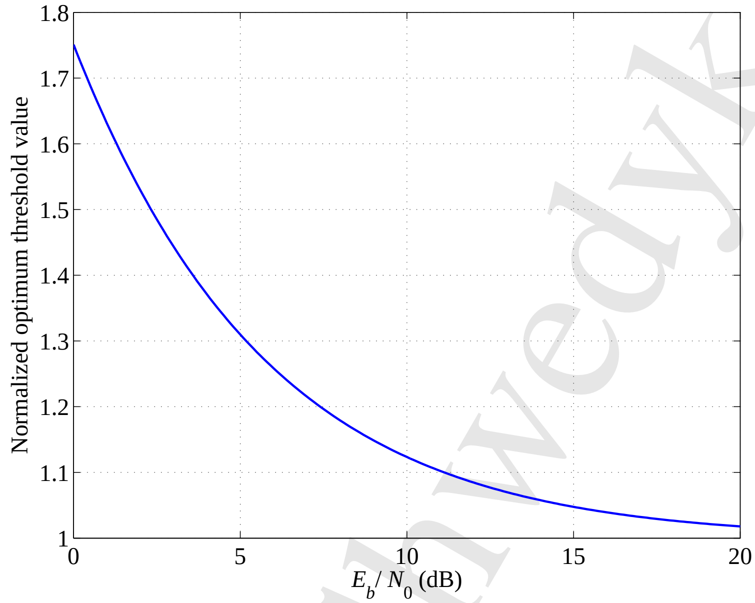
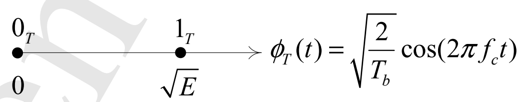


Figure 10.1

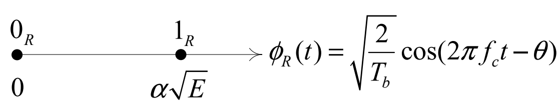
P10.3 The transmitted signal set and signal space are as follows:

$$\begin{aligned} 1_T &: s(t) = \sqrt{E}\cos(2\pi f_c t) \\ 0_T &: s(t) = 0 \end{aligned} \tag{10.3}$$



The received signal set and signal space is

$$\begin{aligned} 1_T &: \mathbf{r}(t) = \sqrt{E}\alpha\cos(2\pi f_c t - \theta) + \mathbf{w}(t) \\ 0_T &: \mathbf{r}(t) = 0 + \mathbf{w}(t) \end{aligned} \tag{10.4}$$



Since we assume α, θ are estimated accurately for each transmission, we use θ to set the basis function $\phi_R(t)$ appropriately and α to set the threshold. The decision rule is

$$\mathbf{r} \underset{0_D}{\overset{1_D}{\gtrless}} \frac{\alpha\sqrt{E}}{2} = \alpha\sqrt{\frac{E_b}{2}}$$

where $E_b = \frac{E}{2}$ and $\mathbf{r} = \int_0^{T_b} \mathbf{r}(t)\phi_R(t)\mathrm{d}t$.

Note that $f(r|0_T) \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$ and $f(r|1_T) \sim \mathcal{N}\left(\alpha\sqrt{2E_b}, \frac{N_0}{2}\right)$. Conditioned on θ and α, the error probability is given by

$$P[\text{error}|\theta,\alpha] = Q\left(\frac{\alpha\sqrt{E_b/2}}{\sqrt{N_0/2}}\right) = Q\left(\alpha\sqrt{\frac{E_b}{N_0}}\right)$$

The error probability depends on the specific value α (but not on the specific value of θ) that the random variable $\boldsymbol{\alpha}$ assumes during the a specific transmission. Now α takes on values from 0 to ∞ according to the Rayleigh pdf, $f_{\boldsymbol{\alpha}}(\alpha) = \frac{2\alpha}{\sigma_F^2}\mathrm{e}^{-\alpha^2/\sigma_F^2}u(\alpha)$. To find the overall average error probability, average the conditional bit error probability over $f_{\boldsymbol{\alpha}}(\alpha)$, i.e., find

$$P[\text{error}] = \int_0^\infty Q\left(\alpha\sqrt{\frac{E_b}{N_0}}\right) f_{\boldsymbol{\alpha}}(\alpha)\mathrm{d}\alpha = \frac{1}{2}\int_0^\infty \left[1 - \operatorname{erf}\left(\alpha\sqrt{\frac{E_b}{N_0}}\right)\right] f_{\boldsymbol{\alpha}}(\alpha)\mathrm{d}\alpha$$

where we have substituted $Q(x) = \frac{1}{2}\left[1 - \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right]$. It follows that

$$P[\text{error}] = \frac{1}{2} - \frac{1}{\sigma_F^2}\int_0^\infty \alpha\cdot\operatorname{erf}\left(\alpha\sqrt{\frac{E_b}{N_0}}\right)\mathrm{e}^{-\frac{\alpha^2}{\sigma_F^2}}\mathrm{d}\alpha$$

Perusing Gradstyn & Ryzhik we notice, page 649, Eqn. 6.287-1, that

$$\int_0^\infty x\Phi(\beta x)\mathrm{e}^{-\mu x^2}\mathrm{d}x = \frac{\beta}{2\mu\sqrt{\mu+\beta^2}} \qquad [\operatorname{Re}(\mu) > -\operatorname{Re}(\beta^2),\ \operatorname{Re}(\mu) > 0]$$

where $\Phi(x) = \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{e}^{-t^2}\mathrm{d}t$ (G&R, p.930, Eqn. 8.250-1), i.e., $\Phi(x) = \operatorname{erf}(x)$.

Identifying $\beta = \sqrt{\frac{E_b}{2N_0}}$, $\mu = \frac{1}{\sigma_F^2}$, and substituting we get

$$P[\text{error}] = \frac{1}{2} - \frac{1}{2}\frac{\sqrt{\sigma_F^2 E_b/2N_0}}{\sqrt{1+\sigma_F^2 E_b/2N_0}} \tag{10.5}$$

Interpreting $\sigma_F^2 E_b$ as the received energy per bit and comparing with coherent BFSK (Eqn. (10.80)) we see that coherent BASK's performance is identical to that of coherent BFSK and both are 3 dB less efficient than coherent BPSK (Eqn. (10.81)).

As aside, the following shows that $\sigma_F^2 E_b$ is indeed the *average received energy per bit*. During any transmission the received energy is $\mathbf{E}_R(\alpha) = \frac{1}{2}(0) + \frac{1}{2}(\boldsymbol{\alpha}^2 E) = \frac{1}{2}\boldsymbol{\alpha}^2(2E_b) = \boldsymbol{\alpha}^2 E_b$. Therefore the average received energy per bit is

$$E\{\mathbf{E}_R(\alpha)\} = \int_0^\infty \alpha^2 E_b \underbrace{\left\{\frac{2}{\sigma_F^2}\alpha\mathrm{e}^{-\frac{\alpha^2}{\sigma_F^2}}\right\}}_{f_{\boldsymbol{\alpha}}(\alpha)}\mathrm{d}\alpha \overbrace{=}^{\lambda=\frac{\alpha^2}{\sigma_F^2}} \sigma_F^2 E_b\int_0^\infty \lambda\mathrm{e}^{-\lambda}\mathrm{d}\lambda = \sigma_F^2 E_b \quad \text{(joules/bit)}.$$

Note that this is true for both BPSK and BFSK.

P10.4 We use the approach developed in Section 10.3.3. The main difference is that not only is the received phase, $\boldsymbol{\theta}$, random but so is the received amplitude which is scaled by $\boldsymbol{\alpha}$. Eqn. (10.42) which expresses the received signals over 2 bit intervals becomes:

$$\mathbf{r}(t) = \begin{cases} \pm\sqrt{\frac{2E_b}{T_b}}\boldsymbol{\alpha}\cos(2\pi f_c t - \boldsymbol{\theta})\left[u(t) - u(t-2T_b)\right] + \mathbf{w}(t), & \text{“}0_T\text{”} \\ \pm\sqrt{\frac{2E_b}{T_b}}\boldsymbol{\alpha}\cos(2\pi f_c t - \boldsymbol{\theta})\left\{\left[u(t) - u(t-T_b)\right] - \left[u(t-T_b) - u(t-2T_b)\right]\right\} + \mathbf{w}(t), & \text{“}1_T\text{”} \end{cases}$$

Rewrite this as

$$\mathbf{r}(t) = \begin{cases} \pm\sqrt{\frac{2E_b}{T_b}}\left[\mathbf{n}_{F,I}\cos(2\pi f_c t) + \mathbf{n}_{F,Q}\sin(2\pi f_c t)\right]\left[u(t) - u(t-2T_b)\right] + \mathbf{w}(t), & \text{“}0_T\text{”} \\ \pm\sqrt{\frac{2E_b}{T_b}}\left[\mathbf{n}_{F,I}\cos(2\pi f_c t) + \mathbf{n}_{F,Q}\sin(2\pi f_c t)\right] & \\ \qquad \left\{\left[u(t) - u(t-T_b)\right] - \left[u(t-T_b) - u(t-2T_b)\right]\right\} + \mathbf{w}(t), & \text{“}1_T\text{”} \end{cases}$$

where $\mathbf{n}_{F,I} = \boldsymbol{\alpha}\cos\boldsymbol{\theta}$, $\mathbf{n}_{F,Q} = \boldsymbol{\alpha}\sin\boldsymbol{\theta}$ are statistically independent, zero-mean Gaussian random variables with identical variance $\frac{\sigma_F^2}{2}$.

Again we need the 4 basis functions of Eqn. (10.43) to generate the sufficient statistics. These are as in (10.44), (10.45), the difference being that rather than $\cos\boldsymbol{\theta}$, $\sin\boldsymbol{\theta}$ we have $\boldsymbol{\alpha}\cos\boldsymbol{\theta}$, $\boldsymbol{\alpha}\sin\boldsymbol{\theta}$, i.e., $\mathbf{n}_{F,I}$, $\mathbf{n}_{F,Q}$.

The sufficient statistics are therefore

$$0_T: \begin{cases} \mathbf{r}_{1,I} = \sqrt{2E_b}\mathbf{n}_{F,I} + \mathbf{w}_{1,I} \\ \mathbf{r}_{1,Q} = \sqrt{2E_b}\mathbf{n}_{F,Q} + \mathbf{w}_{1,Q} \\ \mathbf{r}_{2,I} = \mathbf{w}_{2,I} \\ \mathbf{r}_{2,Q} = \mathbf{w}_{2,Q} \end{cases} \quad \text{OR} \quad \begin{cases} \mathbf{r}_{1,I} = -\sqrt{2E_b}\mathbf{n}_{F,I} + \mathbf{w}_{1,I} \\ \mathbf{r}_{1,Q} = -\sqrt{2E_b}\mathbf{n}_{F,Q} + \mathbf{w}_{1,Q} \\ \mathbf{r}_{2,I} = \mathbf{w}_{2,I} \\ \mathbf{r}_{2,Q} = \mathbf{w}_{2,Q} \end{cases}$$

$$1_T: \begin{cases} \mathbf{r}_{1,I} = \mathbf{w}_{1,I} \\ \mathbf{r}_{1,Q} = \mathbf{w}_{1,Q} \\ \mathbf{r}_{2,I} = \sqrt{2E_b}\mathbf{n}_{F,I} + \mathbf{w}_{2,I} \\ \mathbf{r}_{2,Q} = \sqrt{2E_b}\mathbf{n}_{F,Q} + \mathbf{w}_{2,Q} \end{cases} \quad \text{OR} \quad \begin{cases} \mathbf{r}_{1,I} = \mathbf{w}_{1,I} \\ \mathbf{r}_{1,Q} = \mathbf{w}_{1,Q} \\ \mathbf{r}_{2,I} = -\sqrt{2E_b}\mathbf{n}_{F,I} + \mathbf{w}_{2,I} \\ \mathbf{r}_{2,Q} = -\sqrt{2E_b}\mathbf{n}_{F,Q} + \mathbf{w}_{2,Q} \end{cases}$$

Observe that, regardless of the case considered, $\mathbf{r}_{1,I}, \mathbf{r}_{1,Q}, \mathbf{r}_{2,I}, \mathbf{r}_{2,Q}$ are statistically independent Gaussian random variables. Consider the case of 0_T, i.e., $f(r_{1,I}, r_{1,Q}, r_{2,I}, r_{2,Q}|0_T)$. In this situation $\mathbf{r}_{1,I}$ is a zero-mean Gaussian random variable of variance $\sigma_2^2 = E_b\sigma_F^2 + \frac{N_0}{2}$. So is $\mathbf{r}_{1,Q}$. This is regardless of whether $+\sqrt{2E_b}$ or $-\sqrt{2E_b}$ is considered. $\mathbf{r}_{2,I}, \mathbf{r}_{2,Q}$ are zero-mean Gaussian random variables of variance $\sigma_1^2 = \frac{N_0}{2}$.

With 1_T the situation is reversed, i.e., $\mathbf{r}_{1,I}, \mathbf{r}_{1,Q} \sim \mathcal{N}(0, \sigma_1^2)$ and $\mathbf{r}_{2,I}, \mathbf{r}_{2,Q} \sim \mathcal{N}(0, \sigma_2^2)$.

The likelihood ratio test

$$\frac{f(r_{1,I}, r_{1,Q}, r_{2,I}, r_{2,Q}|1_T)}{f(r_{1,I}, r_{1,Q}, r_{2,I}, r_{2,Q}|0_T)} \underset{0_D}{\overset{1_D}{\gtrless}} 1$$

becomes

$$\frac{\mathrm{e}^{-r_{1,I}^2/2\sigma_1^2}\mathrm{e}^{-r_{1,Q}^2/2\sigma_1^2}\mathrm{e}^{-r_{2,I}^2/2\sigma_2^2}\mathrm{e}^{-r_{2,Q}^2/2\sigma_2^2}}{\mathrm{e}^{-r_{1,I}^2/2\sigma_2^2}\mathrm{e}^{-r_{1,Q}^2/2\sigma_2^2}\mathrm{e}^{-r_{2,I}^2/2\sigma_1^2}\mathrm{e}^{-r_{2,Q}^2/2\sigma_1^2}} \underset{0_D}{\overset{1_D}{\gtrless}} 1$$

Taking the natural logarithm, gathering terms and rearranging, the test simplifies to

$$\left(r_{2,I}^2 + r_{2,Q}^2\right)\left(-\frac{1}{\sigma_2^2} + \frac{1}{\sigma_1^2}\right) \underset{0_D}{\overset{1_D}{\gtrless}} \left(r_{1,I}^2 + r_{1,Q}^2\right)\left(-\frac{1}{\sigma_2^2} + \frac{1}{\sigma_1^2}\right)$$

Now $\sigma_2^2 > \sigma_1^2 \Rightarrow -\frac{1}{\sigma_2^2} + \frac{1}{\sigma_1^2} > 0 \Rightarrow$ can cancel this function from both sides without affecting the inequality relationship. Therefore the decision rule becomes

$$r_{2,I}^2 + r_{2,Q}^2 \underset{0_D}{\overset{1_D}{\gtrless}} r_{1,I}^2 + r_{1,Q}^2$$

The above makes intuitive sense. Since both the phase and the amplitude information is destroyed by the fading channel the only way to distinguish as to whether a 1 or 0 was transmitted is to look at whether there is more energy in the $\{\phi_{2,I}(t), \phi_{2,Q}(t)\}$ plane or in the $\{\phi_{1,I}(t), \phi_{1,Q}(t)\}$ plane, which is what the above decision rule does.

The decision rule can be rewritten as

$$\sqrt{r_{2,I}^2 + r_{2,Q}^2} \underset{0_D}{\overset{1_D}{\gtrless}} \sqrt{r_{1,I}^2 + r_{1,Q}^2}$$

which looks at the envelope of the received signal in the appropriate plane.

To obtain the error performance, the "energy form" of the decision rule is preferred. Also for purposes of error performance the variables in the decision rule are considered to be random, i.e., it is written as

$$\boldsymbol{\ell}_2 \equiv \mathbf{r}_{2,I}^2 + \mathbf{r}_{2,Q}^2 \underset{0_D}{\overset{1_D}{\gtrless}} \mathbf{r}_{1,I}^2 + \mathbf{r}_{1,Q}^2 \equiv \boldsymbol{\ell}_1$$

Due to symmetry the error probability can be written as

$$P[\text{error}] = P[\text{error}|0_T] = P[\boldsymbol{\ell}_2 \geq \boldsymbol{\ell}_1|0_T]$$

Under the condition that a zero is transmitted the random variable $\boldsymbol{\ell}_2$ is Chi-square with 2 degrees of freedom and parameter σ_1^2 while $\boldsymbol{\ell}_1$ is also Chi-square, 2 degrees of freedom, parameter σ_2^2.

Now fix $\boldsymbol{\ell}_1$ at a specific value, say $\boldsymbol{\ell}_1 = \ell_1$. Then

$$P[\text{error}|0_T, \boldsymbol{\ell}_1 = \ell_1] = \int_{\ell_1}^{\infty} f_{\boldsymbol{\ell}_2}(\ell_2|0_T)\mathrm{d}\ell_2 = \int_{\ell_1}^{\infty} \frac{\mathrm{e}^{-\ell_2/2\sigma_1^2}}{2\Gamma(1)\sigma_1^2}\mathrm{d}\ell_2 = \mathrm{e}^{-\ell_1/2\sigma_1^2}$$

Sweep $\boldsymbol{\ell}_1$ over all possible values, weighted, of course, by the probability of that value occurring, i.e.,

$$\begin{aligned} P[\text{error}|0_T] &= \int_0^{\infty} P[\text{error}|0_T, \boldsymbol{\ell}_1 = \ell_1] f_{\boldsymbol{\ell}_1}(\ell_1|0_T)\mathrm{d}\ell_1 = \int_0^{\infty} \mathrm{e}^{-\ell_1/2\sigma_1^2} \frac{\mathrm{e}^{-\ell_1/2\sigma_2^2}}{2\Gamma(1)\sigma_2^2}\mathrm{d}\ell_1 \\ &= \frac{1}{1 + \frac{\sigma_2^2}{\sigma_1^2}} = \frac{1}{1 + \frac{E_b\sigma_F^2 + N_0/2}{N_0/2}} = \frac{1}{2}\frac{1}{1 + \frac{\sigma_F^2 E_b}{N_0}}. \end{aligned} \tag{10.6}$$

Comparison of (10.6) with the performances of coherent BFSK and coherent BPSK (Eqns. (10.80) and (10.81) in Section 10.4.3) is shown in Fig. 10.2. Observe that noncoherently-demodulated DBPSK performs basically the same as coherently demodulated BFSK.

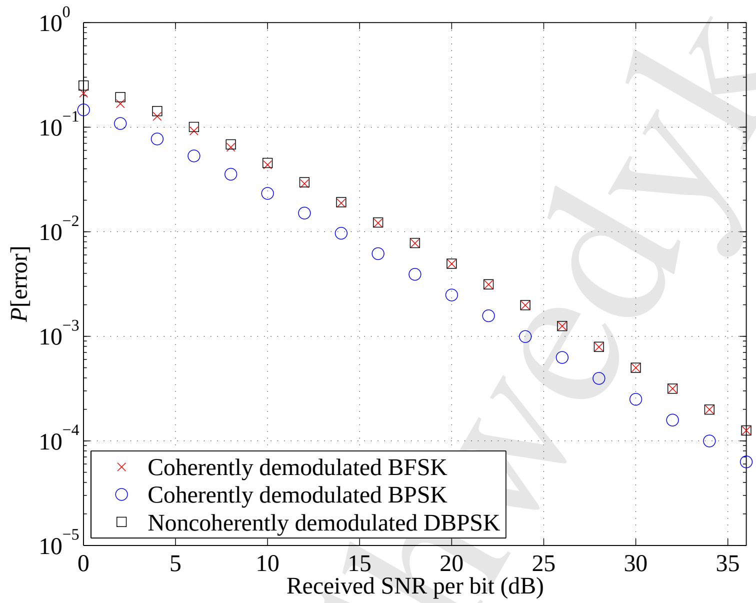


Figure 10.2

P10.5 (a) The received signal is

$$1_T: \quad \mathbf{r}_j(t) = \sqrt{E_b'}\sqrt{\frac{2}{T_b}}\boldsymbol{\alpha}_j \cos(2\pi f_c t - \boldsymbol{\theta}_j) + \mathbf{w}(t)$$

$$0_T: \quad \mathbf{r}_j(t) = -\sqrt{E_b'}\sqrt{\frac{2}{T_b}}\boldsymbol{\alpha}_j \cos(2\pi f_c t - \boldsymbol{\theta}_j) + \mathbf{w}(t)$$

where $(j-1)T_b \leq t \leq jT_b$, $j = 1, 2, \ldots, N$.

Though the terminology is somewhat ambiguous, indeed one might say misleading, coherent demodulation means that not only the phase $\boldsymbol{\theta}_j$ is estimated perfectly (at least perfectly enough for engineering purpose) but so is the attenuation factor $\boldsymbol{\alpha}_j$.

With $\boldsymbol{\theta}_j$ known, a set of sufficient statistics is generated by projecting the received signal(s) onto the N basis functions of $\sqrt{\frac{2}{T_b}}\cos(2\pi f_c t - \theta_j)$, $(j-1)T_b \leq t \leq jT_b$, $j = 1, 2, \ldots, N$. They are:

$$1_T: \quad \mathbf{r}_j = \sqrt{E_b'}\alpha_j + \mathbf{w}_j$$

$$0_T: \quad \mathbf{r}_j = -\sqrt{E_b'}\alpha_j + \mathbf{w}_j$$

where $j = 1, 2, \ldots, N$. The $\mathbf{r}_j$'s are i.i.d. Gaussian random variables of means $\pm\sqrt{E_b}\alpha_j$ and variance $\sigma^2 = N_0/2$. The LRT

$$\frac{f(r_1, r_2, \ldots, r_N|1_T)}{f(r_1, r_2, \ldots, r_N|0_T)} \underset{0_D}{\overset{1_D}{\gtrless}} 1$$

becomes

$$\frac{\Pi_{j=1}^{N}\left(\frac{1}{\sqrt{2\pi}\sigma}\right)\mathrm{e}^{-\left(r_j-\sqrt{E_b'}\alpha_j\right)/2\sigma^2}}{\Pi_{j=1}^{N}\left(\frac{1}{\sqrt{2\pi}\sigma}\right)\mathrm{e}^{-\left(r_j+\sqrt{E_b'}\alpha_j\right)/2\sigma^2}} \underset{0_D}{\overset{1_D}{\gtrless}} 1$$

Taking the ln and simplifying gives the following decision rule:

$$\sum_{j=1}^{N}\alpha_j r_j \underset{0_D}{\overset{1_D}{\gtrless}} 0.$$

Remark: The above decision rule is actually known as the *maximum-ratio-combining* (MRC) detection rule. This is because the signal-to-noise ratio of the decision variable is maximized (the proof is left as a further problem).

(b) In developing the receiver, the α_j's were assumed to be known (i.e., perfectly estimated at the receiver). To determine the error performance, however, we consider many, many transmissions (typically $> 10^8$) and the α_j's will take on a gamut of values, i.e., they are considered to be random variables with a Rayleigh pdf.

Start the calculation of the error performance by observing that due to symmetry

$$P[\text{error}] = P[\text{error}|1_T] = P[\text{error}|0_T]$$

and that

$$P[\text{error}|0_T] = P\left[\boldsymbol{\ell} = \sum_{j=1}^{N}\boldsymbol{\alpha}_j\boldsymbol{r}_j \geq 0\big|0_T\right].$$

To evaluate the above, start by assuming the $\boldsymbol{\alpha}_j$'s take on a specific set of values and then average over all possible values. That is, let $\overrightarrow{\boldsymbol{\alpha}} = (\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\ldots,\boldsymbol{\alpha}_N) = \overrightarrow{\alpha} = (\alpha_1,\alpha_2,\ldots,\alpha_N)$. Then

$$P[\text{error}|0_T,\overrightarrow{\boldsymbol{\alpha}}=\overrightarrow{\alpha}] = P\left[\boldsymbol{\ell} = \sum_{j=1}^{N}\alpha_j\boldsymbol{r}_j \geq 0\big|0_T,\overrightarrow{\boldsymbol{\alpha}}=\overrightarrow{\alpha}\right].$$

Note that the quantities $\alpha_j\mathbf{r}_j = -\sqrt{E_b'}\alpha_j^2 + \alpha_j\mathbf{w}_j$ are statistically independent Gaussian variables of mean $-\sqrt{E_b'}\alpha_j^2$ and variance $\alpha_j^2 N_0/2$. Therefore $\boldsymbol{\ell}$ is a Gaussian random variable whose mean is $m = -\sqrt{E_b'}\sum_{j=1}^{N}\alpha_j^2$ and variance $\sigma^2 = \frac{N_0}{2}\sum_{j=1}^{N}\alpha_j^2$. Therefore,

$$P[\text{error}|0_T,\overrightarrow{\boldsymbol{\alpha}}=\overrightarrow{\alpha}] = \frac{1}{\sqrt{2\pi}\sigma}\int_0^{\infty}\mathrm{e}^{-(\ell+m)^2/2\sigma^2}\mathrm{d}\ell = Q\left(\frac{m}{\sigma}\right) = Q\left(\sqrt{\frac{2E_b'}{N_0}}\sqrt{\sum_{j=1}^{N}\alpha_j^2}\right).$$

But, as mentioned, the above error expression is for a specific set of α_j's and we view the α_j's as random. Therefore the error expression should be averaged over all possible values of the α_j's with a weighting provided by the joint pdf of the $\boldsymbol{\alpha}_j$'s, i.e., $f_{\overrightarrow{\boldsymbol{\alpha}}}(\overrightarrow{\alpha})$. Though this can be done, here we take a (slightly) different approach. Namely let $\boldsymbol{\beta} = \sum_{j=1}^{N}\boldsymbol{\alpha}_j^2$ where $f_{\boldsymbol{\alpha}_j}(\alpha_j) = \frac{2\alpha_j}{\sigma_F^2}\mathrm{e}^{-\alpha_j^2/\sigma_F^2}u(\alpha_j)$ are the statistically independent Rayleigh random variables. Now recognize that each $\boldsymbol{\alpha}_j^2$ can be represented as a sum of squares of two i.i.d. Gaussian random variables. As such, $\boldsymbol{\beta}$ is essentially a sum of squares of $2N$ i.i.d. Gaussian random variables.

The pdf of $\boldsymbol{\beta}$ can be readily determined from Eqn. (10.90), i.e.,

$$f_{\boldsymbol{\beta}}(\beta) = \frac{1}{\sigma^{2N}(N-1)!}\beta^{N-1}\mathrm{e}^{-\beta/\sigma_F^2}u(\beta).$$

Therefore

$$\begin{aligned} P[\text{error}] &= P[\text{error}|0_T] = \int_{\beta=0}^{\infty} Q\left(\sqrt{\frac{2E_b'}{N_0}\beta}\right) f_{\boldsymbol{\beta}}(\beta)\mathrm{d}\beta \\ &= \frac{1}{\sigma^{2N}}\int_{\beta=0}^{\infty} Q\left(\sqrt{\frac{2E_b'}{N_0}\beta}\right)\frac{\beta^{N-1}}{(N-1)!}\mathrm{e}^{-\beta/\sigma_F^2}\mathrm{d}\beta \end{aligned}$$

By changing variables $x = \frac{E_b'}{N_0}\beta$, the above expression can be rewritten as

$$P[\text{error}] = \int_0^{\infty} Q\left(\sqrt{2x}\right)\frac{x^{N-1}\mathrm{e}^{-x/\text{SNR}}}{\text{SNR}^N(N-1)!}\mathrm{d}x$$

where $\text{SNR} = \frac{E_b'\sigma_F^2}{N_0}$. Now $Q\left(\sqrt{2x}\right) = \frac{1}{\sqrt{2\pi}}\int_{y=\sqrt{2x}}^{\infty}\mathrm{e}^{-y^2/2}\mathrm{d}y$ and therefore

$$P[\text{error}] = \int_{x=0}^{\infty}\left\{\frac{1}{\sqrt{2\pi}}\int_{y=\sqrt{2x}}^{\infty}\mathrm{e}^{-y^2/2}\mathrm{d}y\right\}\frac{x^{N-1}\mathrm{e}^{-x/\text{SNR}}}{\text{SNR}^N(N-1)!}\mathrm{d}x$$

The above integral essentially finds the volume under the 2-dimensional function (surface) of $g(x,y) = \frac{1}{\sqrt{2\pi}(N-1)!\text{SNR}^N}x^{N-1}\mathrm{e}^{-x/\text{SNR}}\mathrm{e}^{-y^2/2}$ in the region illustrated as in Fig. 10.3.

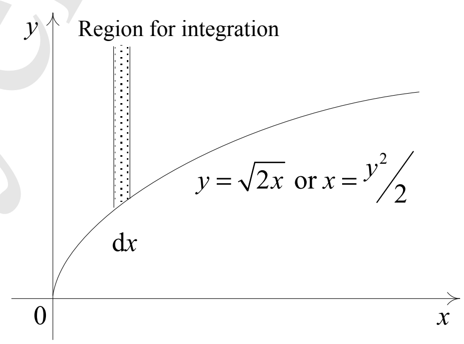


Figure 10.3

The key to performing the integration is to change the order of integration, i.e., sweep first w.r.t. x and then w.r.t. y as illustrated in Fig. 10.4.

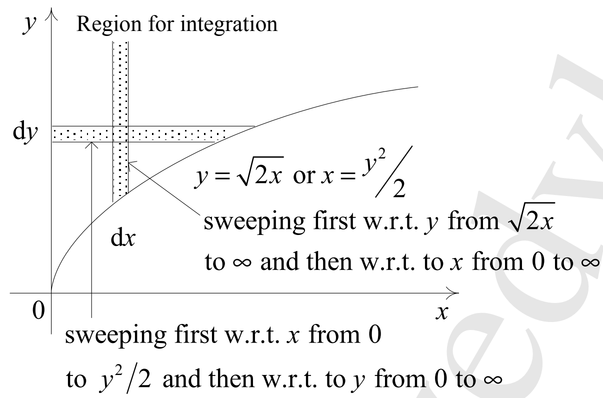


Figure 10.4

Therefore:

$$P[\text{error}] \;=\; \int_{y=0}^{\infty}\left\{\frac{1}{\sqrt{2\pi}}\int_{x=0}^{\frac{y^2}{2}}\frac{x^{N-1}\mathrm{e}^{-x/\mathrm{SNR}}}{\mathrm{SNR}^N(N-1)!}\mathrm{d}x\right\}\mathrm{e}^{-y^2/2}\mathrm{d}y$$

Consider the inner integral $\int_{x=0}^{\frac{y^2}{2}} x^{N-1}\mathrm{e}^{-x/\mathrm{SNR}}\mathrm{d}x$, which looks like an incomplete Gamma function, and from G&R, page 317, Eqn. 3.381.1 with $u = \frac{y^2}{2}$, $\nu = N$, $\mu = \frac{1}{\mathrm{SNR}}$ we get the integral to be $\left(\frac{1}{\mathrm{SNR}}\right)^{-N}\gamma\left(N, \frac{y^2}{2\mathrm{SNR}}\right)$ where $\gamma(\cdot,\cdot)$ is the incomplete Gamma function. Therefore:

$$P[\text{error}] = \frac{1}{(N-1)!}\frac{1}{\sqrt{2\pi}}\int_{y=0}^{\infty}\mathrm{e}^{-y^2/2}\gamma\left(N, \frac{y^2}{2\mathrm{SNR}}\right)\mathrm{d}y$$

Using the relationship from G&R, page 940, Eqn. 8.352.1 with $1 + n \equiv N$, $x \equiv \frac{y^2}{2\mathrm{SNR}}$ the incomplete Gamma function becomes:

$$\gamma\left(N, \frac{y^2}{2\mathrm{SNR}}\right) = (N-1)!\left\{1 - \mathrm{e}^{-\frac{y^2}{2\mathrm{SNR}}}\sum_{k=0}^{N-1}\frac{y^{2k}}{2^k\mathrm{SNR}^k k!}\right\}$$

Therefore the error probability can be written as:

$$P[\text{error}] = \frac{1}{\sqrt{2\pi}}\int_{y=0}^{\infty}\mathrm{e}^{\frac{y^2}{2}}\mathrm{d}y - \frac{1}{\sqrt{2\pi}}\int_{y=0}^{\infty}\mathrm{e}^{-\left(\frac{1}{2}+\frac{1}{2\mathrm{SNR}}\right)y^2}\sum_{k=0}^{N-1}\frac{y^{2k}}{2^k\mathrm{SNR}^k k!}\mathrm{d}y$$

Now $\frac{1}{\sqrt{2\pi}}\int_{y=0}^{\infty}\mathrm{e}^{\frac{y^2}{2}}\mathrm{d}y = \frac{1}{2}$ (half the area of a zero-mean, unit variance Gaussian pdf) and changing variables in the 2nd integral, $\lambda \equiv \frac{y^2}{2}$, simplifying, etc., we get

$$P[\text{error}] = \frac{1}{2} - \frac{1}{2\sqrt{\pi}}\sum_{k=0}^{N-1}\frac{1}{\mathrm{SNR}^k k!}\int_{\lambda=0}^{\infty}\lambda^{k-\frac{1}{2}}\mathrm{e}^{-\left(\frac{\mathrm{SNR}+1}{\mathrm{SNR}}\right)\lambda}\mathrm{d}\lambda$$

The integral looks now like a complete Gamma function and from G&R, page 317, Eqn. 3.381.4 with $\gamma - 1 \equiv k - \frac{1}{2} \Rightarrow \gamma = k + \frac{1}{2}$, $\mu = \frac{\text{SNR}+1}{\text{SNR}}$ we see that the integral is $\frac{1}{\left(\frac{\text{SNR}+1}{\text{SNR}}\right)^{k+\frac{1}{2}}}\Gamma\left(k+\frac{1}{2}\right)$ where $\Gamma(\cdot)$ is the complete Gamma function.

Now $\Gamma\left(k+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}(2k-1)!!$ (Notation: $(2k-1)!! = 1\cdot 3\cdot 5\cdots(2k-3)\cdot(2k-1)$). Using this relationship and after some algebra, one gets:

$$P[\text{error}] = \frac{1}{2} - \frac{1}{2}\sum_{k=0}^{N-1}\frac{(2k-1)!!}{k!2^k\text{SNR}^k}\left(\frac{\text{SNR}}{\text{SNR}+1}\right)^{k+\frac{1}{2}}$$

Define parameter $\mu \equiv \sqrt{\frac{\text{SNR}}{\text{SNR}+1}} \Rightarrow \text{SNR} = \frac{\mu^2}{1-\mu^2}$ and $\left(\frac{\text{SNR}}{\text{SNR}+1}\right)^{k+\frac{1}{2}} = \left(\mu^2\right)^{k+\frac{1}{2}} = \mu^{2k+1}$.

In terms of μ, the error probability is

$$P[\text{error}] = \frac{1}{2} - \frac{1}{2}\sum_{k=0}^{N-1}\frac{(2k-1)!!}{k!2^k}\left(\frac{1-\mu^2}{\mu^2}\right)^k\mu^{2k+1}$$

Now $\left(\frac{1-\mu^2}{\mu^2}\right)^k = \left(\frac{1-\mu}{2}\right)^k\left(\frac{1+\mu}{2}\right)^k 2^k 2^k \frac{1}{\mu^{2k}}$. Therefore

$$P[\text{error}] = \frac{1}{2} - \frac{1}{2}\sum_{k=0}^{N-1}\frac{(2k-1)!!}{k!}2^k\left(\frac{1-\mu}{2}\right)^k\left(\frac{1+\mu}{2}\right)^k\mu$$

Write $\frac{(2k-1)!!}{k!} = \frac{1\cdot3\cdot5\cdots(2k-3)\cdot(2k-1)}{k!}\times\frac{2\cdot4\cdot6\cdots(2k-2)\cdot(2k)}{2\cdot4\cdot6\cdots(2k-2)\cdot(2k)}$. But $2\cdot4\cdot6\cdots(2k-2)\cdot(2k) = 2^k k! \Rightarrow \frac{(2k-1)!!}{k!} = \frac{(2k)!}{k!2^k k!} = \frac{1}{2^k}\binom{2k}{k}$. Finally,

$$P[\text{error}] = \frac{1}{2} - \frac{1}{2}\sum_{k=0}^{N-1}\binom{2k}{k}\left(\frac{1-\mu}{2}\right)^k\left(\frac{1+\mu}{2}\right)^k\mu \tag{10.7}$$

except we want to write it in the form of (P10.1)!

We next show that (10.7) is indeed the same as (P10.1) by induction. This means that we first confirm that (10.7) and (P10.1) are the same for $N = 1$ (quite obvious) and assume that they also the same for $N = L$, i.e.,

$$\frac{1}{2}-\frac{1}{2}\sum_{k=0}^{L-1}\binom{2k}{k}\left(\frac{1-\mu}{2}\right)^k\left(\frac{1+\mu}{2}\right)^k\mu = \left(\frac{1-\mu}{2}\right)^L\left[\frac{1}{2}\sum_{m=0}^{L-1}\binom{L-1+m}{m}\left(\frac{1+\mu}{2}\right)^m\right]. \tag{10.8}$$

Then we need to show that (10.7) and (P10.1) are indeed the same for $N = L+1$. This is accomplished with the following manipulations:

$$\begin{aligned}&\frac{1}{2}\left[1-\sum_{k=0}^{L}\binom{2k}{k}\left(\frac{1-\mu}{2}\right)^k\left(\frac{1+\mu}{2}\right)^k\mu\right]\\ =\ &\frac{1}{2}\left[1-\sum_{k=0}^{L-1}\binom{2k}{k}\left(\frac{1-\mu}{2}\right)^k\left(\frac{1+\mu}{2}\right)^k\mu\right]-\frac{1}{2}\binom{2L}{L}\left(\frac{1-\mu}{2}\right)^L\left(\frac{1+\mu}{2}\right)^L\mu\end{aligned}$$

Using the assumption of (10.8) this can be written as (where the term $\left(\frac{1-\mu}{2}\right)^L$ is factored out):

$$\left(\frac{1-\mu}{2}\right)^L \left[\sum_{m=0}^{L-1} \binom{L-1+m}{m} \left(\frac{1+\mu}{2}\right)^m - \frac{1}{2}\binom{2L}{L}\left(\frac{1+\mu}{2}\right)^L \mu\right] \tag{10.9}$$

Now

$$\begin{aligned}\binom{L-1+m}{m} &= \frac{(L-1+m)!}{m!(L-1)!}\frac{L+m}{L}\frac{L}{L+m} = \frac{(L+m)!}{m!L!}\frac{L}{L+m} \\ &= \binom{L+m}{m}\left(1-\frac{m}{L+m}\right)\end{aligned}$$

Therefore (10.9) becomes

$$\begin{aligned}&\left(\frac{1-\mu}{2}\right)^L \left[\sum_{m=0}^{L-1} \binom{L+m}{m} \left(\frac{1+\mu}{2}\right)^m - \sum_{m=0}^{L-2} \binom{L+m}{m} \left(\frac{1+\mu}{2}\right)^{m+1}\right. \\ &\left. -\frac{1}{2}\binom{2L}{L}\left(\frac{1+\mu}{2}\right)^L \mu\right]\end{aligned} \tag{10.10}$$

where

$$\begin{aligned}\sum_{m=0}^{L-1} \frac{m}{L+m}\binom{L+m}{m}\left(\frac{1+\mu}{2}\right)^m &\overset{k=m-1}{=} \sum_{k=-1}^{L-2} \frac{k+1}{L+k+1}\binom{L+k+1}{k+1}\left(\frac{1+\mu}{2}\right)^{k+1} \\ &= \sum_{k=0}^{L-2} \frac{k+1}{L+k+1}\binom{L+k+1}{k+1}\left(\frac{1+\mu}{2}\right)^{k+1}\end{aligned}$$

But

$$\frac{k+1}{L+k+1}\binom{L+k+1}{k+1} = \frac{k+1}{L+k+1}\frac{(L+k+1)!}{L!(k+1)!} = \frac{(L+k)!}{L!k!} = \binom{L+k}{k}$$

Changing the dummy index k to m we get (10.10) in the following form:

$$\begin{aligned}&\left(\frac{1-\mu}{2}\right)^L \left[\sum_{m=0}^{L-1} \binom{L+m}{m} \left(\frac{1+\mu}{2}\right)^m - \sum_{m=0}^{L-1} \binom{L+m}{m} \left(\frac{1+\mu}{2}\right)^{m+1}\right. \\ &\left. +\frac{1}{2}\binom{2L}{L}\left(\frac{1+\mu}{2}\right)^L - \frac{1}{2}\binom{2L}{L}\left(\frac{1+\mu}{2}\right)^L \mu\right]\end{aligned} \tag{10.11}$$

where the term $\frac{1}{2}\binom{2L}{L}\left(\frac{1+\mu}{2}\right)^L$ is added and subtracted.
Since, obviously

$$\left(\frac{1+\mu}{2}\right)^{m+1} = \left(\frac{1+\mu}{2}\right)^m\left(\frac{1}{2}+\frac{\mu}{2}\right) = \frac{1}{2}\left(\frac{1+\mu}{2}\right)^m + \frac{\mu}{2}\left(\frac{1+\mu}{2}\right)^m$$

(10.11) can be written as

$$\left(\frac{1-\mu}{2}\right)^{L}\left[\frac{1}{2}\sum_{m=0}^{L-1}\binom{L+m}{m}\left(\frac{1+\mu}{2}\right)^{m}-\frac{\mu}{2}\sum_{m=0}^{L-1}\binom{L+m}{m}\left(\frac{1+\mu}{2}\right)^{m}\right.$$
$$\left.+\quad\frac{1}{2}\binom{2L}{L}\left(\frac{1+\mu}{2}\right)^{L}-\frac{1}{2}\binom{2L}{L}\left(\frac{1+\mu}{2}\right)^{L}\mu\right] \tag{10.12}$$

Gathering terms appropriately (10.13) becomes

$$\left(\frac{1-\mu}{2}\right)^{L}\left[\frac{1}{2}\sum_{m=0}^{L}\binom{L+m}{m}\left(\frac{1+\mu}{2}\right)^{m}-\frac{\mu}{2}\sum_{m=0}^{L}\binom{L+m}{m}\left(\frac{1+\mu}{2}\right)^{m}\right],$$

which (finally) equals:

$$\left(\frac{1-\mu}{2}\right)^{L+1}\left[\frac{1}{2}\sum_{m=0}^{L}\binom{L+m}{m}\left(\frac{1+\mu}{2}\right)^{m}\right] \tag{10.13}$$

(c) With approximations, (10.13) with $L=N-1$ becomes

$$P[\text{error}]\approx\left(\frac{1}{4\text{SNR}}\right)^{N}\sum_{k=0}^{N-1}\binom{N-1+k}{k}1^{k}=\frac{1}{4^{N}}\binom{2N-1}{N}\frac{1}{\text{SNR}^{N}}$$

The error probability decreases as the Nth power of the signal-to-noise ratio. N is usually called the diversity gain (or order) of the system. Note that this performance behavior is the same as with noncoherent FSK. But there is a 3 dB saving in power but at the expense of a more complicated receiver since the phase and attenuation need to be estimated.

P10.6 **To be added**.

P10.7 (a) A constant does not vary very much, indeed not at all. Therefore its variance is equal to 0. Further the expected or average value of a constant is the constant itself[1]. Therefore the amount of fading is AF $=0/\text{constant}=0$.

(b) $\text{var}\{\boldsymbol{\alpha}^2\}=E\{\boldsymbol{\alpha}^4\}-E^2\{\boldsymbol{\alpha}^2\}$. $E\{\boldsymbol{\alpha}^2\}=\frac{2}{\sigma_F^2}\int_0^\infty\alpha^3\mathrm{e}^{-\alpha^2/\sigma_F^2}\mathrm{d}\alpha$ and after a change of variable this becomes $\sigma_F^2\int_0^\infty\lambda\mathrm{e}^{-\lambda}\mathrm{d}\lambda=\sigma_F^2$. $E\{\boldsymbol{\alpha}^4\}=\frac{2}{\sigma_F^2}\int_0^\infty\alpha^5\mathrm{e}^{-\alpha^2/\sigma_F^2}\mathrm{d}\alpha$ and with the same change of variable this becomes $\sigma_F^4\int_0^\infty\lambda^2\mathrm{e}^{-\lambda}\mathrm{d}\lambda=2\sigma_F^4$. Therefore $\text{AF}=\frac{2\sigma_F^4-\sigma_F^4}{(\sigma_F^2)^2}=1$.

(c) First find a general expression for $E\{\boldsymbol{\alpha}^n\}$ and then evaluate it for $n=4$ and $n=2$.

$$E\{\boldsymbol{\alpha}^n\}=\frac{2m^m}{\Gamma(m)\sigma^{2m}}\int_0^\infty\alpha^{n+2m-1}\mathrm{e}^{-m\alpha^2/\sigma^2}\mathrm{d}\alpha.$$

[1]For an anecdote about a totally absurd proof that the expected value of a constant is the constant, contact the second author.

Let $\lambda \to \frac{m\alpha^2}{\sigma^2}$. Then

$$E\{\boldsymbol{\alpha}^n\} = \frac{m^m \sigma^{n+2m-2}}{\Gamma(m)\sigma^{2m}\left(m^{\frac{n+2m-2}{2}}\right)m} \int_0^\infty \lambda^{\frac{n+2m-2}{2}} \mathrm{e}^{-\lambda} \mathrm{d}\alpha = \frac{\sigma^n}{\Gamma(m)m^{n/2}} \Gamma\left(m + \frac{n}{2}\right)$$

For $n = 2$,

$$E\{\boldsymbol{\alpha}^2\} = \frac{\sigma^2}{\Gamma(m)m}\Gamma(m+1) = \frac{\sigma^2 m \Gamma(m)}{\Gamma(m)m} = \sigma^2$$

For $n = 4$,

$$E\{\boldsymbol{\alpha}^4\} = \frac{\sigma^4}{\Gamma(m)m^2}\Gamma(m+2) = \frac{\sigma^4(m+1)}{m} = \sigma^4 + \frac{\sigma^4}{m}$$

Therefore AF $= \frac{1}{m}$. Note that when $m = 1$ the Nakagami distribution is Rayleigh and as $m \to \infty$, it tends to a Gaussian pdf. Plots of the Nakagami-m pdf for various values of m are shown in Fig. 10.5.

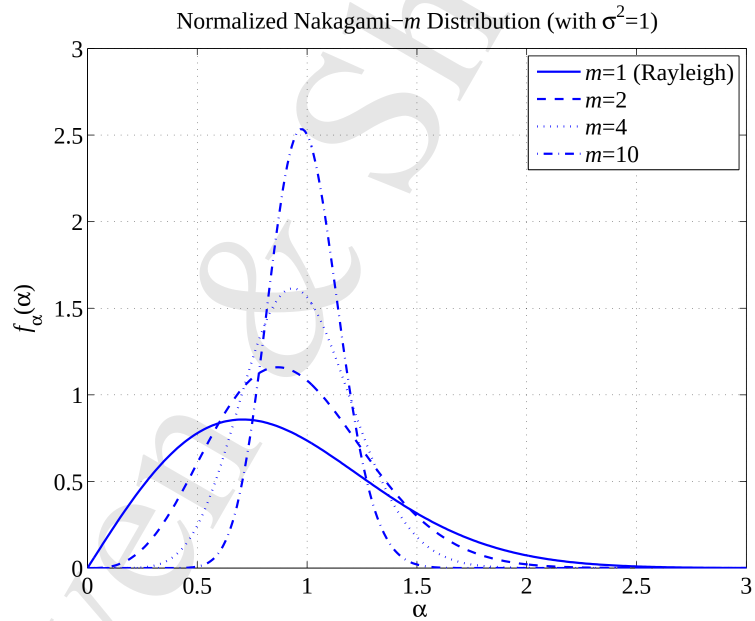


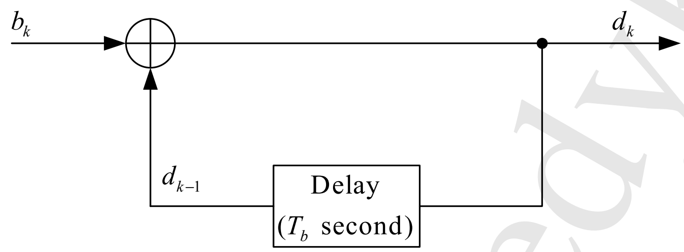
Figure 10.5

P10.8 (a) If the phase is π radians then the sign(s) of the transmitted signal is reversed, i.e., a bit zero is represented by a signal pattern where the transmitted signal in the $(k-1)$th bit interval and kth bit interval are the same: either $(-,-)$ or $(+,+)$. On the other hand, for bit one the pattern is $(+,-)$ or $(-,+)$. In essence, nothing changes.

(b) Start by observing that we want $d_k = d_{k-1}$ if the present bit $b_k = 0$ and $d_k = \overline{d}_{k-1}$ if $b_k = 1$. The truth table for d_k, d_{k-1}, b_k is

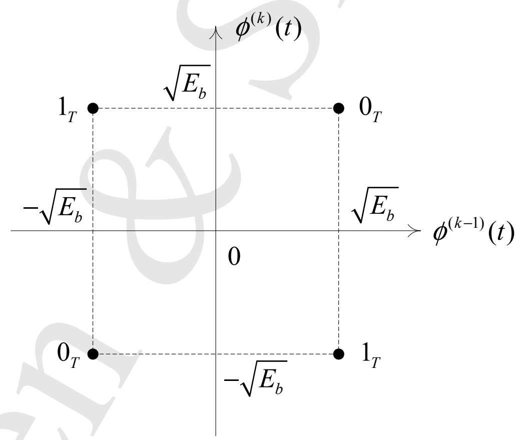
d_{k-1}	b_k	d_k
0 (π rad)	0	0 (π rad)
0 (π rad)	1	0 (0°)
1 (0°)	0	1 (0°)
1 (0°)	1	0 (π rad)

From the truth table one sees (easily) that $d_k = d_{k-1} \oplus b_k$. In block diagram form this means



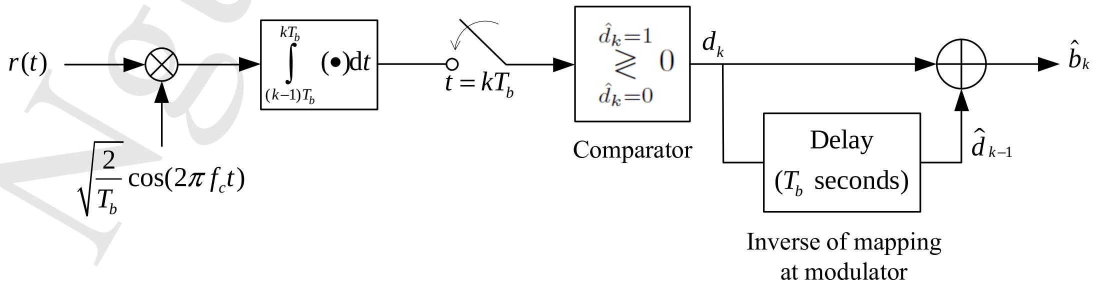
Remark: The above is what is discussed in Chapter 6, Section 6.6, differential modulation, pages 251-252.

(c) Let $\phi^{(k-1)}(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t)$ be the orthogonal basis for the $(k-1)$th bit interval and $\phi^{(k)}(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t)$ for the kth bit interval. Then the signal space plot of the signal set of (10.40) becomes



It is identical (perhaps one should say isomorphic) to the signal space plot for Miller modulation.

(d) Based on (b), consider the following coherent demodulation: first demodulation to $\hat{d}_k$ and then to $\hat{b}_k$. The block diagram looks as follows:



Of course it is possible to demodulate by projecting $r(t)$ onto $\{\phi^{(k-1)}(t), \phi^k(t)\}$ (the signal space of bit d_k) and choosing the closest signal point. But this would require two matched filters (or two correlators).

(e) First, observe that $P[\text{error}] = P[\text{error}|0_T]$. Second, note that $\hat{b}_k$ is in error if (i) $\hat{d}_{k-1}$ is in error and $\hat{d}_k$ is correct or (ii) if $\hat{d}_{k-1}$ is correct and $\hat{d}_k$ is in error, i.e., if both $\hat{d}_k$, $\hat{d}_{k-1}$ are correct then $\hat{b}_k$ is correct and if both $\hat{d}_k$, $\hat{d}_{k-1}$ are incorrect then $\hat{b}_k$ is also correct (who said "two wrongs do not make a right").

Since the events $\hat{d}_{k-1}$, $\hat{d}_k$ are statistically independent (remember noise is white and Gaussian) and the 2 events (i), (ii) above are mutually exclusive, one has

$$P[\text{error}] = 2Q\left(\frac{\sqrt{E_b}}{\sqrt{N_0/2}}\right)\left[1 - Q\left(\frac{\sqrt{E_b}}{\sqrt{N_0/2}}\right)\right] = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\left[1 - Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right].$$

Plots of the error performance for this phase uncertainty model and that of (10.48) are shown in Fig. 10.6. Observe that coherent demodulation saves about 0.5 dB in E_b/N_0.

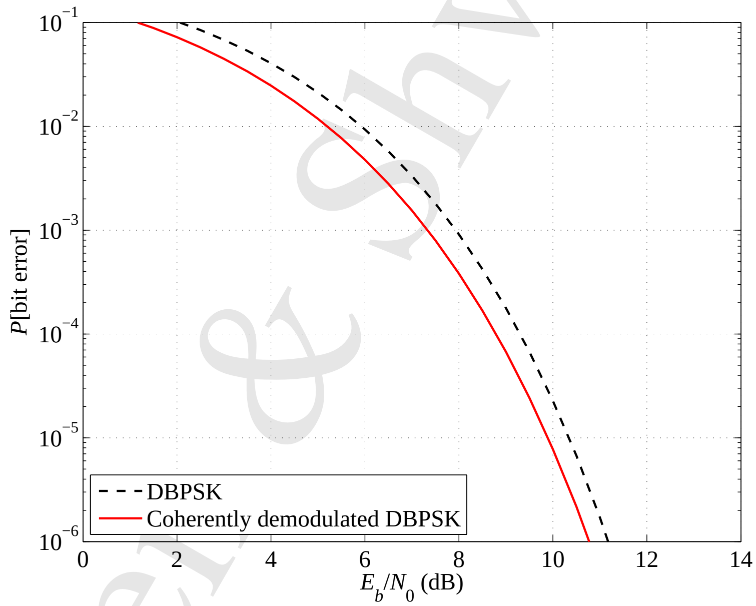


Figure 10.6

(f) CDDBPSK perhaps.

Remark: Because of the memory introduced by the differential mapping one can determine a state diagram, a trellis and use Viterbi's algorithm to perform a sequence demodulation. Left as exercise or convince yourself that Fig. 6.19 is applicable here.

P10.9 (a) One possible relative phase change, $\Delta\phi$, mapping is as follows:

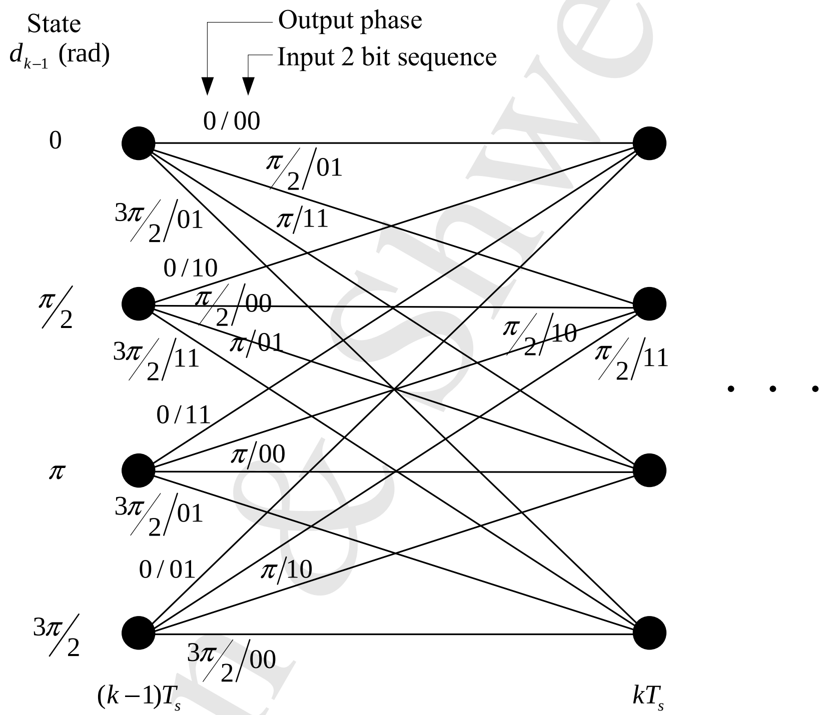
$$\begin{aligned} 00 &\rightarrow \Delta\phi = 0 \quad \text{(radians)} \\ 01 &\rightarrow \Delta\phi = \frac{\pi}{2} \\ 11 &\rightarrow \Delta\phi = \pi \\ 10 &\rightarrow \Delta\phi = \frac{3\pi}{2} \left(\text{or} -\frac{\pi}{2}\right) \end{aligned}$$

(b) Since there is memory, let us assume an initial condition of 0 radians. Parse the sequence into 2 bit sequences and determine the transmitted phase sequence.

	01	11	11	00	01	10	10	...	
0	$\frac{\pi}{2}$	$\frac{3\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	π	$\frac{\pi}{2}$	0	...	(transmitted phase)
$\uparrow$									
initial condition									

As can be seen and as expected, the absolute transmitted phase has no meaning. It is the relative (or differential) phase that conveys the information.

(c) The trellis looks as shown below, where the state is defined as the previous transmitted phase. Note that the trellis is fully developed after one T_s interval.



(d) Because of the memory of one symbol interval we shall look at the signal in the previous symbol interval along with the present symbol interval. A signal in a symbol interval lies in 2 dimensions, that of $\{\sqrt{\frac{2}{T_b}}\cos(2\pi f_c t), \sqrt{\frac{2}{T_b}}\sin(2\pi f_c t)\}$, which means 4 dimensions are needed to represent the signal over 2 symbol intervals. Along each dimension the signal component takes on one of 2 possible value ($\pm\sqrt{E_b}$). Therefore there are $2^4 = 16$ possible signals lying in the 4-dimensional signal space. Each signal lies in one of the 16 quadrants, indeed each quadrant has only 1 signal lying in it.

Remark: More detail as to the actual signal points is provided in the next problem.

(e) Start by considering the following "truth table", perhaps it is more appropriately called a desired table.

$b_{k,Q}$	$b_{k,I}$						$d_{k,Q}$	$d_{k,I}$
0	0	$\Rightarrow$	Phase stays the same as the previous phase			$\Rightarrow$	$d_{k-1,Q}$	$d_{k-1,I}$
0	1	$\Rightarrow$	Phase changes by $+\frac{\pi}{2}$ radians			$\Rightarrow$	Now how $d_{k,Q}$, $d_{k,I}$ behave depends on the bit pattern $d_{k-1,Q}$, $d_{k-1,I}$	
				$d_{k-1,Q}$	$d_{k-1,I}$			
				1	1		$d_{k-1,Q}$	$\overline{d_{k-1,I}}$
				0	0		$d_{k-1,Q}$	$\overline{d_{k-1,I}}$
				1	0		$\overline{d_{k-1,Q}}$	$d_{k-1,I}$
				0	1		$\overline{d_{k-1,Q}}$	$d_{k-1,I}$
1	0	$\Rightarrow$	Phase changes by $-\frac{\pi}{2}$ radians			$\Rightarrow$	Again change in $d_{k,Q}$, $d_{k,I}$ depends on the bit pattern $d_{k-1,Q}$, $d_{k-1,I}$	
				$d_{k-1,Q}$	$d_{k-1,I}$			
				1	1		$\overline{d_{k-1,Q}}$	$d_{k-1,I}$
				0	0		$\overline{d_{k-1,Q}}$	$d_{k-1,I}$
				1	0		$d_{k-1,Q}$	$\overline{d_{k-1,I}}$
				0	1		$d_{k-1,Q}$	$\overline{d_{k-1,I}}$
1	1	$\Rightarrow$	Phase changes by $+\pi$ radians			$\Rightarrow$	$\overline{d_{k-1,Q}}$	$\overline{d_{k-1,I}}$

Remark: To see the above relationships between the $\{d_{k,Q}, d_{k,I}\}$ bits and the $\{d_{k-1,Q}, d_{k-1,I}\}$ bits, especially for the 01, 10 bit patterns for $b_{k,Q}, b_{k,I}$ you may wish to sketch a few (I, Q) signal space diagrams.

Now to form the Boolean expressions. Let us first consider the inphase bit $d_{k,I}$. The formation shall not proceed by formal logic design procedures but more by intuition, in other words by (hopefully) intelligent trial and error.

The first observation is that the pattern for the $d_{k,I}$ bit is similar for two subsets of the $b_{k,Q}, b_{k,I}$ bit pattern, namely $(00, 11)$ and $(01, 10)$. So let us start with the expression $b_{k,Q} \oplus b_{k,I}$ which would create a selector bit of 0 or 1, respectively for the two subsets. Consider subset $(00, 11)$. Expression $\overline{b_{k,Q} \oplus b_{k,I}}$ shall be equal to 1 in this case. Note that $b_{k,I} \oplus d_{k-1,I}$ has as its output $d_{k-1,I}$ when $b_{k,I} = 0$ and $\overline{d_{k-1,I}}$ when $b_{k,I} = 1$. So AND this with $\overline{b_{k,Q} \oplus b_{k,I}}$ and the subset (00,11) is accounted for. Thus far we have $d_{k,I} = (\overline{b_{k,Q} \oplus b_{k,I}})$ AND $(b_{k,I} \oplus d_{k-1,I})$.

To account for the 2nd subset, OR the above expression with an expression to be determined for the 2nd subset. Since when the 2nd subset is relevant, the above expression is 0, we can develop this expression independently.
To see what this expression is, consider the table below where just $d_{k,I}$ is considered and the Boolean expressions $(b_{k,Q} \oplus d_{k-1,Q})$ and $(b_{k,I} \oplus d_{k-1,I})$ are also shown.

$b_{k,Q}$	$b_{k,I}$	$d_{k-1,Q}$	$d_{k-1,I}$	$d_{k,I}$	$(b_{k,Q} \oplus d_{k-1,Q})$	$(b_{k,I} \oplus d_{k-1,Q})$
0	1	1	1	$d_{k,I} = \overline{d_{k-1,I}} = 0$	1	0
		0	0	$d_{k,I} = \overline{d_{k-1,I}} = 1$	0	1
		1	0	$d_{k,I} = d_{k-1,I} = 0$	1	0
		1	1	$d_{k,I} = d_{k-1,I} = 1$	0	1
1	0	1	1	$d_{k,I} = d_{k-1,I} = 1$	0	1
		0	0	$d_{k,I} = d_{k-1,I} = 0$	1	0
		1	0	$d_{k,I} = \overline{d_{k-1,I}} = 1$	0	1
		0	1	$d_{k,I} = \overline{d_{k-1,I}} = 0$	1	0

Note that both "track" the $d_{k,I}$ bit, albeit one in a complementary fashion. Therefore form the following Boolean expression for the 2nd subset: $(b_{k,Q} \oplus b_{k,I})$ AND $(\overline{b_{k,Q} \oplus d_{k-1,Q}})$. The final overall expression for the inphase bit is

$$d_{k,I} = \left[(\overline{b_{k,Q} \oplus b_{k,I}}) \text{ AND } (b_{k,I} \oplus d_{k-1,I})\right] \text{ OR } \left[(b_{k,Q} \oplus b_{k,I}) \text{ AND } (\overline{b_{k,Q} \oplus d_{k-1,Q}})\right]$$

Remark: One could also use $(b_{k,I} \oplus d_{k-1,Q})$. Do it? Also convince yourself that $(b_{k,I} \oplus d_{k-1,I})$ or $(b_{k,Q} \oplus d_{k-1,I})$ are not appropriate to be used. For the quadrature bit reason as above, or use "symmetry", or guess, to get

$$d_{k,Q} = \left[(\overline{b_{k,Q} \oplus b_{k,I}}) \text{ AND } (b_{k,Q} \oplus d_{k-1,Q})\right] \text{ OR } \left[(b_{k,Q} \oplus b_{k,I}) \text{ AND } (\overline{b_{k,I} \oplus d_{k-1,Q}})\right]$$

P10.10 (a) There are four basis functions, namely

$$\left.\begin{aligned} \phi_{k,I}(t) &= \sqrt{\frac{2}{T_s}}\cos(2\pi f_c t) \\ \phi_{k,Q}(t) &= \sqrt{\frac{2}{T_s}}\sin(2\pi f_c t) \end{aligned}\right\} \begin{array}{c} {\scriptstyle (k-1)T_s \le t \le kT_s} \\ \text{(present symbol interval)} \end{array}$$

and

$$\left.\begin{aligned} \phi_{k-1,I}(t) &= \sqrt{\frac{2}{T_s}}\cos(2\pi f_c t) \\ \phi_{k-1,Q}(t) &= \sqrt{\frac{2}{T_s}}\sin(2\pi f_c t) \end{aligned}\right\} \begin{array}{c} {\scriptstyle (k-2)T_s \le t \le (k-1)T_s} \\ \text{(previous symbol interval)} \end{array}$$

(b) 2^4, i.e., 16 possible bit patterns.

(c) 2^4 quadrants.

(d) Using Table 10.2 of P10.9

b_{k-3}	b_{k-2}	b_{k-1}	b_k		Previous phase	Present phase
0	0	0	0	i)	$\frac{\pi}{4}$	$\frac{\pi}{4}$
				ii)	$\frac{3\pi}{4}$	$\frac{3\pi}{4}$
				iii)	$\frac{5\pi}{4}$	$\frac{5\pi}{4}$
				iv)	$\frac{7\pi}{4}$	$\frac{7\pi}{4}$

Since the present 2 bits are 00, the present phase is the same as the previous phase, i.e., does not change.

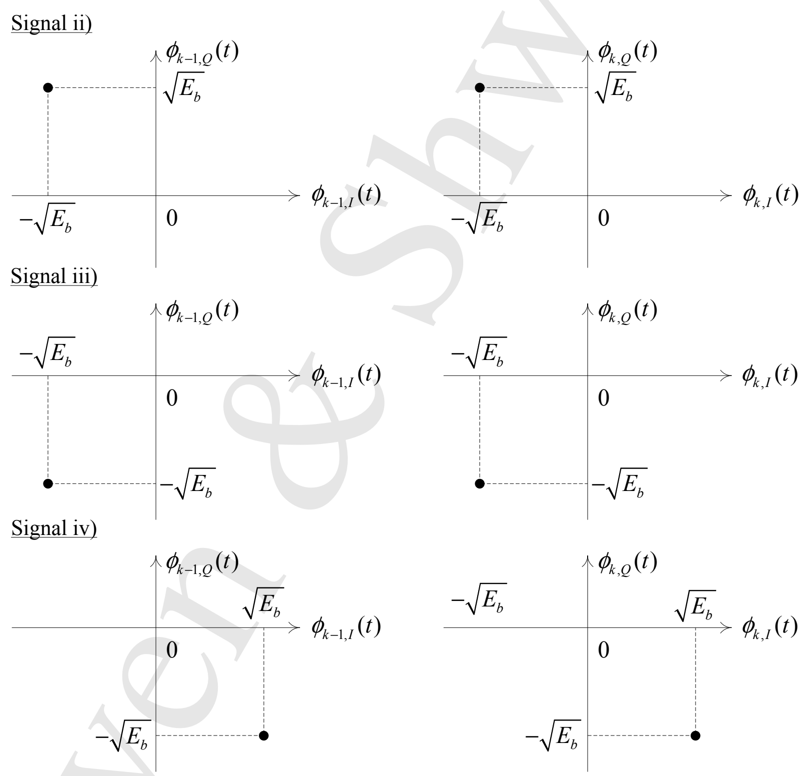


Figure 10.7

To determine which quadrant(s) the signals fall in, the notation needs to be clarified further. Let the components along a specific axis be ± 1 as implied and let the axes be ordered as follows: $(\phi_{k-1,I}, \phi_{k-1,Q}, \phi_{k,I}, \phi_{k,Q})$. Then the quadrants in this notation are:

i) $(+1,+1,+1,+1)$	or if we let	1111	or in decimal,	15
ii) $(-1,+1,-1,+1)$	$-1 \leftrightarrow$ bit 0, $+1 \leftrightarrow$ bit 1	0101	taking the left	5
iii) $(-1,-1,-1,-1)$	then in binary	0000	most bit as being	0
iv) $(+1,-1,+1,-1)$	the quadrants are	1010	the least significant	10

(e) The previous phase can still be any one of the 4 values, $(\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4})$ and since the present 2 bits are 00, which means $\Delta\phi = 0$, the present phase is also one of $(\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4})$ as in (d).

(f)

b_{k-3}	b_{k-2}	b_{k-1}	b_k		Previous phase	Present phase
0	0	0	1	i)	$\frac{\pi}{4}$	$\frac{3\pi}{4}$
				ii)	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$
				iii)	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$
				iv)	$\frac{7\pi}{4}$	$\frac{\pi}{4}$

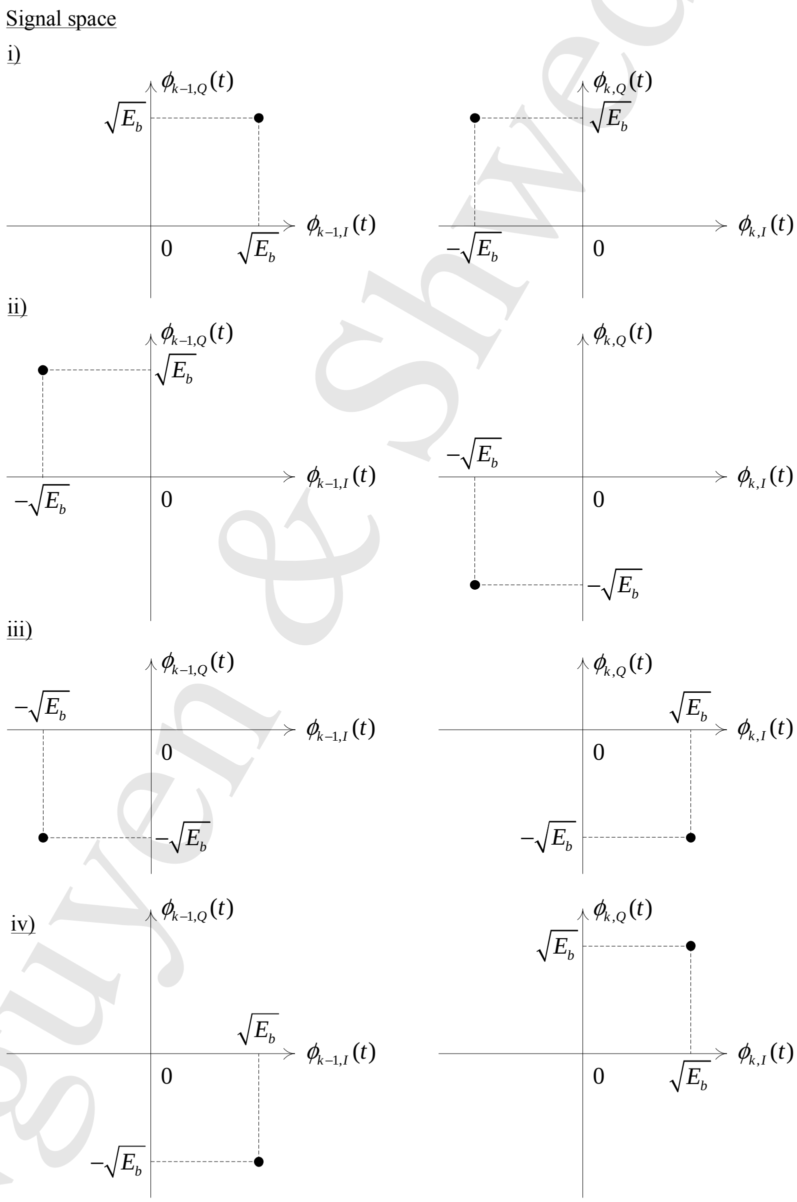


Figure 10.8

The signals fall in the following quadrants

	Quadrant	Binary	Decimal
i)	$(+1,+1,-1,+1)$	1101	13
ii)	$(-1,+1,-1,-1)$	0100	4
iii)	$(-1,-1,+1,-1)$	0010	2
iv)	$(+1,-1,+1,+1)$	1011	11

If the previous bits $b_{k-3}b_{k-2}$ are 01, 10, or 11 we still have the same set of signals. As in (e), the previous phase can still be any of the 4 values and the present phase is a "relative" shift of $\frac{\pi}{4}$ with respect to these 4 values.

Consider now

b_{k-3}	b_{k-2}	b_{k-1}	b_k		Previous phase	Present phase
0	0	1	1	i)	$\frac{\pi}{4}$	$\frac{5\pi}{4}$
				ii)	$\frac{3\pi}{4}$	$\frac{7\pi}{4}$
				iii)	$\frac{5\pi}{4}$	$\frac{\pi}{4}$
				iv)	$\frac{7\pi}{4}$	$\frac{3\pi}{4}$

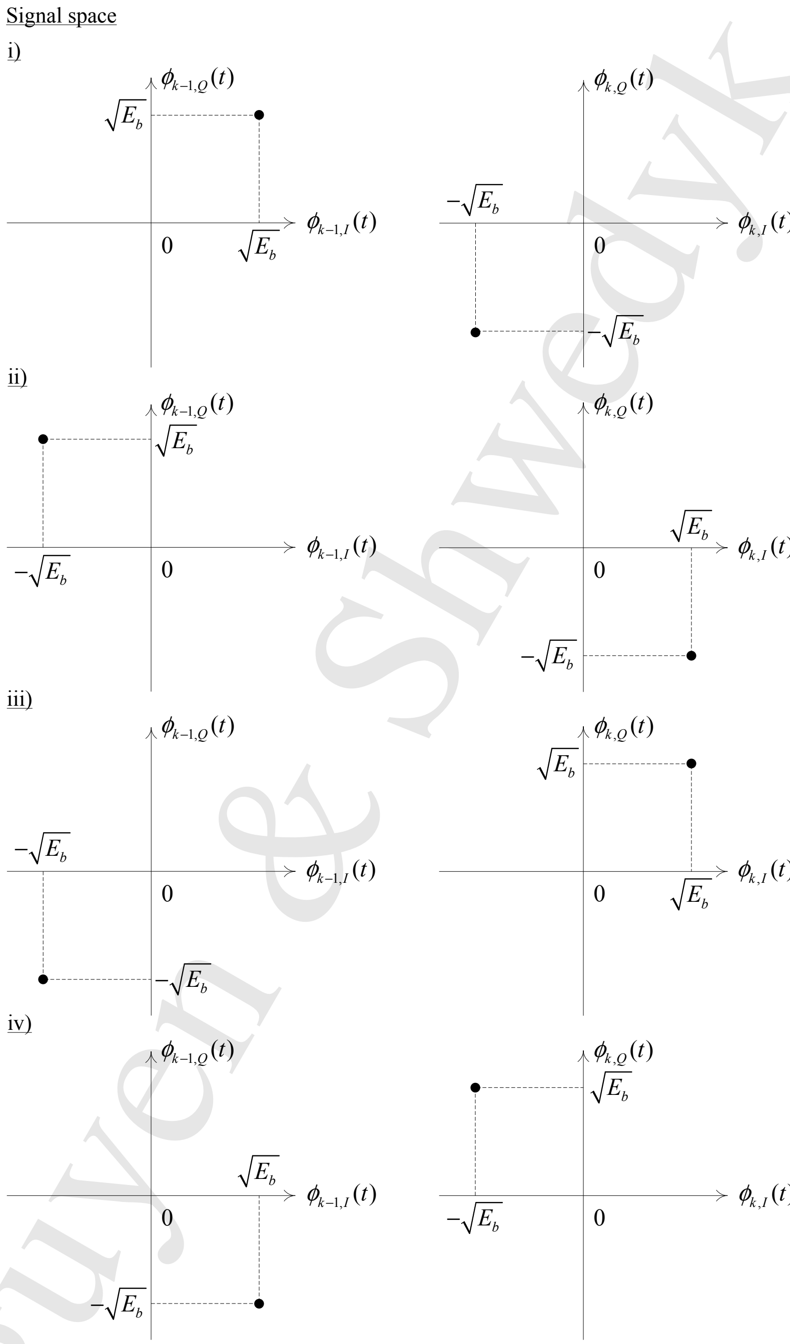


Figure 10.9

Quadrant	Binary	Decimal
i) $(+1,+1,-1,-1)$	1100	12
ii) $(-1,+1,+1,-1)$	0110	6
iii) $(-1,-1,+1,+1)$	0011	3
iv) $(+1,-1,-1,+1)$	1001	9

When $b_{k-3}b_{k-2}$ are 01, 10, or 11, same argument as before, i.e., have the same signal set.

Finally, consider

b_{k-3}	b_{k-2}	b_{k-1}	b_k		Previous phase	Present phase
0	0	1	0	i)	$\frac{\pi}{4}$	$\frac{7\pi}{4}$
				ii)	$\frac{3\pi}{4}$	$\frac{\pi}{4}$
				iii)	$\frac{5\pi}{4}$	$\frac{3\pi}{4}$
				iv)	$\frac{7\pi}{4}$	$\frac{5\pi}{4}$

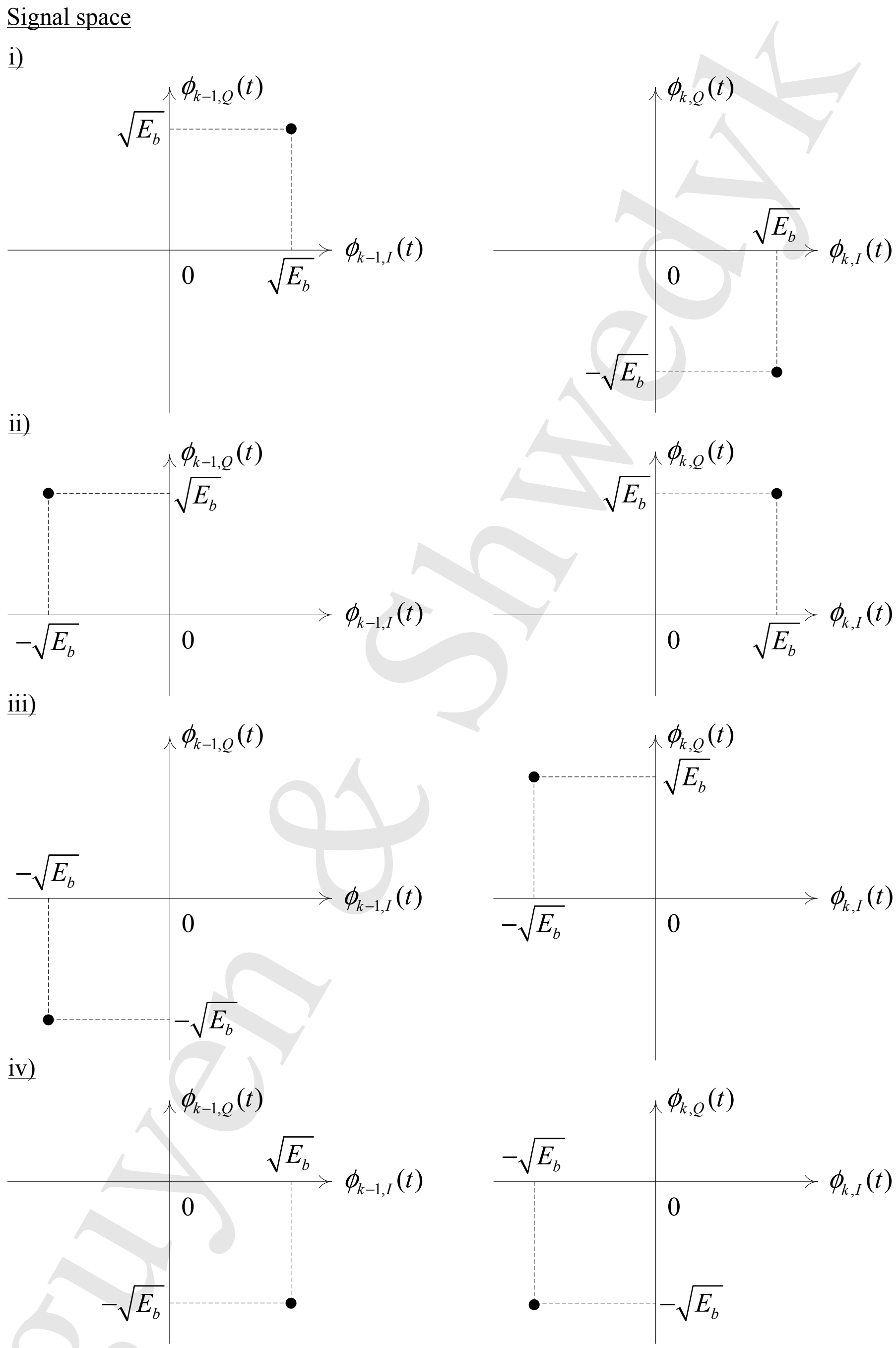


Figure 10.10

Quadrant	Binary	Decimal
i) $(+1,+1,+1,-1)$	1110	14
ii) $(-1,+1,+1,+1)$	0111	7
iii) $(-1,-1,-1,+1)$	0001	1
iv) $(+1,-1,-1,-1)$	1000	8

Again $b_{k-3}b_{k-2} = 01, 10, 11$ results in the same set of signal points for $b_{k-1}b_k = 10$.

(g) To develop the demodulator, observe that the signal points lie as follows for the various possibilities of $b_{k-1}b_k$

$$\begin{array}{ccccc} b_{k-1}b_k & 00 & 01 & 11 & 10 \\ \text{Quadrant} \rightarrow & \begin{pmatrix} 15 \\ 5 \\ 0 \\ 10 \end{pmatrix} & \begin{pmatrix} 13 \\ 4 \\ 2 \\ 11 \end{pmatrix} & \begin{pmatrix} 12 \\ 6 \\ 3 \\ 9 \end{pmatrix} & \begin{pmatrix} 14 \\ 7 \\ 1 \\ 8 \end{pmatrix} \end{array}$$

The observation is that they lie in disjoint quadrants (or disjoint quadrant subsets). Therefore, to make a decision on the symbol $b_{k-1}b_k$ decide which quadrant the received signal lies in, more accurately which quadrant the sufficient statistics lie in, and choose the $b_{k-1}b_k$ symbol that belongs to this quadrant.

The sufficient statistics are the projection of $r(t)$ onto the basis functions $\phi_{k-1,I}(t)$, $\phi_{k-1,Q}(t)$, $\phi_{k,I}(t)$, $\phi_{k,Q}(t)$. The block diagram looks as follows:

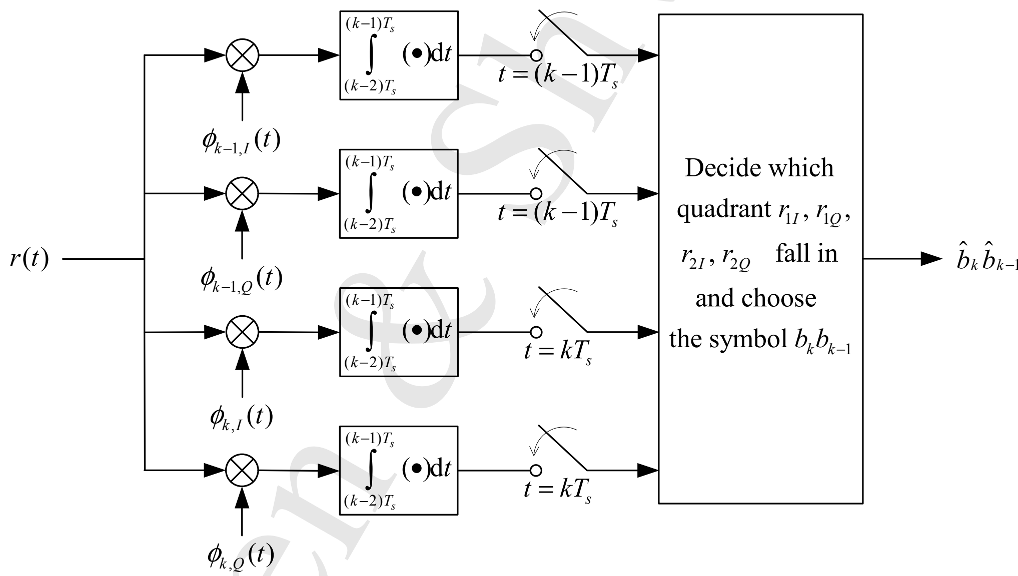


Figure 10.11

Remarks:

(i) The above block diagram is a minimum-distance receiver, i.e., the signal point $r(t)$ is closest to is chosen and the symbol $b_k b_{k-1}$ it corresponds to is decided on. Remember we are in white Gaussian noise.

(ii) As an example quadrant 6 is chosen and symbol $\hat{b}_k\hat{b}_{k-1} = 11$ is decided on if $(r_{1,I} < 0)$ and $(r_{1,Q} > 0)$ and $(r_{2,I} > 0)$ and $(r_{2,Q} < 0)$.

P10.11 From Problems 10.8 and 10.9 let us make the following observations:

(i) The received signal over two symbol intervals can be written as:

$$\mathbf{r}(t) = \pm\sqrt{E_b}\phi_{k-1,I}(t) \pm \sqrt{E_b}\phi_{k-1,Q}(t) \pm \sqrt{E_b}\phi_{k,I}(t) \pm \sqrt{E_b}\phi_{k,Q}(t) + \mathbf{w}(t)$$

Therefore the sufficient statistics $\mathbf{r}_{1,I}, \mathbf{r}_{1,Q}, \mathbf{r}_{2,I}, \mathbf{r}_{2,Q}$ are of the form $\pm\sqrt{E_b} + \mathbf{w}$, i.e., statistically independent Gaussian random variables of mean $\pm\sqrt{E_b}$ and variance $N_0/2$. Note that E_b is readily interpreted as the transmitted energy per bit.

(ii) Because, as usual, the information bits b_k are assumed to be equally probable and statistically independent which leads to symmetry in the transmitted (and received) signal space. We have

$$P[\text{symbol error}] = P[\text{symbol error}|b_k b_{k-1} = 00]$$

Indeed we can refine this further as:

$$P[\text{symbol error}] = P[\text{symbol error}|\text{a specific quadrant signal transmitted}]$$

(iii) Let us choose as a specific quadrant signal the one that corresponds to quadrant 5, or $(-1, +1, -1, +1)$ or the signal $-\sqrt{E_b}\phi_{k-1,I}(t) + \sqrt{E_b}\phi_{k-1,Q}(t) - \sqrt{E_b}\phi_{k,I}(t) + \sqrt{E_b}\phi_{k,Q}(t)$. A symbol error is made if $r_{1,I}, r_{1,Q}, r_{2,I}, r_{2,Q}$ fall in quadrants 1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14.

Note that if $r_{1,I}, r_{1,Q}, r_{2,I}, r_{2,Q}$ fall in quadrants 0, 10, or 15, though an error is made in the transmitted signal, a correct decision is still made on the transmitted information bits $b_k b_{k-1} = 00$.

(iv) So one needs to calculate the volume under $f(r_{1,I}, r_{1,Q}, r_{2,I}, r_{2,Q}|\text{signal in quadrant 5})$ in the quadrants in which a symbol error is made. This can be done (almost) by inspection. Consider the volume in quadrant 2, i.e., quadrant $(-1, -1, +1, -1)$. Note that there is agreement in 2 of the components, in this case specifically $r_{1,I}, r_{2,Q}$ and disagreement in 2 components, namely $r_{1,Q}, r_{1,I}$. The "contributions" to the volume when there are agreement and disagreement are shown below:

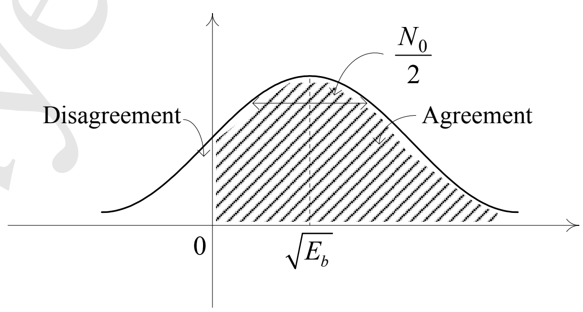


Figure 10.12

Quantified mathematically,

$$\begin{aligned}\text{Agreement "contribution"} &= 1 - Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \\ \text{Disagreement "contribution"} &= Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\end{aligned}$$

The volume in this quadrant is then a product of the four "contributions", 1 agreement and 3 disagreements in this case. The contribution of this quadrant volume to the total volume is therefore $\left[1-Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right]Q^3\left(\sqrt{\frac{2E_b}{N_0}}\right)$. Based on this reasoning, set up the following table where the argument $\sqrt{\frac{2E_b}{N_0}}$ of the $Q(\cdot)$ function is understood.

Quadrant	Decimal	Error volume contribution
$(-1,+1,-1,+1)$	(5)	$(1-Q)Q^3$
$(-1,-1,+1,-1)$	(2)	$(1-Q)Q^3$
$(-1,+1,-1,-1)$	(4)	$(1-Q)^3Q$
$(+1,-1,+1,+1)$	(11)	$(1-Q)Q^3$
$(+1,+1,-1,+1)$	(13)	$(1-Q)^3Q$
$(-1,-1,-1,+1)$	(1)	$(1-Q)^3Q$
$(-1,+1,+1,+1)$	(7)	$(1-Q)^3Q$
$(+1,-1,-1,-1)$	(8)	$(1-Q)Q^3$
$(+1,+1,+1,-1)$	(14)	$(1-Q)Q^3$
$(-1,-1,+1,+1)$	(3)	$(1-Q)^2Q^2$
$(-1,+1,+1,-1)$	(6)	$(1-Q)^2Q^2$
$(+1,-1,-1,+1)$	(9)	$(1-Q)^2Q^2$
$(+1,+1,-1,-1)$	(12)	$(1-Q)^2Q^2$

The probability of symbol error is the sum of the above volumes, remember the events are mutually exclusive. A little algebra yields

$$P[\text{symbol error}] = 4\left[Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - 2Q^2\left(\sqrt{\frac{2E_b}{N_0}}\right) + 2Q^3\left(\sqrt{\frac{2E_b}{N_0}}\right) - Q^4\left(\sqrt{\frac{2E_b}{N_0}}\right)\right]. \tag{10.14}$$

Remark: An error expression is given in "Telecommunication Systems Engineering", W. C. Lindsay & M. K. Simon, Doves, 1973, p. 246. The derivation is quite different. Compare. Also the Viterbi algorithm can be applied here. Left as an exercise.

For comparison, the symbol error probability of coherently-demodulated QPSK is given from Eqns. (7.39) and (7.40) in the text as:

$$P[\text{symbol error}] = 1-\left[1-Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right]^2 = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - Q^2\left(\sqrt{\frac{2E_b}{N_0}}\right) \tag{10.15}$$

Fig. 10.13 shows plots of (10.14) and (10.15). Observe that coherently-demodulated QPSK outperforms coherently-demodulated DQPSK over the whole range of E_b/N_0. At high SNR (i.e., high E_b/N_0), the difference in E_b/N_0 to achieve the same symbol error probability is quite small, about 0.2 dB only.

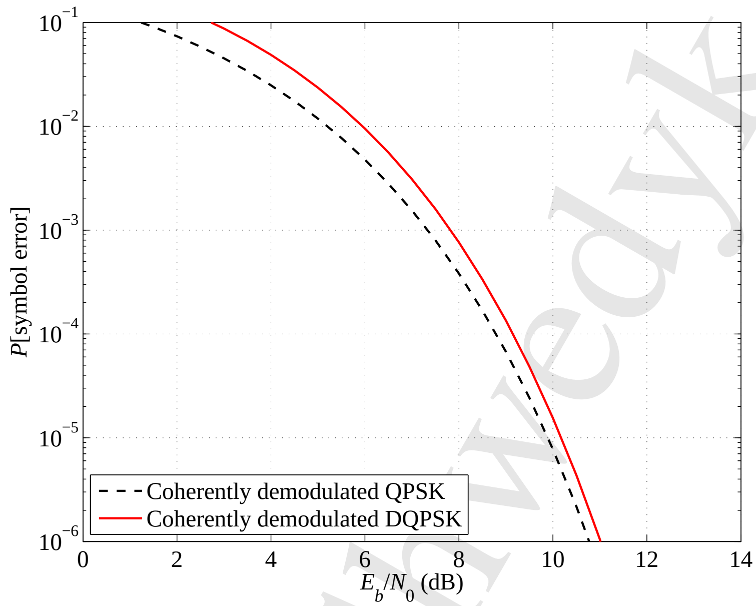


Figure 10.13

P10.12 (a) 16-QAM symbols are 4-bit sequences. Consider 2 contiguous symbols

$$b_{k-7}b_{k-6}b_{k-5}b_{k-4} \,\Big|\, b_{k-3}b_{k-2}b_{k-1}b_k \,\Big|$$
$$(k-1)T_s \qquad\qquad kT_s \qquad T_s = 4T_b$$

Arbitrarily, let $b_{k-1}b_k$ be the differentially encoded bits, which determine the transmitted phase, i.e.,

$b_{k-1}b_k$	
00	Transmitted phase = Previous xmitted phase
01	Transmitted phase = Previous xmitted phase $+\pi/2$
01	Transmitted phase = Previous xmitted phase $+\pi$
01	Transmitted phase = Previous xmitted phase $+3\pi/2$ (or $-\pi/2$)

(b) The remaining 2 bits, $b_{k-3}b_{k-2}$ amplitude modulate the transmitted sinusoid by $+\Delta/2, +3\Delta/2$. Write the transmitted sinusoid as $\sqrt{\frac{1}{T_s}}\cos(2\pi f_c t + \varphi_k)$ where $\varphi_k = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$ radians (φ_k is the relative phase as determined by $b_k b_{k-1}$). In the symbol interval $\{(k-1)T_s, kT_s\}$ the transmitted signal lies in the signal space as follows:

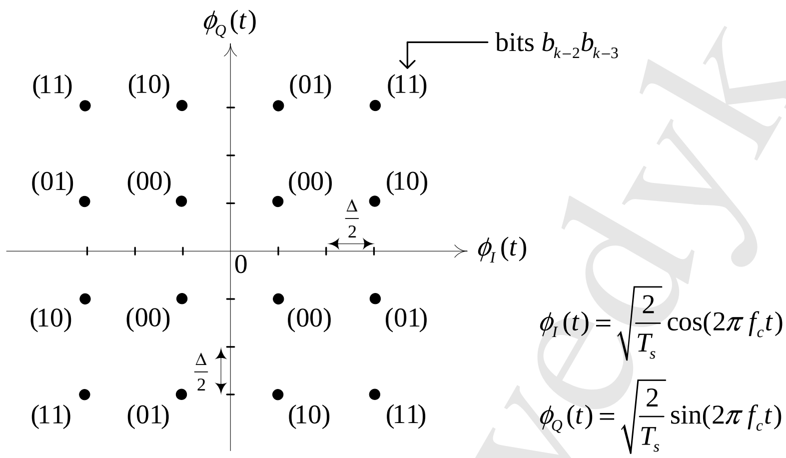


Figure 10.14

Remark: Which quadrant the transmitted signal lies in depends on what the differential encoding tells one.

(c) 64-QAM has 6-bit signals $b_{k-5}b_{k-4}b_{k-3}b_{k-2}b_{k-1}b_k$. Again we let $b_{k-1}b_k$ be the differentially encoded bits while the bits $b_{k-5}b_{k-4}b_{k-3}b_{k-2}$ amplitude modulate the transmitted carrier by $+\Delta/2, +3\Delta/2, +5\Delta/2, +7\Delta/2$. Showing only one quadrant of the transmitted signal space:

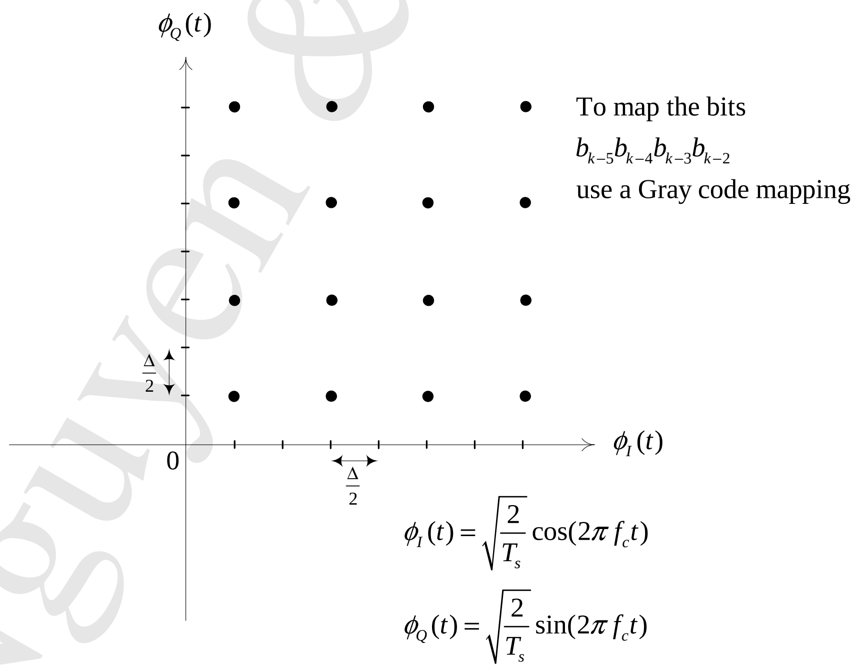


Figure 10.15

Again the transmitted signal space occupies all 4 quadrants. The signal points in the other 3 quadrant are rotated version of the ones shown in the first quadrant. The bits $b_{k-5}b_{k-4}b_{k-3}b_{k-2}$ would amplitude modulate the carrier $\sqrt{\frac{1}{T_s}}\cos(2\pi f_c t + \varphi_k)$, where φ_k is the differential phase determined by $b_{k-1}b_k$.

(d) To generalize to any square QAM, proceed as above. Simply take the first 2 bits and differentially encode the phase. The remaining bits then amplitude modulate the carrier.

Out of curiosity which, if any, of the 16-QAM constellations in Fig. 8.17 lend themselves to differentially encoding (against $k\pi/2$ phase ambiguity)?

(e) Modulator:

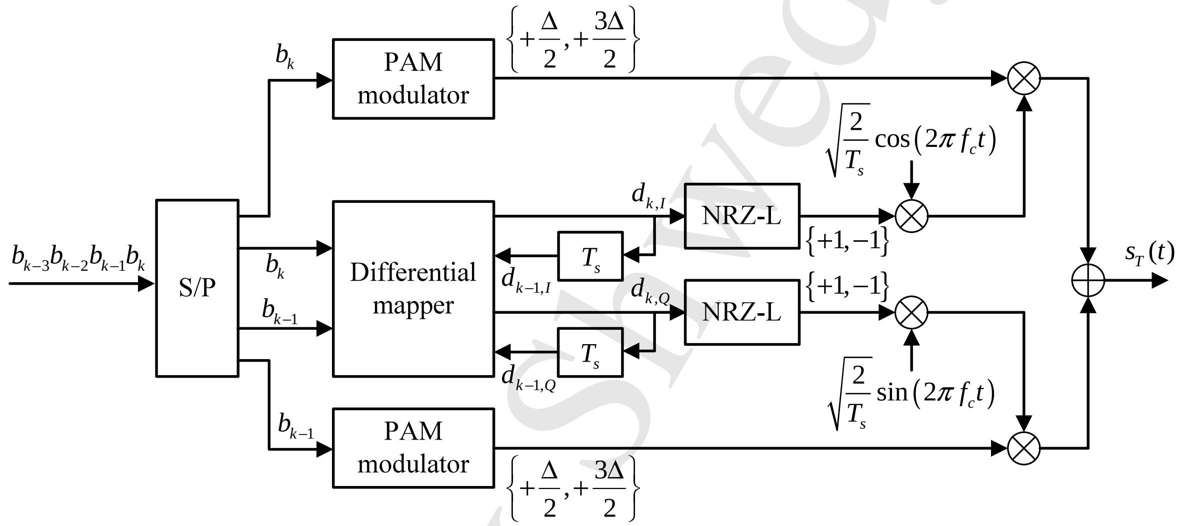


Figure 10.16

Demodulator: The block diagram consists of 2 parts. One of these demodulates the differential bits $b_k b_{k-1}$, and is that of P10.10(g) where one looks at the signal over 2 symbol intervals $(k-2)T_s$ to $(k-1)T_s$. Decide which of the 16 quadrants the (4) sufficient statistics fall in and then determine the differential bits $\hat{b}_k\hat{b}_{k-1}$.
The second part demodulates the "amplitude bits". It carves up the signal space of (b) using a minimum-distance criterion (as we are in AWGN).

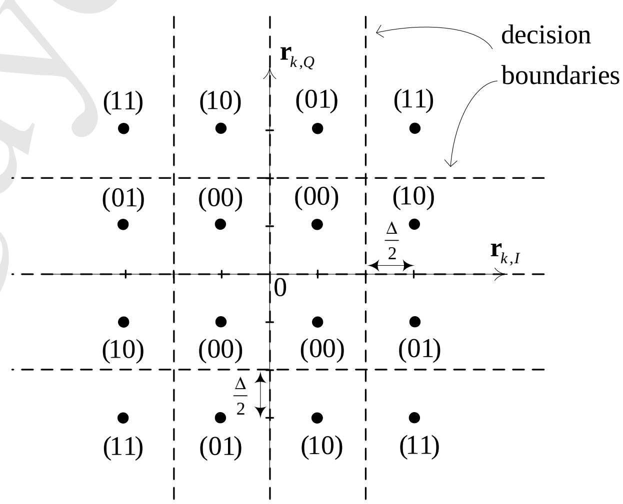


Figure 10.17

P10.13 (a) The received signal space has 4 basis functions, namely

$$\phi_{1,I}(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_1 t); \tag{10.16}$$

$$\phi_{1,Q}(t) = \sqrt{\frac{2}{T_b}}\sin(2\pi f_1 t); \tag{10.17}$$

$$\phi_{2,I}(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_2 t); \tag{10.18}$$

$$\phi_{2,Q}(t) = \sqrt{\frac{2}{T_b}}\sin(2\pi f_2 t). \tag{10.19}$$

In terms of these basis functions the received signal can be written as

$$0_T:\ \mathbf{r}(t) = \sqrt{E_1}\phi_{1,I}(t) + \sqrt{E_2}\mathbf{n}_{F,I}\phi_{1,I}(t) + \sqrt{E_2}\mathbf{n}_{F,Q}\phi_{1,Q}(t) + \mathbf{w}(t), \tag{10.20}$$

$$1_T:\ \mathbf{r}(t) = \sqrt{E_1}\phi_{2,I}(t) + \sqrt{E_2}\mathbf{n}_{F,I}\phi_{2,I}(t) + \sqrt{E_2}\mathbf{n}_{F,Q}\phi_{2,Q}(t) + \mathbf{w}(t), \tag{10.21}$$

where the fading parameters $\mathbf{n}_{F,I} = \boldsymbol{\alpha}\cos\boldsymbol{\theta}$, $\mathbf{n}_{F,Q} = \boldsymbol{\alpha}\sin\boldsymbol{\theta}$ are statistically independent Gaussian random variables, zero mean, variance $\sigma_F^2/2$.

Project $\mathbf{r}(t)$ onto the 4 basis functions to obtain a set of sufficient statistics:

$$\begin{array}{llll|llll}
0_T: & \mathbf{r}_1 = \sqrt{E_1} + & \sqrt{E_1}\mathbf{n}_{F,I} + & \mathbf{w}_{1,I} & 1_T: & \mathbf{r}_1 = & & \mathbf{w}_{1,I} \\
 & \mathbf{r}_2 = & \sqrt{E_2}\mathbf{n}_{F,Q} + & \mathbf{w}_{1,Q} & & \mathbf{r}_2 = & & \mathbf{w}_{1,Q} \\
 & \mathbf{r}_3 = & & \mathbf{w}_{2,I} & & \mathbf{r}_3 = \sqrt{E_1} + & \sqrt{E_2}\mathbf{n}_{F,I} + & \mathbf{w}_{2,I} \\
 & \mathbf{r}_4 = & & \mathbf{w}_{2,Q} & & \mathbf{r}_4 = & \sqrt{E_2}\mathbf{n}_{F,Q} + & \mathbf{w}_{2,Q}
\end{array}$$

where $\mathbf{w}_{1,I}, \mathbf{w}_{1,Q}, \mathbf{w}_{2,I}, \mathbf{w}_{2,Q}$, due to the AWGN, are the usual statistically independent, zero-mean Gaussian random variables, variance $\sigma^2 \equiv N_0/2$. Note that whether 0_T or 1_T the sufficient statistics $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4$ are statistically independent Gaussian random variables.

(b) The LRT is therefore (as usual, bits are assumed equally probable):

$$\frac{\prod_{i=1}^{4} f(\mathbf{r}_i|1_T)}{\prod_{i=1}^{4} f(\mathbf{r}_i|0_T)} \underset{0_D}{\overset{1_D}{\gtrless}} 1 \tag{10.22}$$

Ignoring the $\sqrt{2\pi}$ factor which cancels out top and bottom and letting $\sigma_t^2 \equiv E_2\sigma_F^2/2 + N_0/2$, the LRT is

$$\frac{\frac{1}{\sigma}\mathrm{e}^{-\frac{\mathbf{r}_1^2}{2\sigma^2}}\frac{1}{\sigma}\mathrm{e}^{-\frac{\mathbf{r}_2^2}{2\sigma^2}}\frac{1}{\sigma_t}\mathrm{e}^{-\frac{(\mathbf{r}_3-\sqrt{E_1})^2}{2\sigma_t^2}}\frac{1}{\sigma_t}\mathrm{e}^{-\frac{\mathbf{r}_4^2}{2\sigma_t^2}}}{\frac{1}{\sigma_t}\mathrm{e}^{-\frac{(\mathbf{r}_1-\sqrt{E_1})^2}{2\sigma_t^2}}\frac{1}{\sigma_t}\mathrm{e}^{-\frac{\mathbf{r}_2^2}{2\sigma_t^2}}\frac{1}{\sigma}\mathrm{e}^{-\frac{\mathbf{r}_3^2}{2\sigma^2}}\frac{1}{\sigma}\mathrm{e}^{-\frac{\mathbf{r}_4^2}{2\sigma^2}}} \underset{0_D}{\overset{1_D}{\gtrless}} 1 \tag{10.23}$$

Canceling and taking ln, we get:

$$\frac{(\mathbf{r}_1-\sqrt{E_1})^2}{\sigma_t^2} + \frac{\mathbf{r}_2^2}{\sigma_t^2} + \frac{\mathbf{r}_3^2}{\sigma^2} + \frac{\mathbf{r}_4^2}{\sigma^2} \underset{0_D}{\overset{1_D}{\gtrless}} \frac{\mathbf{r}_1^2}{\sigma^2} + \frac{\mathbf{r}_2^2}{\sigma^2} + \frac{(\mathbf{r}_3-\sqrt{E_1})^2}{\sigma_t^2} + \frac{\mathbf{r}_4^2}{\sigma_t^2} \tag{10.24}$$

Rewrite this as:

$$\frac{\mathbf{r}_3^2}{\sigma^2} - \frac{(\mathbf{r}_3 - \sqrt{E_1})^2}{\sigma_t^2} + \frac{\mathbf{r}_4^2}{\sigma^2} - \frac{\mathbf{r}_4^2}{\sigma_t^2} \underset{0_D}{\overset{1_D}{\gtrless}} \frac{\mathbf{r}_1^2}{\sigma^2} - \frac{(\mathbf{r}_1 - \sqrt{E_1})^2}{2\sigma_t^2} + \frac{\mathbf{r}_2^2}{\sigma^2} - \frac{\mathbf{r}_2^2}{\sigma_t^2} \tag{10.25}$$

After some algebra, namely consisting of completing the square by adding and subtracting terms, the decision rule can be written as:

$$\left(\mathbf{r}_3 + \frac{\sqrt{E_1}}{c\sigma_t^2}\right)^2 + \mathbf{r}_4^2 \underset{0_D}{\overset{1_D}{\gtrless}} \left(\mathbf{r}_1^2 + \frac{\sqrt{E_1}}{c\sigma_t^2}\right) + \mathbf{r}_2^2 \tag{10.26}$$

where $c \equiv \frac{1}{\sigma^2} - \frac{1}{\sigma_t^2} > 0$. Also note that $c\sigma_t^2 = E_2\sigma_F^2/2\sigma^2 \Rightarrow \frac{\sqrt{E_1}}{c\sigma_t^2} = \frac{2\sqrt{E_1}\sigma^2}{E_2\sigma_F^2} \equiv m$. Therefore the decision rule can be written as

$$\therefore (\mathbf{r}_3 + m)^2 + \mathbf{r}_4^2 \underset{0_D}{\overset{1_D}{\gtrless}} \left(\mathbf{r}_1^2 + m\right) + \mathbf{r}_2^2.$$

The parameter m reflects the direct LOS path, which results in a "DC" received component.

P10.14(a-d) Though it is easy enough to find the characteristic functions of what should read $\mathbf{x}^2$ and $\mathbf{y}^2$, i.e., $E\{\mathrm{e}^{j\omega\mathbf{x}^2}\}$ and $E\{\mathrm{e}^{j\omega\mathbf{y}^2}\}$ where $\mathbf{x} \sim \mathcal{N}(m, \sigma^2)$ and $\mathbf{y} \sim \mathcal{N}(0, \sigma^2)$, and then multiply the 2 characteristic functions, the difficulty is in finding the inverse transform.

Specifically,

$$\begin{aligned} \Phi_{\mathbf{x}^2}(\omega) = E\left\{\mathrm{e}^{j2\omega\mathbf{x}^2}\right\} &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \mathrm{e}^{j\omega x^2} \mathrm{e}^{-\frac{(x-m)^2}{2\sigma^2}} \mathrm{d}x \\ &= \frac{1}{\sqrt{1 - j2\omega\sigma^2}} \mathrm{e}^{\frac{m^2}{1-j2\omega\sigma^2}} \end{aligned} \tag{10.27}$$

and

$$\Phi_{\mathbf{y}^2}(\omega) = \frac{1}{\sqrt{1 - j2\omega\sigma^2}} \Rightarrow \Phi_{\mathbf{z}}(\omega) = \frac{\mathrm{e}^{\frac{m^2}{1-j2\omega\sigma^2}}}{(1 - j2\omega\sigma^2)}$$

$$\therefore \; f_{\mathbf{z}}(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\frac{m^2}{(1-j2\omega\sigma^2)}}}{(1 - j2\omega\sigma^2)} \mathrm{e}^{-j\omega z} \mathrm{d}\omega$$

Unfortunately, we are not able to perform this integration.

Well, if one fails in one approach, try another one. Consider random variable $\mathbf{z} = \sqrt{\mathbf{x}^2 + \mathbf{y}^2}$ where $\mathbf{x}$, $\mathbf{y}$ are statistically independent Gaussian random variables, common variance, σ^2, and mean $m_{\mathbf{x}}, m_{\mathbf{y}}$, respectively. First let us find the probability distribution function $F_{\mathbf{z}}(z) = P[\mathbf{z} = \mathbf{x}^2 + \mathbf{y}^2 \le z]$. Graphically this is finding the probability that $\mathbf{z}$ falls in the region indicated in Fig. 10.18.

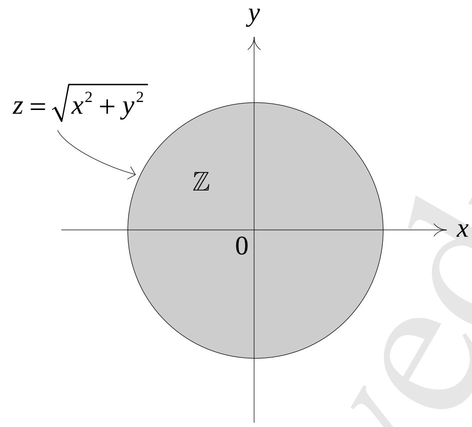


Figure 10.18

This probability is

$$\iint_{\mathbb{Z}} f_{\mathbf{xy}}(x,y)\mathrm{d}x\mathrm{d}y = \frac{1}{2\pi\sigma^2}\iint_{\mathbb{Z}} \mathrm{e}^{-\frac{(x-m_{\mathbf{x}})^2}{2\sigma^2}}\mathrm{e}^{-\frac{(y-m_{\mathbf{y}})^2}{2\sigma^2}}\mathrm{d}x\mathrm{d}y \tag{10.28}$$

Now change to polar coordinates: $\rho^2 = x^2+y^2$; $\mathrm{d}x\mathrm{d}y = \rho\mathrm{d}\rho\mathrm{d}\alpha$; $x=\rho\cos\alpha$; $y=\rho\sin\alpha$ and region $\mathbb{Z}$ is $0\le\rho\le z$; $0\le\alpha<2\pi$.

$$\begin{aligned}
\therefore\ F_{\mathbf{z}}(z) &= \int_{\rho=0}^{z}\int_{\alpha=0}^{2\pi}\frac{1}{2\pi\sigma^2}\mathrm{e}^{-\frac{\rho^2}{2\sigma^2}}\mathrm{e}^{\frac{m_{\mathbf{x}}\rho\cos\alpha+m_{\mathbf{y}}\rho\sin\alpha}{\sigma^2}}\mathrm{e}^{-\frac{(m_{\mathbf{x}}^2+m_{\mathbf{y}}^2)}{2\sigma^2}}\rho\mathrm{d}\rho\mathrm{d}\alpha \\
&= \frac{1}{\sigma^2}\mathrm{e}^{-\frac{(m_{\mathbf{x}}^2+m_{\mathbf{y}}^2)}{2\sigma^2}}\int_{\rho=0}^{z}\rho\mathrm{e}^{-\frac{\rho^2}{2\sigma^2}}\left[\frac{1}{2\pi}\int_{\alpha=0}^{2\pi}\mathrm{e}^{\frac{\rho(m_{\mathbf{x}}\cos\alpha+m_{\mathbf{y}}\sin\alpha)}{\sigma^2}}\mathrm{d}\alpha\right]\mathrm{d}\rho
\end{aligned} \tag{10.29}$$

Write $m_{\mathbf{x}}\cos\alpha+m_{\mathbf{y}}\sin\alpha$ as $\sqrt{m_{\mathbf{x}}^2+m_{\mathbf{y}}^2}\cos\left(\alpha-\tan^{-1}\frac{m_{\mathbf{y}}}{m_{\mathbf{x}}}\right)$ and recognize the inner integral to be $I_0\left(\frac{\rho\sqrt{m_{\mathbf{x}}^2+m_{\mathbf{y}}^2}}{\sigma^2}\right)$. Therefore

$$F_{\mathbf{z}}(z) = \frac{\mathrm{e}^{-\frac{(m_{\mathbf{x}}^2+m_{\mathbf{y}}^2)}{2\sigma^2}}}{\sigma^2}\int_{\rho=0}^{z}\rho\mathrm{e}^{-\frac{\rho^2}{2\sigma^2}}I_0\left(\frac{\rho\sqrt{m_{\mathbf{x}}^2+m_{\mathbf{y}}^2}}{\sigma^2}\right)\mathrm{d}\rho \tag{10.30}$$

But what is desired is the probability density function $f_{\mathbf{z}}(z)=\mathrm{d}F_{\mathbf{z}}(z)/\mathrm{d}z$. Differentiating the expression for $F_{\mathbf{z}}(z)$ with respect to z, (using Leibnitz rule), we get

$$f_{\mathbf{z}}(z) = \frac{z}{\sigma^2}\mathrm{e}^{-\frac{(z^2+m_{\mathbf{x}}^2+m_{\mathbf{y}}^2)}{2\sigma^2}}I_0\left(\frac{z\sqrt{m_{\mathbf{x}}^2+m_{\mathbf{y}}^2}}{\sigma^2}\right)\ \text{(of course it is } = 0 \text{ for } z<0) \tag{10.31}$$

which is (P10.10) with $m_{\mathbf{x}}=m$ and $m_{\mathbf{y}}=0$.

(e) We have $m^2 = \frac{\kappa\sigma_F^2}{(1+\kappa)}$ and $\sigma^2 = \frac{\sigma_F^2}{2(1+\kappa)}$. Substituting and simplifying by letting $z_n \equiv z/\sigma_F$, one has

$$\sigma_F f_{\mathbf{z}}(z_n) = 2z_n(\kappa+1)\mathrm{e}^{-\left(z_n^2+\frac{\kappa}{1+\kappa}\right)(1+\kappa)} I_0\left(2z_n\sqrt{\kappa(1+\kappa)}\right)$$

Plots of $\sigma_F f_{\mathbf{z}}(z_n)$ with $\sigma_F = 1$ and various values of κ are shown in Fig. 10.19. As κ increases the pdf tends to a Gaussian pdf. This is expected since the fading becomes less and less of a factor.

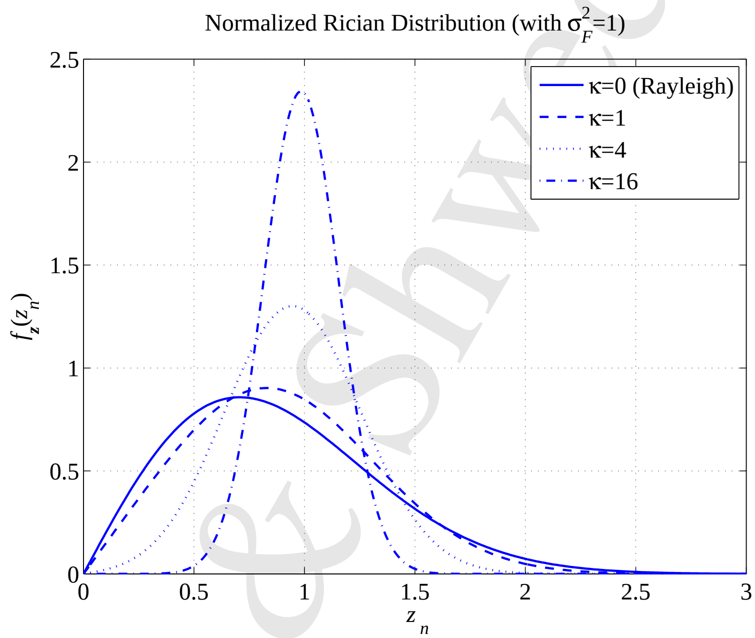


Figure 10.19

Chapter 11

Advanced Modulation Techniques

P11.13 (a) Because in general $s_1(t)$ and $s_2(t)$ are correlated, two orthornormal basis functions $\phi_1(t)$ and $\phi_2(t)$ are required to completely represent them. Since the four signals $y_1(t)$, $y_2(t)$, $y_3(t)$ and $y_4(t)$ are just linear combinations of $s_1(t)$ and $s_2(t)$, they can also be represented as linear combinations of $\phi_1(t)$ and $\phi_2(t)$. Thus the dimensionality of the four signals is **two**. One possible set of orthornormal functions $\phi_1(t)$ and $\phi_2(t)$ are given below (using Gram–Schmidt procedure):

$$\begin{cases} \phi_1(t) &= s_1(t) \\ \phi_2(t) &= \dfrac{1}{\sqrt{1-\rho^2}}s_2(t) - \dfrac{\rho}{\sqrt{1-\rho^2}}s_1(t) \end{cases} \tag{11.1}$$

(b) First the two signature waveforms $s_1(t)$ and $s_2(t)$ can be written in terms of the the basis functions as follows:

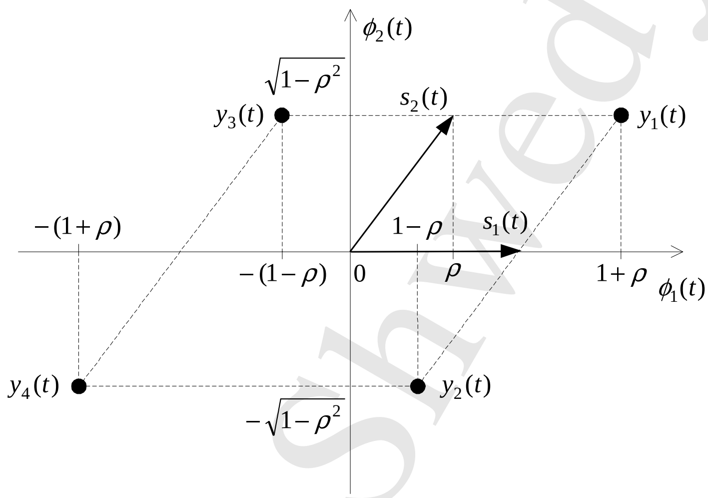
$$\begin{cases} s_1(t) &= \phi_1(t) \\ s_2(t) &= \rho\phi_1(t) + \sqrt{1-\rho^2}\phi_2(t) \end{cases} \tag{11.2}$$

It follows that the four signals $\{y_1(t), y_2(t), y_3(t), y_4(t)\}$ can be written as follows:

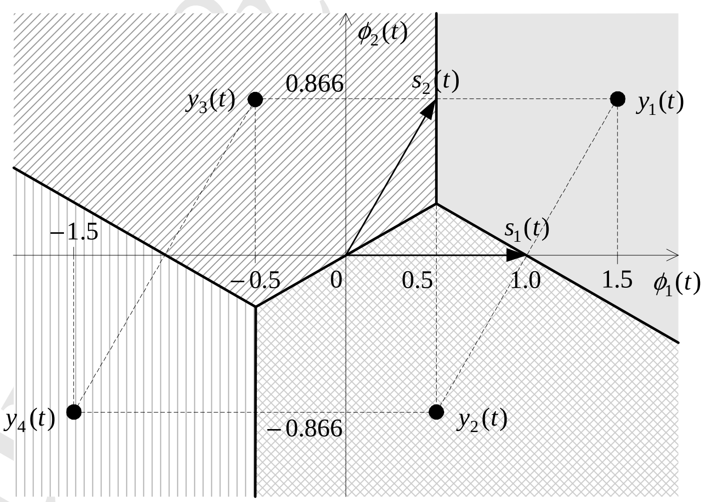
$$\begin{aligned} y_1(t) &= +s_1(t) + s_2(t) &&= (1+\rho)\phi_1(t) + \sqrt{1-\rho^2}\phi_2(t) \\ y_2(t) &= +s_1(t) - s_2(t) &&= (1-\rho)\phi_1(t) - \sqrt{1-\rho^2}\phi_2(t) \\ y_3(t) &= -s_1(t) + s_2(t) &&= -(1-\rho)\phi_1(t) + \sqrt{1-\rho^2}\phi_2(t) \\ y_4(t) &= -s_1(t) - s_2(t) &&= -(1+\rho)\phi_1(t) - \sqrt{1-\rho^2}\phi_2(t) \end{aligned} \tag{11.3}$$

These four signals are plotted in Fig. 11.1-(a)

(c) The *joint* optimum receiver for this CDMA system is a **minimum distance receiver**, which jointly demodulate the transmitted bits of both user 1 and user 2 based on the signal ($y_1(t)$, $y_2(t)$, $y_3(t)$ or $y_4(t)$) that is closest to $r(t)$. The decision boundary and the decision regions for this minimum distance receiver when $\rho = 0.5$ are shown in Fig. 11.1-(b). Note that for $\rho = 0.5$ one has $\sqrt{1-\rho^2} = 0.866$.



(a)



(b)

Figure 11.1: Two-User CDMA: (a) Signal space plot for four possible received signals in the absence of the background noise, (b) Decision regions of the jointly minimum distance receiver when $\rho = 0.5$.

Chapter 12

Synchronization

P12.1 The attenuation affects the received energy, E_b, which is now $\alpha^2 E_b$. From equations (12.1), (12.2) with $\sqrt{E_b} \to \alpha\sqrt{E_b}$.

$$\text{BPSK: } P[\text{bit error}] = Q\left(\alpha\sqrt{\frac{2E_b}{N_0}}\cos\theta\right), \tag{12.1}$$

$$\begin{aligned} \text{QPSK: } P[\text{bit error}] &= \frac{1}{2}Q\left(\alpha\sqrt{\frac{2E_b}{N_0}}(\cos\theta - \sin\theta)\right) \\ &\quad + \frac{1}{2}Q\left(\alpha\sqrt{\frac{2E_b}{N_0}}(\cos\theta + \sin\theta)\right). \end{aligned} \tag{12.2}$$

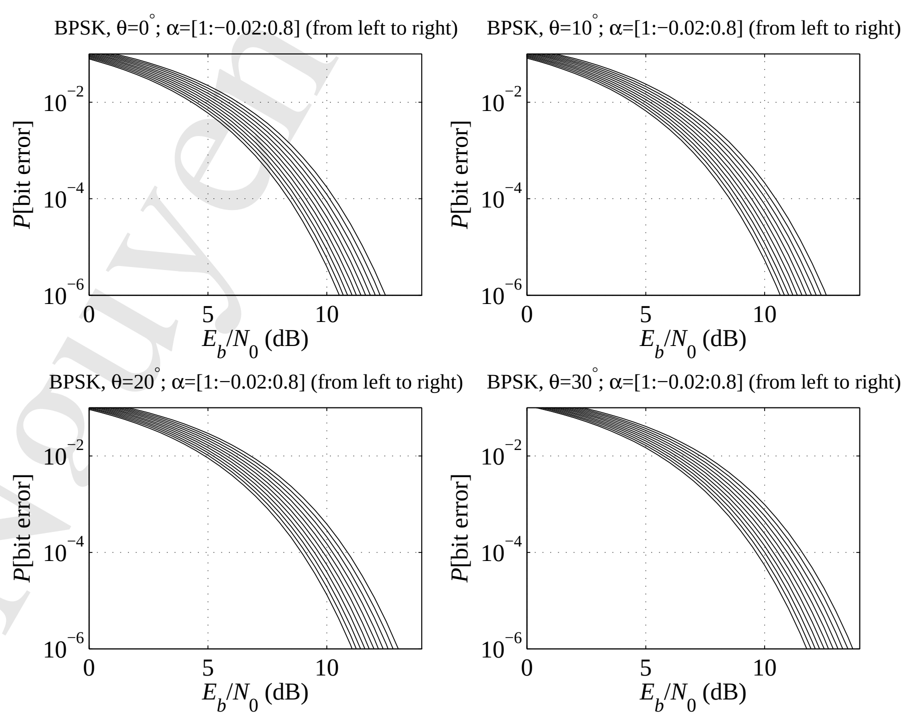


Figure 12.1

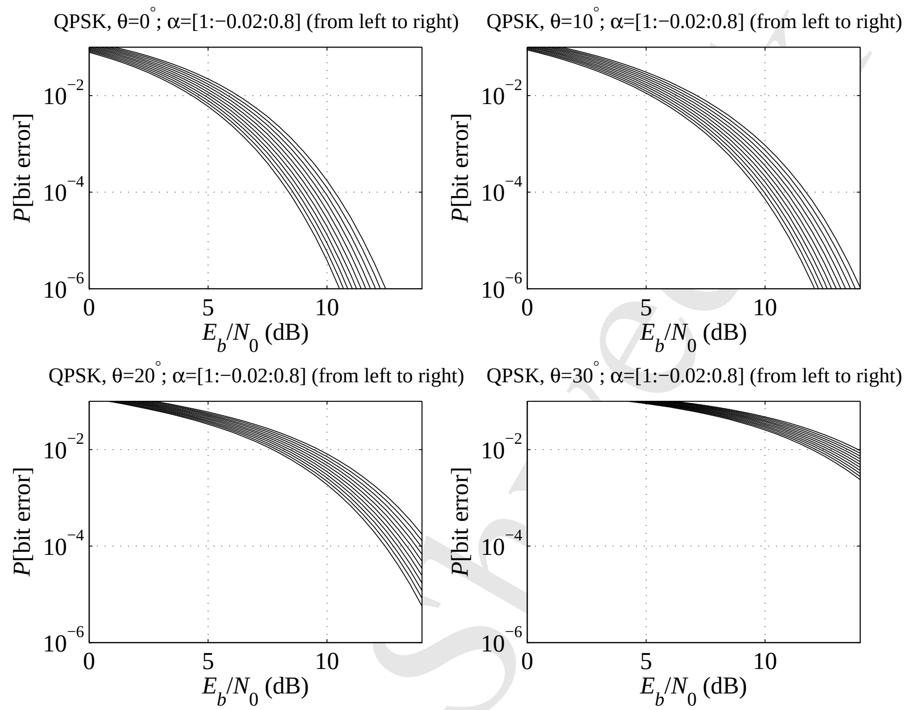


Figure 12.2

The Matlab plots allow one to make design judgements as to the margins needed for transmitted power and phase locked loop performance based usually on worst-case scenarios.

Note that this is always somewhat of a subjective judgement. For instance – what error probability does one choose to make the decision(s)?

P12.2 (a) Refer to Figure 12.3. Note that $x_1 = d - (i-1)\Delta$ where $d = r\cos(\alpha+\theta)$.

$$\therefore x_1 = \frac{\Delta}{2}\left[(2i-1)^2 + (2j-1)^2\right]^{1/2}\cos\left(\tan^{-1}\left(\frac{2j-1}{2i-1}\right)+\theta\right) - \Delta(i-1)$$

$$x_2 + x_1 = \Delta \Rightarrow x_2 = i\Delta - \frac{\Delta}{2}\left[(2i-1)^2 + (2j-1)^2\right]^{1/2}\cos\left(\tan^{-1}\left(\frac{2j-1}{2i-1}\right)+\theta\right)$$

Similarly, $d_1 = r\sin(\alpha+\theta)$, $y_2 = d_1 - (j-1)\Delta$.

$$\therefore y_2 = \frac{\Delta}{2}\left[(2i-1)^2 + (2j-1)^2\right]^{1/2}\sin\left(\tan^{-1}\left(\frac{2j-1}{2i-1}\right)+\theta\right) - \Delta(i-1)$$

$$y_1 = \Delta - y_2 = j\Delta - \frac{\Delta}{2}\left[(2i-1)^2 + (2j-1)^2\right]^{1/2}\sin\left(\tan^{-1}\left(\frac{2j-1}{2i-1}\right)+\theta\right)$$

(b) The error probabilities are determined by finding a volume under a 2-dimensional Gaussian pdf in the appropriate regions. Because the 2 Gaussian random variables ($\mathbf{n}_I$, $\mathbf{n}_Q$ if you wish) are statistically independent (each of variance $N_0/2$ and a mean determined by the signal point under consideration) the volume(s) are a product of 2 Q-functions and basically determined by inspection from the geometrical picture. Let $\sigma^2 \equiv N_0/2$.

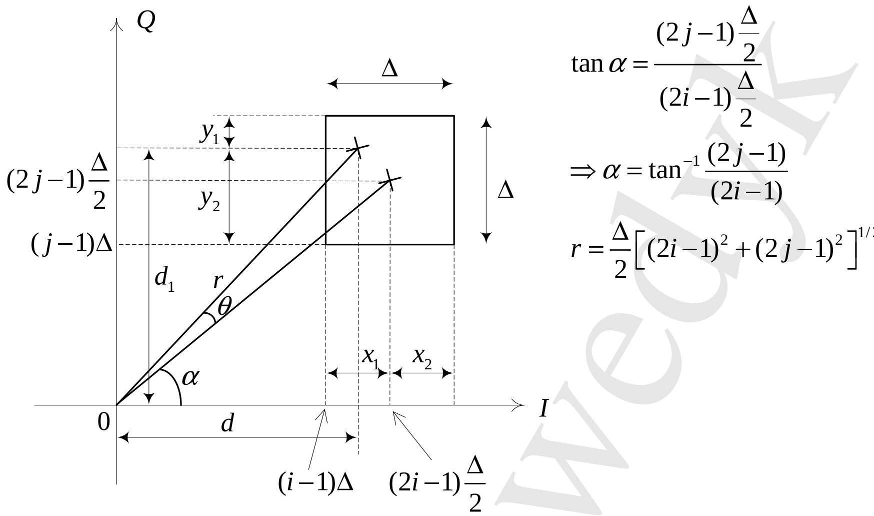


Figure 12.3

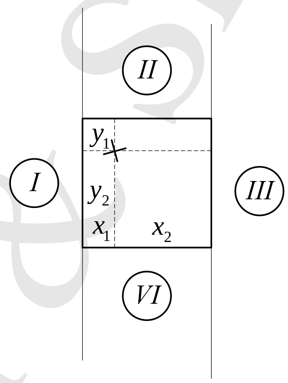


Figure 12.4

Inner signals (refer to Figure 12.4)
Need to find the volumes under the 2-D Gaussian pdf in the 4 regions I, II, III, IV. These are:

$$\text{Volume } I = Q\left(\frac{x_1}{\sigma}\right) \tag{12.3}$$

$$\text{Volume } II = \left[1 - Q\left(\frac{x_1}{\sigma}\right) - Q\left(\frac{x_2}{\sigma}\right)\right] Q\left(\frac{y_1}{\sigma}\right) \tag{12.4}$$

$$\text{Volume } III = Q\left(\frac{x_2}{\sigma}\right) \tag{12.5}$$

$$\text{Volume } IV = \left[1 - Q\left(\frac{x_1}{\sigma}\right) - Q\left(\frac{x_2}{\sigma}\right)\right] Q\left(\frac{y_2}{\sigma}\right). \tag{12.6}$$

The total volume is of course the sum, and therefore:

$$\begin{aligned} P_{\text{inner}}[\text{symbol error}] &= Q\left(\frac{x_1}{\sigma}\right) + Q\left(\frac{x_2}{\sigma}\right) \\ &+ \left[Q\left(\frac{y_1}{\sigma}\right) + Q\left(\frac{y_2}{\sigma}\right)\right]\left[1 - Q\left(\frac{x_1}{\sigma}\right) - Q\left(\frac{x_2}{\sigma}\right)\right]. \end{aligned} \tag{12.7}$$

Outer-vertical signals (refer to Figure 12.5)

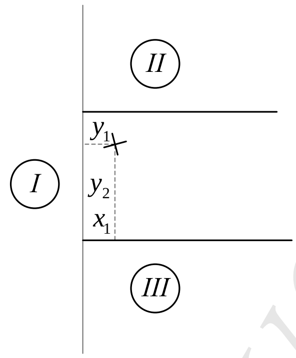


Figure 12.5

$$
\begin{aligned}
\text{Volume } I &= Q\left(\frac{x_1}{\sigma}\right) && (12.8)\\
\text{Volume } II &= \left[1-Q\left(\frac{x_1}{\sigma}\right)\right]Q\left(\frac{y_1}{\sigma}\right) && (12.9)\\
\text{Volume } III &= \left[1-Q\left(\frac{x_1}{\sigma}\right)\right]Q\left(\frac{y_2}{\sigma}\right) && (12.10)\\
P_{\text{outer}-\text{V}}[\text{symbol error}] &= Q\left(\frac{x_1}{\sigma}\right)+\left[Q\left(\frac{y_1}{\sigma}\right)+Q\left(\frac{y_2}{\sigma}\right)\right]\left[1-Q\left(\frac{x_1}{\sigma}\right)\right] && (12.11)
\end{aligned}
$$

Outer-horizonal signals (refer to Figure 12.6)

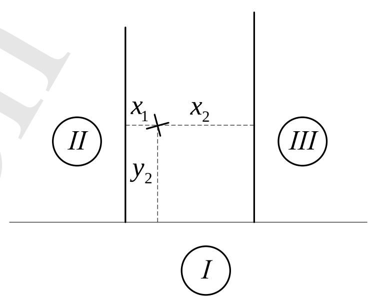


Figure 12.6

$$
\begin{aligned}
\text{Volume } I &= Q\left(\frac{y_2}{\sigma}\right) && (12.12)\\
\text{Volume } II &= \left[1-Q\left(\frac{y_2}{\sigma}\right)\right]Q\left(\frac{x_1}{\sigma}\right) && (12.13)\\
\text{Volume } III &= \left[1-Q\left(\frac{y_2}{\sigma}\right)\right]Q\left(\frac{x_2}{\sigma}\right) && (12.14)\\
P_{\text{outer}-\text{H}}[\text{symbol error}] &= Q\left(\frac{y_2}{\sigma}\right)+\left[Q\left(\frac{x_1}{\sigma}\right)+Q\left(\frac{x_2}{\sigma}\right)\right]\left[1-Q\left(\frac{y_2}{\sigma}\right)\right] && (12.15)
\end{aligned}
$$

Corner signals (refer to Figure 12.7)

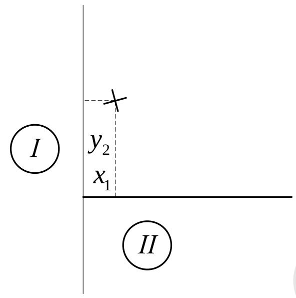


Figure 12.7

$$\text{Volume } I = Q\left(\frac{x_1}{\sigma}\right) \tag{12.16}$$

$$\text{Volume } II = \left[1 - Q\left(\frac{x_1}{\sigma}\right)\right] Q\left(\frac{y_2}{\sigma}\right) \tag{12.17}$$

$$P_{\text{corner}}[\text{symbol error}] = Q\left(\frac{x_1}{\sigma}\right) + Q\left(\frac{y_2}{\sigma}\right)\left[1 - Q\left(\frac{x_1}{\sigma}\right)\right] \tag{12.18}$$

(c) Consider the (i,j)th signal, i.e., the signal at $\left[(2i-1)\frac{\Delta}{2}, (2j-1)\frac{\Delta}{2}\right]$. Its energy is $E_{ij} = \frac{\Delta^2}{4}(2i-1)^2 + \frac{\Delta^2}{4}(2j-1)^2$ joules. (Note - phase rotation does not affect energy, therefore ignore θ).

$$\begin{aligned} E_s &= \underset{(4\text{ quadrants})}{\underset{\uparrow}{4}} \sum_{i=1}^{N_I}\sum_{j=1}^{N_Q} E_{ij}\, \underset{\frac{1}{M}=\frac{1}{2^\lambda}}{\underset{||}{P[\text{signal } ij]}} \qquad \begin{matrix} N_I = 2^{\lambda_I/2}, N_Q = 2^{\lambda_Q/2} \\ \lambda = \lambda_I + \lambda_Q, M = 2^\lambda \end{matrix} \\ &= \frac{4}{M}\sum_{i=1}^{N_I}\sum_{j=1}^{N_Q}\left[\frac{\Delta^2}{4}(2i-1)^2 + \frac{\Delta^2}{4}(2j-1)^2\right] \text{ (joules/symbol)} \\ &= \frac{\Delta^2}{M}\left\{\sum_{i=1}^{N_I}\sum_{j=1}^{N_Q}(4i^2 - 4i + 1) + \sum_{i=1}^{N_I}\sum_{j=1}^{N_Q}(4j^2 - 4j + 1)\right\} \end{aligned} \tag{12.19}$$

Now

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}; \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

After some algebra and by realizing that $\sum_{k=1}^{n} 1 = n$, we get

$$E_s = \frac{\Delta^2 N_I N_Q}{3M}\left[(4N_I^2 - 1) + (4N_Q^2 - 1)\right] \tag{12.20}$$

But $N_I N_Q = 2^{(\lambda_I + \lambda_Q)/2} = \left(2^\lambda\right)^{1/2} = M^{1/2}$

$$\therefore E_s = \frac{2\Delta^2}{3\sqrt{M}}\left(2N_I^2 + 2N_Q^2 - 1\right) \Rightarrow E_b = \frac{E_s}{\lambda} = \frac{2\Delta^2}{3\lambda\sqrt{M}}\left(2N_I^2 + 2N_Q^2 - 1\right)$$

With square QAM, we have $\lambda_I = \lambda_Q = \lambda/2 \Rightarrow N_I^2 = \left(2^{\lambda/4}\right)^2 = \left(2^{\lambda}\right)^{1/2} = \sqrt{M} = N_Q^2$.

$$\therefore E_b = \frac{2\Delta^2}{3\lambda\sqrt{M}}\left(4\sqrt{M}-1\right) \text{ joules/bit} \tag{12.21}$$

$$\text{or } \Delta = \left[\frac{3\lambda\sqrt{M}}{2\left(4\sqrt{M}-1\right)}\right]^{1/2} \tag{12.22}$$

Remark: Can express Δ as a function of λ and M only for square QAM.

(d) The expression for P[symbol error] is quite lengthy. To simplify it, at least notationally note that the distances x_1, x_2, y_1, y_2 are functions of i, j which means the error probabilities in (b) are functions of i, j. So write $P_{\text{inner}}[\text{error}]$ as $P_{\text{inner}}[\text{error}, i, j]$, $P_{\text{outer-V}}[\text{error}]$ as $P_{\text{outer-V}}[\text{error}, i, j]$, and so on. Then

$$P[\text{symbol error}] = \underset{\substack{\uparrow\\ \text{symbols are}\\ \text{equally probable}}}{\overset{\substack{\text{only 1 quadrant}\\ \text{considered}\\ \downarrow}}{\frac{4}{M}}} \left\{ \sum_{i=1}^{N_I-1}\sum_{j=1}^{N_Q-1} P_{\text{inner}}[\text{error}, i, j] \right.$$
$$+ \sum_{j=1}^{N_Q-1} P_{\text{outer-V}}[\text{error}, i = N_I, j]$$
$$\left. + \sum_{i=1}^{N_I-1} P_{\text{outer-H}}[\text{error}, i, j = N_Q] + P_{\text{corner}}[\text{error}, i = N_I, j = N_Q] \right\} \tag{12.23}$$

(e) Use the expression for Δ in terms of E_b/N_0 derived in (c) and the general expression of (d) to guide you in writing the Matlab program.

P12.3 (a) See Fig. 12.8.

(b)

$$\mathbf{P}[\text{error}] = Q\left(\sqrt{\frac{2E_b}{N_0}}\cos\boldsymbol{\theta}\right) \tag{12.24}$$

$$\begin{aligned} E\{\mathbf{P}[\text{error}]\} &= \int_0^{2\pi} Q\left(\sqrt{\frac{2E_b}{N_0}}\cos\theta\right) f_{\boldsymbol{\theta}}(\theta)\mathrm{d}\theta \\ &= \int_0^{2\pi} Q\left(\sqrt{\frac{2E_b}{N_0}}\cos\theta\right) \frac{1}{2\pi I_0(\Lambda_m)} \mathrm{e}^{\Lambda_m\cos\theta}\mathrm{d}\theta \\ &= \frac{1}{2\pi I_0(\Lambda_m)} \int_0^{2\pi} Q\left(\sqrt{\frac{2E_b}{N_0}}\cos\theta\right) \mathrm{e}^{\Lambda_m\cos\theta}\mathrm{d}\theta. \end{aligned} \tag{12.25}$$

There is no "closed-form" integration available. Therefore resort to numerical integration. Specifically, make use of the routine `quadl` in Matlab. Plots of P[bit error] for different values of Λ_m are shown in Fig. 12.9.

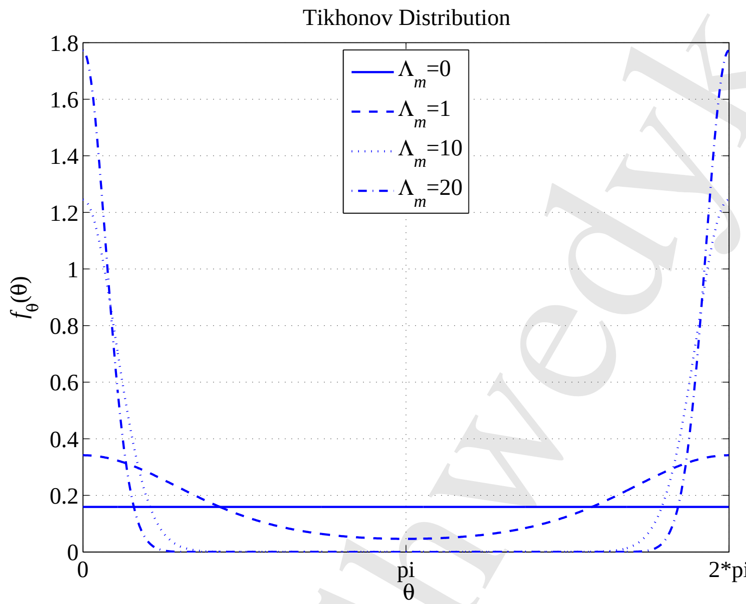


Figure 12.8

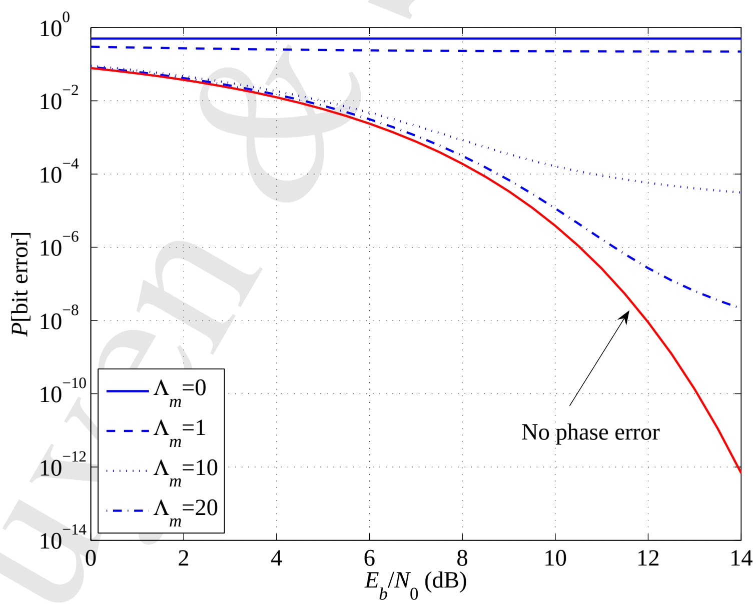


Figure 12.9

Note that for $\Lambda_m = 0$, i.e., the uniform pdf case, the performance breakdowns completely, equivalent to flipping a coin to make a decision. This is expected since the phase (which carries the information) is completely random.

P12.4 The demodulator looks like:

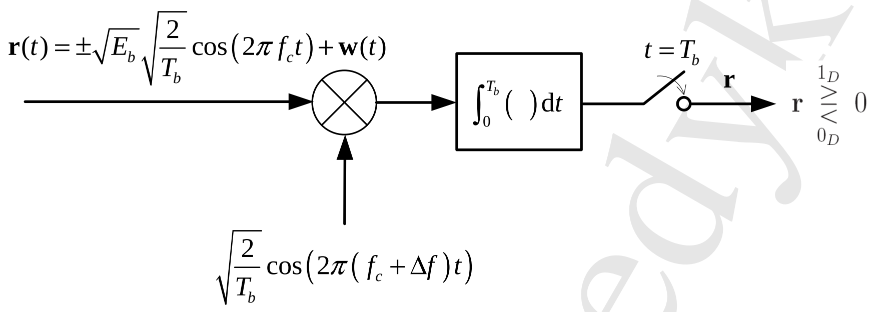


Figure 12.10

Performing the integration one gets:

$$\mathbf{r} = \pm\sqrt{E_b}\frac{\sin(2\pi\Delta f T_b)}{2\pi\Delta f T_b}\left[1+\frac{\Delta f T_b}{2\pi(2f_c+\Delta f)}\right]+\mathbf{w} \tag{12.26}$$

where

$$\mathbf{w} = \int_0^{T_b}\sqrt{\frac{2}{T_b}}\cos\left(2\pi(f_c+\Delta f)t\right)\mathbf{w}(t)\mathrm{d}t$$

is a Gaussian r.v., zero-mean and a variance that, strictly speaking, is not $N_0/2$. We shall take it to be $N_0/2$ as a very good approximation.

Remark: The interested reader may wish to show that the variance, $E\{\mathbf{w}^2\}$, is given by $\frac{N_0}{2}\left[1+\frac{\sin(4\pi(f_c+\Delta f)T_b)}{4\pi(f_c+\Delta f)T_b}\right]$. Since f_c is on the order of 10^6 to 10^9 while T_b on the order of 10^{-3} to 10^{-4}, the second term is neglected.

Therefore the error probability is given by:

$$P[\text{bit error}] = Q\left(\sqrt{\frac{2E_b}{N_0}}\frac{\sin(2\pi\Delta f T_b)}{2\pi\Delta f T_b}\right), \tag{12.27}$$

where the approximation $1+\frac{\Delta f T_b}{2\pi(2f_c+\Delta f)}\approx 1$ is used for the signal part of $\mathbf{r}$.

From the plot, take $\Delta f T_b < 0.1$ (at an error rate of $\approx 10^{-5}$). Then if $T_b \sim 10^{-3} \Rightarrow \Delta f < 0.1(10^3) = 10^2$, i.e., 100Hz. So the oscillator specification would read 16Hz ±100Hz. Need to be quite accurate and stable. Note that as T_b becomes smaller (i.e., a higher bit rate r_b) the allowable frequency deviation is larger.

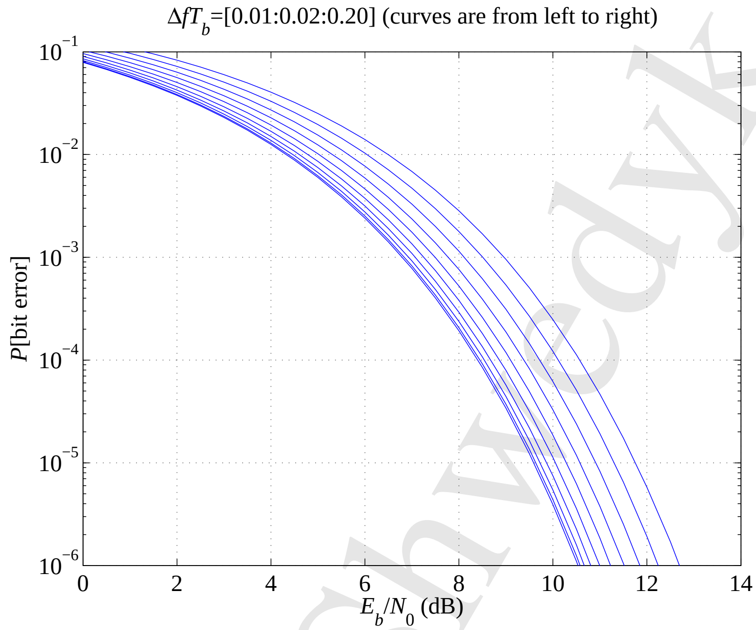


Figure 12.11

P12.5 The demodulator's block diagram looks as shown in Fig. 12.12, where $f_i = f_1$ if 0_T and $f_i = f_2$ if 1_T.

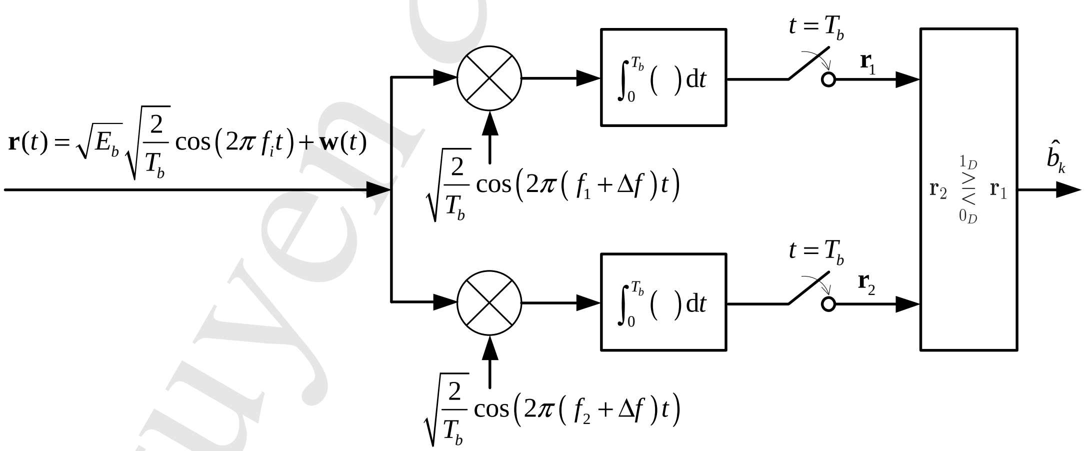


Figure 12.12

The sufficient statistics $\mathbf{r}_1, \mathbf{r}_2$ are:

$$
\begin{aligned}
0_T: \qquad \mathbf{r}_1 &= \sqrt{E_b}\frac{2}{T_b}\int_0^{T_b}\cos(2\pi f_1 t)\cos\left(2\pi(f_1+\Delta f_1)t\right)\mathrm{d}t \\
&\quad + \sqrt{\frac{2}{T_b}}\int_0^{T_b}\cos\left(2\pi(f_1+\Delta f_1)t\right)\mathbf{w}(t)\mathrm{d}t \\
\mathbf{r}_1 &= \sqrt{E_b}\frac{\sin\left(2\pi(f_1+\Delta f_1)T_b\right)}{2\pi(f_1+\Delta f_1)T_b} + \sqrt{E_b}\frac{\sin\left(2\pi\Delta f_1 T_b\right)}{2\pi\Delta f_1 T_b} + \mathbf{w}_1
\end{aligned}
\tag{12.28}
$$

An easy assumption to make is to ignore the 1st term since $f_1 >> 1$, typically $\sim 10^6 - 10^9$. Therefore,

$$
\mathbf{r}_1 = \sqrt{E_b}\frac{\sin\left(2\pi\Delta f_1 T_b\right)}{2\pi\Delta f_1 T_b} + \mathbf{w}_1
$$

On the other hand,

$$
\begin{aligned}
\mathbf{r}_2 &= \sqrt{E_b}\frac{2}{T_b}\int_0^{T_b}\cos(2\pi f_1 t)\cos\left(2\pi(f_2+\Delta f_2)t\right)\mathrm{d}t \\
&\quad + \frac{2}{T_b}\int_0^{T_b}\cos\left(2\pi(f_2+\Delta f_2)t\right)\mathbf{w}(t)\mathrm{d}t \\
&= \sqrt{E_b}\frac{\sin\left(2\pi(f_1+f_2+\Delta f_2)T_b\right)}{2\pi(f_1+f_2+\Delta f_2)T_b} + \sqrt{E_b}\frac{\sin\left(2\pi(f_2-f_1+\Delta f_2)T_b\right)}{2\pi(f_2-f_1+\Delta f_2)T_b} + \mathbf{w}_2
\end{aligned}
\tag{12.29}
$$

Again the 1st term is easily ignored. To ignore the 2nd term is more problematical. Let $f_2 - f_1 = 1/T_b$, the minimum separation for (noncoherent) orthogonality. Then the term becomes $\sqrt{E_b}\frac{\sin 2\pi\Delta f_2 T_b}{2\pi(1+\Delta f_2 T_b)}$. If $\Delta f_2 T_b << 1$, say < 0.1, then $\sqrt{E_b}\frac{\sin 2\pi\Delta f_2 T_b}{2\pi(1+\Delta f_2 T_b)} < 0.09\sqrt{E_b}$ and is ignored in the plots. Therefore $\mathbf{r}_2 = \mathbf{w}_2$.

Similarly for 1_T:

$$
\begin{aligned}
\mathbf{r}_1 &= \sqrt{E_b}\frac{\sin\left(2\pi(f_2+f_1+\Delta f_1)T_b\right)}{2\pi(f_2+f_1+\Delta f_1)T_b} + \sqrt{E_b}\frac{\sin\left(2\pi(f_1-f_2+\Delta f_1)T_b\right)}{2\pi(f_1-f_2+\Delta f_1)T_b} + \mathbf{w}_1 \approx \mathbf{w}_1 \\
\mathbf{r}_2 &= \sqrt{E_b}\frac{\sin\left(2\pi\Delta f_2 T_b\right)}{2\pi\Delta f_2 T_b} + \mathbf{w}_2 \qquad \left(\begin{array}{l}\text{where again the double}\\ \text{frequency term is ignored}\end{array}\right)
\end{aligned}
\tag{12.30}
$$

The received signal space looks as follows:

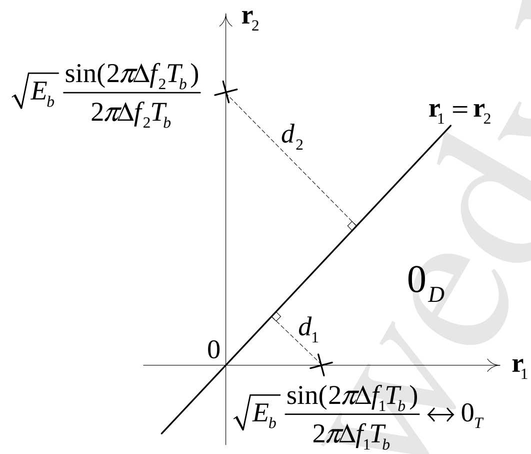


Figure 12.13

The terms $\mathbf{w}_1, \mathbf{w}_2$ are zero-mean Gaussian random variables. However because of the frequency offsets they are not of variance $N_0/2$ (see the solution to P12.4). Further they are correlated because $\cos(2\pi(f_1+\Delta f_1)t)$, $\cos(2\pi(f_2+\Delta f_2)t)$ are not orthogonal. However, this correlation is

$$\begin{aligned}\frac{\sin(2\pi(f_2+f_1+\Delta f_2+\Delta f_1)T_b)}{2\pi(f_2+f_1+\Delta f_2+\Delta f_1)T_b} &+ \frac{\sin(2\pi(f_2-f_1+\Delta f_2-\Delta f_1)T_b)}{2\pi(f_2-f_1+\Delta f_2-\Delta f_1)T_b}\\ &\approx \frac{\sin 2\pi(\Delta f_2-\Delta f_1)T_b}{2\pi\,(1+(\Delta f_2-\Delta f_1)T_b)} \qquad \left(\text{note that } f_2-f_1\approx\frac{1}{T_b}\right)\end{aligned} \tag{12.31}$$

And since it is reasonable to assume that $\Delta f_2 \approx \Delta f_1$ then the correlation is negligible and ignored, i.e., $\mathbf{w}_1, \mathbf{w}_2$ are treated as zero-mean statistically independent Gaussian random variables of variance $N_0/2$.

Therefore

$$P[\text{bit error}] = \frac{1}{2}Q\left(\frac{d_1}{\sqrt{N_0/2}}\right) + \frac{1}{2}Q\left(\frac{d_2}{\sqrt{N_0/2}}\right) \tag{12.32}$$

If we assume (which we do for the plots) that $\Delta f_2 \approx \Delta f_1 \approx \Delta f \Rightarrow d_1 \approx d_2 \approx d$

$$P[\text{bit error}] = Q\left(\frac{d}{\sqrt{N_0/2}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\frac{\sin(2\pi\Delta f T_b)}{2\pi\Delta f T_b}\right) \tag{12.33}$$

where $d = \sqrt{E_b}\frac{\sin 2\pi\Delta f T_b}{2\pi\Delta f T_b}\cos(45^\circ)$. Plots are those of P12.4 except for a 3dB (loss) factor. See Fig. 12.14.

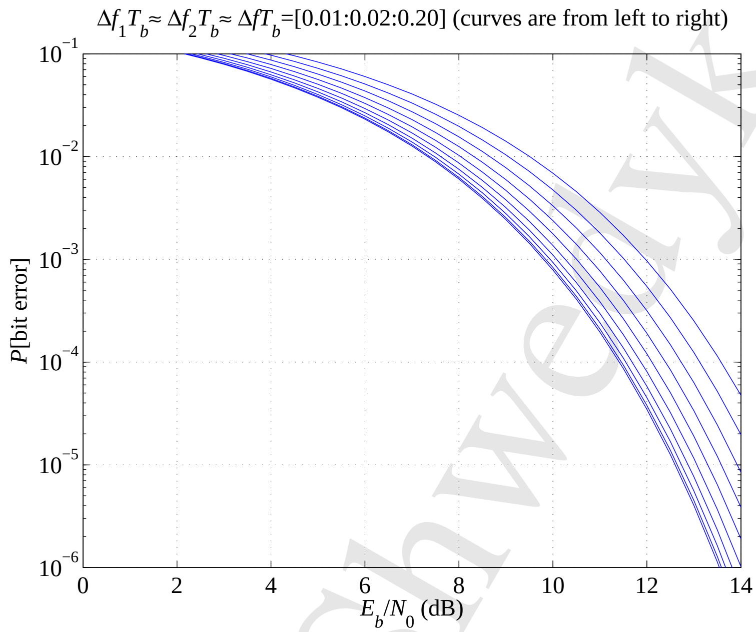


Figure 12.14

P12.6 The block diagram looks as follows:

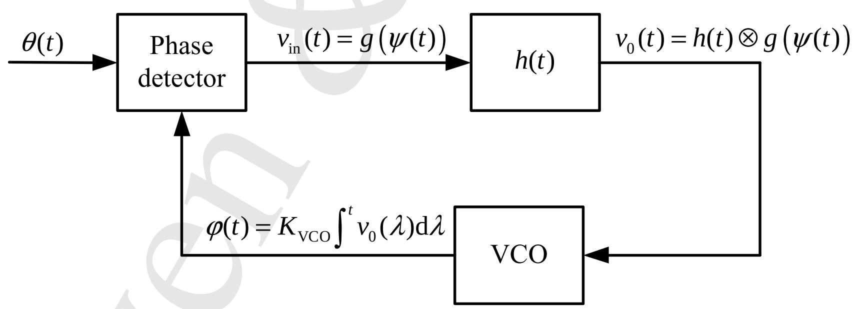


Figure 12.15

Now $\psi(t) = \theta(t) - \varphi(t)$ or $\psi(t) + \varphi(t) = \theta(t)$.

$$\Rightarrow \frac{\mathrm{d}\psi(t)}{\mathrm{d}t} + \frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} = \frac{\mathrm{d}\theta(t)}{\mathrm{d}t}.$$

But $\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} = K_{\text{vco}} v_0(t) = K_{\text{vco}} h(t) * g[\psi(t)]$.

$$\therefore \frac{\mathrm{d}\psi(t)}{\mathrm{d}t} + K_{\text{vco}} h(t) * g[\psi(t)] = \frac{\mathrm{d}\theta(t)}{\mathrm{d}t}.$$

P12.7 The impulse response of the loop filter of Figure 12.7(a) is $h(t) = \delta(t)$. Therefore the differential equation becomes

$$\frac{\mathrm{d}\psi(t)}{\mathrm{d}t} + K_{\text{vco}} g(\psi) = \Delta\omega \text{ or } \frac{\mathrm{d}\psi(t)}{\mathrm{d}t} = \Delta\omega - K_{\text{vco}} g(\psi).$$

(a) The sawtooth characteristic looks as follows:

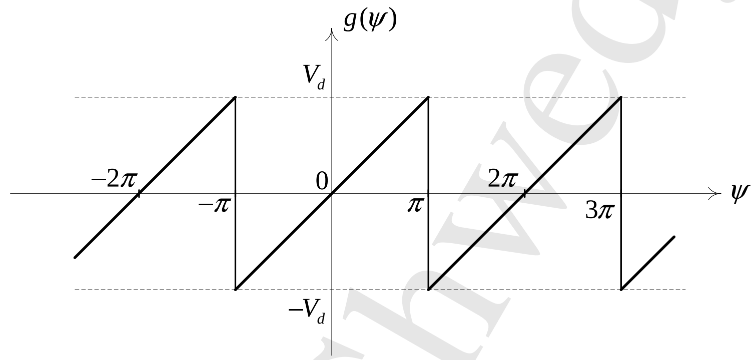


Figure 12.16

The phase plane is

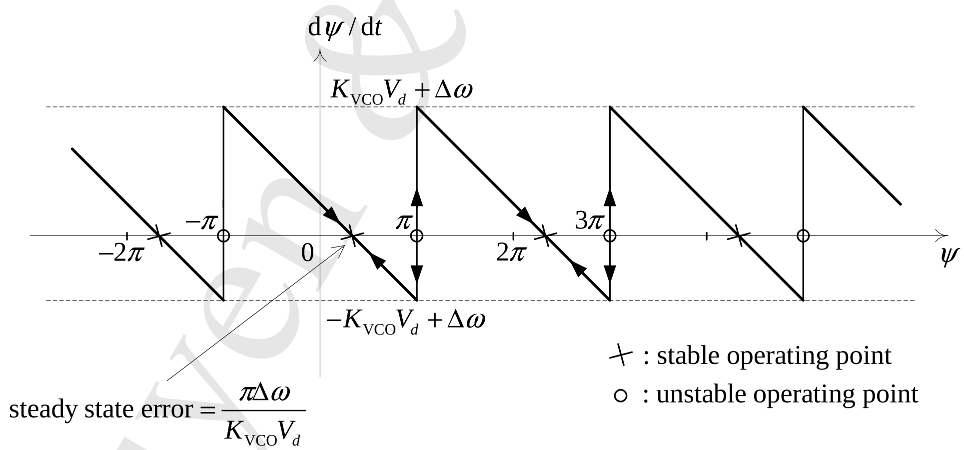


Figure 12.17

Remark: V_d/π can be called the gain of the phase detector.

(b) The triangular characteristic looks as follows:

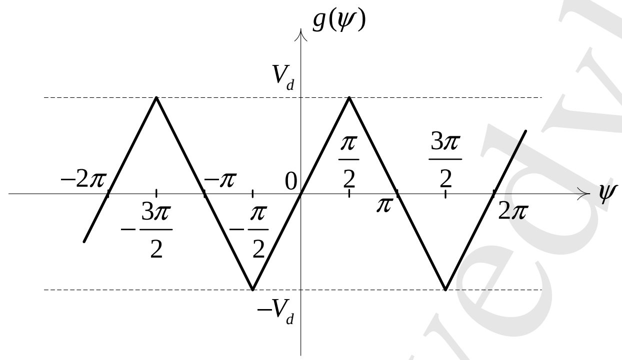


Figure 12.18

The phase plane is

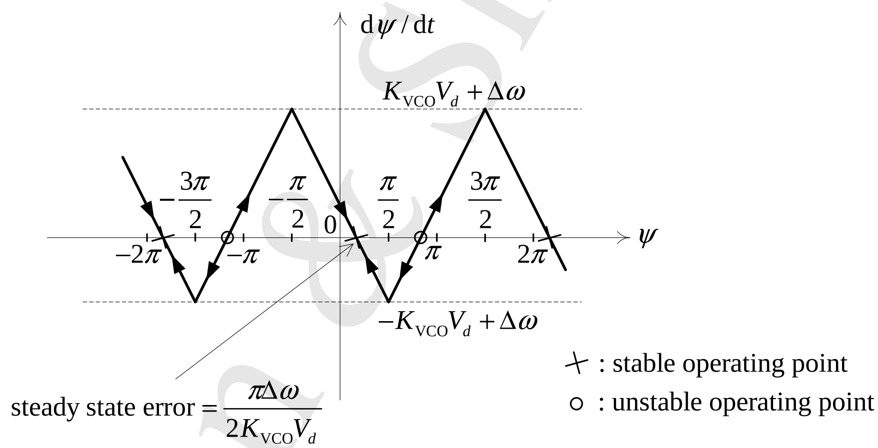


Figure 12.19

Remark: One feature of the above 2 characteristics is that their implementation is accomplished by digital circuits. See Gardner, "Phaselock techniques", for a discussion.

P12.8 (a)

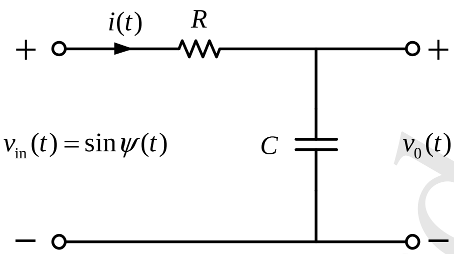


Figure 12.20

Using $i(t) = C\frac{\mathrm{d}v_0(t)}{\mathrm{d}t}$ and KVL, we have

$$v_{\text{in}}(t) = v_0(t) + RC\frac{\mathrm{d}v_0(t)}{\mathrm{d}t} \tag{12.34}$$

Now (refer to Fig. 12.10b): $\psi(t) = \theta(t) - \varphi(t)$ and $\varphi(t) = K_{\text{loop}}\int^t v_0(\lambda)\mathrm{d}\lambda$.

$$\Rightarrow \varphi(t) = \theta(t) - \psi(t);\ \frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} = K_{\text{loop}}v_0(t);\ \frac{\mathrm{d}^2\varphi(t)}{\mathrm{d}t^2} = K_{\text{loop}}\frac{\mathrm{d}v_0(t)}{\mathrm{d}t}$$
$$\Rightarrow \frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} = \frac{\mathrm{d}\theta(t)}{\mathrm{d}t} - \frac{\mathrm{d}\psi(t)}{\mathrm{d}t};\ \frac{\mathrm{d}^2\varphi(t)}{\mathrm{d}t^2} = \frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t^2} - \frac{\mathrm{d}^2\psi(t)}{\mathrm{d}t^2}$$

Therefore

$$RC\frac{\mathrm{d}v_0(t)}{\mathrm{d}t} + v_0(t) = \frac{RC}{K_{\text{loop}}}\frac{\mathrm{d}^2\varphi(t)}{\mathrm{d}t^2} + \frac{1}{K_{\text{loop}}}\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} = \sin\psi(t)$$
$$\Rightarrow \frac{RC}{K_{\text{loop}}}\frac{\mathrm{d}^2\psi(t)}{\mathrm{d}t^2} + \frac{1}{K_{\text{loop}}}\frac{\mathrm{d}\psi(t)}{\mathrm{d}t} + \sin\psi(t) = \frac{RC}{K_{\text{loop}}}\frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t^2} + \frac{1}{K_{\text{loop}}}\frac{\mathrm{d}\theta(t)}{\mathrm{d}t}$$
$$\text{or } \frac{\mathrm{d}^2\psi(t)}{\mathrm{d}t^2} + \frac{1}{RC}\frac{\mathrm{d}\psi(t)}{\mathrm{d}t} + \frac{K_{\text{loop}}}{RC}\sin\psi(t) = \frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t^2} + \frac{1}{RC}\frac{\mathrm{d}\theta(t)}{\mathrm{d}t}$$

(b) The state variables are $\psi(t)$ and $\dot{\psi}(t) = \frac{\mathrm{d}\psi(t)}{\mathrm{d}t}$. We can write the state equation(s) in two different ways. The standard approach is to define $x = \psi(t)$ and $y = \frac{\mathrm{d}\psi(t)}{\mathrm{d}t}$ as the state variables. Then the two state equations are:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}\psi}{\mathrm{d}t} = y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}^2\psi}{\mathrm{d}t^2} = -\frac{1}{RC}\frac{\mathrm{d}\psi(t)}{\mathrm{d}t} - \frac{K_{\text{loop}}}{RC}\sin\psi(t) = \frac{1}{RC}\left[-y - K_{\text{loop}}\sin x\right]$$

One can then solve (numerically) the above two 1st order nonlinear differential equations for state variables x, y (i.e., $\psi, \dot{\psi}$) and plot the phase plane, namely $\dot{\psi}$ versus ψ.

The other approach is to obtain directly a differential equation for $\dot{\psi}$ with ψ as the independent variable and solve this numerically. Proceeding as in the text (see page 518) we get:

$$\frac{\mathrm{d}\dot{\psi}}{\mathrm{d}\psi} = -\frac{1}{RC} - \frac{K_{\text{loop}}}{RC}\frac{\sin\psi}{\dot{\psi}}.$$

Remark: For the state (or phase) plane we assumed that the forcing function $\theta(t)$ is zero. Have only non-zero initial conditions.

(c) The phase plane plots are shown in Fig. 12.21 for $K_{\text{loop}} = 1/2$ and $K_{\text{loop}} = 2$ and with $RC = 1$.

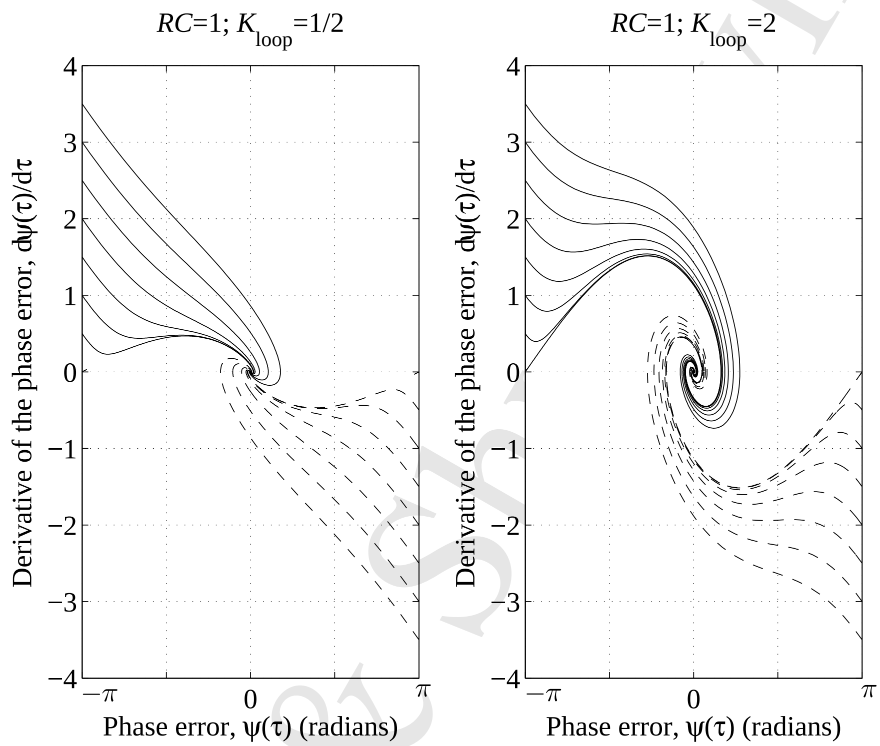


Figure 12.21

P12.9 Use the Laplace transform to obtain the transfer function and then convert to the time domain using the relationship between operators of $s \leftrightarrow \frac{\mathrm{d}}{\mathrm{d}t}$.

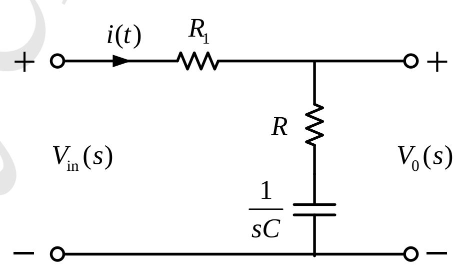


Figure 12.22

$$\frac{V_0}{V_{\text{in}}} = \frac{R + \frac{1}{sC}}{R + R_1 + \frac{1}{sC}} = \frac{1 + sRC}{1 + s(R + R_1)C}$$
$$\Rightarrow (R_1 + R)CsV_0(s) + V_0(s) = RCsV_{\text{in}}(s) + V_{\text{in}}(s)$$
$$\Rightarrow (R_1 + R)C\frac{\mathrm{d}v_0(t)}{\mathrm{d}t} + v_0(t) = RC\frac{\mathrm{d}v_{\text{in}}(t)}{\mathrm{d}t} + v_{\text{in}}(t)$$

As in P12.8, using the relationships: $v_{\text{in}}(t) = \sin\psi(t)$; $\varphi(t) = K_{\text{loop}}\int^t v_0(\lambda)\mathrm{d}\lambda$; $\varphi(t) =$

$\theta(t) - \psi(t)$ and observing that $\frac{\mathrm{d}v_{\text{in}}(t)}{\mathrm{d}t} = \cos\psi(t)\frac{\mathrm{d}\psi(t)}{\mathrm{d}t}$ we get

$$\frac{\mathrm{d}^2\psi(t)}{\mathrm{d}t^2} + \frac{1 + RC\cos\psi(t)}{(R_1+R)C}\frac{\mathrm{d}\psi(t)}{\mathrm{d}t} + \frac{K_{\text{loop}}}{(R_1+R)C}\sin\psi = \frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t^2} + \frac{1}{(R_1+R)C}\frac{\mathrm{d}\theta(t)}{\mathrm{d}t}$$

The state equations are obtained as follows. Let $x = \psi$ and $y = \frac{\mathrm{d}\psi}{\mathrm{d}t}$ be the state variables. Then the two state equations are:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= y \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -\frac{1 + RC\cos x}{(R_1+R)C}y - \frac{K_{\text{loop}}}{(R_1+R)C}\sin x \end{aligned} \tag{12.35}$$

Or the differential equation for $\dot{\psi}$ as a function of ψ is

$$\frac{\mathrm{d}\dot{\psi}}{\mathrm{d}\psi} = -\frac{1 + RC\cos\psi}{(R_1+R)C} - \frac{K_{\text{loop}}}{(R_1+R)C}\frac{\sin\psi}{\dot{\psi}}. \tag{12.36}$$

Remarks:

(i) As a quick (partial) check let $R = 0$ and $R_1 \to R$. Then equations (12.35), (12.36) should agree with those of P12.8.

(ii) One can write the state equations (12.35) with a non-zero forcing function, if desired.

(iii) One can use numerical methods to solve the nonlinear 1st order differential equation of (12.35) to obtain $\dot{\psi}, \psi$ and plot $\dot{\psi}$ versus ψ. It seems preferable to solve (12.36) and obtain $\dot{\psi}$ as a function of ψ. Why?

The phase plane plots are shown in Fig. 12.23 for $K_{\text{loop}} = 1/2$ and $K_{\text{loop}} = 2$ and with $R_1C = RC = 1$.

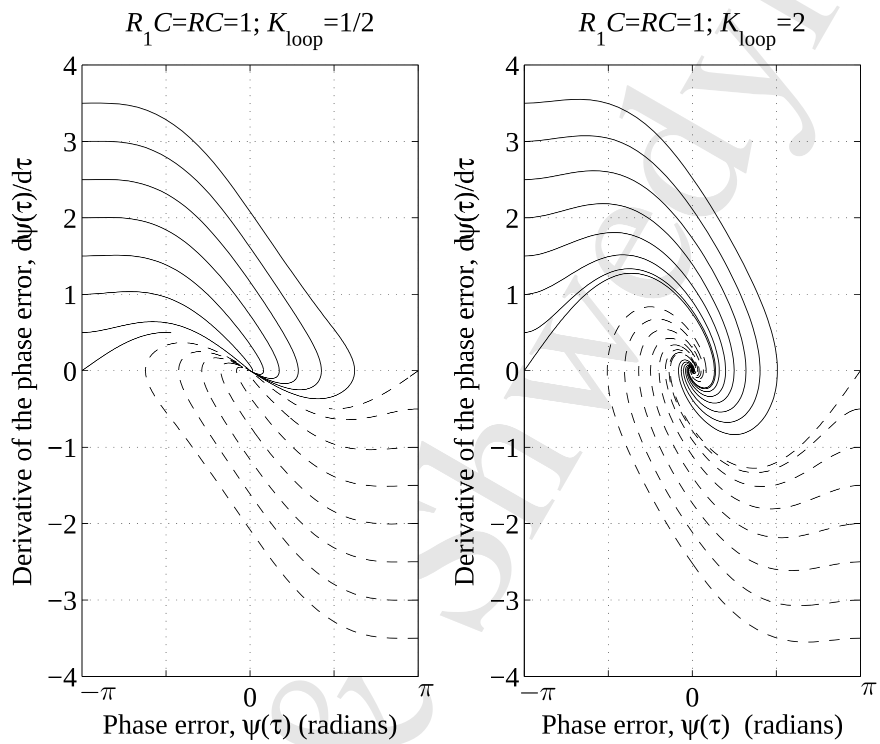


Figure 12.23

Remarks: For the interested reader, generic Matlab codes to generate the phase plane plots of P12.9 are given below.

```
t_range=[0:0.05:20];
d_phase_error=[-3.5:0.5:3.5, -10^(-4),10^(-4)];
PLL=inline('[-0.5*(1+cos(y(2)))*y(1)-(2)*0.5*sin(y(2));y(1)]','t','y');
% Set R_1C=RC=1, parameter K_loop is entered in this expression
for i=1:length(d_phase_error);
        if d_phase_error(i)<0
            phase_error=pi;
        else
            phase_error=-pi;
        end
        ini_con=[d_phase_error(i),phase_error];
        [T,Y]=ode45(PLL,t_range,ini_con);
        x=Y(:,2);fx=Y(:,1);
        figure(1);
        if d_phase_error(i)<0
            plot(x,fx,'k','linewidth',0.5,'linestyle','--');hold on;
        else
            plot(x,fx,'k','linewidth',0.5,'linestyle','-');hold on;
        end
end
```

P12.10 The transfer function of the inverting (ideal) operational amplifier is

$$\frac{V_0(s)}{V_{\text{in}}(s)} = -\frac{Z_f(s)}{Z_{\text{in}}(s)} = \frac{R+\frac{1}{sC}}{R_1} = -\frac{R}{R_1}\left(1+\frac{1}{RCs}\right). \tag{12.37}$$

Let $a \to \frac{1}{RC}$ and $R_1 = R$, we have the desired transfer function of $H(s) = 1+\frac{a}{s}$. The negative sign is easily accounted for by inserting an unity-gain inverting amplifier.

P12.11 The question is what straight line to use. Let us use the simplest one. Choose its slope to be that of $\sin\psi$ at the origin, which is $\frac{\mathrm{d}\sin\psi}{\mathrm{d}\psi}\Big|_{\psi=0} = 1$. Over the range $(-\pi/6, \pi/6)$ the maximum error occurs at $\pi/6$ and is $\pi/6 - \sin(\pi/6) = 0.024 \Rightarrow$ % error $= \frac{0.024}{\pi/6} \times 100 = 4.5\%$.

P12.12 (a) $\varphi(s) = \frac{K_{\text{loop}}H(s)}{s}\Psi(s)$; $\Psi(s) = \Theta(s) - \varphi(s)$.

$$\Rightarrow \varphi(s) = \frac{K_{\text{loop}}H(s)}{s}\left[\Theta(s)-\varphi(s)\right] \Rightarrow \varphi(s) = \frac{K_{\text{loop}}H(s)/s}{1+K_{\text{loop}}H(s)/s}\Theta(s) = T(s)\Theta(s).$$

For the reader knowledgeable in control theory the above result follows immediately by considering $\varphi(s)$ as the output of a (linear) unity feedback system with $\frac{K_{\text{loop}}H(s)}{s}$ as the forward loop transfer function.

(b) $\Psi(s) = \Theta(s) - \varphi(s) = \Theta(s) - T(s)\Theta(s) = [1-T(s)]\,\Theta(s)$.

P12.13

$$\begin{aligned}
\Theta(s) &= \frac{\Delta\omega}{s^2} + \frac{\theta_0}{s};\ T(s) = \frac{K_{\text{loop}}}{s+K_{\text{loop}}};\ 1-T(s) = \frac{s}{s+K_{\text{loop}}} \\
\Psi(s) &= \frac{\Delta\omega}{s(s+K_{\text{loop}})} + \frac{\theta_0}{s+K_{\text{loop}}} = \frac{\Delta\omega}{K_{\text{loop}}s} - \frac{\Delta\omega}{K_{\text{loop}}(s+K_{\text{loop}})} + \frac{\theta_0}{s+K_{\text{loop}}} \\
\psi(t) &= \frac{\Delta\omega}{K_{\text{loop}}}\left(1-\mathrm{e}^{-K_{\text{loop}}t}\right)u(t) + \theta_0\mathrm{e}^{-K_{\text{loop}}t}u(t).
\end{aligned}$$

The steady-state error is $\frac{\Delta\omega}{K_{\text{loop}}}$ which agrees.

Matlab work to be done for $\psi(0) = \pi/4, -\pi/10$; $\omega_d = 0.1, 0.5$.

For the plot of the transient response, write the error expression in terms of $\omega_d \equiv \frac{\Delta\omega}{K_{\text{loop}}}$ and $t_n \equiv K_{\text{loop}}t$. Note that $\theta_0 = \psi(0)$, $\psi(t) = \omega_d(1-\mathrm{e}^{-t_n}) + \psi(0)\mathrm{e}^{-t_n}$.

P12.14 From P12.12 we have $\Psi(s) = [1-T(s)]\,\Theta(s)$ where $T(s) = \frac{K_{\text{loop}}H(s)/s}{1+K_{\text{loop}}H(s)/s}$

$$\begin{aligned}
&\Rightarrow 1-T(s) = \frac{s}{s+K_{\text{loop}}H(s)} = \frac{s^2}{s^2+K_{\text{loop}}(s+a)} \\
&\therefore \Psi(s) = \frac{s^2}{s^2+K_{\text{loop}}(s+a)}\cdot\frac{\Delta\omega+s\theta_0}{s^2} = \frac{s\theta_0+\Delta\omega}{s^2+K_{\text{loop}}(s+a)} \\
&\Rightarrow \lim_{s\to 0} s\Psi(s) = 0 = \lim_{t\to\infty}\psi(t) = \text{steady-state error.}
\end{aligned}$$

Result agrees with that of text - see discussion on page 517.

P12.15 (a) From P12.9 we have

$$H(s) = \frac{R}{R_1+R}\frac{s+\frac{1}{RC}}{s+\frac{1}{(R_1+R)C}}$$

$$\therefore K = \frac{R}{R_1+R};\ a = \frac{1}{RC};\ b = \frac{1}{(R_1+R)C}.$$

(b)

$$\begin{aligned}
T(s) &= \frac{K_{\text{loop}}H(s)}{s+K_{\text{loop}}H(s)} = \frac{K_{\text{loop}}(s+a)}{s(s+b)+K_{\text{loop}}(s+a)} \\
&= \frac{K_{\text{loop}}(s+a)}{s^2+(K_{\text{loop}}+b)s+aK_{\text{loop}}} \\
1-T(s) &= \frac{s(s+b)}{s^2+(K_{\text{loop}}+b)s+aK_{\text{loop}}} \\
\therefore \Psi(s) &= \frac{s(s+b)}{s^2+(K_{\text{loop}}+b)s+aK_{\text{loop}}}\cdot\frac{\Delta\omega+s\theta_0}{s^2} \\
\Rightarrow \lim_{s\to 0} s\Psi(s) &= \lim_{s\to 0}\frac{(s+b)(\Delta\omega+s\theta_0)}{s^2+(K_{\text{loop}}+b)s+aK_{\text{loop}}} = \frac{b\Delta\omega}{aK_{\text{loop}}} \\
&= \frac{R}{R_1+R}\frac{\Delta\omega}{K_{\text{loop}}} \quad \text{(the steady-state error).}
\end{aligned}$$

Compared to the 1st-order loop the steady state error is reduced by the factor $\frac{R}{R_1+R}$. However compared to the perfect integrator it is non-zero since the filter does not "realize" the perfect integrator but only approximates it. This approximation becomes better as R_1 increases relative to R (perfect when $R_1 \to \infty$). But practically R_1 can only be made so large otherwise the high gain amplifier used for compensation shall start giving problems.

P12.16 (a) The phase detector output is:

$$[V\sin(\omega_c t+\theta(t)) + \mathbf{n}_I(t)\cos\omega_c t + \mathbf{n}_Q(t)\sin\omega_c t]\,K\cos(\omega_c t + t\varphi(t))$$

Using the trigonometric identities

$$\cos x\cos y = \frac{\cos(x+y)+\cos(x-y)}{2} \text{ and } \sin x\cos y = \frac{\sin(x+y)+\sin(x-y)}{2}$$

and ignoring the $2\omega_c$ terms which are fairly easily filtered out, we get

$$\mathbf{v}_{\text{out}}(t) = \frac{KV}{2}\sin(\theta(t)-\varphi(t)) + \frac{K}{2}\mathbf{n}_I(t)\cos\varphi(t) - \frac{K}{2}\mathbf{n}_Q(t)\sin\varphi(t)$$

(b) Define the noise term $\mathbf{n}'(t) \equiv \frac{K}{2}\mathbf{n}_I(t)\cos\varphi(t) - \frac{K}{2}\mathbf{n}_Q(t)\sin\varphi(t)$. The phase detector output is $\mathbf{v}_{\text{out}}(t) = \frac{KV}{2}\sin(\psi(t)) + \mathbf{n}'(t)$ and the block diagram follows immediately.

(c) The output of the phase detector is $v_0(t) = A\sin\psi(t)$. The output of the loop filter or input to the VCO is

$$v_{\text{in}}(t) = Kh(t) * A\sin\psi(t) + Kh(t) * \mathbf{n}'(t)$$

$$\varphi(t) = \int_0^t v_{\text{in}}(\lambda)\mathrm{d}\lambda \Rightarrow \frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} = v_{\text{in}}(t)$$

and $$\psi(t) = \theta(t) - \varphi(t) \Rightarrow \frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} = \frac{\mathrm{d}\theta(t)}{\mathrm{d}t} - \frac{\mathrm{d}\psi(t)}{\mathrm{d}t}$$

$$\Rightarrow \quad \frac{\mathrm{d}\psi(t)}{\mathrm{d}t} + KA\int_{-\infty}^{t} h(t-\lambda)\sin\psi(\lambda)\mathrm{d}\lambda = \frac{\mathrm{d}\theta(t)}{\mathrm{d}t} - K\int_{-\infty}^{t} h(t-\lambda)\mathbf{n}(\lambda)\mathrm{d}\lambda$$

The reasoning that $\mathbf{n}'(t)$ can be modeled as a Gaussian process, indeed as a "white" Gaussian process can be found in "Principles of Coherent Communication" by A.J. Viterbi (McGraw-Hill); section 2.7, pages 28–34.

P12.17 (a)

$$E\{\boldsymbol{\varphi}(t)\} = E\left\{\int^{t} Kh(\lambda) * \mathbf{n}'(\lambda)\mathrm{d}\lambda\right\} = K\int^{t} Kh(\lambda) * E\left\{\mathbf{n}'(\lambda)\right\}\mathrm{d}\lambda = 0$$

since $E\{\mathbf{n}'(\lambda)\} = 0$. Remember that expectation is a linear operation, as in integration and convolution, so the operations can be interchanged. Further, $E\{\boldsymbol{\psi}(t)\} = -E\{-\boldsymbol{\psi}(t)\} = -E\{\boldsymbol{\varphi}(t)\} \approx 0$.

(b) The block diagram relating $\boldsymbol{\varphi}(t)$ to $\mathbf{n}'(t)$ is

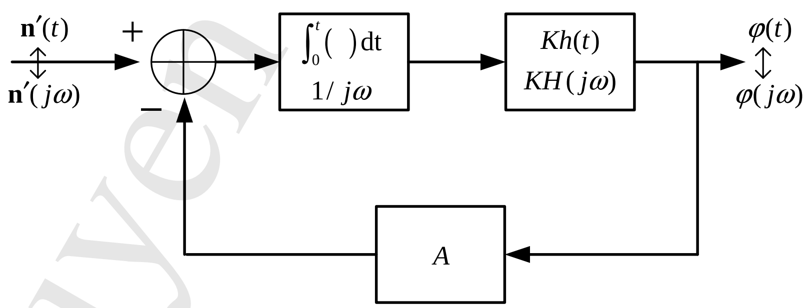


Figure 12.24

The transfer function relating $\boldsymbol{\varphi}(j\omega)$ to $\mathbf{n}'(j\omega)$ is

$$\frac{KH(j\omega)/j\omega}{1+AKH(j\omega)/j\omega} \Rightarrow S_{\boldsymbol{\varphi}}(\omega) = \left|\frac{KH(j\omega)/j\omega}{1+AKH(j\omega)/j\omega}\right|^2 S_{\mathbf{n}'}(\omega) \quad \text{(watts/Hz).}$$

Obviously $S_{\boldsymbol{\psi}}(\omega) = S_{\boldsymbol{\varphi}}(\omega)$.

(b) Multiply the transfer function (top and bottom) by A

$$\begin{aligned} \therefore S_{\boldsymbol{\varphi}}(\omega) &= \left|\frac{AKH(j\omega)/j\omega}{A\left(1+AKH(j\omega)/j\omega\right)}\right|^2 S_{\mathbf{n}'}(\omega) = \frac{1}{A^2}\left|\frac{AKH(j\omega)/j\omega}{1+AKH(j\omega)/j\omega}\right|^2 \frac{N_0}{2} \\ &= \frac{1}{A^2}\left|T(j\omega)\right|^2\frac{N_0}{2} = S_{\boldsymbol{\psi}}(\omega) \end{aligned}$$

where $T(j\omega)$ is the closed loop transfer function as seen by the following block diagram.

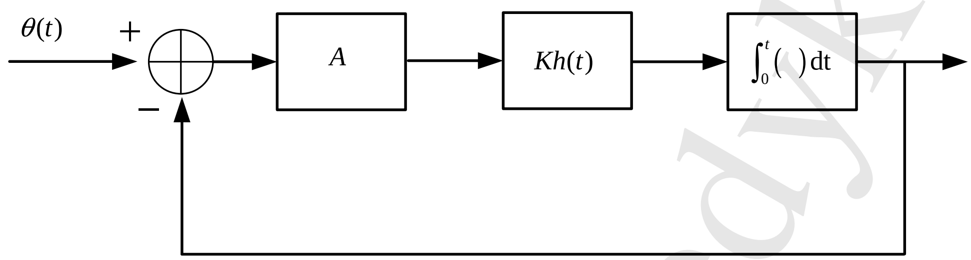


Figure 12.25

(d)

$$\begin{aligned}
\sigma_{\psi}^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{\psi}(\omega)\mathrm{d}\omega = \frac{1}{\pi}\int_{0}^{\infty} S_{\psi}(\omega)\mathrm{d}\omega \quad \text{(using even property of a PSD)} \\
&= \frac{1}{\pi}\int_{0}^{\infty} \frac{N_0}{2A^2}\left|T(j\omega)\right|^2 \mathrm{d}\omega = \frac{N_0}{A^2}\left[\frac{1}{2\pi}\int_{0}^{\infty}\left|T(j\omega)\right|^2 \mathrm{d}\omega\right] = \frac{N_0 B_{\text{loop}}}{A^2} \quad \text{(watts).}
\end{aligned}$$

P12.18 Raising the signal to the nth power (ignore noise) means

$$\left[\sqrt{E_s}\left(\mathrm{e}^{j\frac{2\pi k}{m}}\right)\right]^n = \left(\sqrt{E_s}\right)^n \mathrm{e}^{j\frac{2\pi n k}{m}}.$$

Of course it is desirable to raise it to the smallest possible power to get a (non-zero) constant value, more specifically a (non-zero) constant positive real value for all k. It follows that $n = m$.

P12.19 Again one wishes to create a (non-zero) constant positive real value for all k_l. It follows that $n = \text{LCM}\{k_l\}$. Here LCM means the least common multiple. For example, for Fig. 12.26, $m = \text{LCM}\{4, 8, 12\} = 24$. (Taking $C = 3,\ k_1 = 4,\ k_2 = 8,\ k_3 = 12$).

P12.20 Every woman for herself here.