Communication Systems

Instructor

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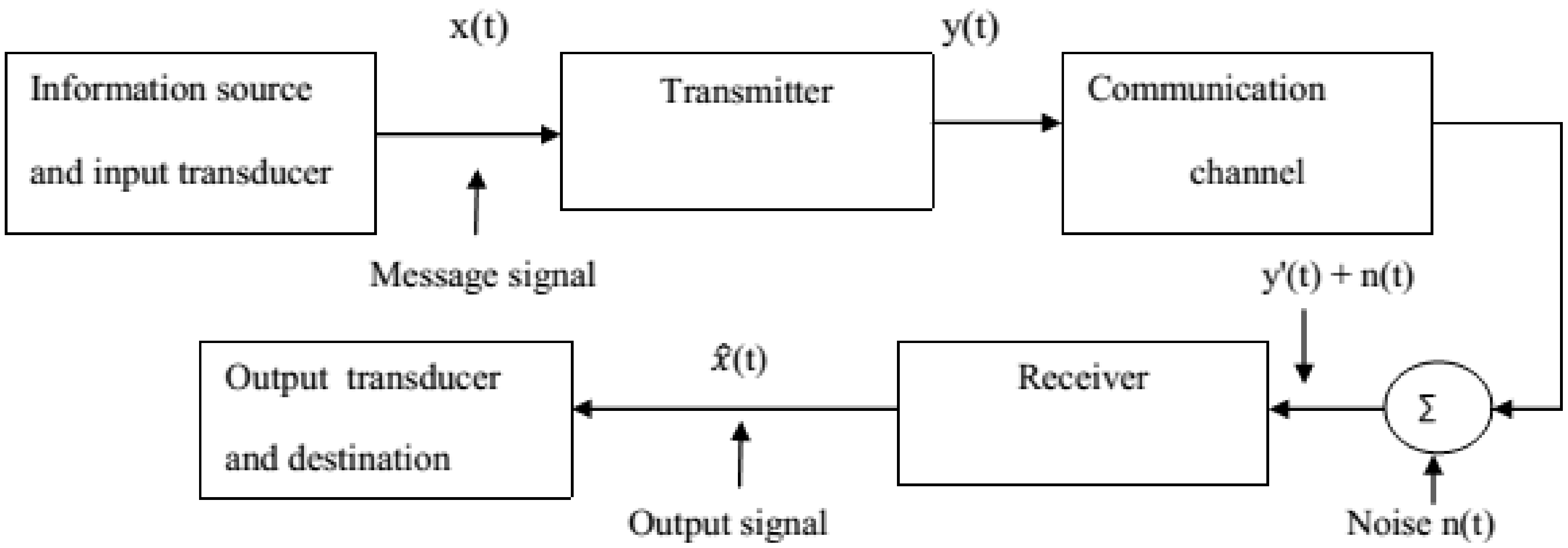
Model of a Communication System

- *Communication* is defined as "exchange of information".
- *Telecommunication* refers to communication over a distance greater than would normally be possible without artificial aids.
- Telephony is an example of point-to-point communication and normally involves a two – way flow of information.
- Broadcast radio and television : Information is transmitted from one location but is received at many locations using different receivers (point to multi-point communication)
- Model of a communication system :



- The purpose of a communication system is to transmit information – bearing signals from a source located at one point to a user located at another end.
- The input transducer is used to convert the physical message generated by the source into a time-varying electrical signal called the *message signal*.
- The original message is recreated at the destination using an output transducer.
- The *transmitter* modifies the message signal into a form suitable for transmission over the channel. *Here modulation takes place.*
- The *channel* is the medium over which signal is transmitted, (like free space, an optical fiber, transmission lines, twisted pair of wires…). Here signal is distorted due to

  A. Nonlinearities and/or imperfections in the frequency response of the channel.
  B. Noise and interference are added to the signal during the course of transmission.

- The purpose of the *receiver* is to recreate the original signal x(t) from the degraded version y'(t) + n(t) of the transmitted signal after propagating through channel . *Here, demodulation takes place.*

Classification of Signals

<u>Definition</u>: A signal may be defined as a single valued function of time that conveys information.

Depending on the feature of interest, we may distinguish four different classes of signals:

<u>1. Periodic Signals, Non-periodic Signals:</u>

A *periodic signal* g(t) is a function of time that satisfies the condition $g(t) = g(t+T_0)$, ∀ t.

The smallest value of T_0 that satisfies this condition is called the period of g(t).

Example: A Periodic Signal

The saw-tooth function shown below is an example of a periodic signal.

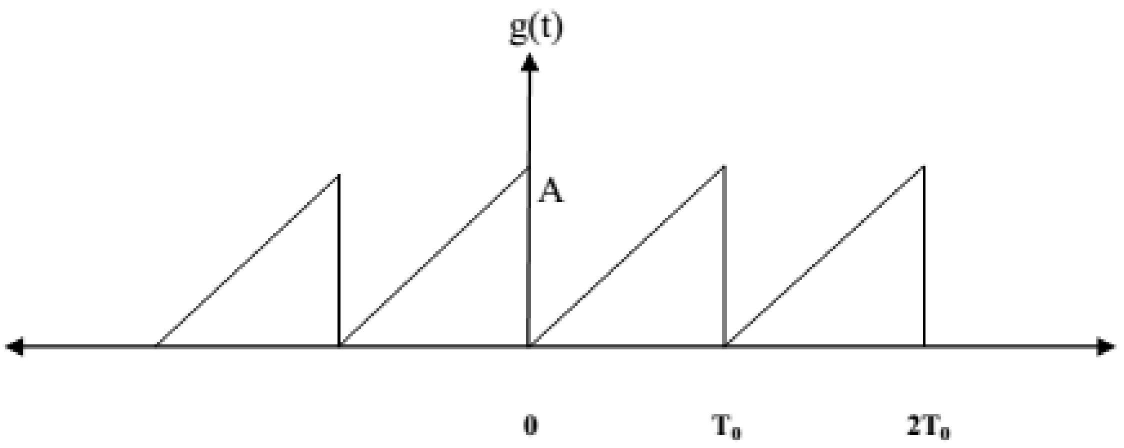


Fig. 1.1: A periodic signal with period T_0

Example: A Non-periodic Signal

The signal

$$g(t) = \begin{cases} A, & 0 \le t \le \tau \\ 0, & otherwise \end{cases}$$

is non-periodic, since there does not exist a T_0 for which the condition $g(t) = g(t+T_0)$ is satisfied.

2. Determinstic Signals, Random Signals:

A *deterministic signal* is one about which there is no uncertainty with respect to its value at any time. It is a completely specified function of time .

Example: A Deterministic Signal

$x(t) = Ae^{-at}u(t)$; A and α are constants.

A *random signal* is one about which there is some degree of uncertainty before it actually occurs. (It involves a random variable)

Example : A Random Signal

x(t) = A e^{-at}u(t) ; α is a constant and A is a random variable with the following probability density function (pdf).

$$F_A(a)= \begin{cases} 1 & 0 \le a \le 1 \\ 0 & otherwise \end{cases}$$

3. Energy Signals, Power Signals:

The *instantaneous power* in a signal g(t) is defined as that power dissipated in a 1-Ω resistor, i.e.,

$$P(t) = |g(t)|^2 .$$

The *average power* is defined as:

$$P_{av} \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |g(t)|^2 \, dt$$

The total energy of a signal g(t) is

$$E \triangleq \lim_{T\to\infty} \int_{-T}^{T} |g(t)|^2 \, dt$$

A signal g(t) is classified as *energy signal* if it has a finite energy, i.e, $0 < E < \infty$

A signal g(t) is classified as *power signal* if it has a finite power, i.e, $0 < P_{av} < \infty$

The average power in a periodic signal is

$$P_{av} = \frac{1}{T_0} \int_{0}^{T_0} |g(t)|^2 \, dt \text{ ; } T_0 \text{ is the period .}$$

Usually, periodic signals and random signals are power signals. Both deterministic and non periodic signals are energy signals.

4. Analog Signals, Digital Signals :

An *analog signal* is a continuous time - continuous amplitude function of time .

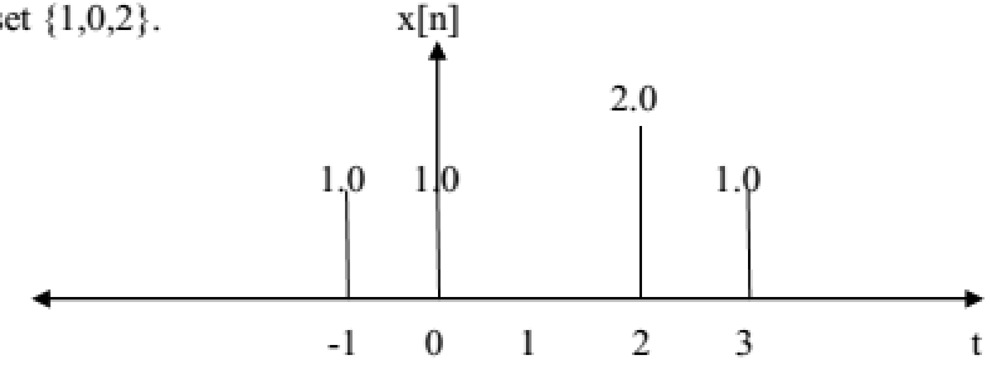
Example:

The sinusoidal signal $x(t) = A\cos 2\pi f t$, $-\infty < t < \infty$, is an example of an analog signal.

A *discrete time- discrete amplitude* (digital) signal is defined only at discrete times. Here, the independent variable takes on only discrete values.

Example:

The sequence x[n] shown below is an examples of a digital signal. The amplitudes are drawn from the finite set {1,0,2}.



More Examples

Example: An Exponential Pulse

Find the energy in the signal g(t) = A $e^{-\alpha t}$ u(t).

E= $\int_0^\infty A^2\ e^{-2\alpha t}dt = A^2 \frac{-e^{-2\alpha t}}{2\alpha}\Big|_0^\infty = \frac{A^2}{2\alpha}$. Since E is finite, then g(t) is an energy signal.

Example: A rectangular Pulse

Find the energy in the signal:

$$g(t) = \begin{cases} A, & 0 < t < \tau \\ 0, & o.w \end{cases}$$

E= $\int_0^\tau A^2\ dt = A^2\tau$. This signal is an energy since E is finite.

Example: A Periodic Sinusoidal Signal

Find the average power in the signal :

$$g(t) = A\cos\omega t \quad , -\infty < t < \infty$$

Since g(t) is periodic, then :

$$P_{av} = \frac{1}{T_0}\int_0^{T_0} A^2\cos^2\omega t\,dt = \frac{A^2}{T_0}\int_0^{T_0}\left(\frac{1+\cos 2\omega t}{2}\right) = \frac{A^2}{T_0}\cdot\frac{T_0}{2} = \frac{A^2}{2} \quad .$$

P_{av} is finite and so g(t) is a power signal.

Example: A Periodic Saw-tooth Signal

Find the average power in the saw-tooth signal g(t) plotted in Fig.1.

$$g(t) = \frac{A}{T_0}t \quad , 0 \le t \le T_0$$

$$P_{av} = \frac{1}{T_0}\int_0^{T_0}\frac{A^2}{T_0^2}\,t^2 dt = \frac{1}{T_0}\,\frac{A^2}{T_0^2}\,\frac{t^3}{3}\Big|_0^{T_0} = \frac{A^2\,T_0^3}{3\,T_0^3} = \frac{A^2}{3} \quad .$$

Example: The Unit Step Function

Consider the signal: g(t) = A u(t).

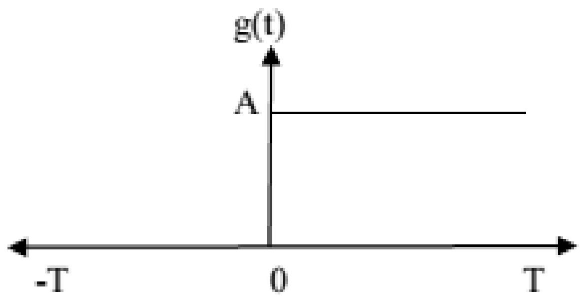


Fig. 1.2

This is a non periodic signal. So let us first try to find its energy:

$$E = \int_0^{\infty} A^2 \, dt = \infty .$$

Therefore, g(t) is not an energy signal (E is not finite).

To find the average power, we employ the definition :

$$P_{av} \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |g(t)|^2 \, dt ,$$

where 2T is chosen to be a symmetrical interval about the origin, as in Fig. 1.2 above.

$$P_{av} = \lim_{T\to\infty} \frac{1}{2T} \int_0^{T} A^2 \, dt \;\; = \lim_{T\to\infty} \frac{A^2\, T}{2T} \;= \frac{A^2}{2} .$$

So, even-though g(t) is non a periodic, it turns out that it is a power signal.

This is an example where the general rule (periodic signals are power signals and energy signals are non periodic signals) fails to hold.

Fourier Series

Let g(t) be a periodic signal with period $T_0 = \frac{1}{f_0}$. The signal g(t) may be expanded in one of three possible Fourier series forms:

The complex form:

$$g(t) = \sum_{n=-\infty}^{\infty} C_n \, e^{jn\omega_0 t}$$

where, $$C_n = \frac{1}{T_0}\int_0^{T_0} g(t)\, e^{-jn\omega_0 t}\, dt \;;$$

C_n : is a complex valued quantity that can be written as:

$$C_n = |\, C_n \,|\, e^{j\theta n}$$

Discrete Amplitude Spectrum: A plot of $|C_n|$ vs. frequency

Discrete Phase Spectrum: A plot of θ_n vs. frequency

The term at ω_0 is referred to as the fundamental frequency. The term at $2\omega_0$ is referred to as the second harmonic, …

The trigonometric form:

$$g(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

Where : $a_0 = \frac{1}{T_0} \int_0^{T_0} g(t)\, dt$ (dc or average value)

$$a_n = \frac{2}{T_0} \int_0^{T_0} g(t) \cos n\omega_0 t\, dt$$

$$b_n = \frac{2}{T_0} \int_0^{T_0} g(t) \sin n\omega_0 t\, dt$$

The polar form :

$$g(t) = c_0 + \sum_{n=1}^{\infty} 2|C_n| \cos(n\omega_0 t + \theta_n)$$

where C_n and θ_n are those terms defined in the complex form.

Remark: The above three forms are equivalent and are representations of the same waveform. If you know one representation, you can easily deduce the other.

Example: Find the trigonometric Fourier series of the periodic rectangular signal defined over one period T_0 as:

$$g(t) = \begin{cases} +A, & -T_0/4 \le t \le T_0/4 \\ 0, & otherwise \end{cases}$$

Solution:

$$a_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\, dt = \frac{1}{T_0}\int_{-T_0/4}^{T_0/4} A\, dt = A/2$$

$$b_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\sin(\frac{2\pi n}{T_0})t\, dt = \frac{2}{T_0}\int_{-T_0/4}^{T_0/4} A\sin(\frac{2\pi n}{T_0})t\, dt = 0$$

$$a_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\cos(\frac{2\pi n}{T_0})t\, dt = \frac{2}{T_0}\int_{-T_0/4}^{T_0/4} A\cos(\frac{2\pi n}{T_0})t\, dt$$

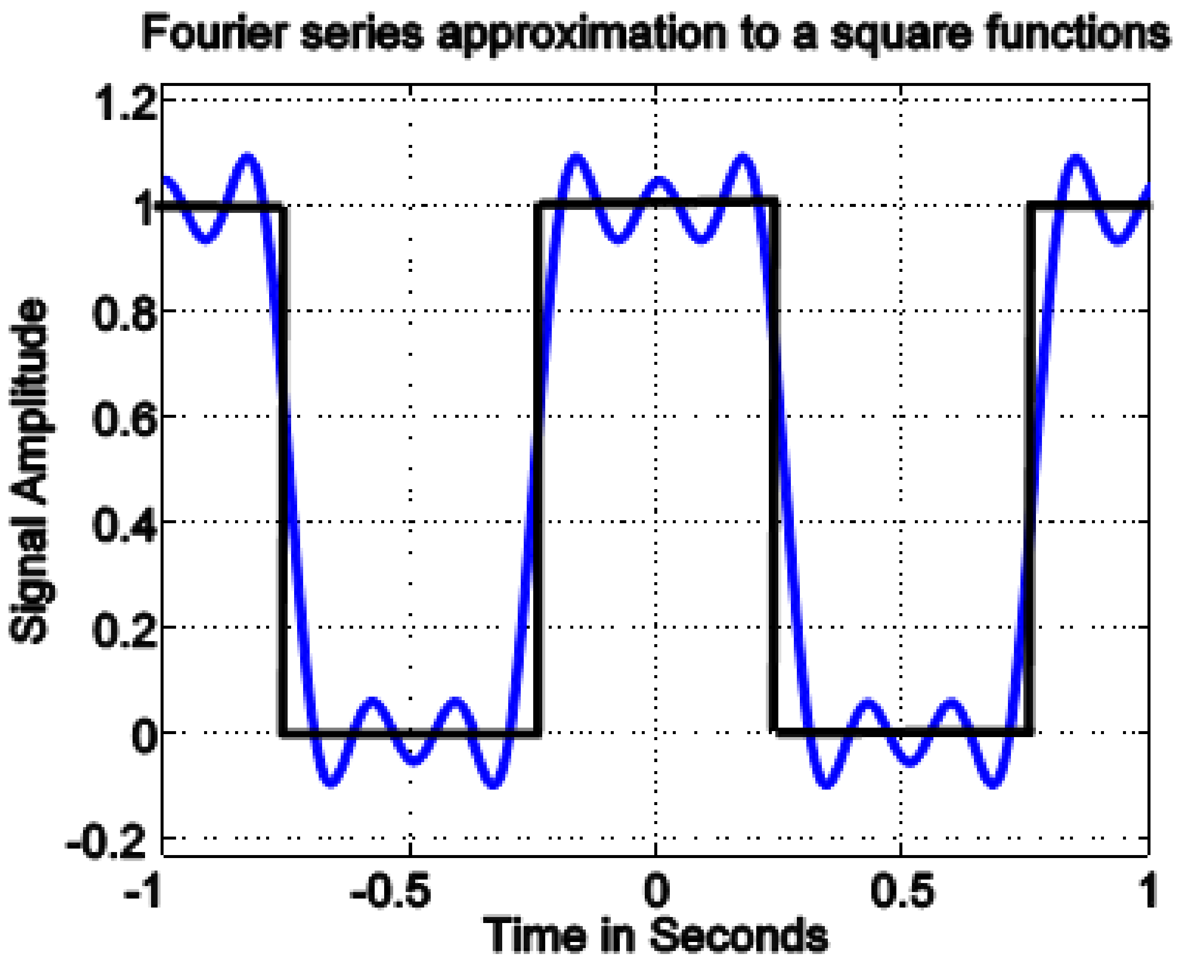
$$a_n = \begin{cases} \frac{2A}{n\pi}, & n = 1, 5, 9, \dots \\ \frac{-2A}{n\pi}, & n = 3, 7, 11, \dots \\ 0, & n = 2, 4, 6 \dots \end{cases}$$

The first four terms in the expansion of $g(t)$ are:

$$\tilde{g}(t) = \frac{A}{2} + \frac{2A}{\pi}\{\cos(2\pi f_0)\, t - \frac{1}{3}\cos(2\pi 3 f_0)\, t + \frac{1}{5}\cos(2\pi 5 f_0)\, t\}$$

The function $\tilde{g}(t)$ along with $g(t)$ are plotted in the figure for $-1 \le t \le 1$

assuming $A = 1$ and $f_0 = 1$



Remark: As more terms are added to $\tilde{g}(t)$, $\tilde{g}(t)$ becomes closer to $g(t)$ and in the limit as $n \to \infty$, $\tilde{g}(t)$ becomes equal to $g(t)$ at all points except at the points of discontinuity.

Parseval's Power Theorem

The average power of a periodic signal g(t) is given by:

$$P_{av} = \frac{1}{T_0}\int_0^{T_0} |g(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |C_n|^2 = |C_n|^2 + 2\sum_{n=1}^{\infty} |C_n|^2$$

$$= |a_0|^2 + \frac{1}{2}\sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2)$$

Power Spectral Density

A plot of $|C_n|^2$ vs. frequency is called the *power spectral density* (PSD). It portrays the power content of each frequency (spectral) component of a signal. For a periodic signal, the PSD consists of discrete values at multiples of the fundamental frequency.

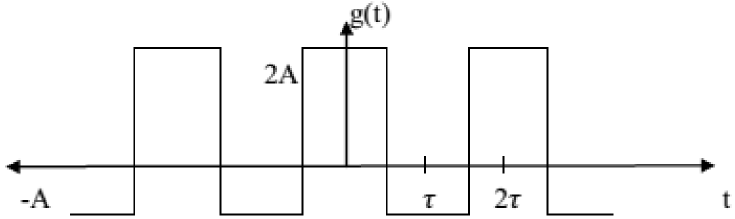
Exercise: Consider again the saw-tooth function defined over one period as $g(t) = t, 0 \le t \le 1$

a. Use matlab to find the dc terms and the first three harmonics (i.e., let n = 3) in the Fourier series expansion

$$\tilde{g}(t) = a_0 + \sum_{n=1}^{3}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

b. Plot $\tilde{g}(t)$ and $g(t)$ versus time for $-1 \le t \le 1$ on the same graph.
c. Find the fraction of the power contained in $\tilde{g}(t)$ to that in $g(t)$.
d. Sketch the power spectral density.

Example : Find the power spectral density of the periodic function g(t) shown in the figure :



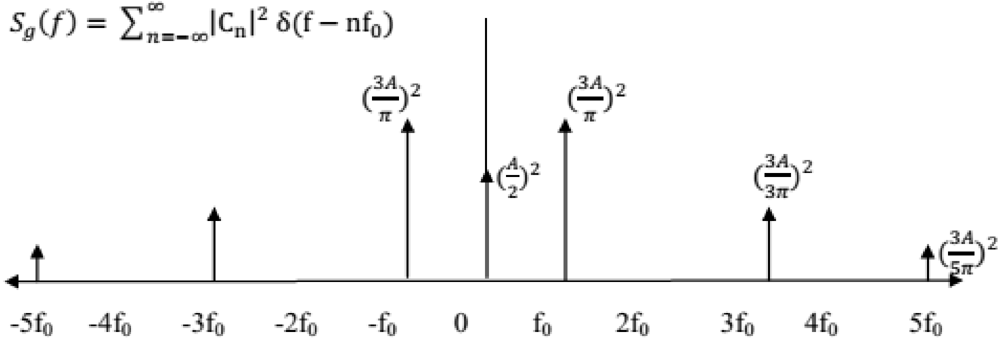
Solution: Here we need to find the complex Fourier series expansion, where the period $T_0 = 2\tau$

$$g(t) = \sum_{n=-\infty}^{\infty} C_n\, e^{jn\omega_0 t}$$

$$C_n = \frac{1}{T_0}\int_0^{T_0} g(t)\, e^{-jn\omega_0 t}\, dt$$

$$C_n = \begin{cases} \frac{A}{2}, & n = 0 \\ \frac{3A}{|n|\pi}, & n = \pm1, \pm5, \pm9, \ldots \\ \frac{-3A}{|n|\pi}, & n = \pm3, \pm7, \pm11, \ldots \\ 0, & n = \pm2, \pm4, \ldots \end{cases} \quad \Rightarrow \quad |C_n|^2 = \begin{cases} (\frac{A}{2})^2, & n = 0 \\ (\frac{3A}{n\pi})^2, & n: odd \\ 0, & n: even \end{cases}$$

$$S_g(f) = \sum_{n=-\infty}^{\infty} |C_n|^2\, \delta(f - nf_0)$$



As can be seen, the power spectral density of this periodic signal is a discrete function in frequency.

Exercise: Verify Parseval's power theorem for this signal, i.e., show that

$$P_{av} = \frac{1}{T_0}\int_0^{T_0} |g(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |C_n|^2 = 2.5A^2$$

Fourier Transform

Let g(t) be a non periodic square integrable function of time. That is one for which

$$\int_{-\infty}^{\infty} |g(t)|^2\, dt < \infty$$

The Fourier transform of g(t) exists and is defined as:

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t}\, dt$$

The time function g(t) can be recovered from $G(f)$ using the inverse Fourier Transform:

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi f t}\, df$$

Remarks:

- All energy signals for which $E = \int_{-\infty}^{\infty} |g(t)|^2\, dt < \infty$ are Fourier transformable.
- G(f) is a complex function of frequency f, which can be expressed as:

  $G(f) = |G(f)|\, e^{j\theta(f)}$

where, | G(f) | : is the *continuous amplitude spectrum* of g(t), (even function of f).

$\theta(f)$: is the *continuous phase spectrum* of g(t), (odd function of f).

Rayleigh Energy Theorem :

The energy in a signal g(t) is given by :

$$E = \int_{-\infty}^{\infty} |g(t)|^2\, dt = \int_{-\infty}^{\infty} |G(f)|^2 df$$

The function $|G(f)|^2$ is called the *energy spectral density*. It illustrates the range of frequencies over which the signal energy extends and the frequency bands which are significant in terms of their energy contents. For a non-period signal energy signal, the energy spectral density is a continuous function of f.

A General Form of the Rayleigh Energy Theorem

For two energy functions g(t) and $v(t)$, the following result holds:

$$\int_{-\infty}^{\infty} g(t)v(t)^*\, dt = \int_{-\infty}^{\infty} G(f)V(f)^*\, df$$

Example: **Energy spectral density of the exponential signal**

$$v(t) = \begin{cases} A\,e^{-bt} & t > 0 \\ 0 & t < 0 \end{cases}$$

$$V(f) = \int_0^{\infty} v(t)e^{-j2\pi ft}\,dt = \int_0^{\infty} Ae^{-bt}\,e^{-j2\pi ft}dt$$

$$V(f) = A\int_0^{\infty} e^{-(b+j2\pi f)t}\,dt = \mathrm{A}\,\frac{e^{-(b+j2\pi f)t}}{-(b+j2\pi f)}\Big|_0^{\infty} = \frac{A}{b+j2\pi f}.$$

$$V(f) = \frac{A}{b+j2\pi f} \Rightarrow |V(f)| = \frac{A}{(b^2+(2\pi f)^2)^{1/2}}$$

The energy spectral density is: $S_v(f) = |V(f)|^2 = \frac{A^2}{b^2+\omega^2}$

Remark: The signal $v(t)$ is called a *baseband signal* since the signal occupies the low frequency part of the spectrum. That is, the energy in the signal is found around the zero frequency. When the signal is multiplied by a high frequency carrier, the spectrum becomes centered around the carrier and the modulated signal is called a *bandpass signal*.

Exercise : For the exponential pulse verify Rayleigh energy theorem, i.e., show that

$$\int_0^{\infty} |v(t)|^2 dt = 2\int_0^{\infty} |V(f)|^2 df = \frac{A^2}{2b}.$$

Example: The Rectangular Pulse $g(t) = Arect(\frac{t}{T})$

$$G(f) = \int_{-T/2}^{T/2} Ae^{-j2\pi ft}dt = \frac{A}{\pi f}\sin \pi fT$$

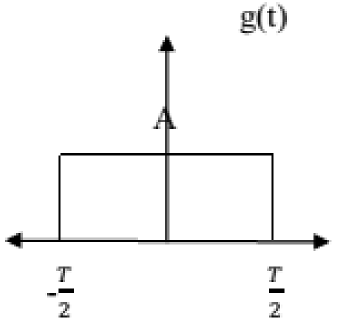
$$= AT\frac{\sin \pi fT}{\pi fT} \triangleq AT\ sinc\ Tf$$

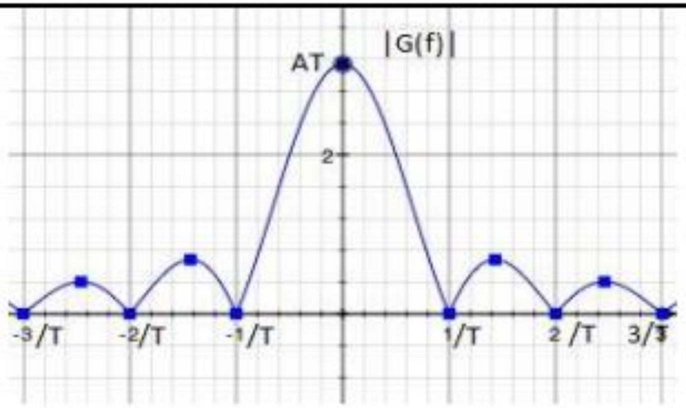
$$|G(f)| = AT\ |sinc\ Tf|$$

$\lim_{x\to 0}\frac{\sin x}{x} = 1$ ⇨ max. of function

$G(f) = 0$ *when* $\sin \pi fT = 0$ or *when* $\pi fT = n\pi$, n=$\pm1, \pm2, \pm3,\ldots$

$fT = n\ ,\therefore f = \frac{n}{T}$





<u>Remark:</u> Time duration and bandwidth :

Note that as the signal time duration T *increases*, the first zero crossing at $f = \frac{1}{T}$ decreases, implying that B.W of signal decreases. More on this will be said later when we discuss the time bandwidth product.

Properties of the Fourier Transform:

1. Linearity (superposition)

   Let $g_1(t) \leftrightarrow G_1(f)$ and $g_2(t) \leftrightarrow G_2(f)$, then :

   $c_1 g_1(t) + c_2 g_2(t) \leftrightarrow c_1 G_1(f) + c_2 G_2(f)$; c_1, c_2 are constants

2. Time scaling

   $$g(at) \leftrightarrow \frac{1}{|a|} G(f/a)$$

3. Duality

   If $g(t) \leftrightarrow G(f)$, then : $G(t) \leftrightarrow g(-f)$

4. Time shifting

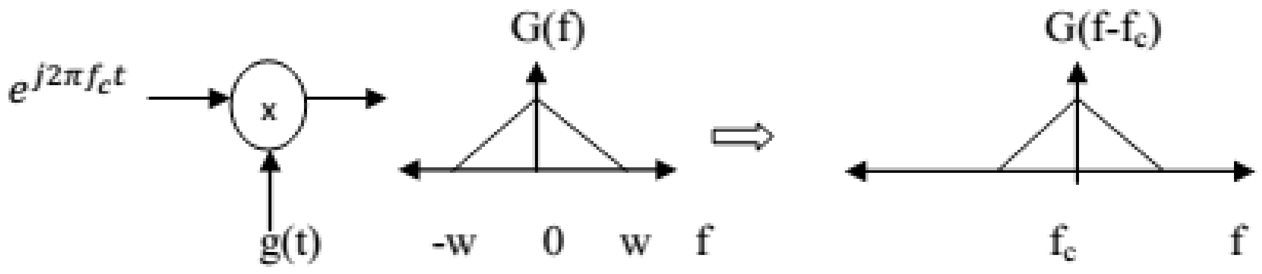
   If $g(t) \leftrightarrow G(f)$, then $g(t - t_0) \leftrightarrow G(f)e^{-j2\pi f t_0}$

   Delay in time $\Longrightarrow$ phase shift in frequency domain

5. Frequency shifting: If $g(t) \leftrightarrow G(f)$, then :

5. Frequency shifting: If $g(t) \leftrightarrow G(f)$, then :

$g(t)\, e^{j2\pi f c t} \leftrightarrow G(f - fc)$; f_c is a real constant



6. Area under $G(f)$: $If\ g(t) \leftrightarrow G(f)$, then:

$$g(0) = \int_{-\infty}^{\infty} G(f)df$$

The value $g(t = 0)$ is equal to the area under its Fourier transform.

7. Area under $g(t)$: If $g(t) \leftrightarrow G(f)$, then:

$$G(0) = \int_{-\infty}^{\infty} g(t)dt$$

The area under a function g(t) is equal to the value of its Fourier transform G(t) at f = 0.

Where $G(0)$ implies the presence of a dc component.

8. Differentiation in the time domain

If $g(t)$ and its derivative $g'(t)$ are Fourier transformable, then :

$$g'(t) \leftrightarrow (j2\pi f)G(f)$$

i.e., differentiation in the time domain $\Longrightarrow$ multiplication by j$2\pi f$ in the frequency domain.

(enhances high frequency components of a signal while attenuates low frequency components)

Also, $\frac{d^n g(t)}{dt^n} \leftrightarrow (j2\pi f)^n G(f)$

9. Integration in the time domain

$$\int_{-\infty}^{t} g(\tau)d\tau \leftrightarrow \frac{1}{j2\pi f} G(f) \text{ ; assuming } G(0) = 0.$$

i.e., integration in the time domain $\Longrightarrow$ division by (j$2\pi f$) in the frequency domain.

(enhancement of low frequency components of the signal).

When $G(0) \neq 0$, the above result becomes :

$$\int_{-\infty}^{t} g(\tau)d\tau \quad \leftrightarrow \quad \frac{1}{j2\pi f} G(f) + \frac{1}{2} G(0)\delta(f).$$

10. Conjugate Functions

For a complex – valued time signal $g(t)$, we have:

$$g^*(t) \quad \leftrightarrow \quad G^*(-f) \quad ;$$

Also, $\quad g^*(-t) \quad \leftrightarrow \quad G^*(f) \quad ;$

Therefore, $\mathrm{Re}\{g(t)\} \quad \leftrightarrow \quad \frac{1}{2}\{ G(f) + G^*(-f)\}$

$$\mathrm{Im}\{g(t)\} \quad \leftrightarrow \quad \frac{1}{2j}\{ G(f) - G^*(-f)\}$$

11. Multiplication in the time domain

$$g_1(t)\, g_2(t) \quad \leftrightarrow \quad \int_{-\infty}^{\infty} G_1(\lambda)\, G_2(f-\lambda)\, d\lambda = G_1(f) * G_2(f)$$

Multiplication of two signals in the time domain is transformed into the convolution of their Fourier transforms in the frequency domain.

12. Convolution in the time domain

$$g_1(t) * g_2(t) \quad \leftrightarrow \quad G_1(f)G_2(f)$$

Convolution of two signals in the time domain is transformed into the multiplication of their Fourier transforms in the frequency domain.

Fourier Transform of Power Signals

For a non-periodic (energy) signal, the Fourier transform exists when

$$E = \int_{-\infty}^{\infty} |g(t)|^2 \, dt < \infty$$

So that $G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t} dt$.

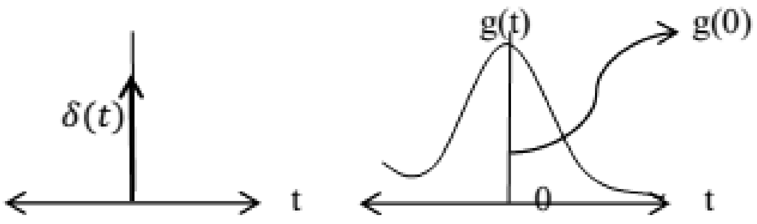
For power signals, the integral $\int_{-\infty}^{\infty} g(t) e^{-j2\pi f t} dt$ **does not exist**.

However, one can still find the Fourier transform of power signals by employing the delta function. This function is defined next.

Dirac – Delta Function (impulse function)

This function is defined as

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$



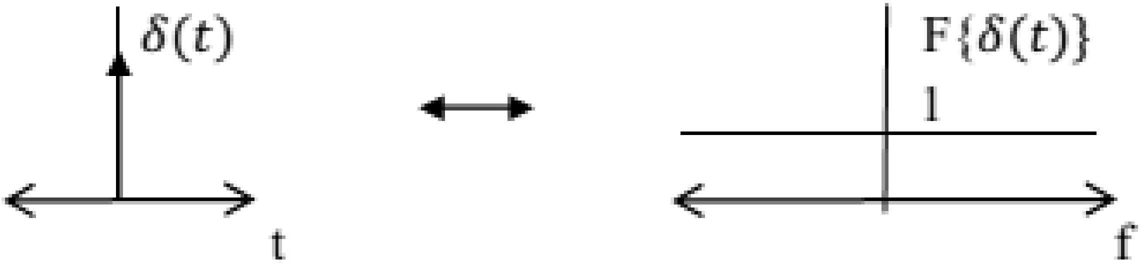
Such that $\int_{-\infty}^{\infty} \delta(t) dt = 1$

and $\int_{-\infty}^{\infty} g(t)\delta(t) dt = g(0)$

(Here, g(t) is a continuous function of time).

Some properties of the delta function:

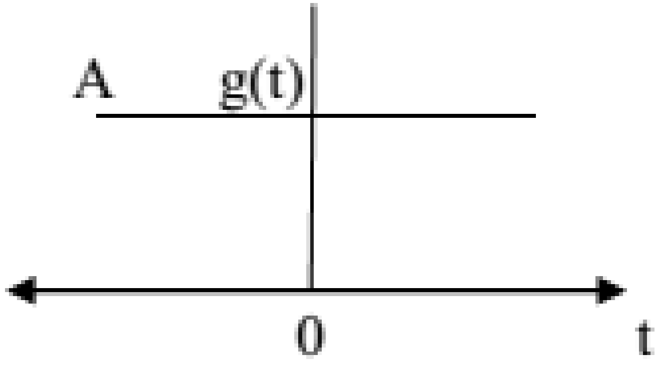
1. $g(t)\delta(t-t_0) = g(t_0)\delta(t-t_0)$; (Multiplication)
2. $\int_{-\infty}^{\infty} g(t)\delta(t-t_0)dt = g(t_0)$; (Shifting)
3. $\delta(\alpha t) = \frac{1}{|\alpha|}\delta(t)$
4. $\delta(t) * g(t) = g(t)$
5. $\delta(t) = \frac{du(t)}{dt} \Longrightarrow u(t) = \int_{-\infty}^{t} \delta(t)dt$
6. $\delta(t) = \delta(-t)$
7. Fourier transform : $F\{\delta(t)\} = 1$

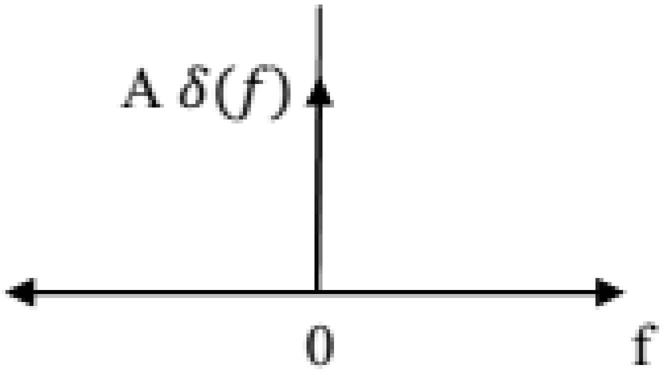


8. 8. $F\{\delta(t-t_0)\} = e^{-j2\pi f t_0}$

Applications of delta functions

1. Dc signal : Since $F\{\delta(t)\} = 1$, then by the duality property $F\{1\} = \delta(f)\}$





2. Complex exponential function

$$F\{A\ e^{j2\pi f_c t}\} = A\ \delta(f - f_c)$$

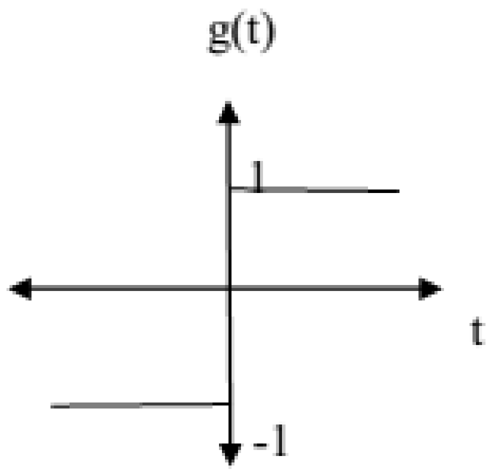
3. Sinusoidal functions

$$F\{\cos 2\pi f_c t\} = \frac{1}{2}\{\delta(f - f_c) + \delta(f + f_c)\}$$

$$F\{\sin 2\pi f_c t\} = \frac{1}{2j}\{\delta(f - f_c) - \delta(f + f_c)\}$$

4. Signum function

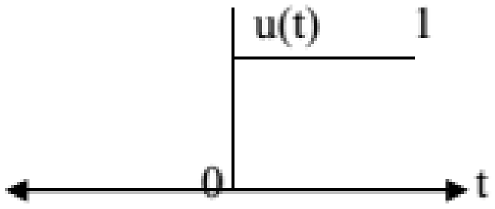
$$\text{sgn}(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$



$$F\{\text{sgn}(t)\} = \frac{1}{j\pi f}$$

5.Unit Step function :

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases}$$



$$sgn(t) = 2u(t) - 1$$

$$u(t) = \frac{1}{2}\{sgn(t) + 1\}$$

$$F\{u(t)\} = \frac{1}{j2\pi f} + \frac{1}{2}\,\delta(f)$$

6.Periodic Signals

A periodic signal g(t) is expanded in the complex form as :

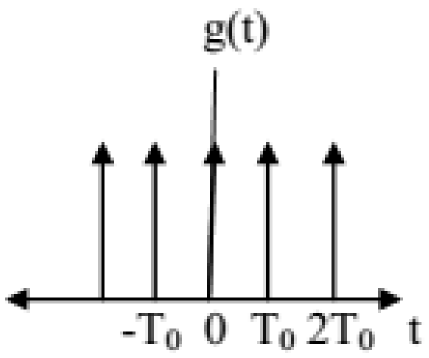
$$g(t) = \sum_{n=-\infty}^{\infty} C_n\, e^{jn\omega_0 t}$$

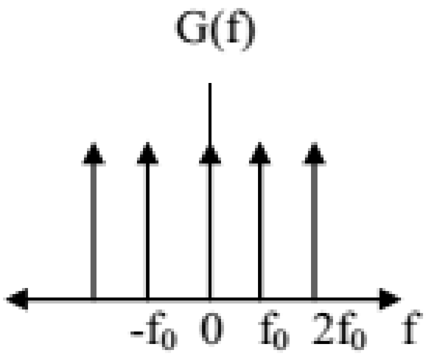
$$F\{g(t)\} = \sum_{n=-\infty}^{\infty} C_c\, \delta(f - nf_0)$$

When $g(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT_0)$; impulse train in time domain

$$C_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\, e^{-jn\omega_0 t}\, dt = \frac{1}{T_0} = f_0$$

$$\Rightarrow F\{g(t)\} = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} \delta(f - nf_0)$$





Note that the signal is periodic in the time domain and its Fourier transform is periodic in the frequency domain. This sequence will be found useful when the sampling theorem is considered.

Transmission of Signals Through Linear Systems

Definition : A system refers to any physical device that produces an output signal in response to an input signal.

Definition : A system is linear if the principle of superposition applies.

If	$x_1(t)$	produces output	$y_1(t)$
	$x_2(t)$	produces output	$y_2(t)$
then	$a_1x_1(t)+ a_2x_2(t)$	produces an output	$a_1y_1(t)+ a_2y_2(t)$
Also,	a zero input	should produce	a zero output.

Example of linear systems include filters and communication channels .

Definition : A filter refers to a frequency selective device that is used to limit the spectrum of a signal to some band of frequencies.

Definition : A channel refers to a transmission medium that connects the transmitter and receivers of a communication system .

Time domain and frequency domain may be used to evaluate system performance.

Time response :

Definition : The impulse response h(t) is defined as the response of a system to an impulse $\delta(t)$ applied to the input at t=0 .

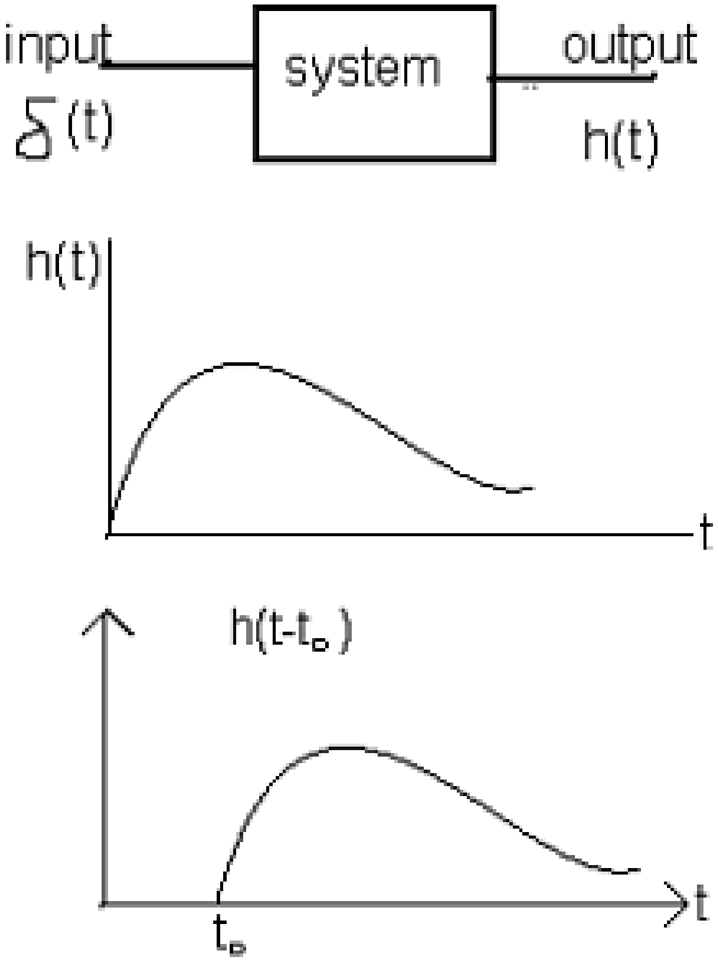
Definition : A system is time-invariant when the shape of the impulse response is the same no matter when the impulse is applied to the system .

$\delta(t) \longrightarrow h(t)$, then $\delta(t - t_d) \longrightarrow h(t - t_d)$

When the input to a linear time-invariant system in a signal x(t) , then the output is given by

$$y(t) = y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t - \lambda)\ d\lambda$$

$$= \int_{-\infty}^{\infty} h(\lambda)x(t - \lambda)\ d\lambda \text{ ; convolution integral}$$



Definition : A system is said to be causal if it doesn't respond before the excitation is applied , i.e. ,

$$h(t) = 0 \qquad t < 0$$

The causal system is physically realizable.

Definition : A system is said to be stable if the output signal is bounded for all bounded input signals .

If $|x(t)| \leq M$; M is the maximum value of the input

then $|y(t)| \leq \int_{-\infty}^{\infty} |h(\tau)||x(t-\tau)|\, d\tau$

$$= M \int_{-\infty}^{\infty} |h(\tau)|\, d\tau$$

$\Longrightarrow$ A necessary and sufficient condition for stability (a bounded output) is

$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$; h(t) is absolutely integrable.

$\therefore$ zero initial conditions assumed .

Frequency Response :

Definition : The transfer function of a linear time invariant system is defined as the Fourier transform of the impulse response .

$$H(f) = F\{h(t)\}$$

Since $y(t) = x(t)*h(t)$, then

$$Y(f) = H(f)\, X(f)$$

or $$\frac{Y(f)}{X(f)} = H(f)$$

The transfer function H(f) is a complex function of frequency, which can be obtained as the ratio of the Fourier transform of the output to that of the input.

$$H(f) = |H(f)|e^{j\,\theta(f)}$$

where

H(f) : amplitude spectrum
θ(f) : phase spectrum.

System Input–Output Energy Spectral Density

Let x(t) be applied to a LTI system , then the Fourier transform of the output is related to the Fourier transform of the input through the relation

$$Y(f) = H(f)\, X(f)$$

Taking the absolute value and squaring both sides, we get

$$|Y(f)|^2 = |H(f)|^2\, |X(f)|^2$$

$$S_Y(f) = |H(f)|^2\, S_X(f)$$

$S_Y(f)$: Output Energy Spectral Density

$S_X(f)$: Input Energy Spectral Density.

Output energy spectral density $= |H(f)|^2$ x Input energy spectral density

The total output energy

$$E_y = \int_{-\infty}^{+\infty} S_Y(f)df$$
$$= \int_{-\infty}^{+\infty} |H(f)|^2\, S_X(f)df.$$

The total input energy is

$$E_x = \int_{-\infty}^{+\infty} S_x(f)df\ .$$

Example: Response of a Filter to a Sinusoidal Input

The signal $x(t) = \cos w_0 t$ is applied to a filter described by the transfer function $H(f) = \frac{1}{1+jf/B}$. Find the filter output $y(t)$.

Solution:

We will find the output using the frequency domain approach.

$$Y(f) = H(f)X(f)$$

$$H(f) = \frac{1}{\sqrt{1+(\frac{f}{B})^2}} e^{-j\theta}; \qquad \theta = \tan^{-1}\frac{f}{B}; \qquad \theta_0 = \tan^{-1}\frac{f_0}{B}$$

$$Y(f) = H(f)[\frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)]$$

$$Y(f) = \frac{1}{2}H(f_0)\delta(f - f_0) + \frac{1}{2}H(-f_0)\delta(f + f_0)$$

$$Y(f) = \frac{1}{2}\frac{1}{\sqrt{1+(\frac{f_0}{B})^2}} e^{-j\theta_0}\delta(f - f_0) + \frac{1}{2}\frac{1}{\sqrt{1+(\frac{f_0}{B})^2}} e^{j\theta_0}\delta(f + f_0)$$

Taking the inverse Fourier transform, we get

$$y(t) = \frac{1}{\sqrt{1+(\frac{f_0}{B})^2}}\frac{1}{2}[e^{j(2\pi f_0 t-\theta_0)} + e^{-j(2\pi f_0 t-\theta_0)}]$$

$$y(t) = \frac{1}{\sqrt{1+(\frac{f_0}{B})^2}}\cos(2\pi f_0 t - \theta_0)$$

Note that in the last step we have made use of the Fourier transform pair

$$e^{j2\pi f_c t} \leftrightarrow \delta(f - f_c)$$

Assume, for instance, that $f_0 = B$. Then $\theta_0 = \tan^{-1}\frac{f_0}{B} = \tan^{-1} 1 = 45°$ and the output can be written as:

$$y(t) = \frac{1}{\sqrt{1+1}}\cos(2\pi f_0 t - 45°)$$

$$y(t) = \frac{1}{\sqrt{2}}\cos(2\pi f_0 t - 45°)$$

Exercise: The signal $x(t) = \cos w_0 t - \frac{1}{\pi} \cos 3w_0 t$ is applied to a filter described by the transfer function $H(f) = \frac{1}{1+jf/B}$. Use the result of the previous example to find the filter output $y(t)$.

Exercise: Consider the periodic rectangular signal $g(t)$ defined over one period T_0 as:

$$g(t) = \begin{cases} +A, & -T_0/4 \le t \le T_0/4 \\ 0, & otherwise \end{cases}$$

If $g(t)$ is applied to a filter described by the transfer function $H(f) = \frac{1}{1+jf/B}$. Use the result of the previous example to find the filter output $y(t)$.

Example:

The signal $g(t) = A rect(\frac{t}{T})$ is applied to the filter $(f) = \frac{1}{1+jf/B}$. Find the output energy spectral density.

Solution:

$$S_Y(f) = |H(f)|^2 S_X(f)$$

$$S_Y(f) = \frac{1}{1+(\frac{f}{B})^2} |AT\,|sinc\,Tf||^2$$

Example:

The signal $g(t) = \delta(t) - \delta(t-1)$ is applied to a channel described by the transfer function $H(f) = \frac{1}{1+jf/B}$. Find the channel output.

Solution:

The impulse response of the channel is obtained by taking the inverse Fourier transform of $H(f)$, which is

$$h(t) = 2\pi B e^{-2\pi B t} u(t)$$

Using the linearity and time invariance property, the output can be obtained as:

$$y(t) = h(t)u(t) - h(t-1)u(t-1)$$

$$y(t) = 2\pi B[e^{-2\pi B t} u(t) - e^{-2\pi B(t-1)} u(t-1)]$$

Exercise: The signal $g(t) = u(t) - u(t-1)$ is applied to a channel described by the transfer function $H(f) = \frac{1}{1+jf/B}$. Find the channel output $y(t)$.

Signal Distortion in Transmission

As we have said before, the objective of a communication system is to deliver to the receiver an almost exact copy of what the source sends. However, communication channels are not perfect in the sense that impairments on the channel will cause the received signal to differ from the transmitted one. During the course of transmission, the signal undergoes attenuation, phase delay, interference from other transmissions, Doppler shift in the carrier frequency, and many other effects. In this introductory discussion we will explain some of the reasons that cause the received signal to be distorted.

a. Linear Distortion

A signal transmission is said to be distortion-less if the output signal y(t) is an exact replica of the input signal x(t) , i.e., y(t) has the same shape as the input, except for a constant amplification (or attenuation) and a constant time delay.

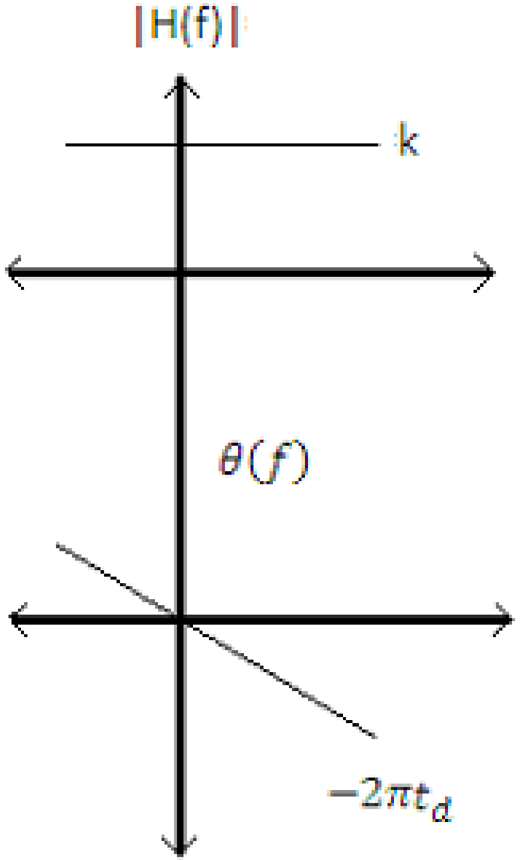
Condition in the time domain for a distortion-less transmission:

$$y(t) = k\, x(t-t_d)$$

where k: is a constant amplitude scaling
t_d: is a constant time delay

In the frequency domain, the condition for a distortion-less transmission becomes

$$Y(f) = k\, X(f)\, e^{-j2\pi f t_d}$$



or $\quad H(f) = \frac{Y(f)}{X(f)} = k\, e^{-j2\pi f t_d} = k e^{-j\theta(f)}$

That is, for a distortion-less transmission, the transfer function should satisfy two conditions:

1. $|H(f)| = k$; where k is a constant amplitude over the frequency range of interest.
2. $\theta(f) = -2\pi f t_d = -(2\pi t_d)f$; linear phase with negative slope that passes through the origin (or multiples of π).

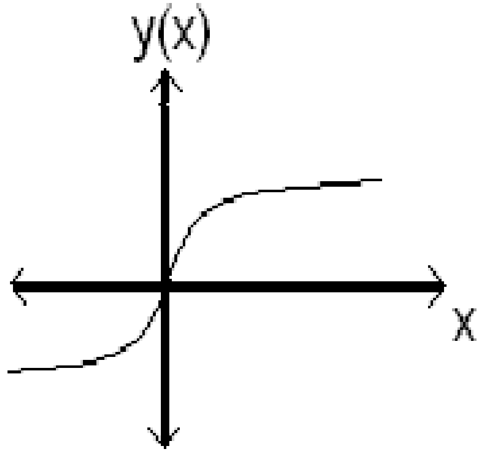
When $|H(f)|$ is not a constant for all frequencies of interest, amplitude distortion results.

When $\theta(f) \neq -2\pi f t_d \pm 180^0$, then we have phase distortion (or delay distortion).

b. Non Linear Distortion

When a system contains nonlinear elements , it is not described by a transfer function , but by a transfer characteristic of the form

$$y(t) = a_1\, x(t) + a_2\, x^2(t) + a_3\, x^3(t) + \ldots \text{ (time domain)}$$



In the frequency domain ,

$$Y(f) = a_1\, X(f) + a_2\, X(f)*X(f) + a_3\, X(f)*X(f)*X(f) + \ldots$$

Here, the output contains new frequencies not originally present in the original signal . The nonlinearity produces undesirable frequency component for $|f| \leq w$.

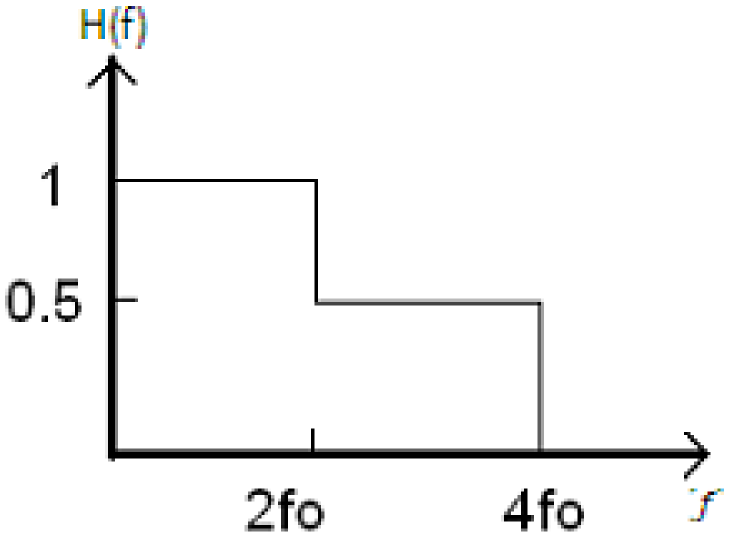
The following examples demonstrate the types of distortion mentioned above.

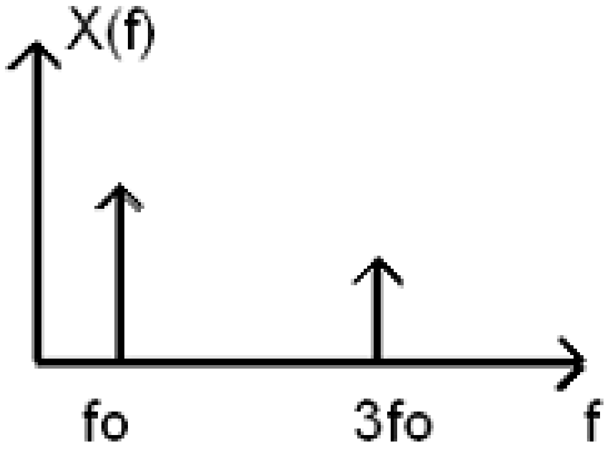
Example : Amplitude Distortion

Consider the signal $x(t) = \cos w_0 t - \frac{1}{3} \cos 3w_0 t$. If this signal passes through a channel with zero time delay (i.e. , $t_d = 0$) and amplitude spectrum as shown in the figure

a. Find y(t)

b. Is this a distortion-less transmission ?





Solution :

x(t) consists of two frequency components, f_0 and $3f_0$. Upon passing through the channel, each one of them will be scaled by a different factor.

a. $y(t) = \cos w_0 t - \frac{1}{2} . \frac{1}{3} \cos 3w_0 t$

b. Since $y(t) \neq k\, x(t)$, this is not a distortion-less transmission .

Example : Phase Distortion

If x(t) in the previous example is passed through a channel whose amplitude spectrum is a constant k. Each component in x(t) suffers a $\frac{\pi}{2}$ phase shift

a. Find y(t).

b. Is this a distortion-less transmission ?

Solution :

$$x(t) = \cos w_o t - \frac{1}{3}\cos 3w_o t$$

$$y(t) = \text{k } \cos(w_o t - \frac{\pi}{2}) - \frac{1}{3} k \cos\left(3w_o t - \frac{\pi}{2}\right)$$

$$y(t) = \text{k } \cos w_o(t - \frac{\pi}{2w_o}) - \frac{1}{3} k \cos\left(3w_o(t - \frac{\pi}{2x3w_o})\right)$$

$$y(t) = \text{k } \cos w_o(t - t_{d1}) - \frac{1}{3} k \cos(3w_o(t - t_{d2}))$$

Note that $t_{d1} \neq t_{d2}$, i.e., each component in $x(t)$ suffers from a different time delay. Hence this transmission introduces phase (delay) distortion.

Harmonic Distortion

Let the input to a nonlinear system be the single tone signal

$x(t)=\cos 2\pi f_0 t$

This signal is applied to a channel with characteristic

$y(t)=a_1x+a_2x^2+a_3x^3$

upon substituting x(t) and arranging terms, we get

$$y(t) = \frac{1}{2}a_2 + \left(a_1 + \frac{3}{4}a_3\right)\cos 2\pi f_0 t + \frac{1}{2}a_2 \cos 4\pi f_0 t + \frac{1}{4}a_3 \cos 6\pi f_0 t$$

Note that the output contains a component proportional to x(t) which is $\left(a_1 + \frac{3}{4}a_3\right)\cos 2\pi f_0 t$, in addition to a second and a third harmonic term (terms at twice and three times the frequency of the input). These new terms are the result of the nonlinear characteristic and are , therefore, considered harmonic distortion.

Define second harmonic distortion

$$D_2 = \frac{|amplitude\ of\ second\ harmonic\,|}{|amplitude\ of\ fundamental\ term|}$$

$$D_2 = \frac{|\frac{1}{2}a_2\,|}{|\left(a_1 + \frac{3}{4}a_3\right)|}\ x\ 100\%$$

In a similar way we can define the third harmonic distortion as:

$$D_2 = \frac{|amplitude\ of\ third\ harmonic\ |}{|amplitude\ of\ fundamental\ term|}$$

Therefore,

$$D_3 = \frac{|\frac{1}{4}a_3|}{|\left(a_1 + \frac{3}{4}a_3\right)|}\ x\ 100\%$$

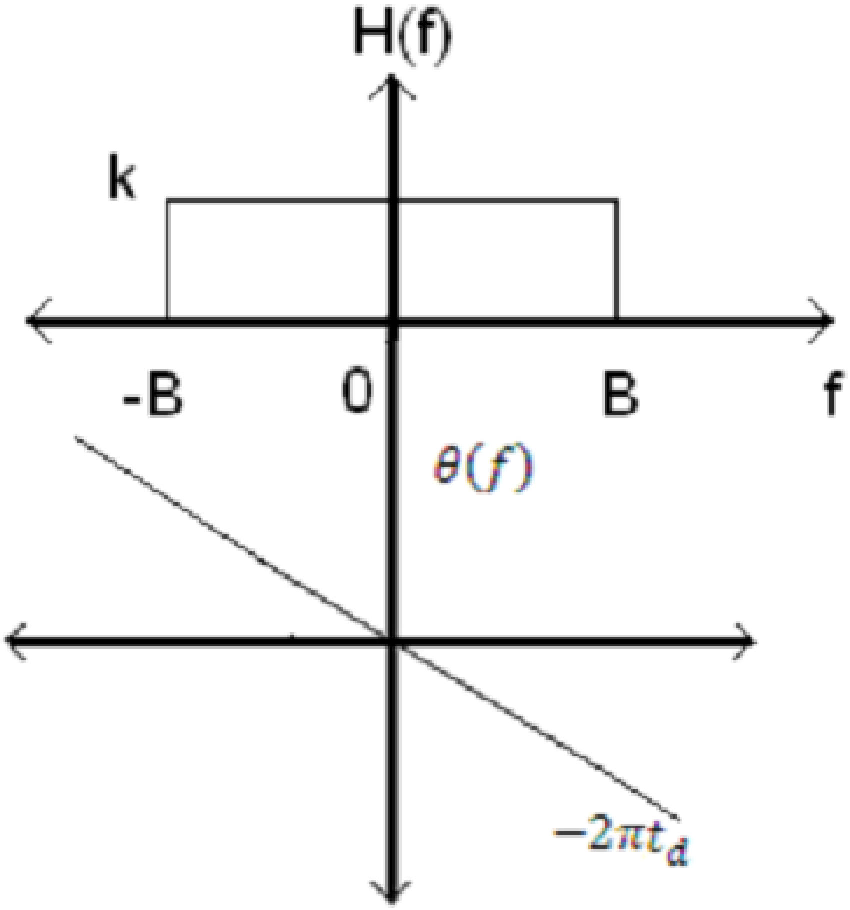
Remark: In the solution above we have made use of the following two identities:

$\mathrm{Cos}^2x = \frac{1}{2}\{1 + cos2x\}$

$\mathrm{Cos}^3x = \frac{1}{4}\{3cosx + cos3x\}$.

Filters and Filtering

A filter is a frequency selective device . It allows certain frequencies to pass almost without attenuation wile it suppresses other frequencies

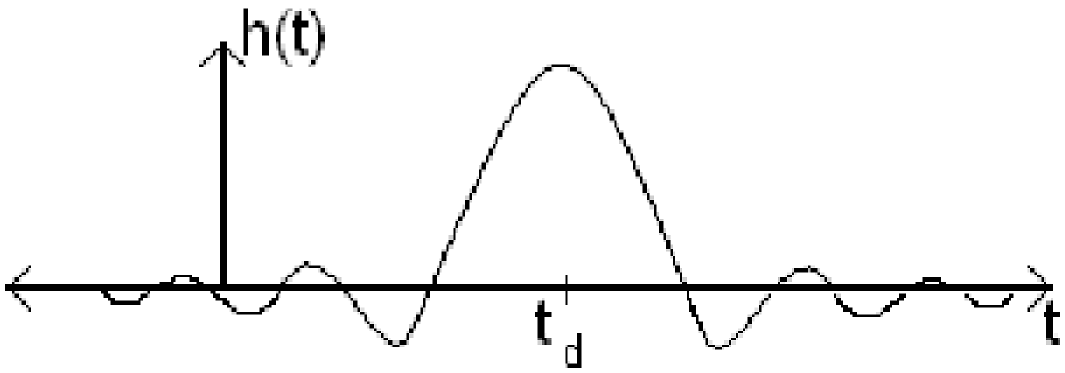


A. Ideal Filter:

Ideal low pass filter :

$$H(f) = \begin{cases} k\, e^{-j2\pi f t_d} & |f| < B \\ 0 & o.w \end{cases}$$

$$h(t) = 2Bk\ sinc\ 2B(t - t_d)$$



since h(t) is the response to an impulse applied at t=0 ,and because h(t) has nonzero values for t<0 , the filter is noncausal (physically non realizable)

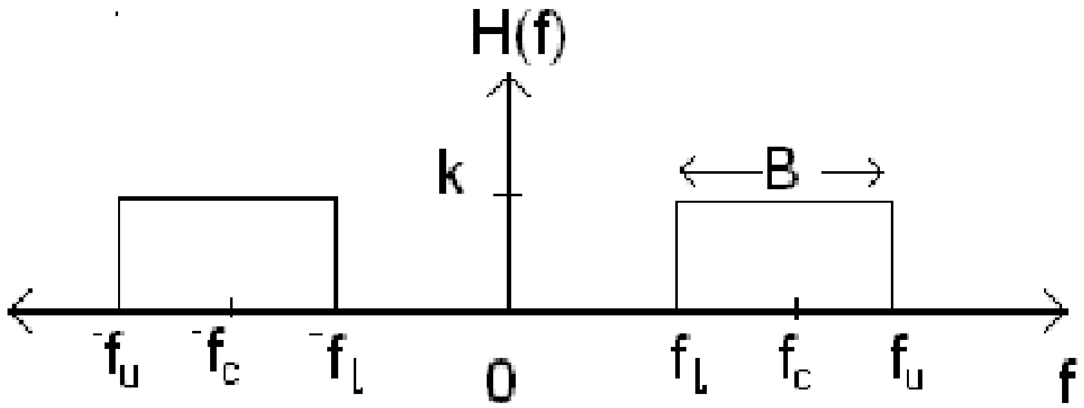
Band Pass Filter

$$H(f) = \begin{cases} k\, e^{-j2\pi f t_d} & f_l < |f| < f_u \\ 0 & o.w \end{cases}$$

Filer bandwidth $B = f_u - f_l$

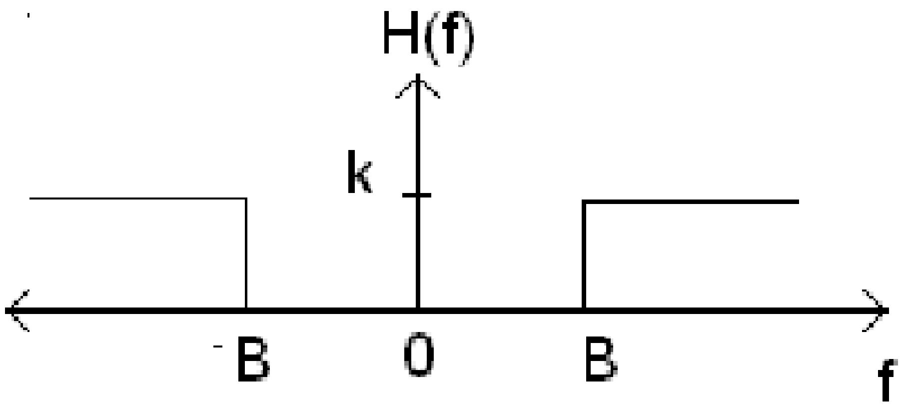
$$f_c = \frac{f_u + f_l}{2}$$

$$h(t) = 2Bk \; sinc\, B(t - t_d) \;\; cos\, w_c(t - t_d)$$



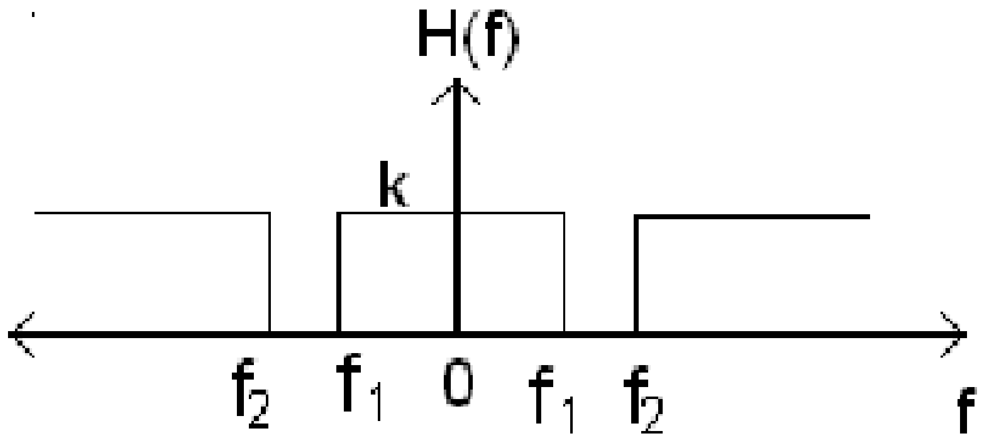
High pass filter :

$$H(f) = \begin{cases} k\, e^{-j2\pi f t_d} & |f| > B \\ 0 & o.w \end{cases}$$



Band Rejection or Notch Filter

$$H(f) = \begin{cases} k\, e^{-j2\pi f t_d} & o.w \\ 0 & f_1 < |f| < f_2 \end{cases}$$



Real Filter:

Here we only consider a Butterworth low pass filter. The transfer function of a low pass Butterworth filter is of the form

$$H(f) = \frac{1}{P_n(\frac{jf}{B})}$$

B is the 3-dB bandwidth of the filter and $P_n(jf/B)$ is a complex polynomial of order n . The family of Butterworth polynomials is defined by the property

$$|P_n\left(\frac{jf}{B}\right)|^2 = 1 + (\frac{f}{B})^{2n}$$

$$|H(f)| = \frac{1}{\sqrt{1+(\frac{f}{B})^{2n}}}$$

The first few polynomials are:

$P_1(x) = 1 + x$

$P_2(x) = 1 + \sqrt{2}x + x^2$

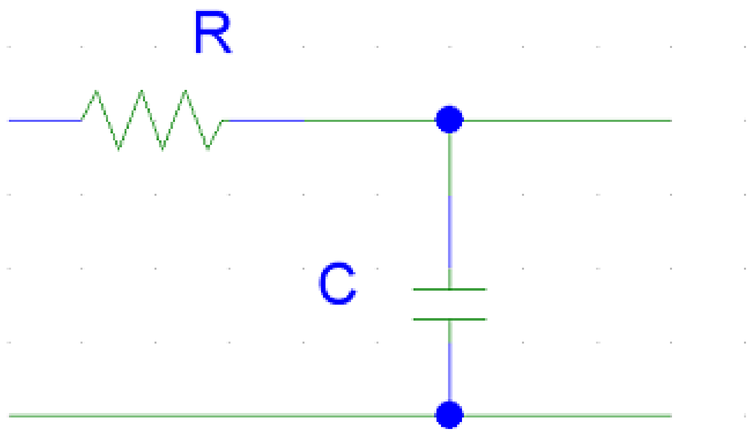
$P_3(x) = (1 + x)(1 + x + x^2)$

A first order LPF :

$$H(f) = \frac{\frac{1}{j2\pi f_c}}{R+\frac{1}{j2\pi f_c}} = \frac{1}{1+j2\pi fRC}$$

Let $B = \frac{1}{2\pi RC}$

$$H(f) = \frac{1}{1 + jf/B} = \frac{1}{P_1(jf/B)} = \frac{1}{P_1(x)}$$



A Second order LPF :

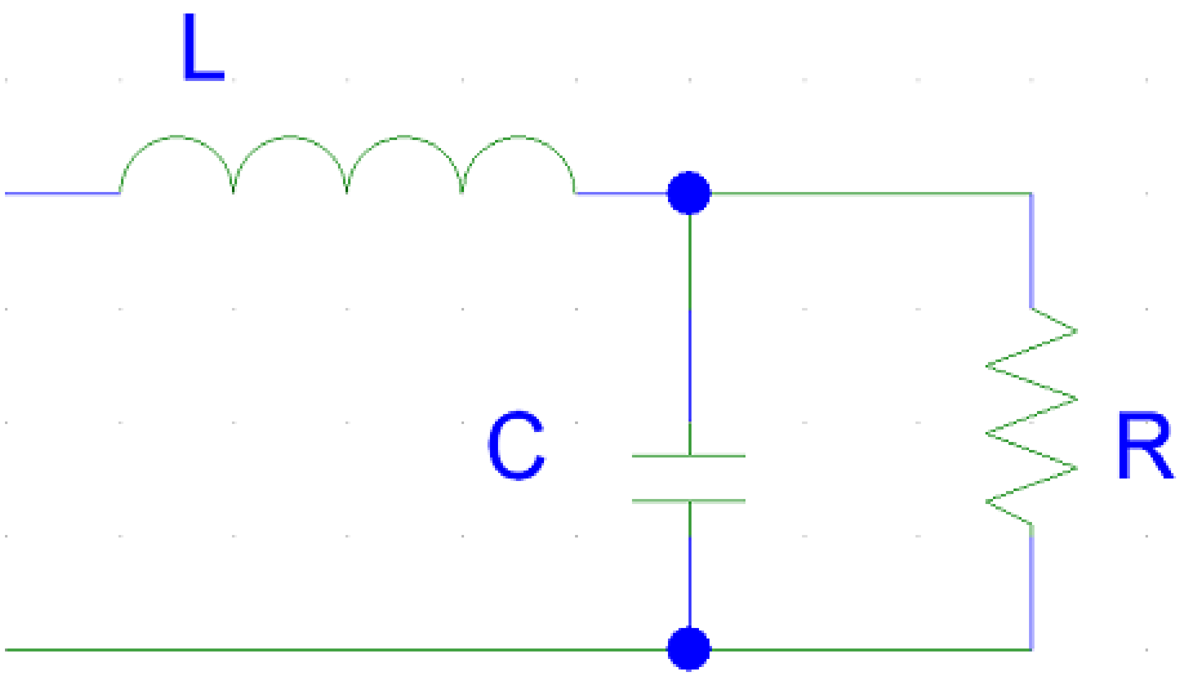
$$H(f) = \frac{1}{1+\frac{jwL}{R}-(2\pi\sqrt{LC}f)^2}$$

$$H(f) = \frac{1}{1+j\sqrt{2}f/B-(f/B)^2}$$

$$where\ \ R = \sqrt{\frac{L}{2C}},\ B = \frac{1}{2\pi\sqrt{LC}}$$

$$H(f) = \frac{1}{1+j\sqrt{2}f/B-(f/B)^2}$$

$$H(f) = \frac{1}{P_2(jf/B)}$$



Hilbert Transform

The quadrature filter : is an all pass filter that shifts the phase of positive frequency by (-90°) and negative frequency by (+90°). The transfer function of such a filter is

$$H(f) = \begin{cases} -j & f > 0 \\ j & f < 0 \end{cases}$$

Using the duality property of Fourier transform the impulse response of the filter is

$$h(t) = \frac{1}{\pi t}$$

The Hilbert transform of a signal g(t) is

$$\hat{g}(t) = \frac{1}{\pi t} * g(t) = \int_{-\infty}^{\infty} \frac{g(\lambda)}{\pi(t-\lambda)}\, d\lambda$$

Note that the Hilbert transform of a signal is a function of time. The Fourier transform of $\hat{g}(t)$ is

$$\hat{G}(f) = -j \operatorname{sgn}(f)\, G(f)$$

Hilbert transform can be found by using either the time domain approach or the frequency domain approach depending on the given problem, that is

- Direct convolution in the time domain of g(t) and $\frac{1}{\pi t}$.
- Find the Fourier transform $\hat{G}(f)$, then find the inverse Fourier transform $\hat{g}(t) = \int_{-\infty}^{\infty} \hat{G}(f)\, e^{j2\pi f t}\, df$

Some properties of the Hilbert transform

1. A signal g(t) and its Hilbert transform $\hat{g}(t)$ have the same energy spectral density

$$|\hat{G}(f)|^2 = |-j\,sgn(f)G(f)|^2 = |-j\,sgn(f)|^2|G(f)|^2$$
$$= |G(f)|^2$$

The consequences of this property are:

- If a signal g(t) is bandlimited, then $\hat{g}(t)$ is bandlimited to the same bandwidth (note that $|\hat{G}(f)| = |G(f)|$)
- $\hat{g}(t)$ and g(t) have the same total energy (or power).
- $\hat{g}(t)$ and g(t) have the same autocorrelation function.

2. A signal g(t) and $\hat{g}(t)$ are orthogonal

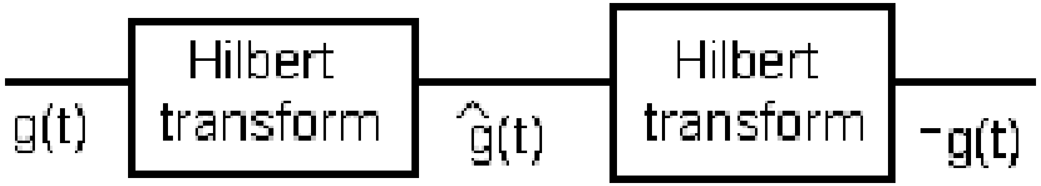
$$\int_{-\infty}^{\infty} g(t)\,\hat{g}(t)dt = 0$$

This property can be verified using the general formula of Rayleigh energy theorem

$$\int_{-\infty}^{\infty} g(t)\,\hat{g}(t)dt = \int_{-\infty}^{\infty} G(f)\,\hat{G}^*(f)df = \int_{-\infty}^{\infty} G(f)\,\{-jsgn(f)\,G(f)\}^* df$$
$$= \int_{-\infty}^{\infty} jsgn(f)\,|G(f)|^2 df = 0$$

The result above follows from the fact that $|G(f)|^2$ is an even function of f while $sgn(f)$ is an odd function of f. Their product is odd. The integration of an odd function over a symmetrical interval is zero.

3. If $\hat{g}(t)$ is a Hilbert transform of g(t) , then the Hilbert transform of $\hat{g}(t)$ is $-g(t)$.



Example on Hilbert Transform

Find the Hilbert transform of the impulse function $g(t) = \delta(t)$

Solution:

Here, we use the convolution in the time domain

$$\hat{g}(t) = \frac{1}{\pi t} * \delta(t)$$

As we know, the convolution of the delta function with a continuous function is the function itself. Therefore,

$$\hat{g}(t) = \frac{1}{\pi t}$$

Example on Hilbert Transform

Find the Hilbert transform of $g(t) = \frac{\sin t}{t}$

Solution :

Here, we will first find the Fourier transform of $g(t)$, find $\hat{G}(f)$, and then find $\hat{g}(t)$

$$A\, rect\left(\frac{t}{\tau}\right) \quad \overset{transform}{\longleftrightarrow} \quad A\tau\, sinc\, f\tau \quad ; \quad when\ \tau = \frac{1}{\pi}$$

$$A\,\mathrm{rect}\left(\frac{t}{1/\pi}\right) \quad \overset{transform}{\longleftrightarrow} \quad A\,\frac{1}{\pi}\,\frac{\sin \pi f\tau}{\pi f\tau} = \frac{1}{\pi}\frac{\sin f}{f}$$

$$\pi\, rect\left(\frac{t}{1/\pi}\right) \quad \overset{transform}{\longleftrightarrow} \quad \frac{\sin f}{f}$$

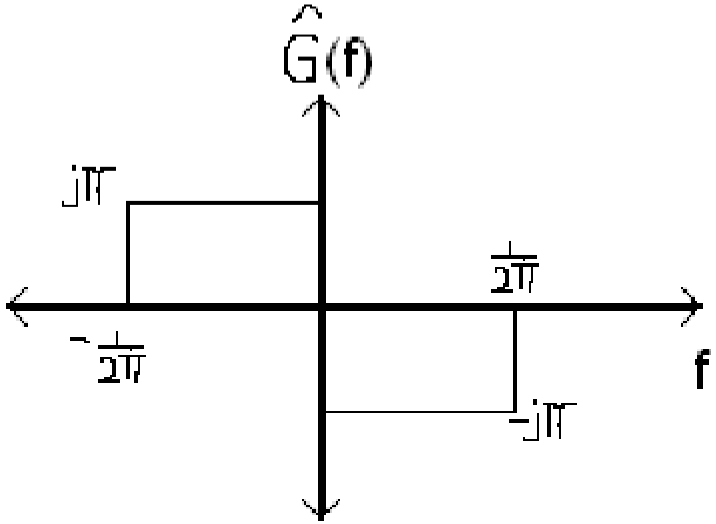
So by the duality property, we get the pair

$$\pi\, rect\left(\frac{f}{1/\pi}\right) \quad \overset{transform}{\longleftrightarrow} \quad \frac{\sin t}{t}$$

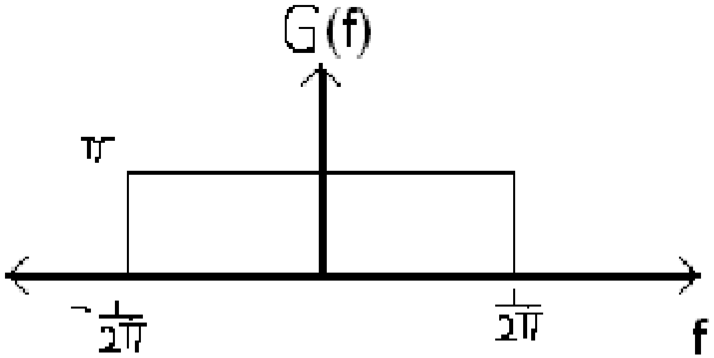
i.e. , $G(f) = \pi\, rect(\frac{f}{1/\pi})$, (See the figure below)

$$\hat{G}(f) = -j sgn(f) G(f) = \begin{cases} -j\pi & 0 < f < 1/2\pi \\ j\pi & -1/2\pi < f < 0 \end{cases}$$

$$\begin{aligned}
\hat{g}(t) &= \int_{-\infty}^{\infty} \hat{G}(f)\, e^{j2\pi ft}\, df \\
&= \int_{-1/2\pi}^{0} j\pi\ \ e^{j2\pi ft}\, df \ - \int_{0}^{1/2\pi} j\pi\ \ e^{j2\pi ft}\, df \\
&= \frac{1}{2t}\left(1 - e^{-jt}\right) - \frac{1}{2t}\left(e^{jt} - 1\right) \\
&= \frac{1}{t} - \frac{1}{t}\,\frac{(e^{jt}+e^{-jt})}{2} \\
&= \frac{1-\cos t}{t}
\end{aligned}$$



Ĝ(f)
jπ
$\frac{1}{2\pi}$
$-\frac{1}{2\pi}$
f
-jπ



G(f)
π
$-\frac{1}{2\pi}$
$\frac{1}{2\pi}$
f

Correlation and Spectral Density

Here we consider the relationship between the autocorrelation function and the power spectral density. In this discussion we restrict our attention to real signals. First, we consider power signals and then energy signals.

Definition: The autocorrelation function of a signal g(t) is a measure of similarity between g(t) and a delayed version of g(t).

a. Autocorrelation function of a power signal

The autocorrelation function of a power signal g(t) is defined as:

$R_g(\tau) = \langle g(t)g(t-\tau)\rangle$; $\langle(.)\rangle$ means time average.

$$R_g(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} g(t)g(t-\tau)dt$$

Exercise: Show that for a periodic signal with period T_0, the above definition becomes

$$R_g(\tau) = \frac{1}{T_0}\int_0^{T_0} g(t)g(t-\tau)dt$$

Exercise: Show that if $g(t)$ is periodic with period T_0, then $R_g(\tau)$ is also periodic with the same period T_0.

Hint: Expand $g(t)$ in a complex Fourier series $g(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0\tau}$. Form the delayed signal $g(t-\tau)$, and then perform the integration over a complete period T_0. You should get the following result:

$$R_g(\tau) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0\tau}$$

This formula bears two results

a. $R_g(\tau)$ is periodic with period T_0.

b. The Complex Fourier coefficients D_n of $R_g(\tau)$ are related to the complex Fourier coefficients C_n of $g(t)$ by the relation $D_n = |C_n|^2$.

Properties of R(τ)

- $R_g(0) = \frac{1}{T_0}\int_0^{T_0} g(t)^2 dt$; is the total average signal power.
- $R_g(\tau)$ is an even function of τ, i.e., $R_g(\tau) = R_g(-\tau)$.
- $R_g(\tau)$ has a maximum (positive) magnitude at $\tau = 0$, i.e. $|R_g(\tau)| \leq R_g(0)$.
- If g(t) is periodic with period T_0 , then $R_g(\tau)$ is also periodic with the same period T_0.
- The autocorrelation function of a periodic signal and its power spectral density (represented by a discrete set of impulse functions) are Fourier transform pairs

$$S_g(f) = \mathrm{F}\{R_g(\tau)\}$$

$$S_g(f) = \sum_{n=-\infty}^{\infty} |C_n|^2 \delta(f - nf_0)$$

Cross Correlation Function

The cross correlation function of two periodic signals $g_1(t)$ and $g_2(t)$ with period T_0 is defined as;

$$R_{1,2}(\tau) = \frac{1}{T_0}\int_0^{T_0} g_1(t)g_2(t-\tau)dt$$

b- **Autocorrelation function of an energy signal**

When g(t) is an energy signal, $R_g(\tau)$ is defined as:

$$R_g(\tau) = \int_{-\infty}^{\infty} g(t)g(t-\tau)dt$$

Properties of R(τ)

- $R_g(0) = \int_{-\infty}^{\infty} g(t)^2 dt;$ is the total signal energy.
- $R_g(\tau)$ is an even function of τ, i.e., $R_g(\tau) = R_g(-\tau)$.
- $R_g(\tau)$ has a maximum (positive) magnitude at $\tau = 0$, i.e. $|R_g(\tau)| \le R_g(0)$.

- The autocorrelation function of an energy signal and its energy spectral density (a continuous function of frequency) are Fourier transform pairs, i.e.,

$$S_g(f) = \mathrm{F}\{R_g(\tau)\}$$

$$S_g(f) = \int_{-\infty}^{\infty} R_g(\tau) e^{-j2\pi f\tau} d\tau$$

$$R_g(\tau) = \int_{-\infty}^{\infty} S_g(f) e^{j2\pi f\tau} df.$$

Proof:

The autocorrelation function is defined as:

$$R_g(\tau) = \int_{-\infty}^{\infty} g(\lambda)g(\lambda - \tau)d\lambda$$

In this integral we have replaced t by λ. With this substitution, we can rewrite the integral as

$$R_g(\tau) = \int_{-\infty}^{\infty} g(\lambda)g(-(\tau - \lambda))d\lambda$$

One can realize that $R_g(\tau)$ is nothing but the convolution of $g(\tau)$ and $-g(\tau)$. That is,

$$R_g(\tau) = g(\tau) * g(-\tau)$$

Taking the Fourier transform of both sides, we get

$$F\{R_g(\tau)\} = G(f)G^*(f)$$

Therefore, $S_g(f) = \mathrm{F}\{R_g(\tau)\} = |G(f)|^2$.

Cross Correlation Function

The cross correlation function of two energy signals $g_1(t)$ and $g_2(t)$ is defined as;

$$R_{1,2}(\tau) = \int_{-\infty}^{\infty} g_1(t) g_2(t-\tau) dt$$

Example:

Find the auto-correlation function of the sine signal $g(t) = A\cos(2\pi f_0 t + \theta)$, where A and θ are constants.

Solution:

As we know, this is a periodic signal. So, we find $R_g(\tau)$ using the definition

$$R_g(\tau) = \frac{1}{T_0}\int_0^{T_0} g(t)g(t-\tau)dt$$

$$R_g(\tau) = \frac{1}{T_0}\int_0^{T_0} A\cos(2\pi f_0 t + \theta)A\cos(2\pi f_0 t - 2\pi f_0 \tau + \theta)dt$$

$$R_g(\tau) = \frac{A^2}{2T_0}\int_0^{T_0}[\cos(4\pi f_0 t - 2\pi f_0 \tau + 2\theta) + \cos(2\pi f_0 \tau)]dt$$

$$R_g(\tau) = \frac{A^2}{2T_0}[0 + \cos(2\pi f_0 \tau)T_0]$$

$$R_g(\tau) = \frac{A^2}{2}\cos(2\pi f_0 \tau)$$

Example:

Determine the autocorrelation function of the sinc pulse $g(t) = Asinc2Wt$.

Solution:

Using the duality property of the Fourier transform, you can deduce that

$$G(f) = \frac{A}{2W} rect(\frac{f}{2W})$$

The energy spectral density of $g(t)$ is

$$S_g(f) = |G(f)|^2 = (\frac{A}{2W})^2 rect(\frac{f}{2W})$$

Taking the inverse Fourier transform, we get the autocorrelation function

$$R_g(\tau) = \frac{A^2}{2W} sinc2Wt$$

Exercise:

a. Find and plot the cross correlation function of the two signals

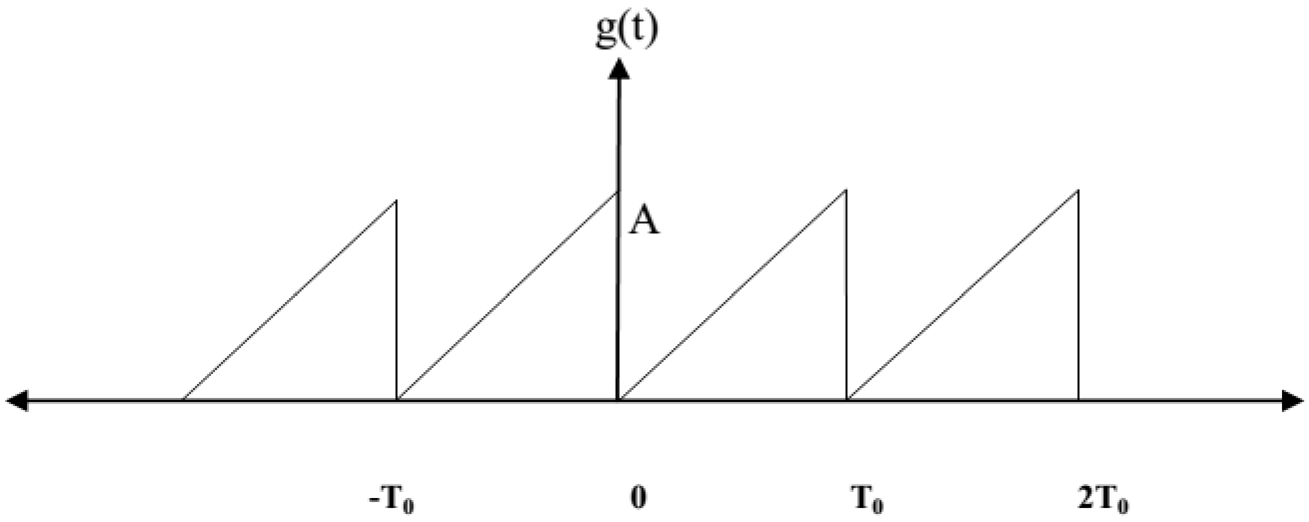
$$g_1(t) = \begin{cases} 1 & 0 \le t \le 2 \\ 0, otherwise \end{cases}$$

$$g_2(t) = \begin{cases} 1 & 0 \le t \le 1 \\ -1 & 1 < t \le 2 \end{cases}$$

b. Are $g_1(t)$ and $g_2(t)$ orthogonal?

Exercise:

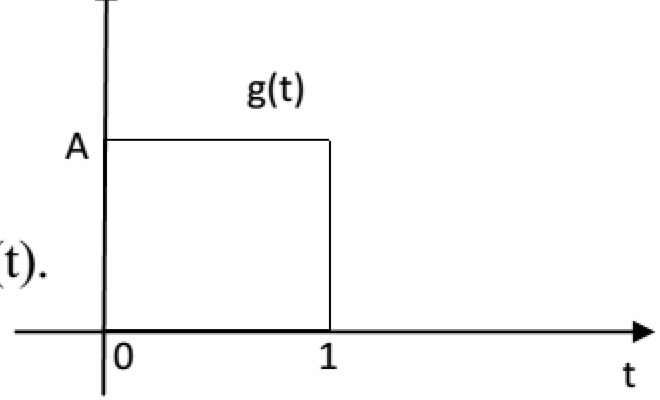
Find and plot the autocorrelation function for the periodic saw-tooth signal shown below:



g(t)
A
-T0
0
T0
2T0

Example:

Find the autocorrelation function of the rectangular pulse g(t).



Solution:

As we saw earlier, this pulse is an energy signal and , therefore, we can find its $R_g(\tau)$ as:

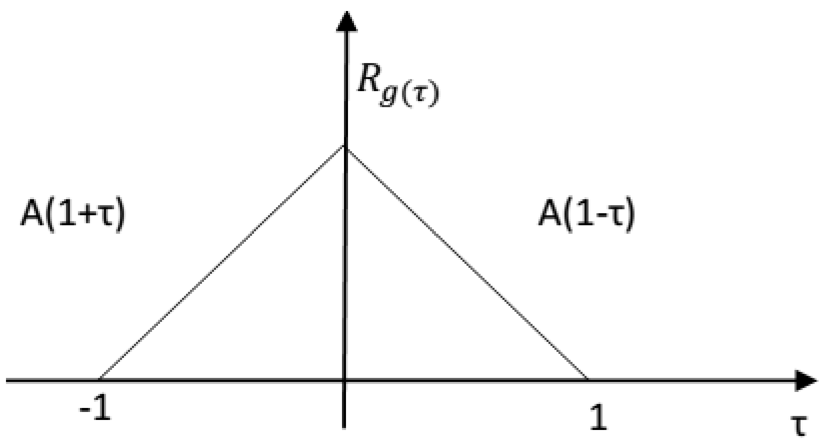
$$R_g(\tau) = \int_{\tau}^{1}(A)(A)dt = A^2\,(1-\tau)\ \ ;\ \ 0< \tau <1$$

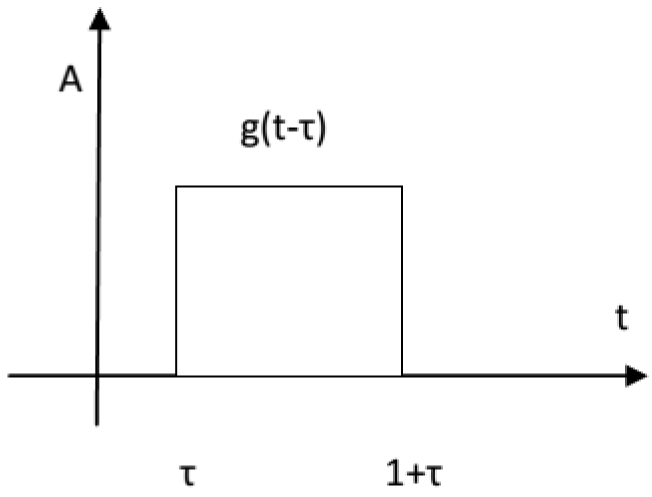
Using the even symmetry property of the autocorrelation function, we can find $R_g(\tau)$ for – ve values of τ as:

$$R_g(\tau) = = A^2\,(1+\tau)\ \ ;\ \ -1< \tau <0$$

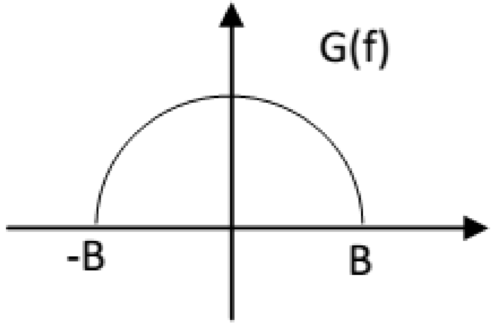
This function is sketched below. Note that that the maximum value occurs at $\tau = 0$ and that g(t) and g(t-τ) become decorrelated for $\tau = 1$ sec, which is the duration of the pulse.

The energy spectral density is $S_g(f) = \mathcal{F}\{R_g(\tau)\} = A^2 \text{sinc}^2 f$





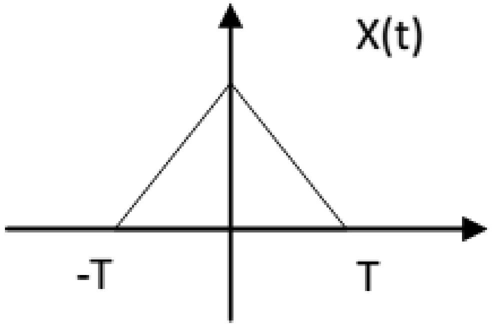
Bandwidth of Signals and Systems:



Def: A signal g(t) is said to be (absolutely) band-limited to BHz if

$G(f) = 0 \qquad \text{for } |f| > B$

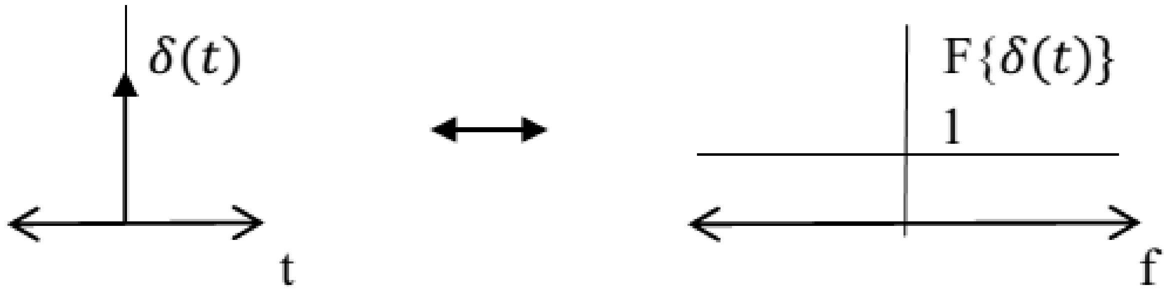
Def: A signal x(t) is said to be (absolutely) time-limited if



$x(t) = 0 \qquad \text{for } |t| > T$

Theorem: An absolutely band-limited waveform cannot be absolutely time-limited (theoretically has an infinite time duration) and vice versa.

We have earlier examples that support this theorem. For example, the delta function, which has an almost zero time duration, has a Fourier transform which extends uniformly over all frequencies. Also, a constant value in the time domain has a Fourier transform, which is an impulse in the frequency domain. This is repeated here for convenience.



In general, there is an inverse relationship between the signal bandwidth and the time duration. The bandwidth and the time duration are related through a relation of the form, called the *time bandwidth product*

$$Bandwidth * Time\ Duration \geq constant$$

The value of the constant depends on the way the bandwidth and the time duration of a signal are defined as will be illustrated later (Possible values of the constant $= \frac{1}{2}, \frac{1}{4\pi}$).

Remarks:

1. The bandwidth of a signal provides a measure of the extent of significant frequency content of the signal.
2. The bandwidth of a signal is taken to be the width of a positive frequency band.
3. For baseband signals or networks , where the spectrum extends from –B to B, the bandwidth is taken to be B Hz.
4. For bandpass signals or systems where the spectrum extends between (f_1, f_2) and ($-f_1$, $-f_2$), the B.W= $f_2 - f_1$.

<u>**Some Definitions of Bandwidth:**</u>

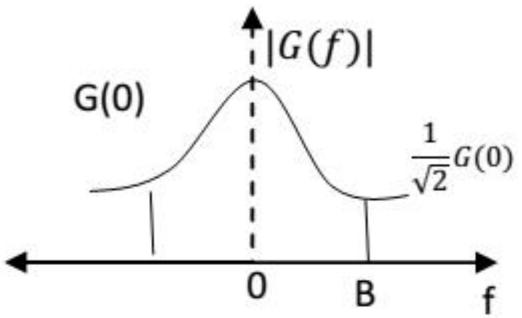
1- Absolute bandwidth

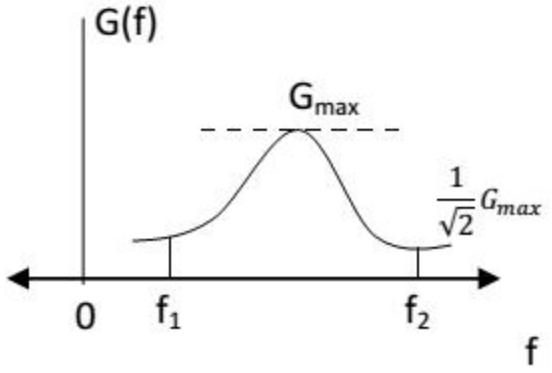
Here, the Fourier transform of a signal is non zero only within a certain frequency band. If $G(f) = 0$ for $|f| > B$, then $g(t)$ is absolutely band-limited to BHz. When $G(f) \neq 0$ for $f_1 < |f| < f_2$, then the absolute bandwidth is $f_2 - f_1$.

2- 3-dB (half power points) bandwidth

The range of frequencies from 0 to some frequency B at which $|G(f)|$ drops to $\frac{1}{\sqrt{2}}$ of its maximum value (for a low pass signal).

As for a band pass signal, the B.W = $f_2 - f_1$





3- The 95 % (energy or power) bandwidth.

Here , the B.W is defined as the band of frequencies where the area under the energy spectral density (or power spectral density) is at least 95% (or 99%) of the total area .

$$\text{Total Energy} = \int_{-\infty}^{\infty} |G(f)|^2\, df = 2\int_{0}^{\infty} |G(f)|^2 df$$

$$\int_{-B}^{B} |G(f)|^2 df = 0.95 \int_{-\infty}^{\infty} |G(f)|^2\, df$$

4- Equivalent Rectangular Bandwidth.

It is the width of a fictitious rectangular spectrum such that the power in that rectangular band is equal to the power associated with the actual spectrum over positive frequency

Area under rectangle = Area under curve

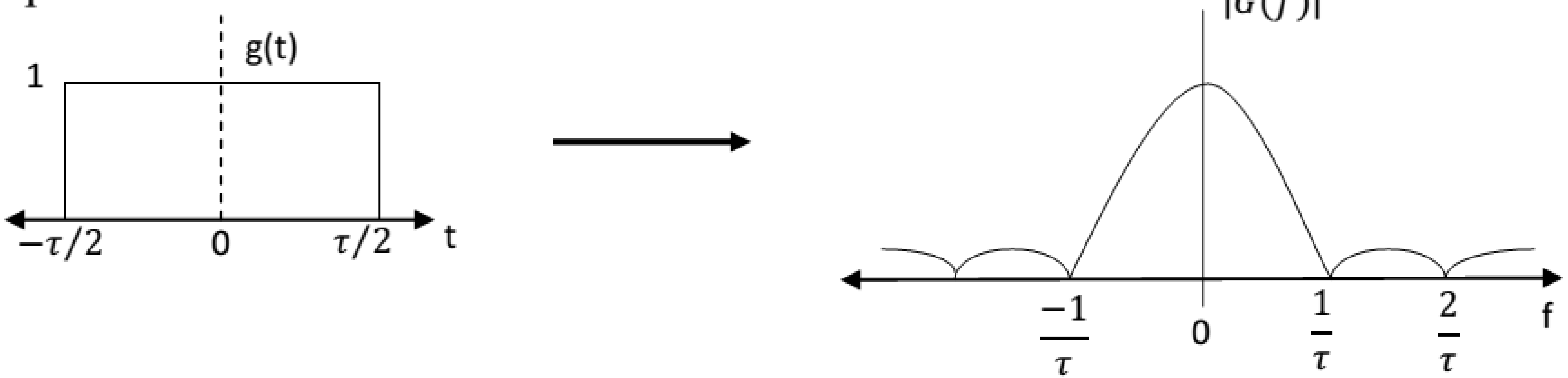
$$|G(0)|^2 * 2B_{eq} = \int_{-\infty}^{\infty} |G(f)|^2 df$$

$$|G(0)|^2 * 2B_{eq} = 2\int_{0}^{\infty} |G(f)|^2 df$$

$$B_{eq} = \frac{1}{|G(0)|^2} \int_{0}^{\infty} |G(f)|^2 df$$

5- <u>Null – to –null bandwidth:</u>

For baseband signals , B.W is the first null in the envelope of the magnitude spectrum above zero.



$$\text{rect}\left(\frac{t}{\tau}\right) \rightarrow \tau\ \text{sinc} f\tau = \tau \frac{\sin\pi f\tau}{\pi f\tau}$$

Zero crossing take place when $\sin\pi f\tau = 0$

$$\pi f\tau = n\pi \rightarrow f = \frac{n}{\tau} \ ; n = 1,2,\ldots\ldots\ldots$$

$B.W = \frac{1}{\tau}$ smaller τ large bandwidth.

For a band pass signal, $B.W = f_2 - f_1$

6- **Bounded spectrum bandwidth:**

Range of frequencies as (0,B) such that outside the band , the power spectral density must be down by say 50 dB below the maximum value

$$-50 \text{ dB} = 10 \log \frac{|G(B)|^2}{|G(0)|^2}$$

7- **RMS Bandwidth:**

$$B_{rms} = \left(\frac{\int_{-\infty}^{\infty} f^2 |G(f)|^2 \mathrm{df}}{\int_{-\infty}^{\infty} |G(f)|^2 \mathrm{df}}\right)^{1\backslash 2}$$

The corresponding rms duration of g(t) is

$$T_{rms} = = \left(\frac{\int_{-\infty}^{\infty} t^2 |g(t)|^2 \mathrm{dt}}{\int_{-\infty}^{\infty} |g(t)|^2 \mathrm{dt}}\right)^{1\backslash 2}$$

(here g(t) is assumed to be centered around the origin).

Remark: The time bandwidth product is $T_{rms}\, B_{rms} \geq \frac{1}{4\pi}$

Time – Bandwidth Product :

To illustrate the time – bandwidth product, consider the equivalent rectangular bandwidth defined earlier as

$$B_{eq} = \frac{\int_{-\infty}^{\infty} |G(f)|^2 df}{2|G(0)|^2}$$

Analogous to this definition, we define an equivalent rectangular time duration as :

$$T_{eq} = \frac{\left(\int_{-\infty}^{\infty} |g(t)| dt\right)^2}{\int_{-\infty}^{\infty} |g(t)|^2 dt}$$

The time bandwidth product is

$$B_{eq}T_{eq} = \frac{\int_{-\infty}^{\infty} |G(f)|^2 df}{2|G(0)|^2} \cdot \frac{\left(\int_{-\infty}^{\infty} |g(t)| dt\right)^2}{\int_{-\infty}^{\infty} |g(t)|^2 dt}$$

Note $\int_{-\infty}^{\infty} |g(t)|^2 \, dt = \int_{-\infty}^{\infty} |G(f)|^2 \, df$; Rayleigh energy theorem. Note also that $G(0) = \int_{-\infty}^{\infty} g(t) dt$. Using these relations, we get

$$B_{eq}T_{eq} = \frac{1}{2}\frac{\left(\int_{-\infty}^{\infty}|g(t)|dt\right)^2}{|\int_{-\infty}^{\infty}g(t)dt|^2}$$

Case 1:

When g(t) is positive for all time t, then |g(t)| = g(t) and $B_{eq}T_{eq}$ becomes

$$B_{eq}T_{eq} = \frac{1}{2}$$

Case 2 :

For a general g(t) that can take on positive as well as negative values, $B_{eq}T_{eq}$ satisfies the inequality

$$B_{eq}T_{eq} \geq \frac{1}{2}$$

Note : For B_{rms} and T_{rms} , the time – bandwidth satisfies the inequality

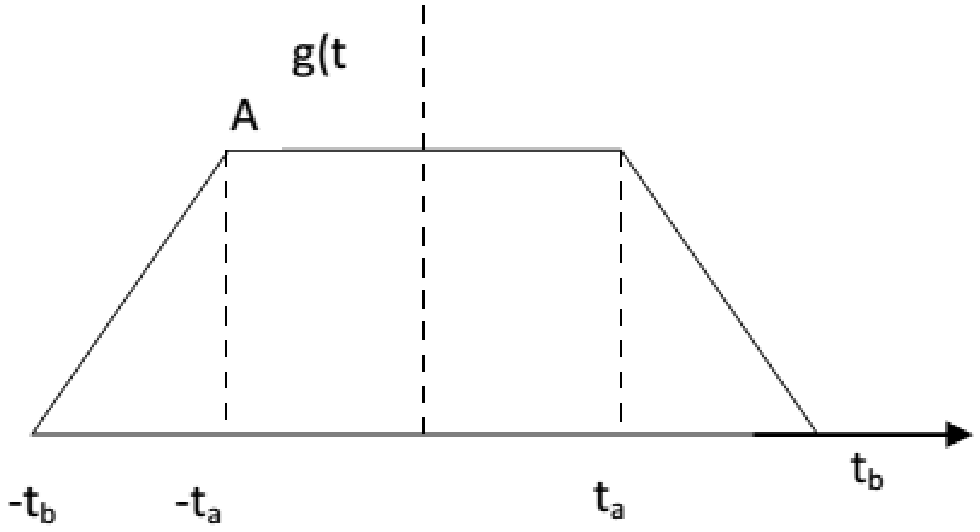
$$B_{rms}\ T_{rms} \geq \frac{1}{4\pi}$$

Example : Bandwidth of a trapezoidal signal

Find the equivalent rectangular bandwidth, B_{eq} , for the trapezoidal pulse shown.

Solution :

$$T_{eq} = \frac{\left(\int_{-\infty}^{\infty} |g(t)| dt\right)^2}{\int_{-\infty}^{\infty} |g(t)|^2 dt}$$



$$\int_{-\infty}^{\infty} |g(t)| dt = A\,(t_a + t_b)$$

$$\int_{-\infty}^{\infty} |g(t)|^2 dt = \frac{2A^2}{3}\,(2t_a + t_b)$$

$$T_{eq} = \frac{3}{2}\frac{(t_a + t_b)^2}{(2t_a + t_b)}$$

$$B_{eq} = \frac{0.5}{T_{eq}} = \frac{2t_a + t_b}{3(t_a + t_b)^2}.$$

Remark: Note that using this method we were able to determine the signal bandwidth without the need to go through the Fourier transform.

Exercise: Use the above method to find the equivalent rectangular bandwidth for the triangular signal $g(t) = tri(\frac{t}{T})$.

Example: Bandwidth of a periodic signal:

Find the bandwidth For the periodic square function define over one period as

$$g(t) = \begin{cases} 2A, & \frac{-T}{4} \le t \le \frac{T}{4} \\ -A, & o.w \end{cases}$$

Solution:

The average power, computed using the time average, is

$$P_{av} = \frac{1}{T_0}\int_0^{T_0} |g(t)|^2 \; dt$$

$$= \frac{1}{T_0}[4A^2\tau + A^2\tau] = \frac{5A^2\tau}{2\tau} = \frac{5A^2}{2} = 2.5A^2$$

Also, by using the Parseval's theorem, the average power can be computed as:

$$P_{av} = |C_0|^2 + 2\sum_{n=1}^{\infty}|C_n|^2$$

We recall that the Fourier coefficients for this signal were found in Chapter 1. Using these values we get

$$P_{av} = \left(\frac{A}{2}\right)^2 + 2\sum_{n=1}^{\infty}\frac{(3A)^2}{(n\pi)^2}$$

$$P_{av} = \frac{A^2}{4} + 2A^2 \sum_{n=1}^{\infty} \frac{(3)^2}{(n\pi)^2}$$

Let us take n = 1

$$P_1 = A^2 \left\{0.25 + 2 \ . \ \frac{9}{\pi^2}\right\} = 2.073A^2$$

$$\frac{P_1}{P_{av}} = \frac{2.073A^2}{2.5A^2} = 82.95\%$$

(This is the percentage of the total power that lies in the dc and the fundamental frequency).

For n = 3

$$P_3 = A^2 \left\{0.25 + 2 \left(\frac{3^2}{\pi^2} + \frac{3^2}{3^2\pi^2}\right)\right\} = 2.276A^2$$

$$\frac{P_3}{P_{av}} = \frac{2.276A^2}{2.5A^2} = 91.05\%$$

(Fraction of power in the dc, fundamental and third harmonic terms)

For n = 5

$$P_5 = A^2 \left\{0.25 + 2 \left(\left(\frac{3}{\pi}\right)^2 + \left(\frac{3}{3\pi}\right)^2 + \left(\frac{3}{5\pi}\right)^2\right)\right\} = 2.349A^2$$

$$\frac{P_5}{P_{av}} = \frac{2.349A^2}{2.5A^2} = 93.97\%$$

Here , the 93% power band width is $5f_0$.

Example: Bandwidth of an energy signal .

If the signal $g(t) = Ae^{-\propto t}\ u(t)$ is passed through an ideal LPF with B.W = B Hz, find the fraction of the signal energy contained in B.

Solution

The Fourier transform of g(t) is:

$$G(f) = \frac{A}{\propto + j2\pi f}$$

The energy in g(t), using the time domain, is

$$E_g = \int_0^\infty |g(t)|^2\, dt = \int_0^\infty A^2 e^{-2\propto t}\, dt = \frac{A^2}{2\propto}$$

Energy contained in the filter output y(t) is

$$E_y = \int_{-B}^{B} |G(f)|^2\, df = \int_{-B}^{B} \frac{A^2}{(\propto^2 + (2\pi f)^2)}\, df$$

$$E_y = \frac{2A^2}{2\pi\propto} tan^{-1}\frac{2\pi B}{\propto}$$

The ratio of E_y to the total energy is

$$\frac{E_y}{E_g} = \frac{2}{\pi} tan^{-1}\frac{2\pi B}{\propto}$$

The table below shows this ratio for various values of B .

B	$(E_y/E_g) \times 100$
$\frac{\propto}{4}$	63.9
$\frac{\propto}{2}$	80.38
$\propto$	89.95
$2\propto$	94.94

Thus, the 95% energy bandwidth is $2\propto$.

Exercise: Find the 98% energy bandwidth.

Pulse Response and Risetime

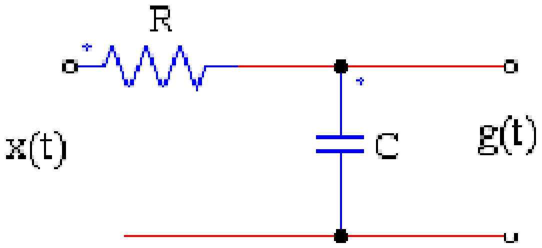
A rectangular pulse contains significant high frequency components. When that pulse is passed through a LPF, the high frequency components will be attenuated resulting in signal distortion.

We need to investigate the relationship that should exist between the pulse bandwidth and the channel bandwidth. This subject is of particular importance, especially, when we study the transmission of data over band-limited channels. In the simplest form, a binary digit 1 may be represented by a pulse $A,\ \ 0 \leq t \leq T_b$, while binary digit 0 may be represented by the negative pulse $-A,\ \ 0 \leq t \leq T_b$. So, in order to retrieve the transmitted data, the channel bandwidth must be wide enough to accommodate the transmitted data.

To convey this idea in a simple form, we first consider the response of a first order low pass filter to a unit step function and then to a pulse.

Step response of a first order LPF (channel)

Let x(t) = u(t) be applied to a first order RC circuit. This first order filter is a fair representation of a low pass communication channel.



The system D.E is:

$$x(t) = Ri + g(t) = Rc\,\frac{dg(t)}{dt} + g(t)$$

where g(t) is the channel output.

$$Rc\,\frac{dg(t)}{dt} + g(t) = u(t)$$

The solution to this first order system is

$$g(t) = (1\text{-}e^{-t/RC})\ u(t)$$

The 3-db B.W of the channel is

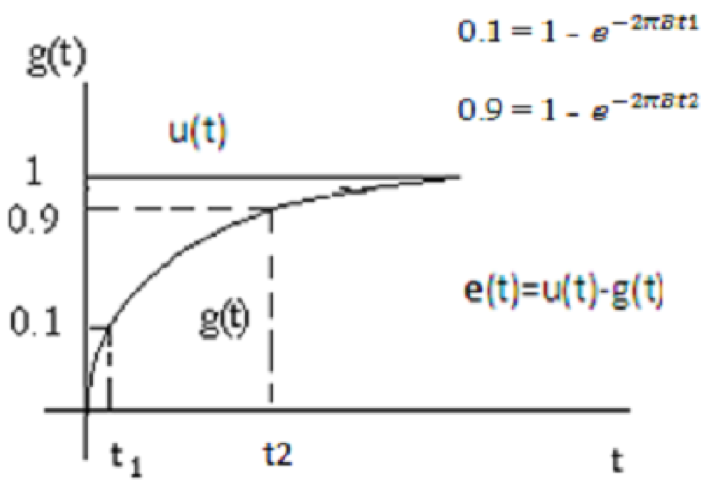
$B = \frac{1}{2\pi Rc}$ (to be derived shortly)

$g(t) = (1 - e^{-2\pi Bt})\, u(t)$

Define the difference between the input and the output as:

$\mathbf{e}(t) = u(t) - g(t) = e^{-2\pi Bt}$

Note that e(t) decreases as B increases. Meaning that as the channel bandwidth increases, the output becomes closer and closer to the input. In the ideal case, when the channel bandwidth becomes infinity, the output becomes a step function. In essence, to reproduce a step function (or a rectangular pulse), a channel with infinite bandwidth is needed.



The Risetime

The Rise time is a measure of the speed of a step response. One common measure is the 10-90 % rise time defined as the time it takes for the output to rise between 10% to 90% of the final (steady state) value (1) when a step function is applied to a LIT system. For the step response g(t) and the first order RC circuit considered above, the rise time can be easily calculated as:

$$t_r = t_2 - t_1 \approx \frac{0.35}{B}$$

From this result we conclude that: increasing the bandwidth of the channel will decrease the rise time (a faster response).

<u>**Exercise:**</u> For the system above, verify that the rise time is given as $t_r = \frac{0.35}{B}$

Exercise: Find the 10-90% rise time for a second order low pass filter with 3-dB bandwidth B and transfer function

$$H(f) = \frac{1}{P_2(\frac{jf}{B})}$$

Where $P_2(x) = 1 + \sqrt{2}x + x^2$.
(Hints: You may let B=10, for example, use matlab to find the step response, and then find the rise time).

Pulse response

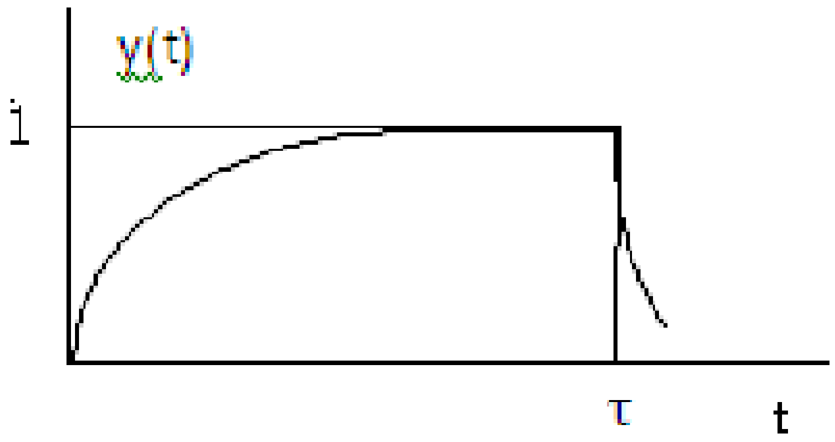
It is the response of the circuit to a pulse of duration τ. For the same circuit let us apply the pulse

$$x(t) = u(t) - u(t-\tau)$$

Using the linearity and time invariance properties, the output can be obtained from the step response as:

$$y(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-t/RC} & 0 < t < \tau \\ \left(1 - e^{-\frac{\tau}{RC}}\right).e^{-\frac{t-\tau}{RC}} & t > \tau \end{cases}$$

This is sketched in the figure below.



Bandwidth considerations:

The transfer function of the RC circuit is

$$H(f) = \frac{1/j2\pi fc}{R + 1/j2\pi fc} = \frac{1}{1 + j2\pi fRc}$$

$$|H(f)| = \frac{1}{\sqrt{1+(2\pi fRc)^2}}$$

Let B = $\frac{1}{2\pi Rc}$; 3-db bandwidth ; $2\pi fRc = 1$; f = $\frac{1}{2\pi Rc}$

Then , $H(f) = \frac{1}{1+jf/B}$

$$|H(f)| = \frac{1}{\sqrt{1+\left(\frac{f}{B}\right)^2}}$$

For the rectangular pulse x(t), we have

$$X(f) = sincf\tau$$

The first null frequency of X(f) is an estimate of the bandwidth B_x of x(t), which is of the order of $\approx \frac{1}{\tau}$.

1. When τ is large, such that signal bandwidth $B_x = \frac{1}{\tau} << B$ (channel B.W)

$$Y(f) = X(f)H(f) \approx X(f)$$

and the output resembles the input .There is enough time for x(t) to reach the maximum value .

2. When τ is small, such that signal $B_x = \frac{1}{\tau} >> B$ (channel B.W)

$$Y(f) = X(f)H(f) \approx H(f)$$

The signal suffers a considerable amount of distortion and $Y(f)$ is no longer proportional to X(f).

Band pass Signals and Systems

A signal g(t) is called a *band pass signal* if its Fourier transform G(f) is non-negligible only in a band of frequencies of total extent 2W centered about f_c .

A signal is called *narrowband* if 2W is small compared with f_c.

A band pass signal g(t) represented in the form:

$$g(t) = g_I(t)\cos\omega_c t - g_Q(t)\sin\omega_c t.$$

$g_I(t)$ is a low pass signal of B.W = W Hz called the *in phase component* of g(t) .

$g_Q(t)$ is a low pass signal of B.W = W Hz called the *quadrature component* .

g(t) is a modulated signal in which $g_I(t)$ and $g_Q(t)$ are the low pass signals. Recall the modulation property of the Fourier transform :

$$x(t)\cos\omega_c t \rightarrow \frac{1}{2}(X(f-f_c)+X(f+f_c))$$

$$x(t)\sin\omega_c t \rightarrow \frac{1}{j2}(X(f-fc)-X(f+fc))$$

Define the *complex envelope* of a signal g(t) as:

$$\tilde{g}(t) = g_I(t) + j\,g_Q(t)$$

$\tilde{g}(t)$ is a low pass signal of B.W =W. The signals g(t) and $\tilde{g}(t)$ are related by :

$$g(t) = \mathrm{Re}\{\tilde{g}(t)\, e^{j\omega ct}\}$$

<u>How to get $g_I(t)$ and $g_Q(t)$ from g(t):</u>

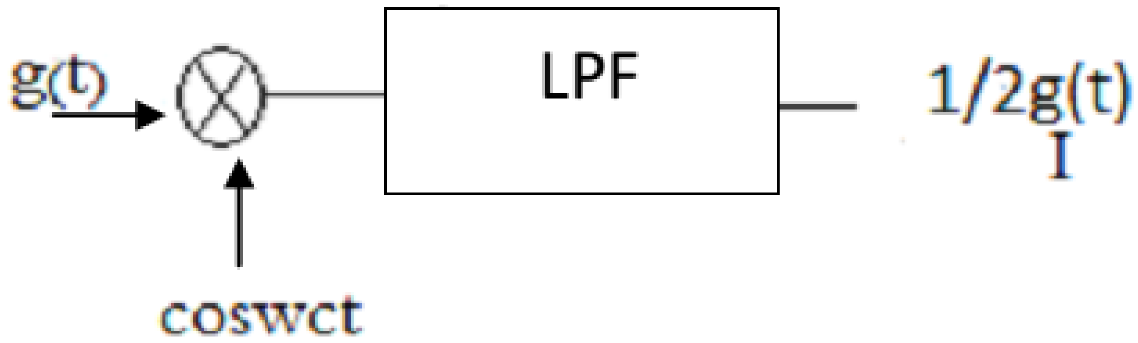
If we multiply g(t) by cos ωct, we get

$$g(t)\cos\omega ct = g_I(t)\cos^2\omega_c t - g_Q(t)\sin\omega_c t\cos\omega_c t$$

$$= \frac{1}{2} g_I(t) + \frac{1}{2} g_I(t)\cos 2\omega ct - \frac{1}{2} g_Q(t)\sin 2\omega_c t\ .$$

The first term is the desired low pass signal. The second and third terms are high frequency components centered about 2 f_c.

$$g_I(t) = \text{low pass}\{2g(t)\cos\omega_c t\}$$



Or, in the frequency domain

$$G_I(f) = \begin{cases} G(f - fc) + G(f + fc) & -w \leq f \leq w \\ 0 & otherwise \end{cases}$$

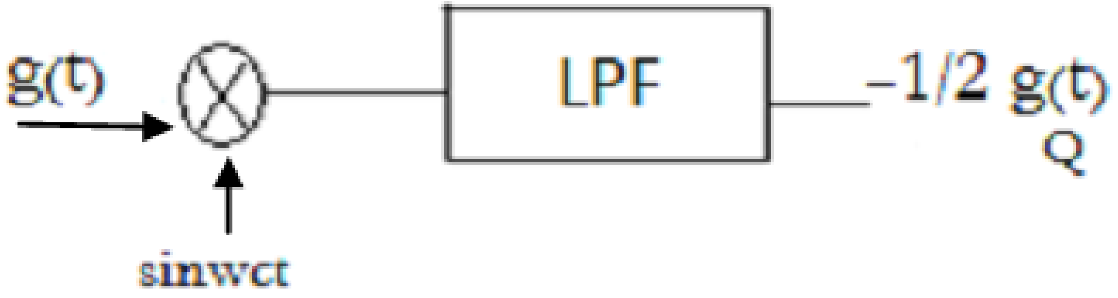
Now if we multiply g(t) by $\sin\omega_c t$, we get

$$g(t)\sin\omega_c t = g_I(t)\sin\omega_C t\cos\omega_C t - g_Q(t)\sin^2\omega_c t$$

$$= -\frac{1}{2}g_Q(t) + \frac{1}{2}g_I(t)\sin 2\omega_c t + \frac{1}{2}g_Q(t)\cos 2\omega_c t$$

Again, the first term is a low pass signal, while the second and third are high frequency terms centered about 2 f_c.

$$g_Q(t) = -\text{ low pass}\{2g(t)\sin\omega_c t\}$$



In the frequency domain, this is equivalent to

$$G_Q(t)=\begin{cases} j[G(f-fc)-G(f+fc) & -w \le f \le w \\ 0 & otherwise \end{cases}$$

Band pass systems:

The analysis of band pass systems can be simplified by using the complex envelope concept. Here, results and techniques from low pass systems can be easily applied to band pass systems .

The problem to be addressed is :

The input x(t) is a band pass signal

$$x(t)=x_I(t)\cos\omega_c t - x_Q(t)\sin\omega_c t$$

x(t) is applied to a band pass filter represented as:

$$h(t) = h_I(t)\cos\omega_c t - h_Q(t)\sin\omega_c t$$

The objective is to find the filter output y(t). The output is of course, the convolution of x(t) and h(t) (y(t) = x(t)*h(t)) which can also be expressed as:

$$y(t)=y_I(t)\cos\omega_c t - y_Q(t)\sin\omega_c t$$

But due to the band-pass nature of the problem, carrying out the direct convolution will be a tedious task. The complex envelope concept simplifies the problem to a very great extent. The procedure is summarized as follows:

a. Form the complex envelope for both the input and the channel:

$\tilde{x}(t) = x_I(t) + jx_Q(t)$

$\tilde{h}(t) = h_I(t) + jh_Q(t)$

b. Carry out the convolution between $\tilde{x}(t)$ and $\tilde{h}(t)$. Note that both signals are low pass signals and so $\tilde{y}(t)$ is also low pass.

$$2\,\tilde{y}(t) = \tilde{h}(t) * \tilde{x}(t)$$

$$\tilde{y}(t) = y_I(t) + jy_Q(t)$$

c. The band-pass filter output is obtained from the low pass signal $\tilde{y}(t)$ through the relation

$$y(t) = \mathrm{Re}\{\tilde{y}(t)\, e^{jw_c t}\}$$

or the relation

$$y(t) = y_I(t)\cos\omega_c t - y_Q(t)\sin\omega_c t$$

Example :

The rectangular radio frequency (RF) pulse

$$x(t) = \begin{cases} A\cos 2\pi f_c t & 0 \le t \le T \\ 0 & otherwise \end{cases}$$

is applied to a linear filter with impulse response (We will see later that this is a filter matched to x(t), called the *matched filter*).

$$h(t) = x(T - t)$$

Assume that T= nT_C; n is an integer, $T_c = \frac{1}{fc}$. Determine the response of the filter and sketch it.

Solution: We follow the three steps outlined above.

$h(t) = A\cos 2\pi f_c(T - t)$

$= A\cos 2\pi f_c T \cos 2\pi f_c t + A\sin 2\pi f_c T \sin 2\pi f_c$ t

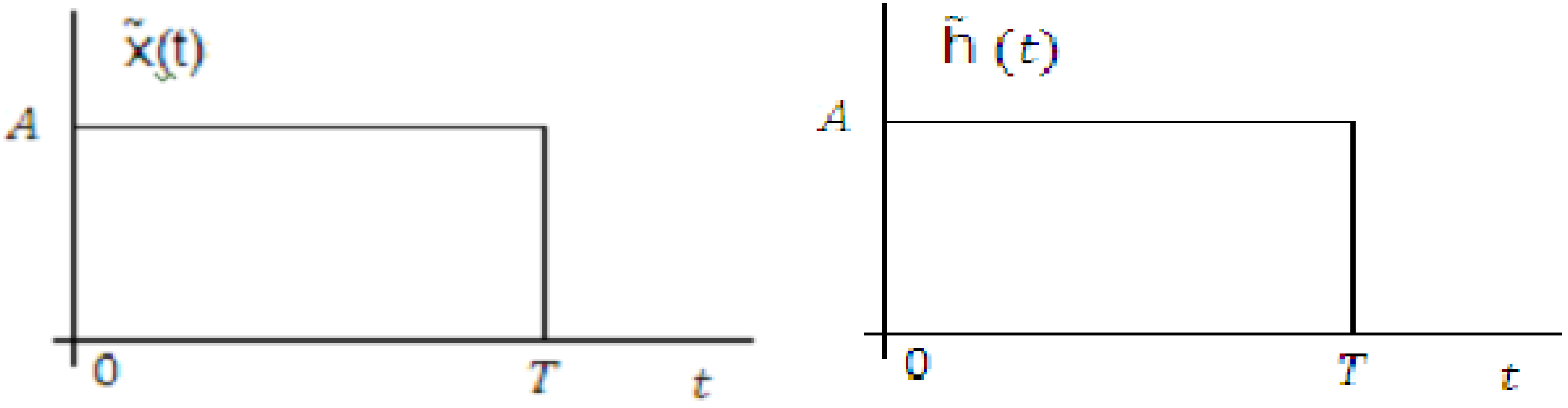
$$= \underbrace{A\cos 2\pi\left(\frac{nT_c}{T_c}\right)\cos 2\pi f_c t}_{\cos 2n\pi = 1} + \underbrace{A\sin 2\pi\left(\frac{nT_c}{T_c}\right)\sin 2\pi f_c t}_{\sin 2n\pi = 0}$$

Therefore, $h(t) = \begin{cases} A\cos 2\pi f_c t & 0 \le t \le T \\ 0 & otherwise \end{cases}$

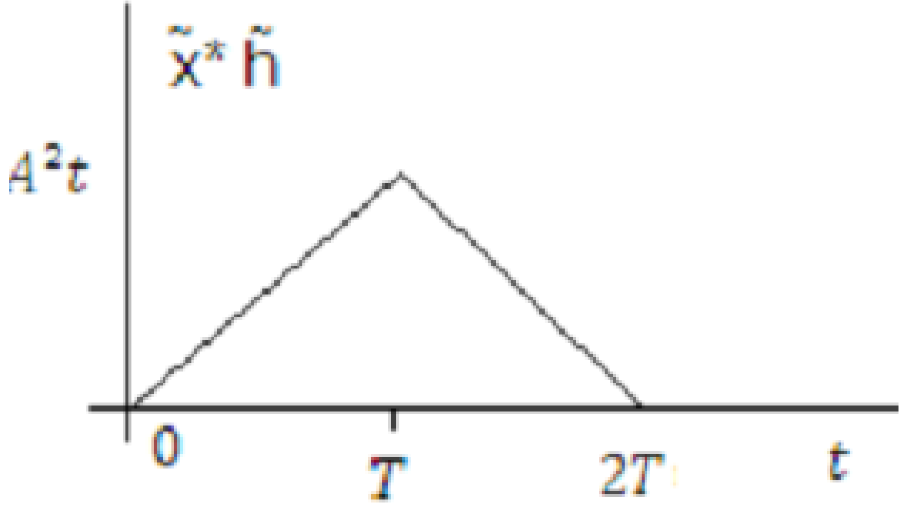
The complex envelopes of x(t) and h(t) are (step a)

$$\tilde{x}(t) = \begin{cases} A & 0 \le t \le T \\ 0 & o.w \end{cases}$$

$$\tilde{h}(t) = \begin{cases} A & 0 \le t \le T \\ 0 & o.w \end{cases}$$



$\tilde{y}(t) = \tilde{x}(t) * \tilde{h}(t)$ is the triangular signal shown in the Figure (step b).

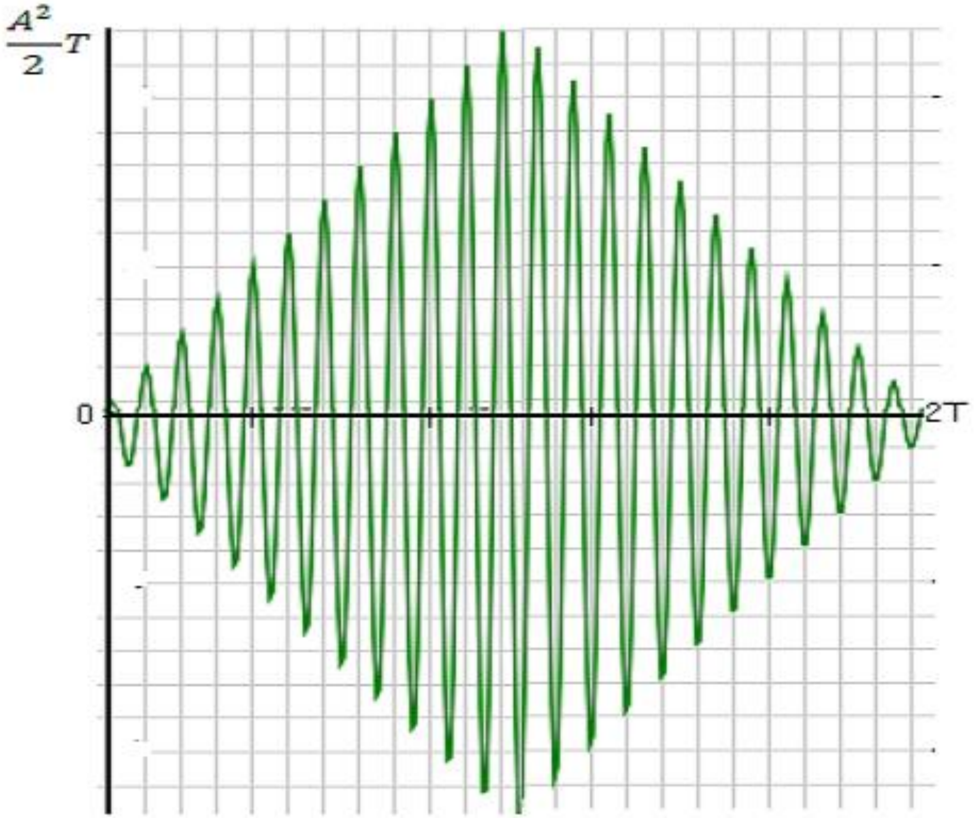


$$2\tilde{y}(t)=\begin{cases} A^2t & 0 \le t \le T \\ A^2(2T-t) & T \le t \le 2T \end{cases}$$

The bandpass signal is obtained as (step c)

$$y(t)=\begin{cases} \frac{A^2}{2}t\, cosw_c t & 0 \le t \le T \\ \frac{A^2}{2}(2T-t)\, cosw_c t & T \le t \le 2T \end{cases}$$

and is sketched as in the figure below.



Exercise

The band-pass signal $x(t) = e^{-\frac{t}{\tau}}\cos(2\pi f_c t)\,u(t)$ is applied to a band-pass filter with impulse response $h(t)$ given as:

$$h(t) = \begin{cases} A\,cos2\pi f_c t & 0 \le t \le T \\ 0 & otherwise \end{cases}$$

Find and sketch the filter output.