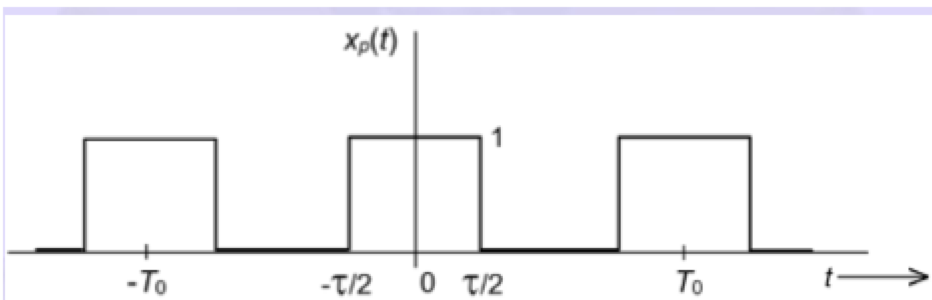
Problem Set 1

Signals and Systems

1. For the periodic signal,
   a. show that

$$x_n = \left(\frac{\tau}{T_0}\right)\text{sinc}(nf_0\tau)$$

   b. Find the average power in the signal
   c. Find the discrete power spectral density
   d. Find the fraction of the total power contained in the first five harmonics assuming $\tau = T_0/4$.
   e. Use matlab to plot both $x_p(t)$ and the first five harmonics assuming $\tau = T_0/4$.



2. A. Use the duality property, to find the Fourier transform of

$$z(t) = A\,\text{sinc}\,2Wt$$

   B. Find the total energy in z(t).

3. Make use of the result of Problem 2 to find the energy in the signal

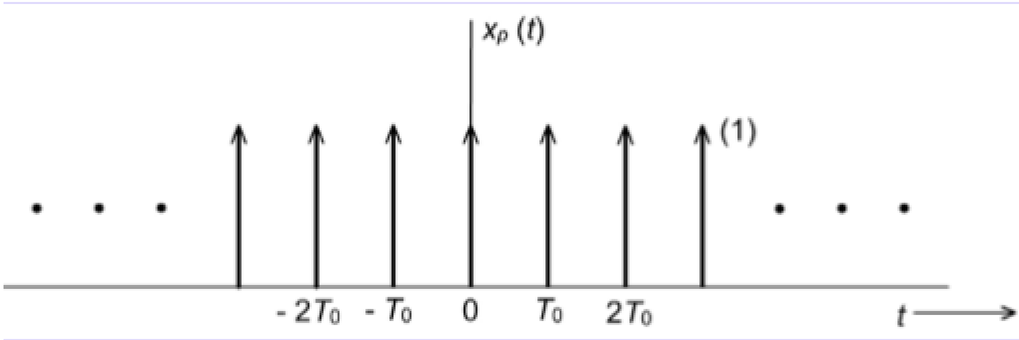
$$x(t) = 2AW\,\text{sinc}\,(2Wt).$$

4. Evaluate the integrals:

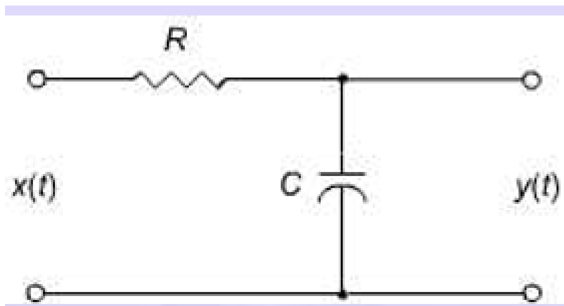
$$\text{(a)} \quad \int_{-4}^{4} t^3\,\delta(t-5)\,dt$$

$$\text{(b)} \quad \int_{4.9}^{5.1} t^3\,\delta(t-5)\,dt$$

5. Find the Fourier transform of the uniform impulse train

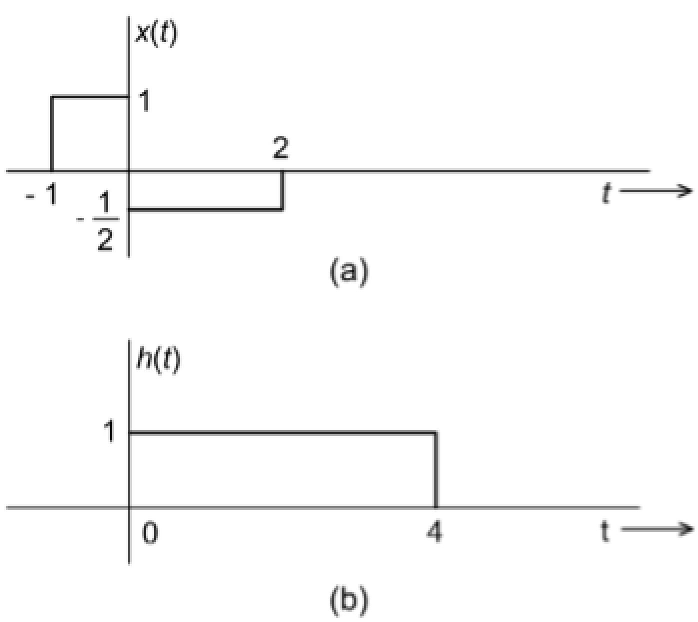


6. A. Find the frequency response H(f) and impulse response h(t) of the circuit:



B. Find the 3-dB bandwidth of the circuit

7. The input x(t) and the impulse response h(t) of a LTI system are as shown in the figure below, find the output y(t)



a. Find the Fourier transfor of x(t) and h(t)
b. Use the convolution intergral to find y(t)

c. Find the equivalent time duration of of x(t)
d. Find the equivalent rectangular bandwidth of x(t)

8. The input x(t) and the impulse response h(t) of a LTI system are

$$x(t) = \begin{cases} e^{-\alpha t}, & t \geq 0 \\ 0, & otherwise \end{cases}$$

$$h(t) = \begin{cases} e^{-\beta t}, & t \geq 0 \\ 0, & otherwise \end{cases}$$

a. Find X(f) and H(f)
b. Find the output energy spectral density
c. Use the convolution integral to find y(t)

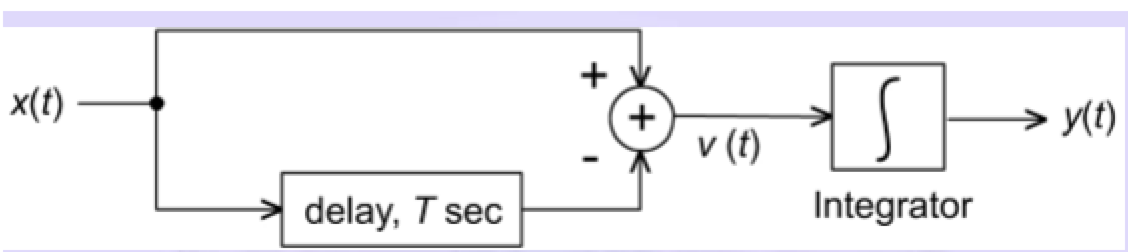
9. The input x(t) and the impulse response h(t) of a LTI system are

$$x(t) = \begin{cases} 2, & |t| < 2 \\ 0, & outside \end{cases}$$

$$h(t) = \begin{cases} 2e^{-t}, & t \geq 0 \\ 0, & outside \end{cases}$$

find the output y(t)

10. Find the impulse response of the system:



11. Fnd the period, and find the Fourier series representation of the following periodic signals

a) $x(t) = 2\cos(200\pi t) + 5\sin(400\pi t)$

b) $x(t) = 2\cos(200\pi t) + 5\sin(300\pi t)$

c) $x(t) = 2\cos(150\pi t) + 5\sin(250\pi t)$

12. Consider the following signal

$$x(t) = \cos(2\pi f_1 t) + a\sin(2\pi f_1 t)$$
$$= X_A(a)\cos(2\pi f_1 t + X_p(a))$$

a) Find $X_A(a)$.

b) Find $X_p(a)$.

c) What is the power of $x(t)$, P_x?

d) Is $x(t)$ periodic? If so what is the period and the Fourier series representation of $x(t)$?

13. The input-output characteristic of a channel is described by the differential equation:

$$dy(t)/dt + 2y(t) = 4x(t)$$

a. Find the transfer function, *H*(*f*), of the channel.
b. Find the 3-dB bandwidth of the channel.

14. Let m(t) be a baseband signal with Fourier transform

$$M(f) = \begin{cases} m_0 & -f_m \le f \le f_m \\ 0 & otherwise \end{cases}$$

Let $\hat{m}(t)$ be the Hilbert transform of m(t), find the energy in $\hat{m}(t)$.

15. The impulse response of a linear time-invariant system is given by:

$$h(t) = e^{-2\pi Bt}u(t)$$

a. Is this system causal? Explain
b. Is this system stable? Explain
c. Find $\int_0^5 h(t)\delta(t-1)dt$

16. The Fourier transform of a time signal $m(t)$ is given by:

$$M(f) = \frac{1}{1+j(f/B)}$$

a. Find the 6-dB bandwidth of the message
b. Find $M(f)\delta(f-B)$

17. Consider the signal $g(t) = e^{-a|t|}$.

a. Explain why this signal is an energy signal.
b. Find and sketch the energy spectral density of *g* (*t*).
c. Find the total energy in *g* (*t*).
d. Find the 3-dB bandwidth of *g* (*t*).
e. Find the fraction of the signal energy contained in the bandwidth of Part d relative to the total signal energy.

18. A periodic signal x(t) defined over one period is:

$$x(t) = \begin{cases} a|t| & -T_0/2 \le t \le -T_0/2 \\ \\ 0 & |f| > W \end{cases};$$

Find the Fourier series coefficient an, n = 1, 3, 5.

19. Consider the signal $g(t) = e^{-2\pi Bt}u(t)$.

a. Find the autocorrelation function $R_g(\tau)$.
b. Find the energy spectral density.
c. Find the energy in the signal.
d. Find the 3-dB bandwidth of the signal.

SIGNALS AND SYSTEMS PROBLEM SET SOLUTION

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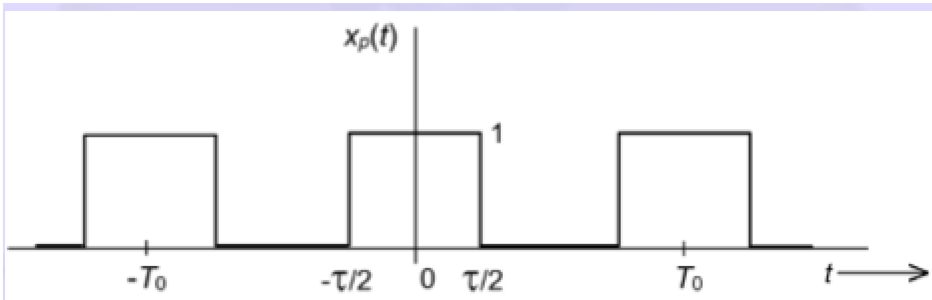
DECEMBER 21, 2020

Problem 1

Q: For the periodic signal,



a- Show that

$$\boldsymbol{x_n = \left(\frac{\tau}{T_0}\right) \cdot sinc(n f_0 \tau)}$$

$$x_n = \frac{1}{T_0} \cdot \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x_p(t)\, e^{-j2\pi n f_0 t} dt$$

$$x_n = \frac{1}{T_0} \cdot \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{-j2\pi n f_0 t} dt$$

$$x_n = \frac{1}{T_0} \cdot \frac{e^{-j\pi n f_0 \tau} - e^{j\pi n f_0 \tau}}{-j2n\pi f_0}$$

$$x_n = \frac{e^{-j\pi n f_0 \tau} - e^{j\pi n f_0 \tau}}{-j2\pi n f_0 T_0}$$

$using\ the\ identity$:

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}$$

$$x_n = \frac{\sin(\pi n f_0 \tau)}{\pi n f_0 T_0} \cdot \frac{\tau}{\tau}$$

$$x_n = \frac{\sin(\pi n f_0 \tau)}{\pi n f_0 \tau} \cdot \frac{\tau}{T_0}$$

$$using\ the\ identity:$$

$$sinc(x) = \frac{\sin(\pi x)}{\pi x}$$

$$x_n = \frac{\tau}{T_0} \cdot sinc(n f_0 \tau)$$

b- **Find the average power in the signal**

$$P = \frac{1}{T_0} \cdot \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \left|x_p(t)\right|^2 dt$$

$$P = \frac{1}{T_0} \cdot \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} dt$$

$$P = \frac{\tau}{T_0}$$

c- **Find the discrete power spectral density**

$$S_x(f) = \sum_{n=-\infty}^{\infty} |x_n|^2 \delta(f - n f_0)$$

d- Find the fraction of the total power contained in the first five harmonics assuming $\boldsymbol{\tau = \frac{T_0}{4}}$

$$Assume\ \tau = \frac{T_0}{4}$$

$$P_5 = \left(\frac{\tau}{T_0}\right)^2 \cdot \sum_{n=-5}^{5} sinc^2(nf_0\tau)$$

$$P_5 = 2 \cdot \left(\frac{1}{4}\right)^2 \cdot \sum_{n=0}^{5} sinc^2(\frac{n}{4})$$

$$P_5 = \frac{1}{16} \cdot [sinc^2(0) + sinc^2(0.25) + sinc^2(0.5) + sinc^2(0.75)$$

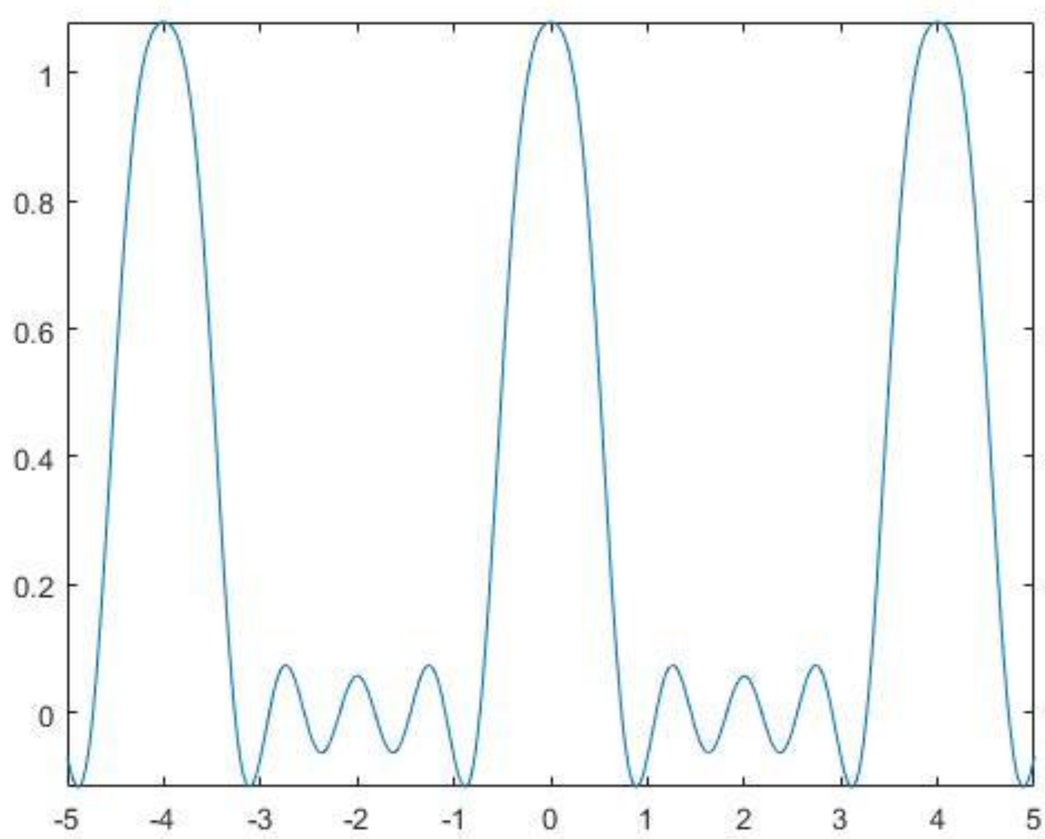
$$+ sinc^2(1) + sinc^2(1.25)]$$

$$P_5 = 2.65697826401$$

e- Use MATLAB to plot both $x_p(t)$ and the first five harmonics assuming

$$\tau = \frac{T_0}{4}$$

```
% Add Symbolic Library %
syms t x n c f0;
% Frequency (unspecified in the problem 1/T0) %
f0 = 0.25;
% Complex Fourier Series Constant %
c(n) = (1/4)*sinc(n/4);
% Building up the series (up to 5 terms) %
x(t) = 1/4
+ symsum(c(n)*exp(1j*2*pi*n*f0*t), n, -5, -1)
+ symsum(c(n)*exp(1j*2*pi*n*f0*t), n, 1, 5);
% Plotting the results %
fplot(t, x);
```



Problem 2

Q:

a- Use the duality property, to find the Fourier transform of

$$\boldsymbol{z(t) = Asinc(2Wt)}$$

$$from\ the\ Fourier\ Transform\ pairs\ we\ have:$$

$$\mathcal{F}\{rect(t)\} = sinc(f)$$

$$By\ frequency\ scaling\ property:$$

$$\mathcal{F}\left\{rect(\frac{t}{2W})\right\} = 2Wsinc(2Wf)$$

$$By\ amplitude\ scaling\ property:$$

$$\mathcal{F}\left\{\frac{A}{2W} \cdot rect\left(\frac{t}{2W}\right)\right\} = Asinc(2Wf)$$

$$By\ the\ duality\ property:$$

$$\mathcal{F}\{Asinc(2Wt)\} = \frac{A}{2W} \cdot rect\left(\frac{-f}{2W}\right)$$

$$As\ the\ rectangular\ pulse\ is\ an\ even\ function:$$

$$\mathcal{F}\{Asinc(2Wt)\} = \frac{A}{2W} \cdot rect\left(\frac{f}{2W}\right)$$

b- Find the total energy in z(t).

$$Using\ Parsevla's\ Theorem:$$

$$E = \int_{-\infty}^{\infty} |X^2(f)| \cdot df$$

$$E = \int_{-W}^{W} \left(\frac{A}{2W}\right)^2 \cdot df$$

$$E = \left(\frac{A}{2W}\right)^2 \cdot [W - (-W)]$$

$$E = \frac{A^2}{2W}$$

Problem 3

Q: Make use of the result of Problem 2 to find the energy in the signal

$$\boldsymbol{z(t) = 2AWsinc(2Wt)}$$

$Assume\ A = 2AW, From\ Problem\ 2:$

$$E = \frac{A^2}{2W}$$

$We\ get:$

$$E = \frac{(2W)^2 A^2}{2W} = 2A^2 W$$

Problem 4

Q: Evaluate the integrals:

a- $\int_{-4}^{4} t^3 \cdot \delta(t-5)dt$

$$Using\ the\ sampling\ property\ 5\ isn't\ in\ the\ interval\ [-4,4]\ so:$$

$$\int_{-4}^{4} t^3 \cdot \delta(t-5)dt = 0$$

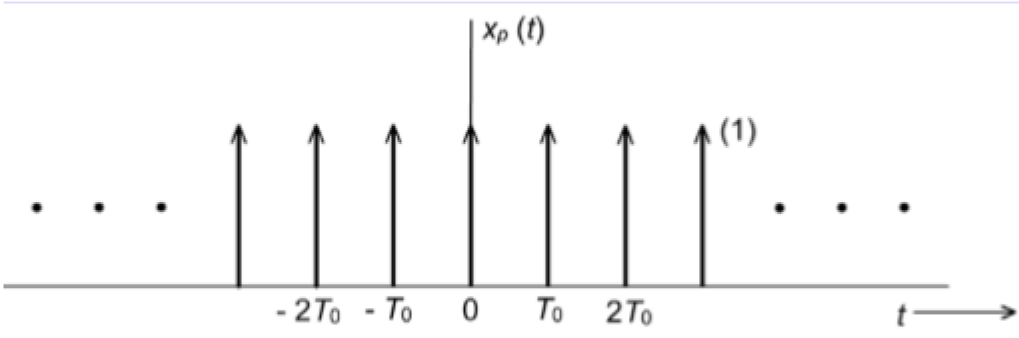
b- $\int_{4.9}^{5.1} t^3 \cdot \delta(t-5)dt$

$$Using\ the\ sampling\ property\ 5\ is\ in\ the\ interval\ [4.9, 5.1]\ so:$$

$$\int_{4.9}^{5.1} t^3 \cdot \delta(t-5)dt = {t^3}_{t=5} = 5^3 = 125$$

Problem 5

Q: Find the Fourier transform of the uniform impulse train



$Let's\ define\ x_p(t)\ as$:

$$x_p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$$

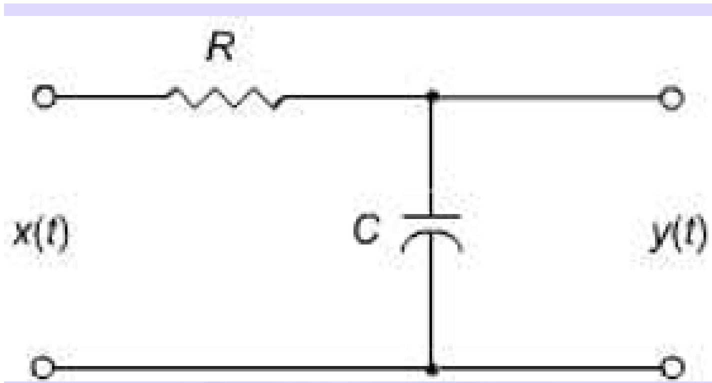
$$\mathcal{F}\{x_p(t)\} = \sum_{n=-\infty}^{\infty} \mathcal{F}\{\delta(t - nT_0)\}$$

$$\mathcal{F}\{x_p(t)\} = \sum_{n=-\infty}^{\infty} e^{-j2\pi n f T_0}$$

Problem 6

Q:

a- **Find the frequency response H(f) and impulse response h(t) of the circuit:**



$$Z_C = \frac{-j}{2\pi f C} \quad Z_R = R$$

$$using\ VDR:$$

$$Y(f) = \frac{Z_C}{Z_R + Z_C} \cdot X(f)$$

$$Y(f) = H(f) \cdot X(f)$$

$$H(f) = \frac{Z_C}{Z_R + Z_C} = \frac{\frac{-j}{2\pi f C}}{R + \frac{-j}{2\pi f C}}$$

$$H(f) = \frac{\frac{1}{RC}}{\frac{1}{RC} + j2\pi f} = \frac{1}{1 + j2\pi f RC}$$

$$from\ the\ Fourier\ Transform\ pairs:$$

$$\mathcal{F}\{ke^{-at}u(t)\} = \frac{k}{a + j2\pi f}$$

$$\mathcal{F}^{-1}\{H(f)\} = \mathcal{F}^{-1}\left\{\frac{\frac{1}{RC}}{\frac{1}{RC} + j2\pi f}\right\} = \frac{1}{RC}e^{-\frac{t}{RC}} \cdot u(t)$$

$$h(t) = \frac{1}{RC}e^{-\frac{t}{RC}} \cdot u(t)$$

b- Find the 3-dB bandwidth of the circuit

$$20\log\left(\frac{|H(f)|}{|H(0)|}\right) = -3dB$$

$$|H(f)| = \frac{1}{\sqrt{1 + (2\pi fRC)^2}}$$

$$|H(0)| = 1$$

$$20\log\left(\frac{1}{\sqrt{1 + (2\pi fRC)^2}}\right) = -3dB$$

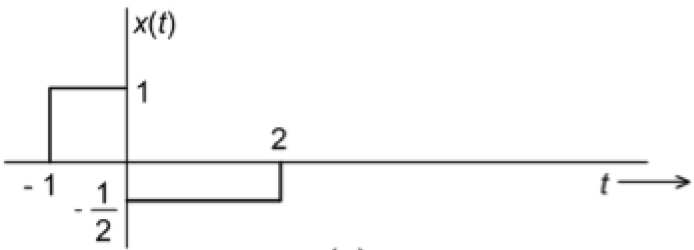
$$-10\log(1 + (2\pi fRC)^2) = -3dB$$

$$1 + (2\pi fRC)^2 = 10^{0.3}$$

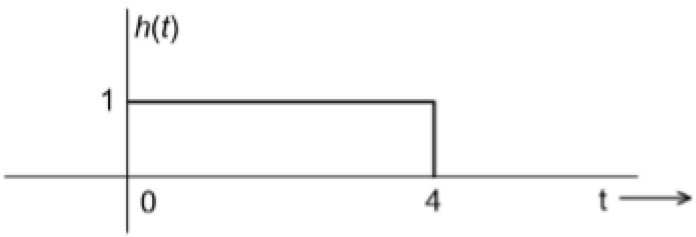
$$BW = f = \frac{\sqrt{10^{0.3} - 1}}{2\pi RC} = \frac{1}{2\pi RC}$$

Problem 7

Q: The input x(t) and the impulse response h(t) of a LTI system are as shown in the figure below, find the output y(t)



(a)



(b)

a- Find the Fourier transform of x(t) and h(t)

$$\mathcal{F}\{x(t)\} = \mathcal{F}\left\{rect\left(t+\frac{1}{2}\right) - \frac{1}{2}rect\left(\frac{t-1}{2}\right)\right\}$$

$$\mathcal{F}\{x(t)\} = sinc(f)\cdot e^{j\pi f} - sinc(2f)\cdot e^{-j\pi f}$$

$$\mathcal{F}\{h(t)\} = \mathcal{F}\left\{rect\left(\frac{t-2}{4}\right)\right\}$$

$$\mathcal{F}\{h(t)\} = 4sinc(4f)\cdot e^{-j\pi f}$$

b- Use the convolution integral to find y(t)

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\cdot h(t-\tau)d\tau$$

$$y(1 \leq t) = 0$$

$$y(0 \leq t < 1) = \int_{-1}^{-t} d\tau = 1 - t$$

$$y(-2 \leq t < 0) = \int_{-1}^{0} d\tau - \frac{1}{2}\int_{0}^{-t} d\tau = 1 + \frac{t}{2}$$

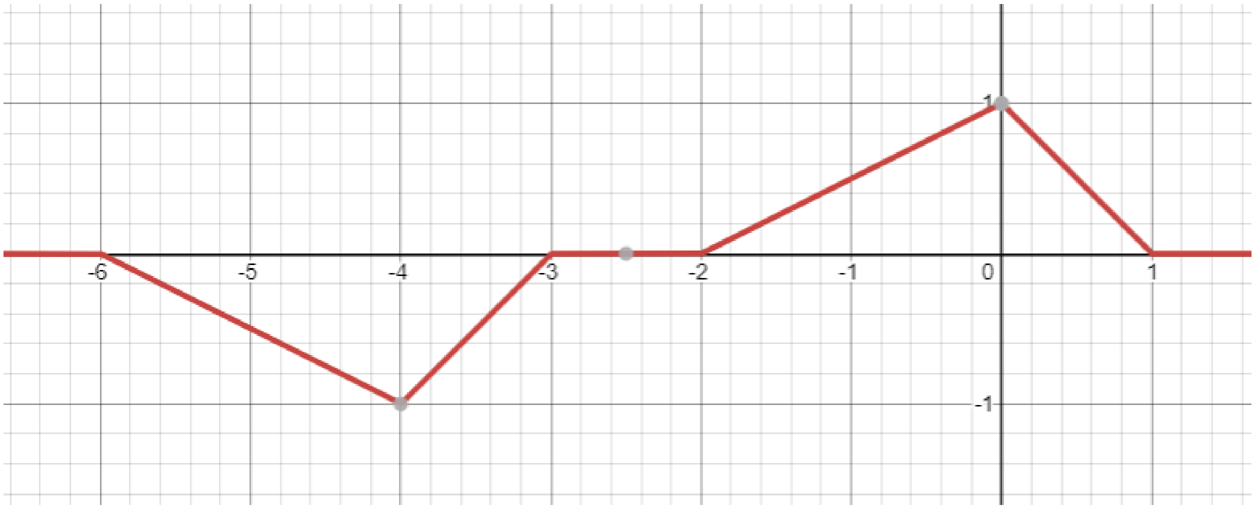
$$y(-3 \leq t < -2) = \int_{-1}^{0} d\tau - \frac{1}{2}\int_{0}^{2} d\tau = 1 - \frac{1}{2} \cdot 2 = 0$$

$$y(-4 \leq t < -3) = \int_{-t-4}^{0} d\tau - \frac{1}{2}\int_{0}^{2} d\tau = t + 3$$

$$y(-6 \leq t < -4) = -\frac{1}{2}\int_{-t-4}^{2} d\tau = \frac{-t}{2} - 3$$

$$y(t < -6) = 0$$

$$y(t) = \begin{cases} 0, & t < -6 \\ \dfrac{-t}{2} - 3, & -6 \leq t < -4 \\ t + 3, & -4 \leq t < -3 \\ 0, & -3 \leq t < -2 \\ 1 + \dfrac{t}{2}, & -2 \leq t < 0 \\ -1 - t, & 0 \leq t < 1 \\ 0, & 1 \leq t \end{cases}$$



c- Find the equivalent time duration of x(t)

$$T_{EQ} = \frac{\left(\int_{-\infty}^{\infty} |x(t)| dt\right)^2}{\int_{-\infty}^{\infty} |x(t)|^2 dt}$$

$$T_{EQ} = \frac{\left((0 - -1) \cdot |1| + (2 - 0) \cdot \left|\frac{-1}{2}\right|\right)^2}{\left((0 - -1) \cdot 1 + (2 - 0) \cdot \frac{1}{4}\right)} = \frac{4}{\frac{3}{2}} = 2.67\ s$$

d- Find the equivalent rectangular bandwidth of x(t)

Usually we use the following formula:

$$B_{EQ} = \frac{\int_{-\infty}^{\infty} |X(f)|^2 dt}{2|X(0)|^2}$$

But in that case the signal is not a baseband, hence we have to define the equivalent band to be the band containing the peak energy, and the equivalent rectangle has a height of that peak value and a width (bandwidth) which will result in having the same amount of energy in both the rectangle and the original ESD (as the bandwidth is defined as the width of the single sideband we will calculate for half the energy). Hence the equivalent bandwidth is given by:

$$B_{EQ} = \frac{\int_{-\infty}^{\infty} |X(f)|^2 dt}{2|X_{max}|^2}$$

$$\int_{-\infty}^{\infty} |X(f)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{3}{2}$$

$$|X_{max}|^2 = 1.264$$

$$B_{EQ} = \frac{1.5}{2.528} = 0.59\ Hz$$

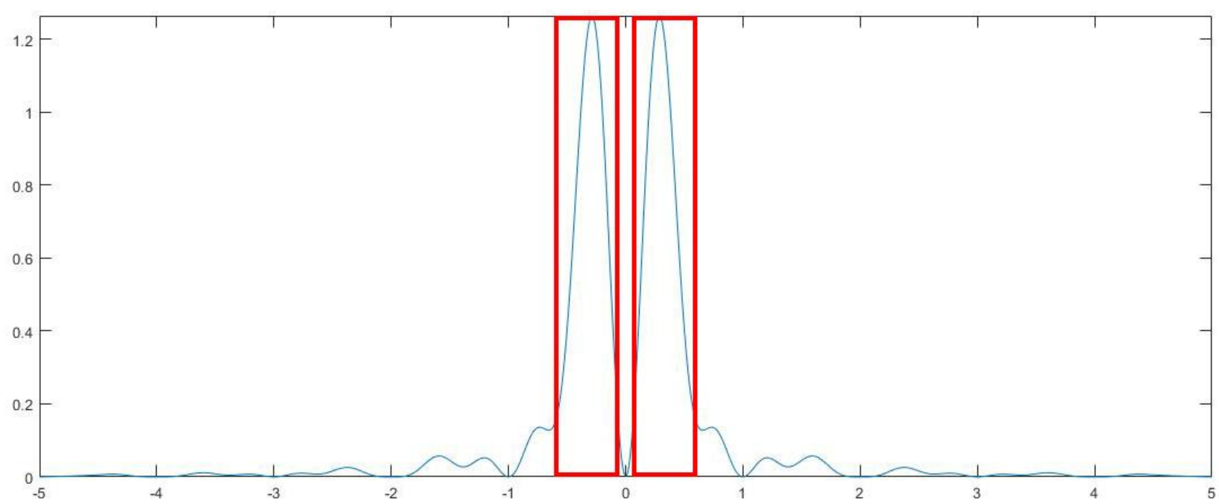


Figure 1: ESD showing the equivalent band as red rectangles

Problem 8

Q: The input x(t) and the impulse response h(t) of a LTI system are

$$\boldsymbol{x(t)} = \begin{cases} \boldsymbol{e^{-\alpha t}} & \boldsymbol{t \geq 0} \\ \boldsymbol{0} & \boldsymbol{otherwise} \end{cases}$$

$$\boldsymbol{h(t)} = \begin{cases} \boldsymbol{e^{-\beta t}} & \boldsymbol{t \geq 0} \\ \boldsymbol{0} & \boldsymbol{otherwise} \end{cases}$$

a- **Find X(f) and H(f)**

$$\mathcal{F}\{ke^{-at}u(t)\} = \frac{k}{a + j2\pi f}$$

$$X(f) = \frac{1}{\alpha + j2\pi f}$$

$$H(f) = \frac{1}{\beta + j2\pi f}$$

b- **Find the output energy spectral density**

$$S_Y(f) = |Y(f)|^2 = |H(f)|^2 \cdot |X(f)|^2$$

$$S_Y(f) = \frac{1}{\alpha^2 + (2\pi f)^2} \cdot \frac{1}{\beta^2 + (2\pi f)^2}$$

c- **Use the convolution integral to find y(t)**

$$y(t) = \int_{-\infty}^{\infty} e^{-\alpha\tau}u(\tau) \cdot e^{-\beta(t-\tau)}u(t-\tau)d\tau$$

$$for\ \tau < 0, u(\tau)\ becomes\ 0$$

$$for\ \tau > t, u(t-\tau)\ becomes\ 0$$

$as\ a\ result, for\ t < 0, \tau \leq t\ so\ \tau \leq 0, u(\tau)\ becomes\ 0$

$$y(t \geq 0) = \int_0^t e^{-\alpha\tau} \cdot e^{-\beta(t-\tau)} d\tau$$

$$y(t \geq 0) = e^{-\beta t} \int_0^t e^{(\beta-\alpha)\tau} d\tau$$

$$y(t \geq 0) = e^{-\beta t} \frac{e^{(\beta-\alpha)t} - e^0}{\beta - \alpha}$$

$$y(t \geq 0) = \frac{e^{-\alpha t} - e^{-\beta t}}{\beta - \alpha}$$

$$y(t) = \frac{e^{-\alpha t} - e^{-\beta t}}{\beta - \alpha} u(t)$$

Problem 9

Q: The input x(t) and the impulse response h(t) of a LTI system are

$$\boldsymbol{x(t)} = \begin{cases} \mathbf{2}, & |\boldsymbol{t}| < \mathbf{2} \\ \mathbf{0}, & \boldsymbol{otherwise} \end{cases}$$

$$\boldsymbol{h(t)} = \begin{cases} \mathbf{2}\boldsymbol{e^{-t}}, & \boldsymbol{t} \geq \mathbf{0} \\ \mathbf{0}, & \boldsymbol{otherwise} \end{cases}$$

Find the output y(t).

$$y(t) = 8\int_{-\infty}^{\infty} rect(4\tau) \cdot e^{-(t-\tau)} u(t-\tau) d\tau$$

$for\ |\tau| > 2, rect(4\tau)\ becomes\ 0$

$for\ \tau > t, u(t-\tau)\ becomes\ 0$

$as\ a\ result, for\ t < -2, \tau \leq t\ so\ \tau \leq -2, rect(4\tau)\ becomes\ 0$

$and\ for\ t \geq 2, rect(4t)\ becomes\ 0\ for\ \tau > 2$

$$y(-2 \leq t < 2) = 8\int_{-2}^{t} e^{\tau - t} d\tau$$

$$y(-2 \leq t < 2) = 8(e^{t-t} - e^{-2-t})$$

$$y(-2 \leq t < 2) = 8(1 - e^{-2-t})$$

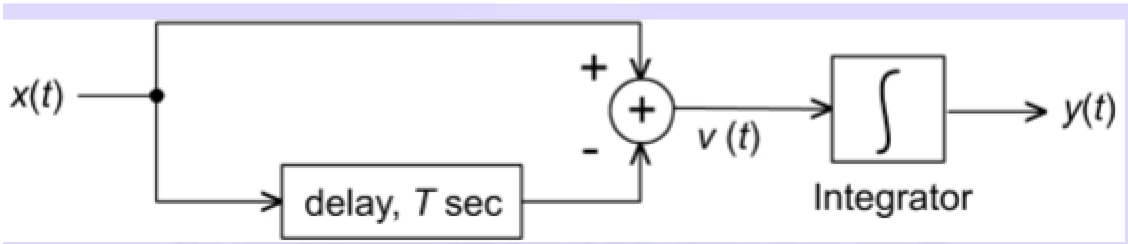
$$y(t < -2) = 0$$

$$y(2 \leq t) = 8(1 - e^{-4}) = 7.8535$$

$$y(t) = \begin{cases} 0, & t < -2 \\ 8(1 - e^{-2-t}), & -2 \leq t < 2 \\ 7.8535 & 2 \leq t \end{cases}$$

Problem 10

Q: Find the impulse response of the system:



$$x(t) = \delta(t)$$

$$v(t) = \delta(t) - \delta(t - T)$$

$$h(t) = y(t) = \int v(t)dt = \int \big(\delta(t) - \delta(t - T)\big)dt$$

$$h(t) = u(t) - u(t - T)$$

Problem 11

Q: Find the period, and find the Fourier series representation of the following periodic signals:

a- $x(t) = 2 \cdot \cos 200\pi t + 5 \cdot \sin 400\pi t$

$$T_0 = lcm(T_1, T_2)$$

$$T_0 = lcm\left(\frac{1}{200}, \frac{1}{400}\right) = \frac{1}{200}$$

$$a_n = \begin{cases} 2, & x = 1 \\ 0, & otherwise \end{cases}$$

$$b_n = \begin{cases} 5, & x = 2 \\ 0, & otherwise \end{cases}$$

b- $x(t) = 2 \cdot \cos 200\pi t + 5 \cdot \sin 300\pi t$

$$T_0 = lcm\left(\frac{1}{200}, \frac{1}{300}\right) = \frac{1}{100}$$

$$a_n = \begin{cases} 2, & x = 2 \\ 0, & otherwise \end{cases}$$

$$b_n = \begin{cases} 5, & x = 3 \\ 0, & otherwise \end{cases}$$

c- $x(t) = 2 \cdot \cos 150\pi t + 5 \cdot \sin 250\pi t$

$$T_0 = lcm\left(\frac{1}{150}, \frac{1}{250}\right) = \frac{1}{50}$$

$$a_n = \begin{cases} 2, & x = 3 \\ 0, & otherwise \end{cases}$$

$$b_n = \begin{cases} 5, & x = 5 \\ 0, & otherwise \end{cases}$$

Problem 12

Q: Consider the following signal

$$\boldsymbol{x(t) = \cos 2\pi f_1 t + a \sin 2\pi f_1 t = X_A(a) \cos(2\pi f_1 t + X_P(a))}$$

a- **Find $X_A(t)$**

b- **Find $X_P(t)$**

$$using\ the\ identities:$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{j2}$$

$$x(t) = \frac{e^{j2\pi f_1 t} + e^{-j2\pi f_1 t}}{2} + a \cdot \frac{e^{j2\pi f_1 t} - e^{-j2\pi f_1 t}}{j2}$$

$$x(t) = \frac{(1 - ja) \cdot e^{j2\pi f_1 t} + (1 + ja) \cdot e^{-j2\pi f_1 t}}{2}$$

$$x(t) = \sqrt{1 + a^2} \cdot \frac{e^{-j\tan^{-1} a} \cdot e^{j2\pi f_1 t} + e^{j\tan^{-1} a} \cdot e^{-j2\pi f_1 t}}{2}$$

$$x(t) = \sqrt{1 + a^2} \cdot \frac{e^{j(2\pi f_1 t - \tan^{-1} a)} + e^{-j(2\pi f_1 t - \tan^{-1} a)}}{2}$$

$$x(t) = \sqrt{1 + a^2} \cdot \cos(2\pi f_1 t - \tan^{-1} a)$$

$$X_A(a) = \sqrt{1 + a^2}, \qquad X_P(a) = \tan^{-1} a$$

c- What is the power of x(t), P_x?

$$P_x = f_1 \cdot \int_{\frac{-1}{2f_1}}^{\frac{1}{2f_1}} (\cos(2\pi f_1 t) + a\sin(2\pi f_1 t))^2 dt$$

$$P_x = f_1 \cdot \int_{\frac{-1}{2f_1}}^{\frac{1}{2f_1}} \left(\sqrt{1+a^2} \cdot \cos(2\pi f_1 t - \tan^{-1} a)\right)^2 dt$$

$$P_x = (1+a^2) \cdot f_1 \cdot \int_{\frac{-1}{2f_1}}^{\frac{1}{2f_1}} \cos^2(2\pi f_1 t - \tan^{-1} a)\, dt$$

$$P_x = \frac{(1+a^2) \cdot f_1}{2}$$

d- Is x(t) periodic? Define the Fourier Series representation.

$$x(t)\ is\ periodic\ for\ constant\ a, period\ is \frac{1}{f_1}\ and\ the\ fourier\ series:$$

$$a_n = \begin{cases} 1, & x = 1 \\ 0, & otherwise \end{cases}$$

$$b_n = \begin{cases} a, & x = 1 \\ 0, & otherwise \end{cases}$$

Problem 13

Q: The input-output characteristic of a channel is described by the differential equation:

$$\frac{dy}{dt} + 2y(t) = 4x(t)$$

a- Find the transfer function, H(f), of the channel.

$$\mathcal{F}\left\{\frac{dy}{dt} + 2y(t)\right\} = \mathcal{F}\{4x(t)\}$$

$$Y(f) \cdot j2\pi f + 2Y(f) = 4X(f)$$

$$Y(f) \cdot (2 + j2\pi f) = 4X(f)$$

$$H(f) = \frac{Y(f)}{X(f)} = \frac{4}{2 + j2\pi f}$$

b- Find the 3-dB bandwidth of the channel.

$$\frac{|H(f)|}{|H(0)|} = \frac{2}{\sqrt{4 + (2\pi f)^2}} = \frac{1}{\sqrt{2}}$$

$$BW = f = \frac{1}{\pi} Hz$$

Problem 14

Q: Let m(t) be a baseband signal with Fourier transform

$$\boldsymbol{M(f)} = \begin{cases} \boldsymbol{m_0}, & \boldsymbol{-f_m \le f \le f_m} \\ \boldsymbol{0}, & \boldsymbol{otherwise} \end{cases}$$

Let m^(t) be the Hilbert transform of m(t), find the energy in m^(t).

$$E_m = E_{\hat{m}}$$

$$using\ Rayleigh\ Rnergy\ Theorem:$$

$$E_m = \int_{-\infty}^{\infty} |m(t)|^2 dt = \int_{-\infty}^{\infty} |M(f)|^2 df$$

$$E_m = \int_{-f_m}^{f_m} m_0^2 df = 2 f_m m_0^2$$

Problem 15

Q: The impulse response of a linear time-invariant system is given by:

$$\boldsymbol{h(t) = e^{-2\pi Bt} u(t)}$$

- **a- Is this system causal? Explain**
  Yes, it doesn't depend on future values.
- **b- Is this system stable? Explain**
  Yes, because if bounded input is applied we get a bounded output
  $$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$
- **c- Find $\boldsymbol{\int_0^5 h(t) \cdot \delta(t-1)\, dt}$**
  $using\ sampling\ property:$
  $$\int_0^5 h(t) \cdot \delta(t-1)\, dt = h(1) = e^{-2\pi B}$$

Problem 16

Q: The Fourier transform of a time signal $m(t)$ is given by:

$$\boldsymbol{M(f) = \frac{1}{1 + j\left(\frac{f}{B}\right)}}$$

a- **Find the 6-dB bandwidth of the message**

$$20\log\left(\frac{|M(f)|}{|M(0)|}\right) = -6$$

$$20\log\left(\frac{1}{\sqrt{1 + \left(\frac{f}{B}\right)^2}}\right) = -6$$

$$-10\log\left(1 + \left(\frac{f}{B}\right)^2\right) = -6$$

$$\frac{f}{B} = \sqrt{10^{0.6} - 1}$$

$$BW = f = 1.72658\,B$$

b- **Find $M(f)\delta(f - B)$**

$using\ sifting\ property$:

$$M(f)\cdot\delta(f - B) = M(f - B)$$

$$M(f)\cdot\delta(f - B) = \frac{1}{1 + j\left(\frac{f}{B} - 1\right)}$$

Problem 17

Q: Consider the signal $g(t) = e^{-a|t|}$

a- Explain why this signal is an energy signal.

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt$$

$$E_g = \int_{-\infty}^{0} e^{2at}\, dt + \int_{0}^{\infty} e^{-2at}\, dt$$

$$E_g = \frac{e^0 - e^{-\infty}}{2a} + \frac{e^{-\infty} - e^0}{-2a}$$

$$E_g = \frac{1-0}{2a} + \frac{0-1}{-2a}$$

$$E_g = \frac{1}{a}$$

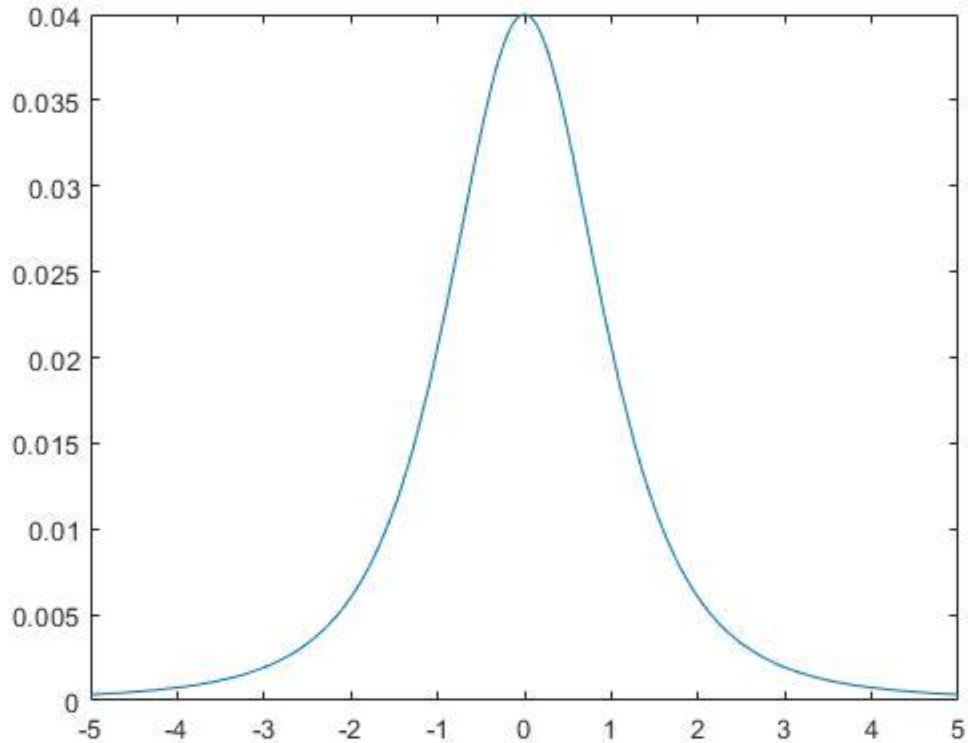
It's an Energy signal for finite a.

b- Find and sketch the energy spectral density of g (t).

$$S_G(f) = |G(f)|^2 = \left|\mathcal{F}\left\{e^{-a|t|}\right\}\right|^2$$

$$S_G(f) = \left|\frac{2a}{a^2 + (2\pi f)^2}\right|^2$$

$$S_G(f) = \frac{4a^2}{a^4 + 16\pi^4 f^4 + 8\pi^2 a^2 f^2}$$



c- **Find the total energy in g (t).**
Solved in part a.

d- **Find the 3-dB bandwidth of g (t).**

$$G(f) = \frac{2a}{a^2 + (2\pi f)^2}$$

$$\frac{|G(BW)|}{|G(0)|} = \frac{1}{\sqrt{2}}$$

$$\frac{a^2}{a^2 + (2\pi BW)^2} = \frac{1}{\sqrt{2}}$$

$$BW = \frac{\sqrt{\sqrt{2} - 1} \cdot a}{2\pi}$$

e- Find the fraction of the signal energy contained in the bandwidth of Part d relative to the total signal energy.

$$E_{BW} = \int_{-\frac{\sqrt{\sqrt{2}-1}\cdot a}{2\pi}}^{\frac{\sqrt{\sqrt{2}-1}\cdot a}{2\pi}} \frac{4a^2}{a^4 + 16\pi^4 f^4 + 8\pi^2 a^2 f^2} df$$

$$E_{BW} = \left[\frac{1}{\pi a} \cdot \tan^{-1}\left(\frac{2\pi f}{a}\right) + \frac{2f}{a^2 + 4\pi^2 f^2}\right]_{-\frac{\sqrt{\sqrt{2}-1}\cdot a}{2\pi}}^{\frac{\sqrt{\sqrt{2}-1}\cdot a}{2\pi}}$$

$$\%E_{BW} = \frac{E_{BW}}{E_G} = \%65.38$$

This part has been calculated using the following MATLAB code:

```
% Add Symbolic Library %
syms f a BW S E_BW E_T;
% Define the Bandwidth (found in part d) %
BW = sqrt(sqrt(2) - 1)*a/(2*pi);
% Define the total Energy (found in part c) %
E_T = 1/a;
% Define the Energy Spectral Density function (found in part b) %
S(f) = (2*a/(a^2 + (2*pi*f)^2))^2;
% Find the total energy in the bandwidth %
E_BW = int(S, f, -1*BW, BW);
% Evaluate the ratio E_BW/E_T %
fprintf('Fraction of Energy contained in bandwidth = %%%0.2f\n',
vpa(E_BW/E_T)*100);
```

Problem 18

Q: A periodic signal x(t) defined over one period is:

$$\boldsymbol{x(t)} = \begin{cases} \boldsymbol{a|t|}, & -\frac{\boldsymbol{T_0}}{\boldsymbol{2}} \le \boldsymbol{t} \le \frac{\boldsymbol{T_0}}{\boldsymbol{2}} \\ \boldsymbol{0}, & |\boldsymbol{f}| > \boldsymbol{W} \end{cases}$$

Find the Fourier series coefficient an, n = 1, 3, 5.

Since it's a definition of one period in a periodic signal, so the other periods are same but shifted in time by a multiple of the period.

$$x(t) = \sum_{n=-\infty}^{\infty} a|t - nT_0| \cdot rect\left(\frac{t - nT_0}{T_0}\right)$$

$$a_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\cos(n\omega_0 t)\, dt$$

$$a_n = \frac{2a}{T_0} \int_{0}^{\frac{T_0}{2}} t\cos(n\omega_0 t)\, dt$$

$$a_n = \begin{cases} \frac{-2a}{\pi\omega_0 n^2} & n\ is\ odd \\ 0, & n\ is\ even \end{cases}$$

$$a_1 = \frac{-2a}{\pi\omega_0}, a_3 = \frac{-2a}{9\pi\omega_0}, a_5 = \frac{-2a}{25\pi\omega_0}$$

Problem 19

Q: Consider the signal $g(t) = e^{-2\pi Bt}u(t)$

a- **Find the autocorrelation function $R_g(\tau)$**

$$R_g(\tau) = \int_{-\infty}^{\infty} g(t) \cdot g(t-\tau)\,dt$$

$$R_g(\tau) = \int_{-\infty}^{\infty} e^{-2\pi Bt}u(t) \cdot e^{-2\pi B(t-\tau)}u(t-\tau)\,dt$$

$$for\ t < 0, u(t) = 0$$

$$for\ t < \ \tau, u(t-\tau) = 0$$

$$R_g(\tau < 0) = \int_{0}^{\infty} e^{-2\pi Bt} \cdot e^{-2\pi B(t-\tau)}\,dt$$

$$R_g(\tau < 0) = e^{2\pi B\tau} \int_{0}^{\infty} e^{-4\pi Bt}\,dt = e^{2\pi B\tau} \frac{e^{-\infty} - e^{0}}{-4\pi B}$$

$$R_g(\tau < 0) = \frac{e^{2\pi B\tau}}{4\pi B}$$

$$R_g(\tau \geq 0) = \int_{\tau}^{\infty} e^{-2\pi Bt} \cdot e^{-2\pi B(t-\tau)}\,dt$$

$$R_g(\tau \geq 0) = e^{2\pi B\tau} \int_{\tau}^{\infty} e^{-4\pi Bt}\,dt = e^{2\pi B\tau} \frac{e^{-\infty} - e^{\tau}}{-4\pi B}$$

$$R_g(\tau \geq 0) = \frac{e^{(2\pi B+1)\tau}}{4\pi B}$$

$$R_g(\tau) = \frac{e^{2\pi B\tau}}{4\pi B} \cdot \left(1 + e^{\tau} u(\tau)\right)$$

b- Find the energy spectral density.

$$S_G(f) = |G(f)|^2 = \frac{1}{(2\pi B)^2 + (2\pi f)^2}$$

c- Find the energy in the signal.

$$E_G = \int_{-\infty}^{\infty} |e^{-2\pi Bt} u(t)|^2 \, dt$$

$$E_G = \int_{0}^{\infty} e^{-4\pi Bt} \, dt = \frac{1}{4\pi B}$$

d- Find the 3-dB bandwidth of the signal.

$$\frac{|G(BW)|}{|G(0)|} = \frac{1}{\sqrt{2}}$$

$$\frac{2\pi B}{\sqrt{(2\pi B)^2 + (2\pi BW)^2}} = \frac{1}{\sqrt{2}}$$

$$BW = B$$