Random Process

A random process $X(t)$ is defined as an ensemble of time functions together with a probability rule that assigns a probability to any meaningful event associated with an observation of one of the sample functions of the random process.

Consider the following experiment: An oscillator produces a waveform of the form $A_m\, cos(w_m t + \theta)$; where θ is a discrete R.V with a probability mass function

$P(\theta = 0) = 0.2 \qquad P\left(\theta = \frac{\pi}{2}\right) = 0.2$

$P(\theta = \pi) = 0.3 \qquad P\left(\theta = \frac{3\pi}{2}\right) = 0.3$

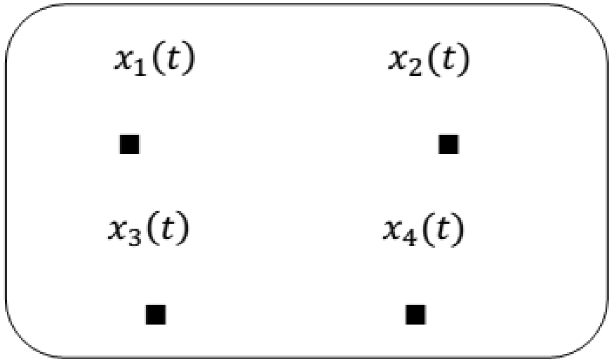
Here the sample space of the experiment consists of four time functions:

$x_1(t) = A_m \cos(\, w_m t\ ) \qquad P(x_1(t)) = 0.2$

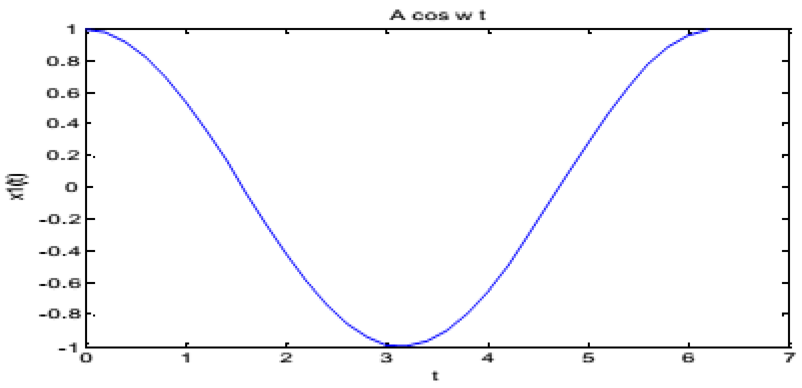
$x_2(t) = A_m \cos\left(w_m\, t + \frac{\pi}{2}\right) \qquad P(x_2(t)) = 0.2$

$x_3(t) = A_m \cos(\, w_m t + \pi\ ) \qquad P(x_3(t)) = 0.3$

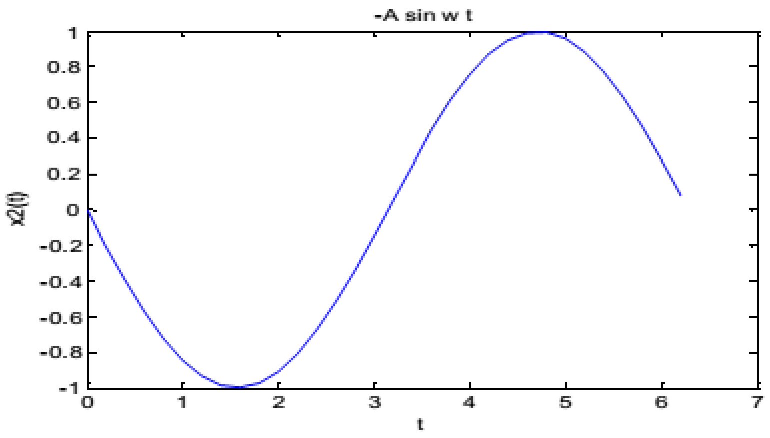
$x_4(t) = A_m \cos(\, w_m t + \frac{3\pi}{2}) \qquad P(x_4(t)) = 0.3$



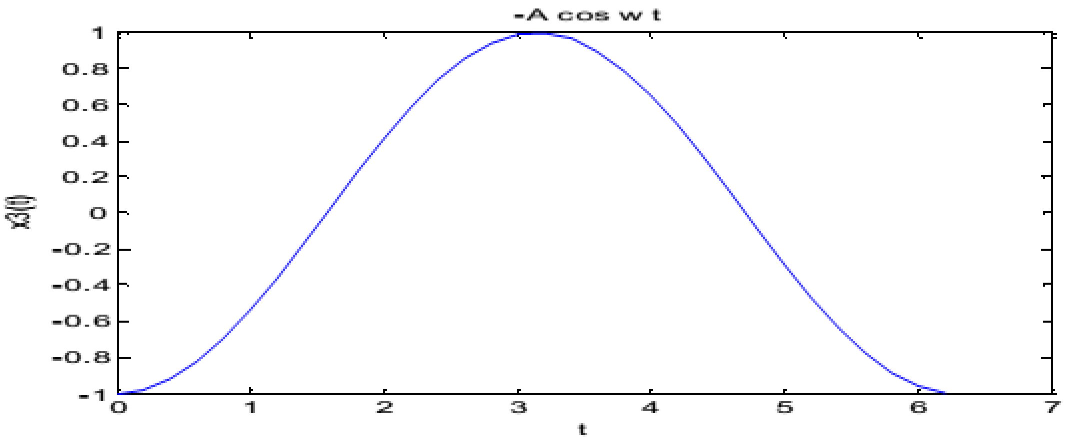
$$x_1(t) = A\,cos(2\pi f_m t)$$



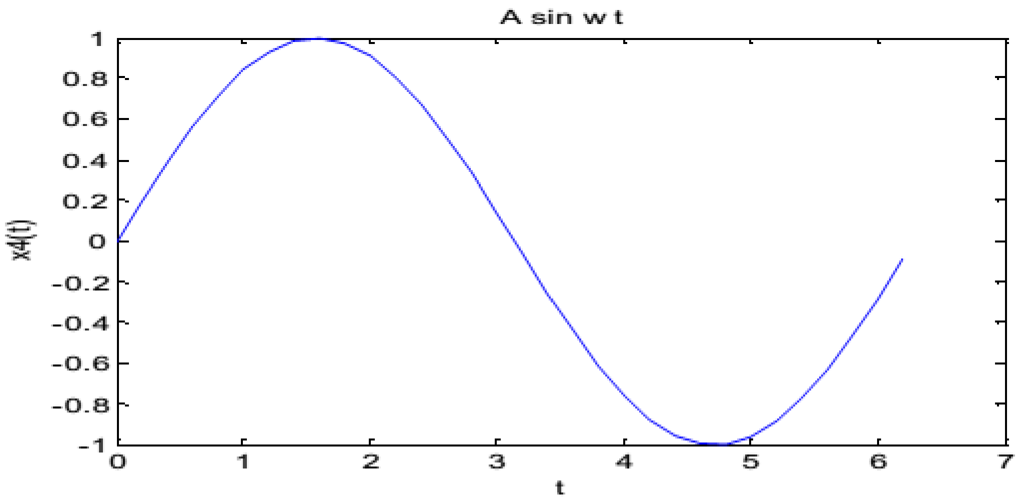
$$x_2\,(t) = -A\,\sin(2\pi f_m t)$$



$$x_3(t) = -A\,cos(2\pi f_m t)$$



$$x_4(t) = +A\,sin(2\pi f_m t)$$



Each realization of the experiment is called a *sample function* $x(t)$. The sample space (ensemble) composed of functions is called *a random or stochastic process* denoted by $X(t)$. The value assumed by a random process at a particular time is a random variable with a certain probability density function.

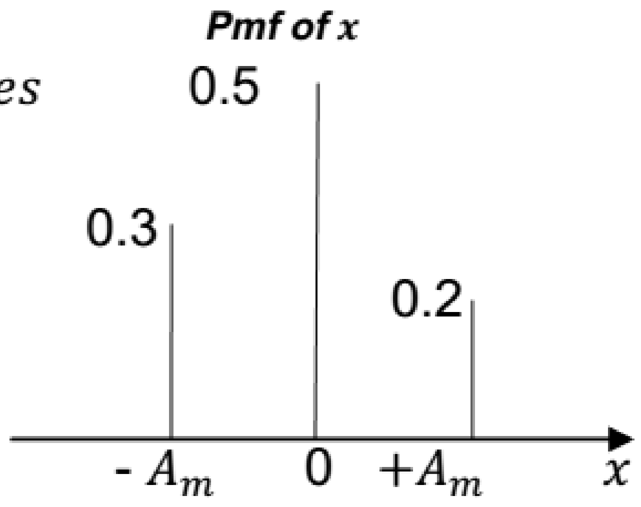
For the example above, $X(0)$ $assumes\ three\ values$

P $\{X(0) = A_m\} = 0.2$

P $\{X(0) = -A_m\} = 0.3$

$P\{X(0) = 0\} = 0.2 + 0.3 = 0.5$

(Corresponding to $= \frac{\pi}{2}, \frac{3\pi}{2}$)



Pmf of X at t = 0.

$P(X = 0)$ = 0.2+0.3 = 0.5

$P(X = +A_m) = 0.2$

$P(x = -A_m) = 0.3$

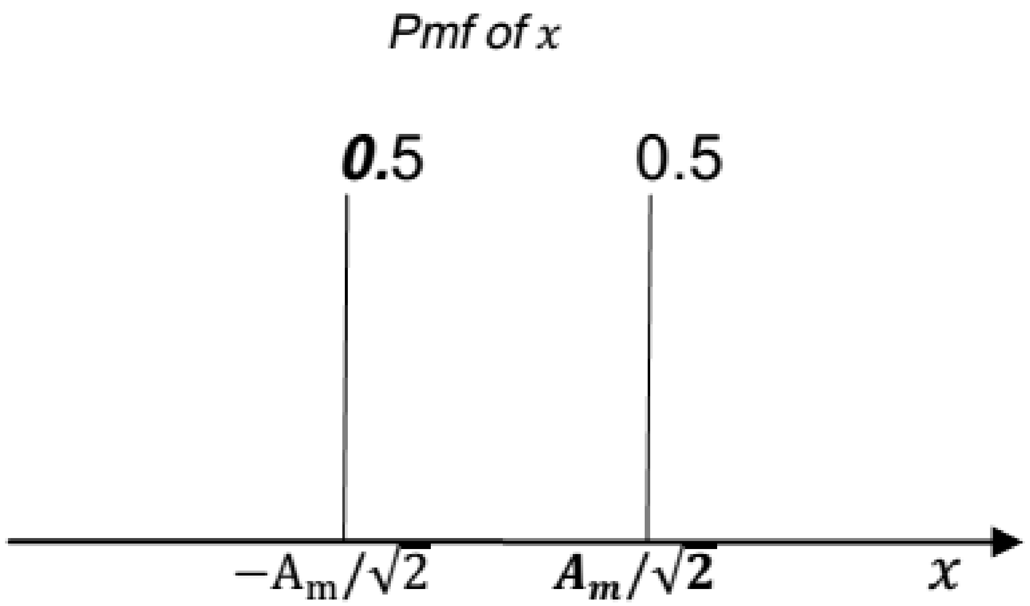
The mean value of the random variable X is

$$E(X) = -A_m \times 0.3 + 0 \times 0.5 + 0.2 \times A_m$$
$$E(X) = -0.3A_m + 0.2A_m = -0.1A_m$$

Pmf of X at $w_m t_n = \pi/4$

	$X(\mathbf{w_m\, t} = \frac{\pi}{4})$	Prob.
Possible values:	$A_m \cos\left(\frac{\pi}{4} + 0\right) = +A_m/\sqrt{2}$	0.2
	$A_m \cos\left(\frac{\pi}{4} + \frac{\pi}{2}\right) = -A_m/\sqrt{2}$	0.2
	$A_m \cos\left(\frac{\pi}{4} + \pi\right) = -A_m/\sqrt{2}$	0.3
	$A_m \cos\left(\frac{\pi}{4} + \frac{3\pi}{2}\right) = +A_m/\sqrt{2}$	0.3

The Pmf of X at $w_m\ t = \frac{\pi}{4}$ is sketched here.



$$E\{X\left(w_m t = \frac{\pi}{4}\right) = 0$$

⇛ Process is not stationary [mean at $t = 0$ is not the same as the mean at $w_m\ t = \frac{\pi}{4}$]

In general, $X(t) = A_m \cos(w_m t + \theta)$

$$E\{X(t)\} = \sum X(t, \theta_i)\, P(\theta = \theta_i)$$

$$= A_m \cos w_m t \times 0.2 + 0.2 \times A_m \cos\left(w_m t + \frac{\pi}{2}\right) + 0.3 \cos(w_m t + \pi) + 0.3 \times A_m \cos(w_m t + \frac{3\pi}{2}).$$

Mean is not a constant (function of time).

$\Rightarrow$ **Process is non stationary.**

Stationarity of a random process:

The mean of a process $X(t)$ is defined as the expectation of the r.v obtained by observing the process at some time t as

$$\mu_x(t) = E\{X(t)\} = \int_{-\infty}^{\infty} x\, f_x(x) dx$$

$f_x(x)$ *is the first order pdf of the process* $X(t)$.

The autocorrelation function of the process $X(t)$ is defined as the expectation of the product of two r.v $X(t_1)$ *and* $X(t_2)$ obtained by observing the process $X(t)$ at times t_1 and t_2.

$$R_x\,(t_1, t_2) = E\{X(t_1)X(t_2)\} = \iint_{-\infty}^{\infty} x_1\, x_2\, f(x_1, x_2) dx_1\, dx_2$$

$f(x_1, x_2\,)$ is the second order (joint pdf) of x_1 and x_2.

A random process is said to be wide sense stationary or (stationary) when the following two conditions hold:

1) $E\{X(t)\} = \mu_x$ = constant for all t

2) $R_x(t_1, t_2) = E\{X(t_1)X(t_2)\} = R_x(t_2 - t_1\,)$
   i.e., R_x *is* a function of the time difference and not on the absolute values of t_1 and t_2. i.e.,

$$R_x(\tau) = E\{X(t)X(t+\tau)\};\ where\ \tau = t_2 - t_1$$

Properties of the autocorrelation function of a stationary process:

1) $Rx(0) = E\{X^2(t)\}$; the mean square value (second moment of x) { total power in $X(t)$ }

2) $R_x(\tau) = R_x(-\tau); R_x(\tau)$ *is an even function of* τ.

3) $R_x(\tau)$attains its maximum value at $\tau = 0$

$$|R_x(\tau)| \leq R_x(0)$$

Decorrelation Time : The decorrelation time τ_0 of the a stationary process X(t) of zero mean is taken as the time taken for the magnitude of the autocorrelation function $R_x(\tau)$ to decrease say 1% of its maximum value $R_x(0)$.

A Result we Recall from ENEE 331: If θ is a r.v with pdf $f_\theta(\theta)$ and $Y = g(\theta)$, then $E\{Y\} = \int g(\theta)f(\theta)\, d\theta$

$$E\{g(\theta)\} = \int g(\theta)f_\theta(\theta)\, d\theta$$

Example: A sinusoidal signal with random phase

$$\text{Let } X(t) = A\,cos(2\pi f_c t + \theta)$$

A, f_c are constants, θ is a continuous r.v uniformly distributed over $(-\pi, \pi)$

$$f_\theta(\theta) = \begin{cases} \frac{1}{2\pi} & -\pi < \theta < \pi \\ 0 & o.w \end{cases}$$

The mean value of $X(t)$ is

$$E\{\,X(t)\,\} = \int_{-\pi}^{\pi} \underbrace{Acos(2\pi f_c\, t + \theta)}_{g(\theta)} \cdot \underbrace{\frac{1}{2\pi}}_{f(\theta)} d\theta = 0$$

Which is a constant (independent of time). The autocorrelation function is:

$$R_x(\tau) = E\{X(t)X(t+\tau)\}$$

$$= \int_{-\pi}^{\pi} \overbrace{Acos(2\pi f_c\, t + \theta) \cdot Acos[2\pi f_c(t+\tau) + \theta]}^{g(\theta)} \cdot \overbrace{1/2\pi}^{f(\theta)}\, d\theta$$

$$= \frac{A^2}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2} \{cos 2\pi f_c\, \tau + \ cos(2\pi(2) f_c t + 2\pi f_c \tau + 2\theta)\}\, d\theta$$

We can easily recognize that the second integral is zero, leaving only the first term. Hence, $R_x(\tau)$ becomes

$$R_x(\tau) = \frac{A^2}{2\pi} \cdot \frac{\cos 2\pi f_c\tau}{2} \cdot 2\pi = \frac{A^2}{2} cos2\pi f_c\tau$$

Note that:

- The mean value is a constant and $R_x(\tau)$ is a function of τ. These are the two conditions necessary for the process to be stationary. So $X(t)$ is a stationary process.
- The process $X(t)$ is periodic with period $T_c = \frac{1}{f_c}$ .The autocorrelation function $R_x(\tau) = \frac{A^2}{2} \cos 2\pi f_c\tau$ is also periodic with period $T_c = \frac{1}{f_c}$.

Exercise: show that the first order pdf of X is

$$f_x(x) = \begin{cases} \frac{1}{\pi\sqrt{A^2 - x^2}} & -A < x < A \\ 0 & o.w \end{cases}$$

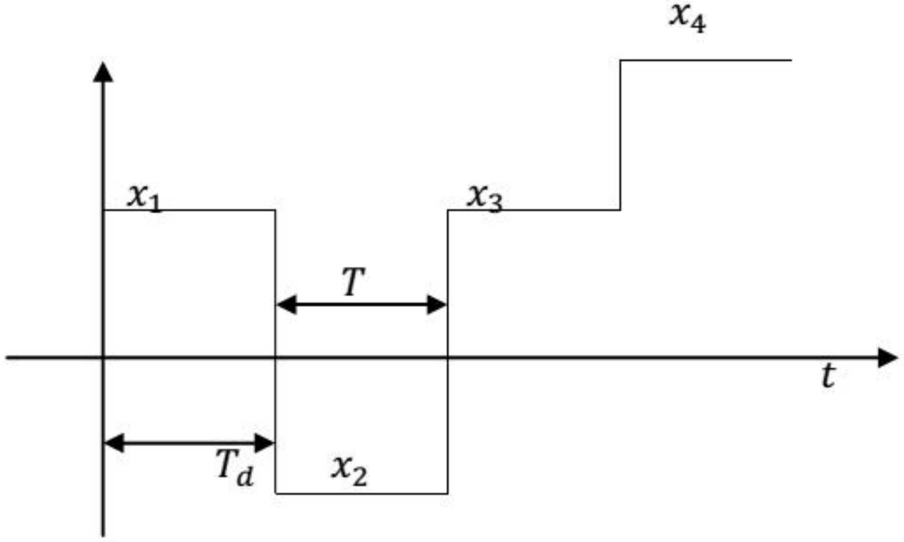
Exercise: Let X and Y be two independent Gaussian random variables each with mean zero and variance σ^2. Define the random process

$$Z(t) = X\cos 2\pi f_c t + Y\sin 2\pi f_c t$$

a. Find the mean and variance of $Z(t)$.
b. Find the autocorrelation function $R_Z(\tau)$.
c. Is this process stationary?

Example: Random digital signal

The figure shows a random sample $x(t)$ of a process $X(t)$ consisting of a random sequence $X_1 X_2 \ldots$ of pulses each with m possible amplitudes (symbols) $a_1, a_2, \ldots, a_m$ within each signaling interval T. The possible symbols occur with probabilities $P_1, P_2, \ldots, P_m$.



- The time delay T_d is a continuous r.v uniformly distributed over $0 < T_d < T$; where T is the symbol duration $f\ (t_d) = \begin{cases} \frac{1}{T} & 0 < t_d < T \\ 0 & o.w \end{cases}$
- The amplitudes in different intervals are independent.

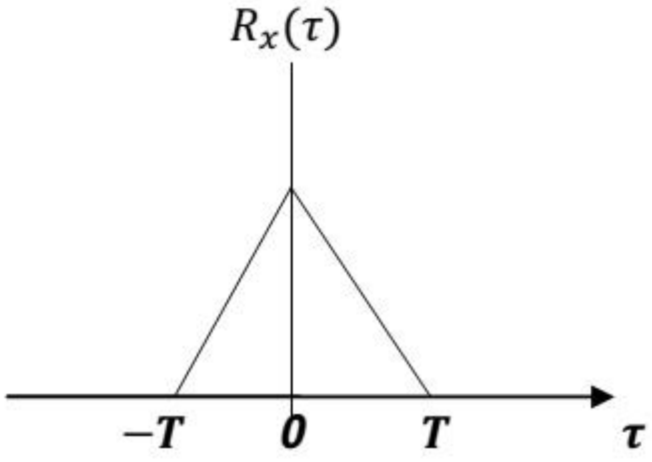
- The mean value of the process is
$$E\big(X(t)\big) = a_1P_1 + aP_2 + \cdots\cdots\cdots a_mP_m$$
Without loss of generality, let $E\big(X(t)\big) = 0$

  The variance of the process is
$$Var\big(X(t)\big) = \sigma^2 = a_1^2P_1 + \ a_2^2P_2 \ + \cdots\cdots\cdots + a_m^2P_m$$
- The average power of the process is also $R_x(0) = \sigma^2$.

- The autocorrelation function of the process is

$$R_x(\tau) = \sigma^2 tri(\frac{\tau}{T}).$$

This function is sketched below



Example: Random Binary Signal (also known as polar non return to zero)

Here, the possible symbols of X(t) in each signaling time interval T are:

$+A$	with probability	½	for	$0 \le t \le T$
$-A$	with probability	½	for	$0 \le t \le T$

Find the mean, variance and autocorrelation function of X(t).

Solution

The mean value is $E\big(X(t)\big) = +A \times \frac{1}{2} - A \times \frac{1}{2} = 0$

The variance is $\sigma^2 = A^2 \times \frac{1}{2} + (-A)^2 \times \frac{1}{2} = A^2$

Therefore, $R_x(\tau) = A^2 tri(\frac{\tau}{T})$

$$R_x(\tau) = \begin{cases} A^2\left(1 - \left|\frac{\tau}{T}\right|\right) & |\tau| < T \\ 0 & |\tau| > T \end{cases}$$

Exercise: Unipolar non return to zero signaling

Let the transmitted symbols of Z(t) in each signaling time interval T be:

$+A$	with probability	½	for	$0 \le t \le T$
0	with probability	½	for	$0 \le t \le T$

a. Show that Z(t) is related to the polar NZR in the previous example by

$$Z(t) = (X(t) + A)/2$$

b. Find the mean and variance of Z(t).

c. Show that $R_Z(\tau) = \begin{cases} \frac{A^2}{4} + \frac{A^2}{4}\left(1 - \left|\frac{\tau}{T}\right|\right) & |\tau| < T \\ \frac{A^2}{4} & |\tau| > T \end{cases}$

Exercise: Polar non return to zero signaling with non-equal symbol probabilities
Let the possible symbols of X(t) in each signaling time interval T be:

$+A$	with probability	p	for	$0 \leq t \leq T$
$-A$	with probability	(1-p)	for	$0 \leq t \leq T$

a. Find the mean and variance of X(t).
b. Find the autocorrelation function of X(t).

Exercise: M-ary pulse amplitude signal
Let the possible symbols of X(t) in each signaling time interval T be ($-3A$, $-A$, $+A$, $+3A$) with equal probabilities

a. Find the mean and variance of X(t).
b. Find the autocorrelation function of X(t).

Ergodic processes:

Given a sample function $x(t)$ of a random process $X(t)$, we define the following two time averages:

$$\langle X(t)\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)dt$$

$$\langle X(t)X(t+\tau)\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt$$

Here, $2T$ is an observation interval.

<u>Def:</u> A random process is said to be ergodic if statistical properties can be determined from a sample function representing one realization of the process.

$$Statistical\ average = Time\ Average.$$

The two quantities of interest are the mean value and the autocorrelation function. For an ergodic process, they can be computed using time average as:

$E\{X(t)\} = \langle x(t)\rangle = constant.$

$R_x(\tau) = E\{X(t)X(t+\tau)\} = \langle x(t)\, x(t+\tau)\rangle \to$ function of τ.

Remark: An ergodic process in stationary, but a stationary process is not necessarily ergodic.

Example: consider again the process $X(t) = Acos(2\pi f_c\, t + \theta)$, θ is uniformly distributed over (-π < θ < π).

The two time averages are calculated as follows:

$$\langle x(t)\rangle = \frac{1}{T_c}\int_0^{T_c} Acos(2\pi f_c\, t + \theta)dt = 0 \;\; ; \qquad T_c = 1/f_c$$

$$\langle x(t)x(t+\tau)\rangle = \frac{1}{T_c}\int_0^{T_c} Acos(2\pi f_c t + \theta)\cdot Acos(2\pi f_c t + 2\pi f_c\,\tau + \theta)dt$$

$$= \frac{A^2}{2T}\int_0^{T_c} \cos(2\pi f_c\, t + 2\pi f_c \tau + 2\theta) + cos2\pi f_c \tau\, dt$$

$$= \frac{A^2}{2} cos2\pi f_c \tau$$

These are the same values found in the previous example.

⇒ process is ergodic.

Power spectral Density and Autocorrelation Function:

Consider a stationary random process $X(t)$ that is ergodic. Consider a truncated segment of $x(t)$ defined over the observation interval $-T < t < T$. Let X_{2T} be the truncated signal:

$$x_{2T}(t) = \begin{cases} x(t) & -T < t < T \\ 0 & o.w \end{cases}$$

The Fourier transform of x_{2T} is:

$$X_{2T}(f) = \int_{-T}^{T} x(t)e^{-j2\pi ft}dt$$

The energy spectral density of $X_{2T}(t)$ is $|X_{2T}(f)|^2$. Since $x(t)_{2T}$ is only one realization of a random process, then we need to find its mean value $E\{|X_{2T}(f)|^2\}$. Dividing this by the observation interval 2T, and letting T becomes very large, we get the power spectral density of the whole process, averaged over all sample functions and over all time.

The power spectral density, of a stationary process, may then be defined as:

$$S_X(f) = \lim_{T\to\infty} \frac{1}{2T} E\{|X_{2T}(f)|^2\}$$

The Wiener –Khintchine Thorem:

The power spectral density $S_X(f)$ and the autocorrelation function $R_X(\tau)$ of a stationary random process $X(t)$ form a Fourier transform pairs:

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} d\tau$$
$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) e^{j2\pi f\tau} df$$

Properties of the power spectral Density:

1. The zero frequency value of the power spectral density of a stationary process equals the total area under the graph of the autocorrelation function ;

$$S_X(0) = \int_{-\infty}^{\infty} R_X(\tau) d\tau$$

2. The mean squared value (the total signal power) of a stationary process equals the area under the power spectral density curve.

$$E\{X(t)^2\} = R_X(0) = \int_{-\infty}^{\infty} S_X(f)df$$

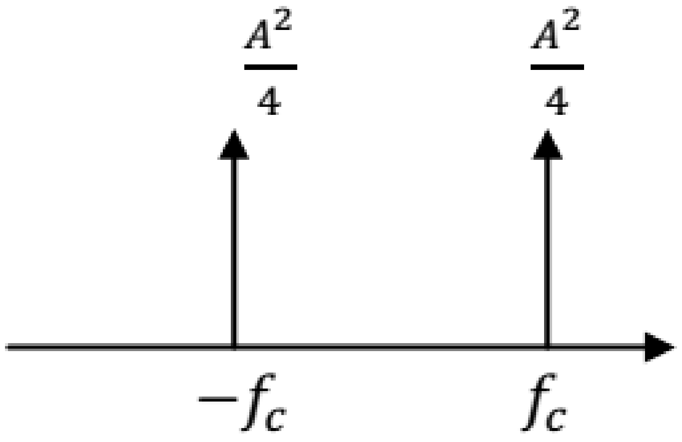
3. The power spectral density of a stationary process is always nonnegative ; i.e., $S_X(f) \geq 0$ for all f.

4. The power spectral density of a real-valued random process is an even function of f.

$$S_X(f) = S_X(-f)$$

Find the power spectral density of the random process $X(t) = A\cos(2\pi f_c t + \theta)$;

θ is a uniform r.v over $(-\pi, \pi)$.



For this process, we found earlier that $R_X(\tau) = \frac{A^2}{2} cos2\pi f_c \tau$

Since $S_X(f) = F\{\, R_X(\tau)\,\}$, then

$$S_X(f) = \frac{A^2}{4}\{\, \delta(f - f_c) + \delta(f + f_c)\,\}$$

The total average power is obtained by integrating $S_X(f)$ over all frequencies.

$$\int_{-\infty}^{\infty} S_X(f)df = \int_{-\infty}^{\infty} \frac{A^2}{4}[\delta(f - f_c) + \delta(f + f_c)]df$$

$$= \frac{A^2}{4} + \frac{A^2}{4} = \frac{A^2}{2}$$

$$= R_X(0)$$

Example: Random Binary Signal (revisited)

The autocorrelation function of the random binary signal was found to be

$$R_X(\tau) = \begin{cases} A^2\left(1 - \left|\frac{\tau}{T}\right|\right) & |\tau| < T \\ 0 & |\tau| > T \end{cases}$$

The power spectral density is the Fourier transform of $R_X(\tau)$ which is

$$S_X(f) = F\{\, R_X(\tau)\} = A^2 T \, sinc^2 fT$$

Exercise: For the random binary signal X(t), find the total signal power.

Exercise: Unipolar non return to zero signaling (revisited)

Let the transmitted symbols of X(t) in each signaling time interval T be:

$+A$	with probability	½	for	$0 \leq t \leq T$
0	with probability	½	for	$0 \leq t \leq T$

a. Find the power spectral density of X(t).
b. Find the null to null bandwidth of X(t).

Example: Random Binary Signal (Revisited)

Here, the possible symbols of X(t) in each signaling time interval T are represented by a pulse + g(t) and – g(t):

$+g(t)$	with probability ½	for	$0 < t < T$
$-g(t)$	with probability ½	for	$0 < t < T$

The power spectral density for this signal is

$$S_X(f) = F\{ R_X(\tau)\} = \frac{1}{T}|G(f)|^2$$

Where $G(f)$ if the Fourier transform of g(t). As we will see later, the transmission of digital data by means of signals with opposite polarity is called *antipodal signaling*.

Exercise: Manchester Coding

Let g(t) in the previous example be given by

$$g(t) = \begin{cases} A, & T/2 \leq t < T \\ -A, & T/2 \leq t < T \end{cases}$$

Find the power spectral density of the transmitted signal.

Example: Mixing of a random process with a sinusoidal signal.

A random process $X(t)$ with an autocorrelation function $R_X(\tau)$ and a power spectral density $S_X(f)$ is mixed with a sinusoidal function $\cos(2\pi f_c t + \theta)$; θ is a r.v uniformly distribution over $(0, 2\pi)$ to form a new process

$$Y(t) = X(t)\cos(2\pi f_c t + \theta)$$

Find $R_Y(\tau)$ and $S_Y(f)$

Solution

We first find $R_Y(\tau)$

$$R_Y(\tau) = E\{\,Y(t)Y(t+\tau)\}$$

$$= E\{X(t)\cos(2\pi f_c t + \theta)\cdot X(t+\tau)\cos(2\pi f_c t + 2\pi f_c \tau + \theta)\}$$

When $X(t)$ and θ are independent, then

$$= E\{X(t)\,X(t+T)\}E\{\cos(2\pi f_c t + \theta)\cdot\cos(2\pi f_c t + 2\pi f_c \tau + \theta)\}$$

$$= R_X(\tau)E\{\frac{\cos(4\pi f_c t + 2\pi f_c \tau + 2\theta) + \cos 2\pi f_c \tau}{2}\}$$

$$R_Y(\tau) = \frac{R_X(\tau)}{2}\cdot cos2\pi f_c \tau$$

The power spectral density is

$$S_Y(f) = \frac{1}{4}\{S_X(f-f_c) + S_X(f+f_c)\}$$

Which is quite similar to the modulation property of the Fourier transform.

Exercise: Binary Phase Shift Keying

Consider again the random binary signal m(t) which assumes the values $+1$ and -1 in each signaling time interval T as:

$+1$	with probability	½	for	$0 < t < T$
-1	with probability	½	for	$0 < t < T$

A new modulated signal $Y(t)$ is generated from X(t) as:

$$Y(t) = Am(t)\cos(2\pi f_c t + \theta)$$

where θ is a random variable uniformly distribution over $(0, 2\pi)$ and independent of m(t),

a. Find the null to null bandwidth of $m(t)$
b. Find the autocorrelation of $Y(t)$
c. Find and sketched the power spectral density of $Y(t)$
d. Find the null to null bandwidth of $Y(t)$.

Exercise: Binary Amplitude Shift Keying

Consider again the unipolar NRZ signal m(t) which assumes the values +1 and 0 in each signaling time interval T as:

+1	with probability	½	for	$0 < t < T$
0	with probability	½	for	$0 < t < T$

A new modulated signal $Y(t)$ is generated from X(t) as:

$$Y(t) = Am(t)\cos(2\pi f_c t + \theta)$$

where θ is a r.v uniformly distribution over $(0, 2\pi)$ and independent of m(t)

a. Find the null to null bandwidth of $m(t)$
b. Find the autocorrelation function of $Y(t)$.
c. Find and sketched the power spectral density of $Y(t)$
d. Find the null to null bandwidth of $Y(t)$.

Exercise: Binary Frequency Shift Keying

Consider again the random binary signal m(t) which assumes the values $+1$ and 0 in each signaling time interval T as:

$+1$	with probability	½	for	$0 < t < T$
0	with probability	½	for	$0 < t < T$

A new modulated signal $Y(t)$ is generated from X(t) as:

$$Y(t) = Am(t)\cos(2\pi f_1 t + \theta_1) + A\acute{m}(t)\cos(2\pi f_2 t + \theta_2)$$

where θ_1 and θ_2 are independent random variables uniformly distribution over $(0, 2\pi)$ and $\acute{m}(t) = (1 - m(t))$

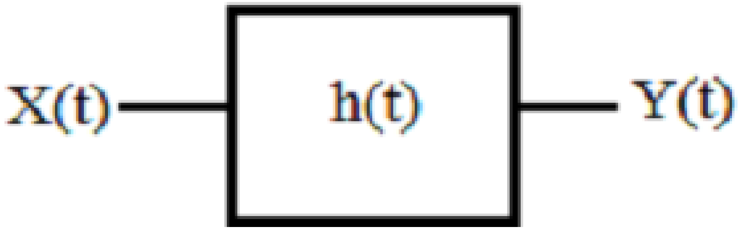
a. Show that $\acute{m}(t)$ *and* $m(t)$ have the same autocorrelation function.
b. Show that

$$R_Y(\tau) = \frac{A^2}{2}R_m(\tau)cos2\pi f_1\tau + \frac{A^2}{2}R_m(\tau)cos2\pi f_2\tau$$

c. Find and sketched the power spectral density of $Y(t)$
d. Find the null to null bandwidth of $Y(t)$.

Transmission of a Random Process Through a LTI Filter

Suppose that a stationary process X(t) is applied to a LTI filter of impulse response h(t) producing a new process Y(t) at the filter output.



The input Signal X (t) is a stationary process characterized by an auto-correlation function $R_x(\tau)$ and a power spectral density S_x (f).

Mean Value of Y(t)

Y (t) is related to X(t) through the convolution integral

$$Y(t) = \int_{-\infty}^{\infty} h(\lambda) X(t-\lambda) d\lambda$$

The mean value of Y(t) is

$$E\{Y(t)\} = \int_{-\infty}^{\infty} h(\lambda) E\{X(t-\lambda)\} d\lambda$$

Since X (t) is a stationary process, then E{X (t-λ)} = μ_x, a constant, so

$$E\{Y(t)\} = \int_{-\infty}^{\infty} h(\lambda)\ \mu_x\ d\lambda = \mu_x \int_{-\infty}^{\infty} h(\lambda) d\lambda$$

$$\boxed{\mu_y = \mu_x \int_{-\infty}^{\infty} h(\lambda) d\lambda = \mu_x H(0)}$$

where, H(0) is the value of the transfer function evaluated at f = 0.

Autocorrelation Function of Y(t)

The autocorrelation function of Y(t) can be evaluated as:

$$R_Y(t,u) = E\{Y(t).\ Y(u)\} \qquad ;\ u = t+\tau$$

$$= E\{\int_{-\infty}^{\infty} h(\lambda_1) X(t-\lambda_1) d\lambda_1 . \int_{-\infty}^{\infty} h(\lambda_2) X(u-\lambda_2) d\lambda_2$$

$$= \iint_{-\infty}^{\infty} h(\lambda_1) h(\lambda_2) E\{X(t-\lambda_1) X(u-\lambda_2)\} d\lambda_1 d\lambda_2$$

$$= \iint_{-\infty}^{\infty} h(\lambda_1) h(\lambda_2) R_X [(t-\lambda_1) - (u-\lambda_2] d\lambda_1 \lambda_2$$

$$R_X[(t-\lambda_1)-(u-\lambda_2] = R_X[(t-\lambda_1-u+\lambda_2] = R_X[(\tau-\lambda_1-\lambda_2]$$

Where, $\tau = t - u$. With this, $R_X(t,u)$ becomes

$$R_Y(\tau) = \iint_{-\infty}^{\infty} h(\lambda_1)h(\lambda_2)\, R_X\ (\tau-\lambda_1+\lambda_2)d\lambda_1\, d\lambda_2$$

Which can be expressed in a compact form as:

$$R_y(\tau) = h(\tau) * h(-\tau) * R_x(\tau)$$

Mean Square Value of Y(t)

Setting $\tau = 0$ in the expressions for $R_X(\tau)$, we get

$$E\{Y(t)^2\} = R_y(0) = \iint_{-\infty}^{\infty} h(\lambda_1) h(\lambda_2) R_x(\lambda_2 - \lambda_1)\, d\lambda_1\, d\lambda_2$$

Power Spectral density $S_y(f)$ of Y(t)

The power spectral density of Y(t) is related to the autocorrelation function through the relations

$$S_y(f) = F\{R_y(\tau)\} = F\{h(\tau) * h(-\tau) * R_x(\tau)\}$$

$$= H(f) \cdot H^*(f) \cdot S_x(f)$$

$$S_Y(f) = \int_{-\infty}^{\infty} |H(f)|^2 S_X(f) df$$

Total Input and Output Power

The total input and output powers can be found as the total area under the power spectral density curve.

$$E\{X(t)^2\} = \int_{-\infty}^{\infty} S_x(f)df = R_x(0)$$

$$E\{Y(t)^2\} = \int_{-\infty}^{\infty} S_y(f)df = R_y(0)$$

The Gaussian Random Process

A random variable X is said to be Gaussian if its probability density function is :

$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-(x-\mu_x)^2/2\sigma_x^2}$$

where,

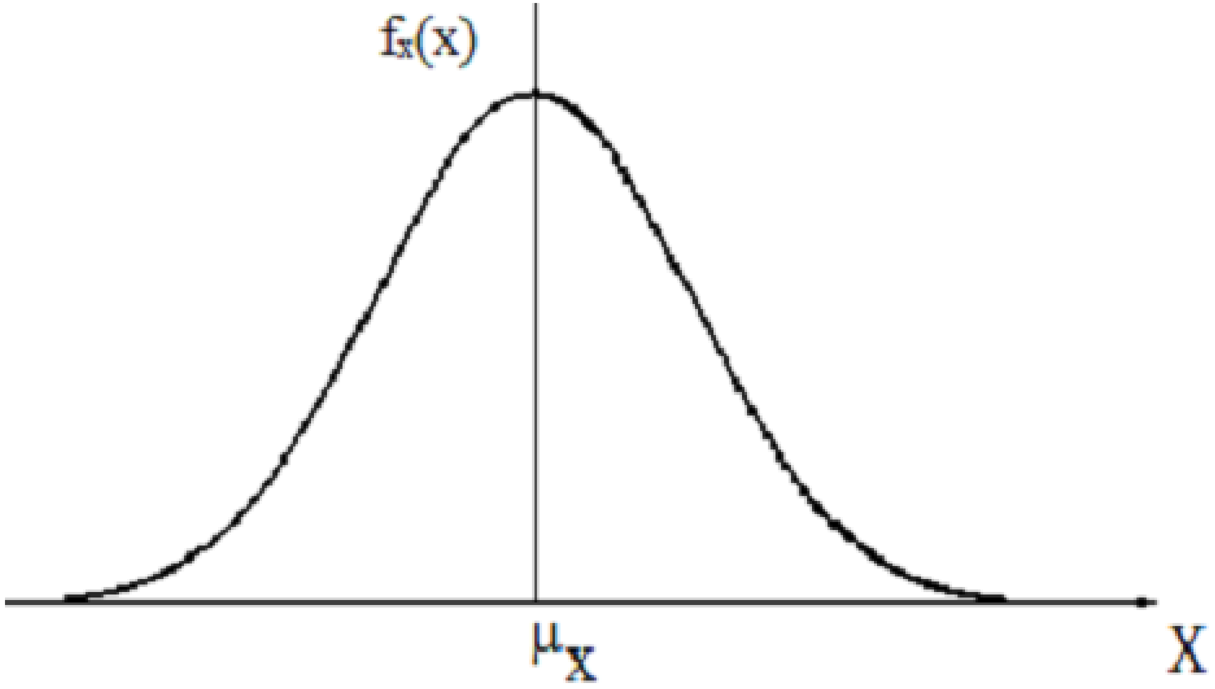
$\mu_x = E(x)$ is the mean value of X

$\sigma_x^2 = E\{(X-\mu_x)^2\}$ is the variance of X.

Defenition

A random process $X(t)$ is said to Gaussian if the random variables X_1, X_2,., X_n (obtained by observing the process at times $t_1, t_2, \ldots, t_n$) have a jointly Gaussian probability density function for all possible values of n and all times $t_1, t_2, \ldots, t_n$.

Two Virtues of the Gaussian Process



First : The process has many properties that make analytic results possible (easy to handle mathematically).

Second: The random process produced by physical phenomena is often such that a Gaussian model is appropriate . The use of a Gaussian model to describe the physical phenomena is usually confirmed by experiments.

The Central Limit Theorem

Let $X_1, X_2, ., X_n$ be a set of independent and identically distributed (iid) random variables such that $E(X_i) = \mu_x$ and $Var\ (X_i) = \sigma_x^2$. Define the random variable

$$U = \frac{\sum_1^n X_i}{n}$$

The probability distribution of U approaches a Gaussian distribution with mean μ_x and variance ${\sigma_x}^2/n$ in the limit as n → ∞ .

The theorem provides a justification for using a Gaussian process as a model for a large number of physical phenomena in which the observed random variable at a particular instant of time, is a result of a large number of individual events.

Properties of the Gaussian Process

1- If a Gaussian process X(t) is applied to a stable linear filter, then the random process Y(t) at the output of the filter is also Gaussian .

To see that we consider the convolution integral relating Y(t) to X(t)

$$\mathrm{Y(t)} = \int_{-\infty}^{\infty} X(\lambda)h\,(t - \lambda)d\lambda$$

Which comes from the approximation

$$\mathrm{Y(t)} = \sum X(\lambda_i)h\,(t - \lambda_i)$$

Note that Y(t) is a linear combination of Gaussian random variables, and so Y(t) is Gaussian for any value of t (any linear operation on X(t) produces another Gaussian process).

2- Consider the set of random variables $X(t_1)$, $X(t_2)$,…., $X(t_n)$, obtained by observing a random process X(t) at times t_1 , t_2 , … , t_n . If the process X(t) is Gaussian , then this set of random variables is jointly Gaussian for any n. The joint pdf is completely determined by specifying

the mean vector

$$\mu = [\ \mu_1 , \mu_2 , \dots \mu_n]^T$$

and the covariance matrix

$$\Sigma = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix}, c_{ij} = E\{(X_i - \mu_i)(X_j - \mu_j)\}, i, j = 1, \ldots, n$$

The joint pdf of the n random variables is

$$f(X_1, \ldots, X_n) = \frac{1}{(2\pi)^{n/2}\sqrt{\det \Sigma}} e^{-0.5 (X-\mu)^T \Sigma^{-1} (X-\mu)}$$

3- If a Gaussian process is stationary in the wide sense, then it is also stationary in the strict sense (this follows from property 2 above).

4- If the random variables $X(t_1), X(t_2), \ldots, X(t_n)$ obtained by sampling a Gaussian process X(t) at times $t_1, t_2, \ldots, t_n$ are uncorrelated, that is

$$E\{(X_i - \mu_i)(X_j - \mu_j)\} = 0; \qquad i \neq j$$

then these random variables are statistically independent. Here the covariance matrix is diagonal.

$$\Sigma = \begin{bmatrix} \sigma_1^2 & 0 \ldots & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & \cdots & \sigma_n^2 \end{bmatrix}$$

The joint pdf becomes a product of the marginal pdf's.

$$f_x(x) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-(x_i-\mu_i)^2/2\sigma_i^2}$$

A diagonal covariance matrix is a necessary and sufficient condition for statistical independence .

Noise in Communication Systems

The term noise is used to designate unwanted signals that tend to disturb the transmission and processing of signals in communication system and over which we have no control .

The noise may be external (man-made noise, galactic noise) or internal (arising from spontaneous fluctuation of current or voltage in electronic devices), example of which are shot noise and thermal noise .

Shot Noise

The noise arises in electronic devices such as diodes, transistors and photo-detector circuits, due to the discrete nature of current flow in these devices. Remember that current is a result of the flow of electrons, which have a discrete nature. The Poisson distribution is often used to model this type of noise. Using this model the number of electrons emitted in an interval of length T is a random variable with the pdf

$$P(X=x) = e^{-\lambda T} \frac{(\lambda T)^x}{x!} \quad , x = 0, 1, 2, \ldots$$

λ: average number of electrons emitted /unit time (rate of emission).

Thermal Noise :-

Voltages and currents that exist in a network due to the random motion of electrons in conductors is referred to as thermal noise (Johnson's noise).

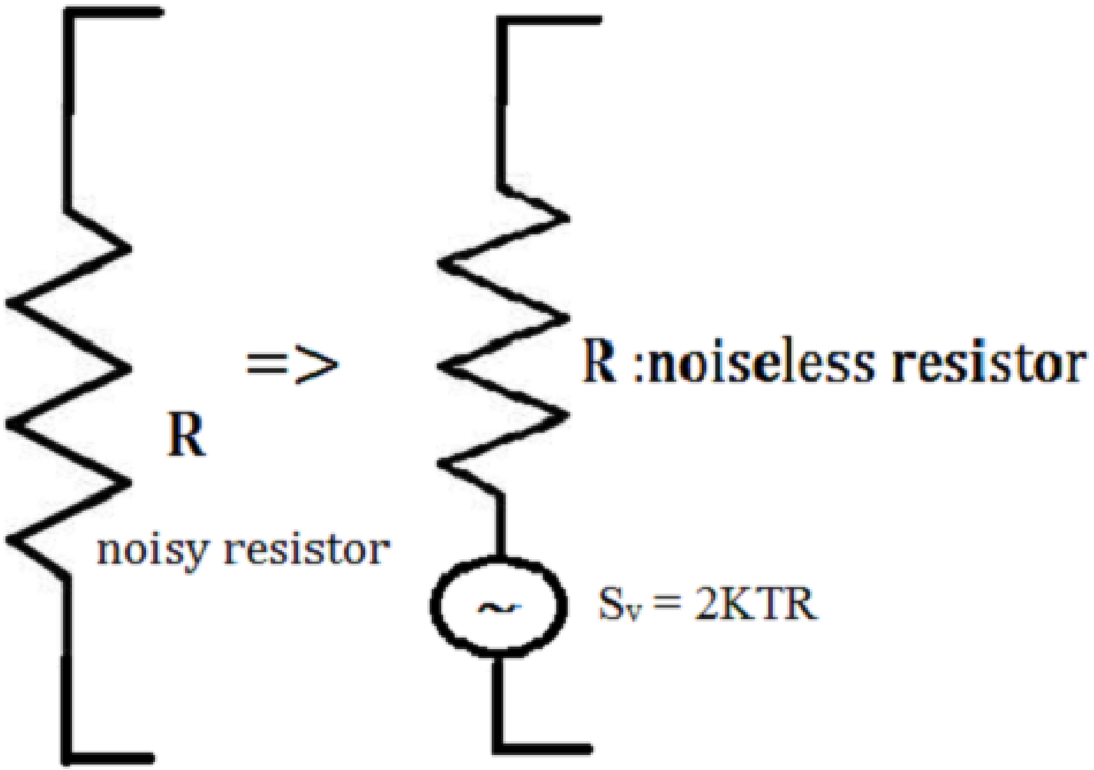
Quantum mechanics shows that the power spectral density of the thermal noise associated with a resistor with a resistance R is given by

$$S_v(f) = \frac{2Rh|f|}{e^{\frac{h|f|}{kT}} - 1} \qquad V^2/Hz$$

K: Boltsman constant ($1.38 * 10^{-23}$ J/degree)

h : Planck constant ($6.62 * 10^{-34}$ Joules-sec)

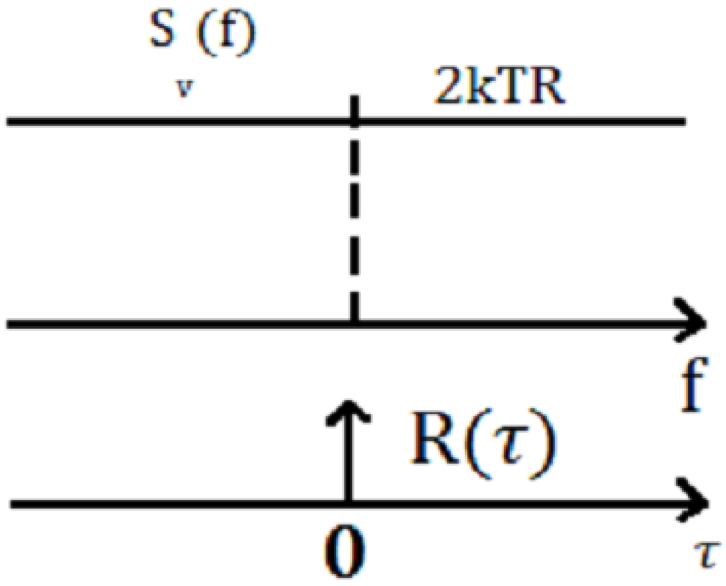
T: degree in Kelvin



For frequencies up to 10^{12} =1000 GHz, this power spectral density is almost constant having the value

$$S_v(f) = 2kTR \quad V^2/Hz\ .$$

The thermal noise voltage in a zero –mean Gaussian random process.



$E\{V(t)\}=0;$ zero mean.

$S_v(f) = 2kTR;$ constant power spectral density.

The autocorrelation function corresponding to this constant power spectral density is:

$$R_v(\tau) = 2kTR\,\delta(\tau);$$

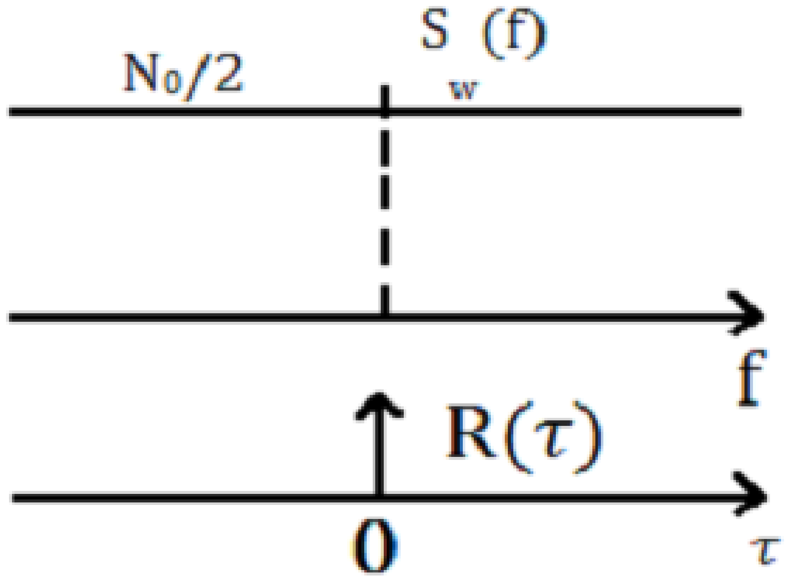
This result shows that two random variables V_{t1} , V_{t2} taken at times t_1 and t_2 are statistically independent for any value of $\tau = t_2 - t_1$ $\tau > 0$.

White Noise

White noise is one whose power spectral density is constant over all frequencies. The power spectral density and autocorrelation function for this type of noise are:

$$S_w(f) = N_0/2$$

$$R_w(\tau) = N_0/2\ \delta(\tau)$$



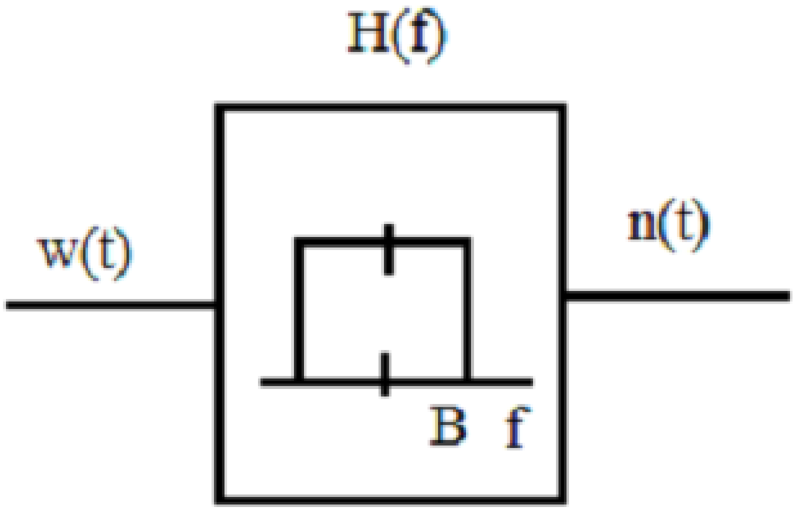
This is the type of noise (model) that we will use in the analysis of communication systems.

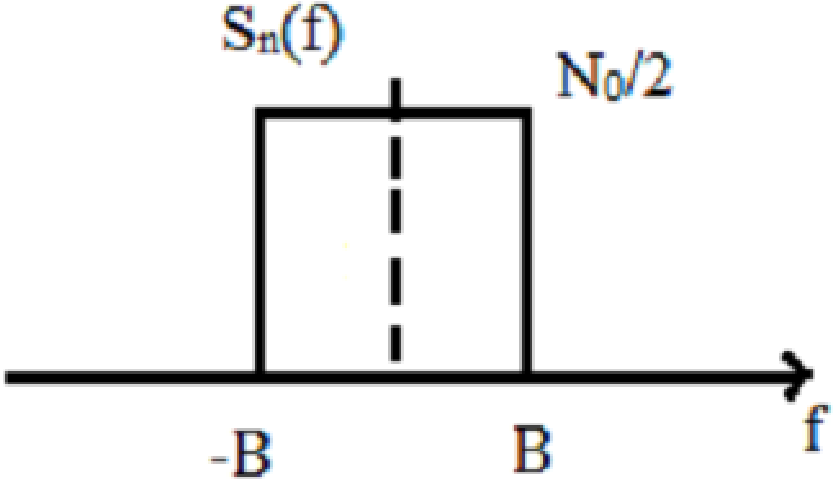
The assumption made is that this noise is additive white Gaussian (AWGN). If s(t) is the transmitted signal and r(t) is the received signal, then

$$r(t) = s(t) + w(t); \quad \text{additive channel noise .}$$

Filtered White Noise

Assume that a white Gaussian noise w(t) of zero mean and psd = $N_0/2$ is applied to an ideal LPF of B.W = B. Let n(t) denote the filtered noise, then





$S_n(f) = |H(f)|^2 S_w(f)$

$S_n(f)= \begin{cases} N_0/2 & , -B < f < B \\ 0 & , o.w \end{cases}$; Output psd

$R_n(\tau) = \int_{-B}^{B} \frac{N_0}{2} e^{(j2\pi f\tau)} df$

$R_n(\tau) = N_0B \text{ sinc } 2B\tau$; Output autocorrelation function

$E\{n(t)\} = 0$; zero mean noise

$E\{n(t)^2\} = \int_{-B}^{B} S_n(f)df = \frac{N_0}{2}(2B) = N_0B$; Total output noise power

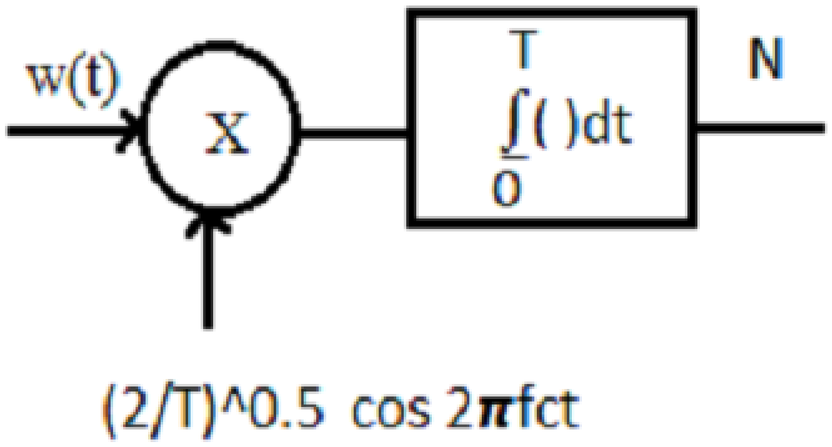
The pdf of the filtered noise at any particular time t is

$$f_n(n) = \frac{1}{\sqrt{2\pi(N_oB)}} e^{\frac{-n^2}{2(N_0B)}} \quad , -\infty < n < \infty$$

Remark: note that the pdf is not a function of time indicating that this filtered Gaussian process in stationary in the strict sense.

Correlation of White Noise with a Sinusoidal Signal

Let w(t) be a white Gaussian noise with zero mean. This noise is multiplied by a sinusoidal basis function and integrated over an interval of duration T to produce the scalar N. This scheme is repeatedly used in the coherent demodulation of digital signals. The interval T corresponds to one symbol interval and f_c is the frequency of the carrier. The carrier period and the symbol period are related by $T=nT_c$ where n is an integer. We wish to study the properties of N.



Mathematically, the correlation process is represented as:

$$N = \int_0^T w(t)\sqrt{\frac{2}{T}} \cos 2\pi f_c t \ dt$$

The mean value of N is:

$$E\{N\} = \int_0^T E\{w(t)\}\sqrt{\frac{2}{T}} \cos 2\pi f_c t \ dt = 0.$$

The variance of N is:

$$E\{N^2\} = \frac{2}{T} \iint_0^T E\{w(t_1)w(t_2)\} cos 2\pi f_c t_1 \ \cos 2\pi f_c t_2 \ dt_1 \ dt_2$$

$$E\{w(t_1)w(t_2)\} = R_w(t_2 - t_1) = N_0/2 \ \delta(t_2 - t_1)$$

$$\Rightarrow E\{ N^2 \} = \frac{2}{T}\frac{No}{2} \int_0^T \left(\int_0^T \delta(t_2 - t_1) \cos 2\pi f_c \ t_1 \ dt_1\right) \cos 2\pi f_c t_2 dt_2$$

$$= \frac{2}{T}\frac{No}{2} \int_0^T \cos^2 2\pi f_c t_2 \ dt_2; \qquad T = nT_c$$

The last step comes by virtue of the sifting property of the delta function. By performing the integration, we get

$$E\{N^2\} = \frac{No}{2} = \sigma^2$$

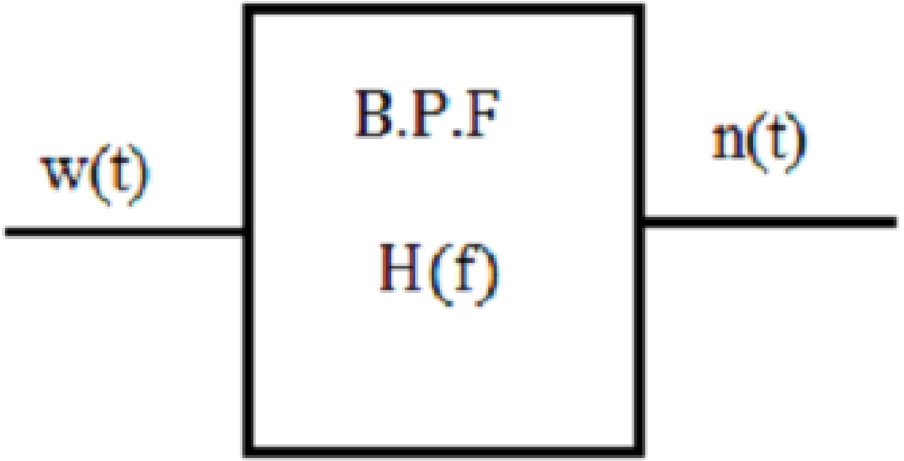
=> N is a zero mean Gaussian r.v with variance $\sigma^2 = \frac{No}{2}$. Its pdf can be written as

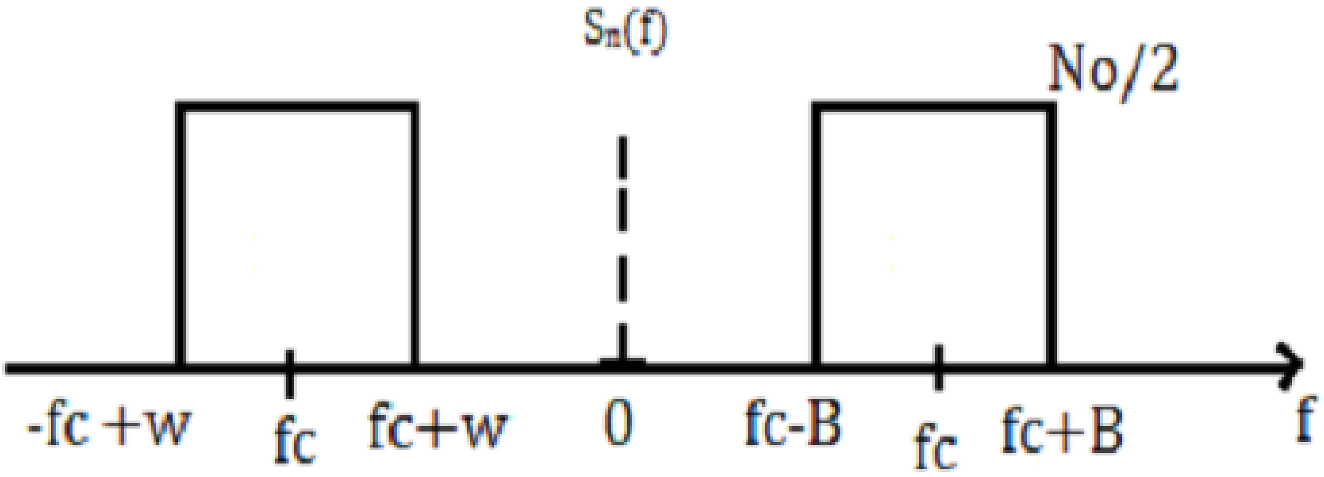
$$f_N(n) = \frac{1}{\sqrt{2\pi(\frac{N_o}{2})}}\ e^{\frac{-n^2}{2.(N_0/2)}}$$

$$f_N(n) = \frac{1}{\sqrt{\pi N_o}}\ e^{\frac{-n^2}{No}}$$

Narrow-band Noise

Now let the white Gaussian noise w(t) of psd $S_w(f) = N_0 / 2$ be applied to an ideal band pass filter with center frequency f_c and bandwidth 2B.





The noise is described as narrow band when $2B \ll f_c$. The analysis is similar to that done for the LPF and the results are summarized as follows:

$S_n(f) = N_0/2 \quad \text{for} \quad f_c\text{-}B < |f| < f_c+B;$ Output psd

$E\{n(t)\} = 0;$ zero mean noise

$E\{n(t)^2\} = 2 \cdot \frac{N_o}{2} \cdot 2B = N_o(2B) = \sigma^2;$ Total output power

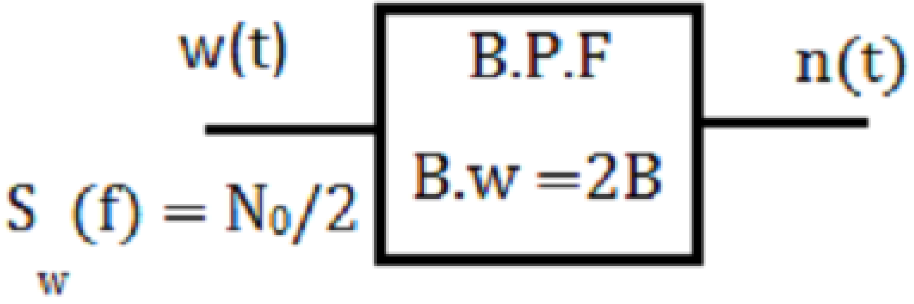
$f_N(n) = \frac{1}{\sqrt{2\pi(2BN_0)}} \; e^{\frac{-n^2}{2(2BN_0)}} \quad , -\infty \le n \le \infty;$ output noise pdf.

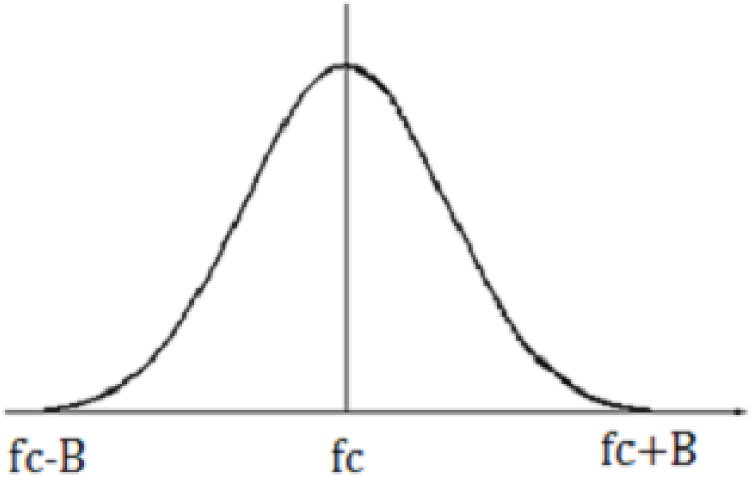
$R_n(0) = E\{n(t)^2\} = \int_{-\infty}^{\infty} S_n(f)df \;;$ Mean square value.

Narrow-band Noise: In-phase and Quadrature Representation

Let w(t) be applied to a bandpass filter of B.w = 2B centered at f_c to produce a narrow band noise n(t).

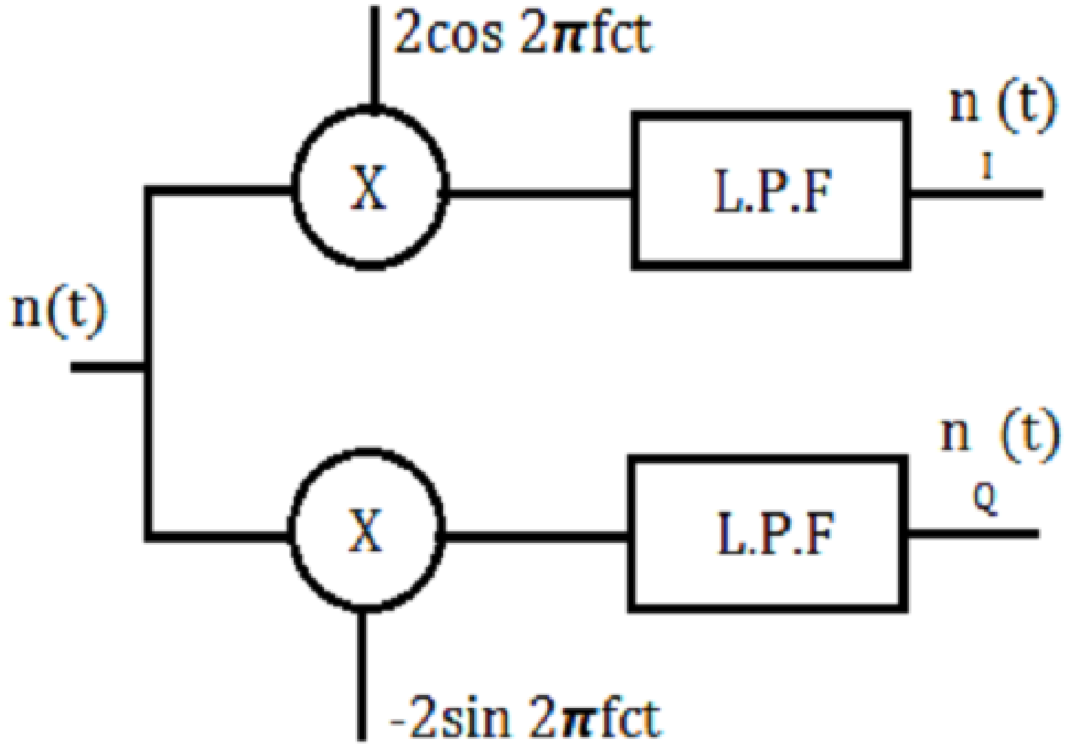
The narrow band noise n(t) can be represented in terms of an in-phase $n_I(t)$ and a quadrature component $n_Q(t)$ as:





$$n(t) = n_I(t) \cos 2\pi f_c t - n_Q(t) \sin 2\pi f_c t$$

The in-phase and quadrature components $n_I(t)$ and $n_Q(t)$ can be recovered from n(t) as demonstrated in the block diagram.



$n_I(t) = Lp\{2n(t)\cos 2\pi f_c t\}$; in-phase noise component

$S_{NI}(f) = Lp\{S_n(f-f_c)+S_n(f+f_c)\}$; in-phase noise psd.

$n_Q(t) = -\ Lp\{2n(t)\sin 2\pi f_c t\}$; quadrature noise component

$S_{NI}(f) = S_{NQ}(f)$;   both components have the same psd

Finally, $n_I(t)$ and $n_Q(t)$ can be retrieved from n(t) as:

$$n_I(t) = n(t)\cos 2\pi f_c t + \widehat{n(t)} sin 2\pi f_c t$$

$$n_I(t) = \widehat{n(t)}\cos 2\pi f_c t - n(t) sin 2\pi f_c t$$

Properties of the Noise Components

- The in-phase component $n_I(t)$ and the quadrature component $n_Q(t)$ of narrow band noise n(t) have zero mean .
- If the narrow band noise n(t) is Gaussian, then $n_I(t)$ and $n_Q(t)$ are jointly Gaussian .
- If n(t) is wide sense stationary, then $n_I(t)$ and $n_Q(t)$ are jointly wide sense stationary .
- Both $n_I(t)$ and $n_Q(t)$ have the same power spectral density

$$S_{NI}(f) = S_{NQ}(f) = \begin{cases} S_N(f - f_c) + S_N(f + f_c), & -B < f < B \\ 0 & , o.w \end{cases}$$

- $n_I(t)$, $n_Q(t)$ and n(t) have the same variance

$$E\{n(t)^2\} = E\{n_I(t)^2\} = E\{n_Q(t)^2\} = \sigma^2$$

- The cross-spectral densities of $n_I(t)$ and $n_Q(t)$ are imaginary

$$S_{N_IN_Q}(f) = - S_{N_QN_I}(f) = \begin{cases} j[S_N(f + f_c) - S_N(f - f_c)], & -B < f < B \\ 0 & , o.w \end{cases}$$

- If n(t) is Gaussian with zero mean and a power spectral density $S_n(f)$ that is symmetric about f_c, then $n_I(t)$ and $n_Q(t)$ are statistically independent. The joint pdf of $n_I(t)$ and $n_Q(t)$ is the product of the marginal pdf's

$$f(n_I, n_Q) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-n_I^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-n_Q^2}{2\sigma^2}}$$

(i.e, when the cross spectral density = 0 ∀ f, then n_I and n_Q are independent)

Polar Representation of Narrow-band Noise

Let n(t) be a narrow band zero-mean, white Gaussian noise with a symmetric psd about some center frequency f_c .

$$n(t) = n_I(t) \cos 2\pi f_c t - n_Q(t) \sin 2\pi f_c t.$$

Because $S_n(f)$ is symmetric, it follows that $n_I(t)$ and $n_Q(t)$, observed at a fixed time t, are independent Gaussian r.v with zero mean and variance σ^2 .

n(t) can also be represented as

$$n(t) = R(t) \cos (2\pi f_c t + \emptyset(t))$$

where the envelope R(t) and the phase Ø(t) are given as:

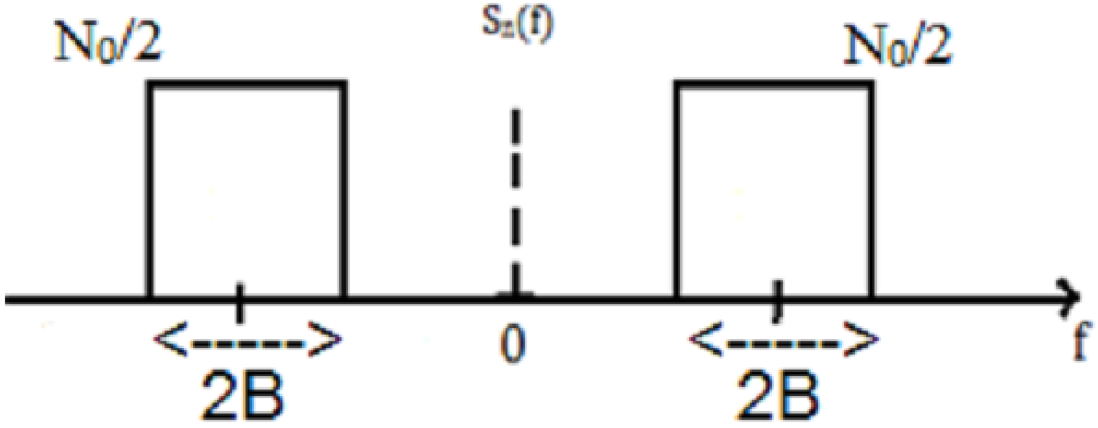
$$R(t) = [n_I(t)^2 + n_Q(t)^2]^{1/2}$$

$$\emptyset(t) = \tan^{-1} (n_Q(t) / n_I(t))$$

It can be shown (Go back to your ENEE 331 lecture notes and go over the proof) that R and Ø are independent random variables with pdf 's

$$f_\Phi(\Phi) = \begin{cases} \frac{1}{2\pi} & ,0 \le Ø \le 2\pi \\ 0 & ,o.w \end{cases}; \quad \text{(Uniform pdf)}$$

$$f_R(r) = \begin{cases} \frac{r}{\sigma^2} \exp[-(r^2/2\sigma^2)] & ,\ r \ge 0 \\ 0 & ,o.w \end{cases}; \quad \text{(Rayleigh distribution)}$$



If $S_n(f)$ has the psd shown then , $\sigma^2 = 2\,(N_0/2)(2B) = 2N_0\,B$ and the pdf of R is as given above .

A Test Sine Signal Plus Narrow-band Noise

If a test sine signal $A\cos 2\pi f_c t$ is added to the narrow band filtered noise, then the signal plus noise can be expressed (using the in-phase and quadrature representation of the noise) as:

$$X(t) = A_c \cos 2\pi f_c t + n_I(t)\cos 2\pi f_c t - n_Q(t)\sin 2\pi f_c t$$

$$X(t) = (A_c + n_I(t))\cos 2\pi f_c t - n_Q(t)\sin 2\pi f_c t$$

The noise components N_I and N_Q are independent zero mean Gaussian r.v (psd of n(t) is symmetric) each with variance σ^2. X(t) can also be represented in polar form as:

$$X(t) = R(t)\cos(2\pi f_c t + \emptyset(t))$$

Where

$$R(t) = \sqrt{(A_c + n_I(t))^2 + n_Q(t)^2}$$

$$\emptyset(t) = tan^{-1} \frac{n_Q(t)}{(A_c + n_I(t))}$$

It can be shown that the pdf of R is

$f_R(r) = \{\frac{r}{\sigma 2}$ Exp -[$(r^2 + A^2) / 2\,\sigma^2$] $I_o\,(Ar/\,\sigma^2)$; (Rician distribution)

$I_o\,(.)$ is the modifies Bessel function of the first kind of zero order, and

$\sigma^2 = E\{n(t)^2\} = E\{n_i(t)^2\} = E\{n_q(t)^2\}$.

The Rician distribution arises in the study of the performance of some digital communication applications like the noncoherent demodulation of ASK and FSK.