Noise in Analog Communication

Random Process

A random process $X(t)$ is defined as an ensemble of time functions together with a probability rule that assigns a probability to any meaningful event associated with an observation of one of the sample functions of the random process.

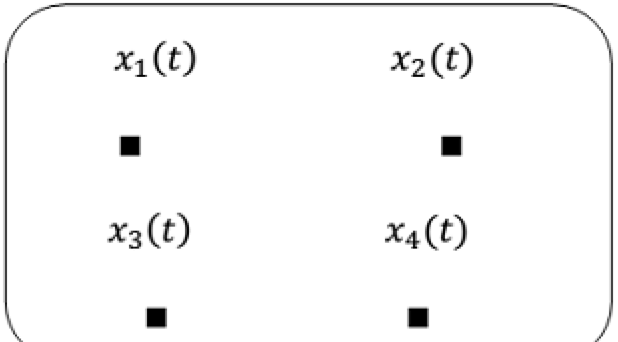
Consider the following experiment: An oscillator produces a waveform of the form $A_m\, cos(w_m t + \theta)$; where θ is a discrete R.V with a probability mass function

$$P(\theta = 0) = 0.2 \qquad P\left(\theta = \frac{\pi}{2}\right) = 0.2$$

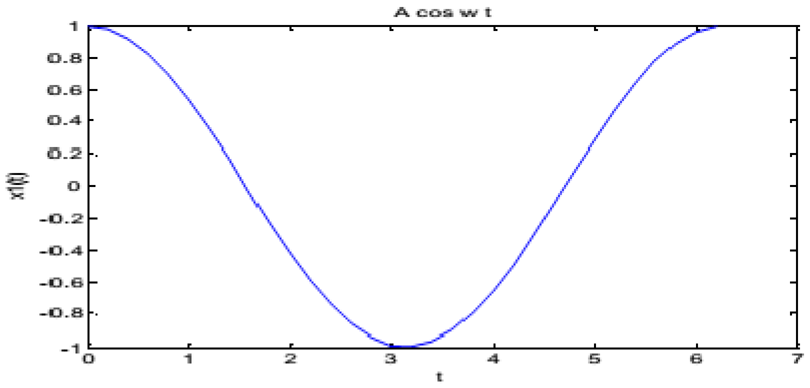
$$P(\theta = \pi) = 0.3 \qquad P\left(\theta = \frac{3\pi}{2}\right) = 0.3$$

Here the sample space of the experiment consists of four time functions:

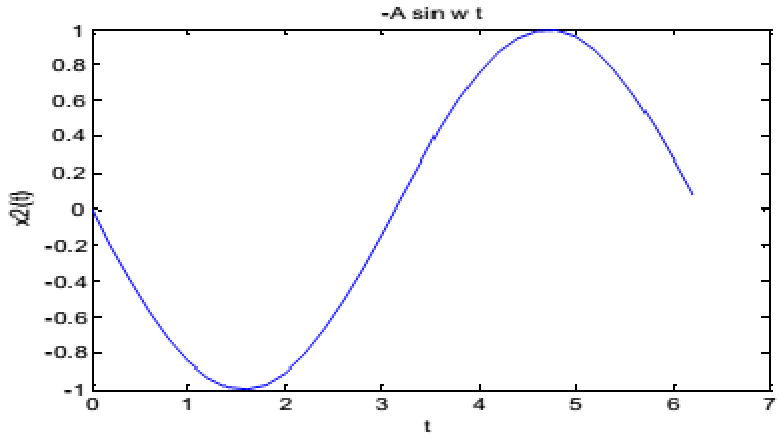
$$x_1(t) = A_m \cos(\, w_m t\ ) \qquad P(x_1(t)) = 0.2$$
$$x_2(t) = A_m \cos\left(w_m\, t + \frac{\pi}{2}\right) \qquad P(x_2(t)) = 0.2$$
$$x_3(t) = A_m \cos(\, w_m t + \pi\ ) \qquad P(x_3(t)) = 0.3$$
$$x_4(t) = A_m \cos(\, w_m t + \frac{3\pi}{2}) \qquad P(x_4(t)) = 0.3$$



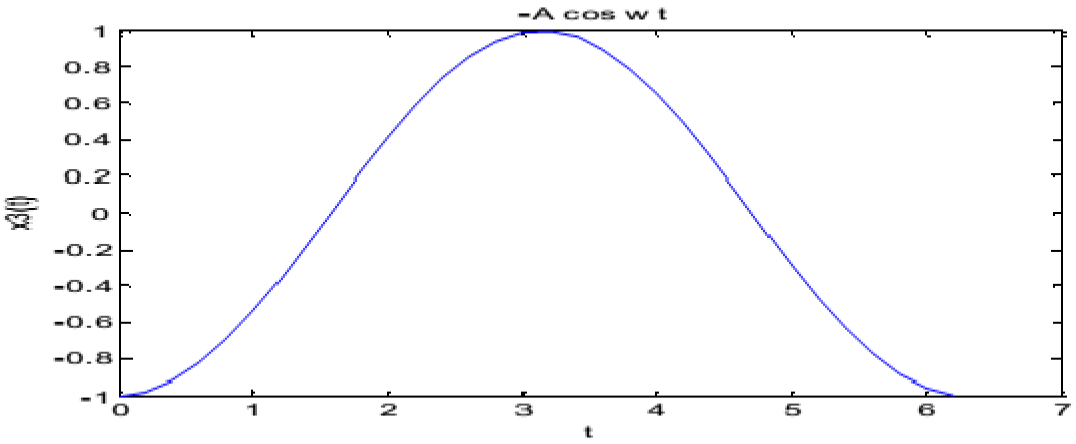
$$x_1(t) = A\,cos(2\pi f_m t)$$



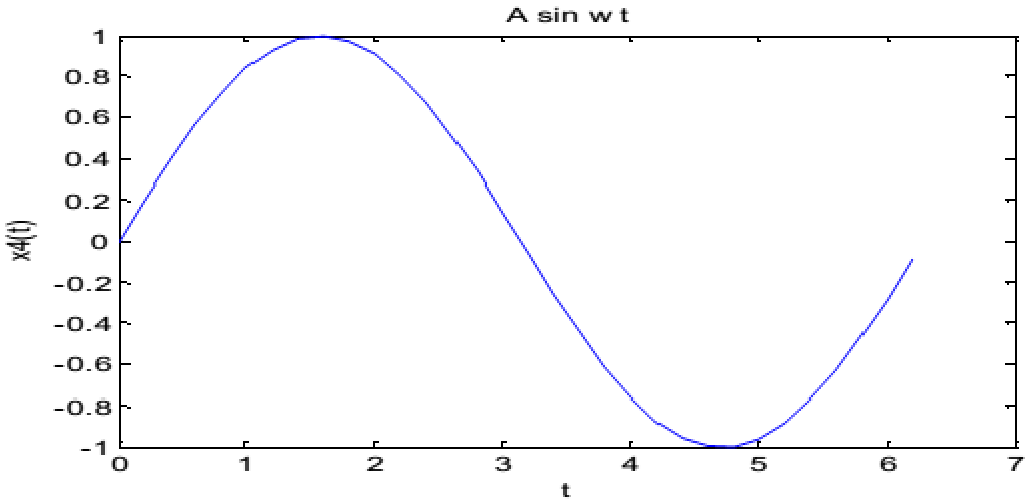
$$x_2(t) = -A\,\sin(2\pi f_m t)$$



$$x_3(t) = -A\,cos(2\pi f_m t)$$



$$x_4(t) = +A\,sin(2\pi f_m t)$$



Each realization of the experiment is called a *sample function* $x(t)$. The sample space (ensemble) composed of functions is called *a random or stochastic process* denoted by $X(t)$. The value assumed by a random process at a particular time is a random variable with a certain probability density function.

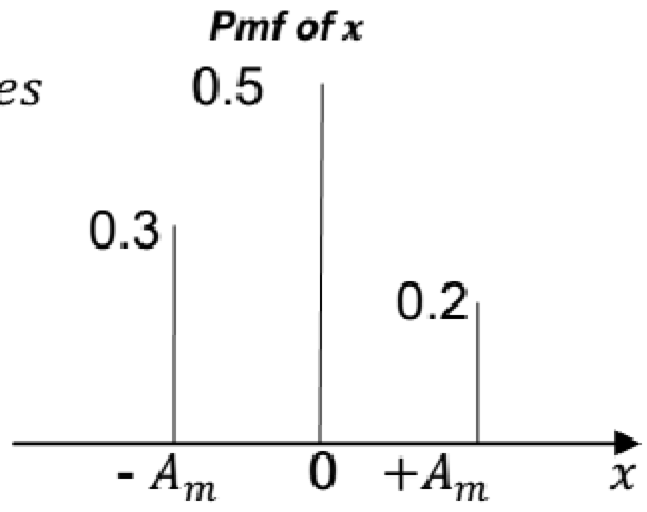
For the example above, $X(0)$ *assumes three values*

P $\{X(0) = A_m\} = 0.2$

P $\{X(0) = -A_m\} = 0.3$

$P\{X(0) = 0\} = 0.2 + 0.3 = 0.5$

(Corresponding to $= \frac{\pi}{2}, \frac{3\pi}{2}$)



Pmf of X at t = 0.

$P(X = 0)$ = 0.2+0.3 = 0.5

$P(X = +A_m) = 0.2$

$P(x = -A_m) = 0.3$

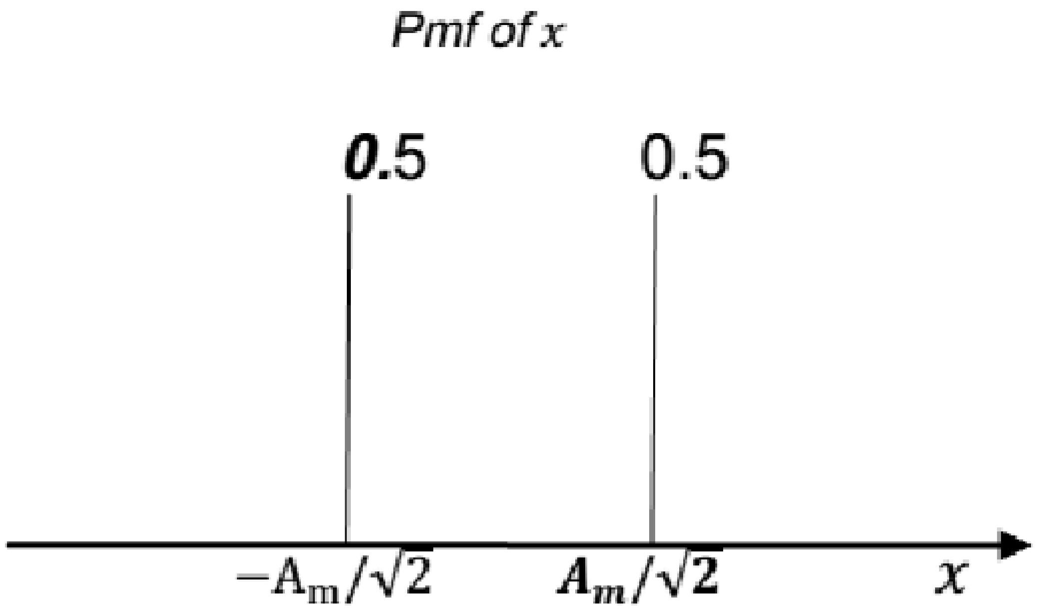
The mean value of the random variable X is

$$E(X) = -A_m \times 0.3 + 0 \times 0.5 + 0.2 \times A_m$$
$$E(X) = -0.3A_m + 0.2A_m = -0.1A_m$$

Pmf of X at $w_m t_n = \pi/4$

	$X(\mathbf{w_m}\ \mathbf{t} = \frac{\boldsymbol{\pi}}{\mathbf{4}})$	Prob.
Possible values:	$A_m \cos\left(\frac{\pi}{4} + 0\right) = +A_m/\sqrt{2}$	0.2
	$A_m \cos\left(\frac{\pi}{4} + \frac{\pi}{2}\right) = -A_m/\sqrt{2}$	0.2
	$A_m \cos\left(\frac{\pi}{4} + \pi\right) = -A_m/\sqrt{2}$	0.3
	$A_m \cos\left(\frac{\pi}{4} + \frac{3\pi}{2}\right) = +A_m/\sqrt{2}$	0.3

The Pmf of X at $w_m t = \frac{\pi}{4}$ is sketched here.



$$E\{X\left(w_m t = \frac{\pi}{4}\right) = 0$$

⇛ Process is not stationary [mean at $t = 0$ is not the same as the mean at $w_m\, t = \frac{\pi}{4}$]

In general, $X(t) = A_m \cos(w_m t + \theta)$

$$E\{X(t)\} = \sum X(t, \theta_i)\, P(\theta = \theta_i)$$

$$= A_m \cos w_m t \times 0.2 + 0.2 \times A_m \cos\left(w_m t + \frac{\pi}{2}\right) + 0.3 \cos(w_m t + \pi) +$$

$$0.3 \times A_m \cos(w_m t + \frac{3\pi}{2}).$$

Mean is not a constant (function of time).

$\Rightarrow$ **Process is non stationary.**

Stationarity of a random process:

The mean of a process $X(t)$ is defined as the expectation of the r.v obtained by observing the process at some time t as

$$\mu_x(t) = E\{X(t)\} = \int_{-\infty}^{\infty} x\, f_x(x)dx$$

$f_x(x)$ *is the first order pdf of the process* $X(t)$.

The autocorrelation function of the process $X(t)$ is defined as the expectation of the product of two r.v $X(t_1)$ *and* $X(t_2)$ obtained by observing the process $X(t)$ at times t_1 and t_2.

$$R_x(t_1, t_2) = E\{X(t_1)X(t_2)\} = \iint_{-\infty}^{\infty} x_1\, x_2\, f(x_1, x_2) dx_1\, dx_2$$

$f(x_1, x_2)$ is the second order (joint pdf) of x_1 and x_2.

A random process is said to be wide sense stationary or (stationary) when the following two conditions hold:

1) $E\{X(t)\} = \mu_x =$ constant for all t

2) $R_x(t_1, t_2) = E\{X(t_1)X(t_2)\} = R_x(t_2 - t_1)$
   i.e., R_x *is* a function of the time difference and not on the absolute values of t_1 and t_2. i.e.,

$R_x(\tau) = E\{X(t)X(t+\tau)\}$; *where* $\tau = t_2 - t_1$

Properties of the autocorrelation function of a stationary process:

1) $Rx(0) = E\{X^2(t)\}$; the mean square value (second moment of x) { total power in $X(t)$ }

2) $R_x(\tau) = R_x(-\tau); R_x(\tau)$ *is an even function of* τ.

3) $\mathrm{R_x}(\tau)$attains its maximum value at $\tau = 0$

$$|\mathrm{R_x}(\tau)| \leq \mathrm{R_x}(0)$$

Decorrelation Time : The decorrelation time τ_0 of the a stationary process $X(t)$ of zero mean is taken as the time taken for the magnitude of the autocorrelation function $R_x(\tau)$ to decrease say 1% of its maximum value $R_x(0)$.

A Result we Recall from ENEE 331: If θ is a r.v with pdf $f_\theta(\theta)$ and $Y = g(\theta)$, then $E\{Y\} = \int g(\theta)f(\theta)\, d\theta$

$$E\{g(\theta)\} = \int g(\theta)f_\theta(\theta)\, d\theta$$

Example: A sinusoidal signal with random phase

$$\text{Let } X(t) = A\,cos(2\pi f_c t + \theta)$$

A, f_c are constants, θ is a continuous r.v uniformly distributed over $(-\pi, \pi)$

$$f_\theta(\theta) = \begin{cases} \dfrac{1}{2\pi} & -\pi < \theta < \pi \\ 0 & o.w \end{cases}$$

The mean value of $X(t)$ is

$$E\{\,X(t)\,\} = \int_{-\pi}^{\pi} \underbrace{Acos(2\pi f_c\, t + \theta)}_{g(\theta)} \cdot \underbrace{\frac{1}{2\pi}}_{f(\theta)} d\theta = 0$$

Which is a constant (independent of time). The autocorrelation function is:

$$R_x(\tau) = E\{X(t)X(t+\tau)\}$$

$$= \int_{-\pi}^{\pi} \overbrace{Acos(2\pi f_c\, t + \theta) \cdot Acos[2\pi f_c(t+\tau) + \theta]}^{g(\theta)} \cdot \overbrace{1/2\pi}^{f(\theta)}\, d\theta$$

$$= \frac{A^2}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2} \{cos 2\pi f_c\, \tau + \ cos(2\pi(2) f_c t + 2\pi f_c \tau + 2\theta)\}\, d\theta$$

We can easily recognize that the second integral is zero, leaving only the first term. Hence, $R_x(\tau)$ becomes

$$R_x(\tau) = \frac{A^2}{2\pi} \cdot \frac{\cos 2\pi f_c \tau}{2} \cdot 2\pi = \frac{A^2}{2} cos 2\pi f_c \tau$$

Note that:

- The mean value is a constant and $R_x(\tau)$ is a function of τ. These are the two conditions necessary for the process to be stationary. So $X(t)$ is a stationary process.
- The process $X(t)$ is periodic with period $T_c = \frac{1}{f_c}$ .The autocorrelation function $R_x(\tau) = \frac{A^2}{2} \cos 2\pi f_c \tau$ is also periodic with period $T_c = \frac{1}{f_c}$.

Ergodic processes:

Given a sample function $x(t)$ of a random process $X(t)$, we define the following two time averages:

$$\langle X(t)\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)dt$$

$$\langle X(t)X(t+\tau)\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt$$

Here, $2T$ is an observation interval.

Def: A random process is said to be ergodic if statistical properties can be determined from a sample function representing one realization of the process.

Statistical average = Time Average.

The two quantities of interest are the mean value and the autocorrelation function. For an ergodic process, they can be computed using time average as:

$E\{X(t)\} = \langle x(t)\rangle = constant.$

$R_x(\tau) = E\{X(t)X(t+\tau)\} = \langle x(t)\, x(t+\tau)\rangle \rightarrow$ function of τ.

Remark: An ergodic process in stationary, but a stationary process is not necessarily ergodic.

Example: consider again the process $X(t) = Acos(2\pi f_c\, t + \theta)$, θ is uniformly distributed over (-π < θ < π).

The two time averages are calculated as follows:

$$\langle x(t)\rangle = \frac{1}{T_c}\int_0^{T_c} Acos(2\pi f_c\, t + \theta)dt = 0 \;\; ; \qquad T_c = 1/f_c$$

$$\langle x(t)x(t+\tau)\rangle = \frac{1}{T_c}\int_0^{T_c} Acos(2\pi f_c t + \theta)\cdot Acos(2\pi f_c t + 2\pi f_c\,\tau + \theta)dt$$

$$= \frac{A^2}{2T}\int_0^{T_c} \cos(2\pi f_c\, t + 2\pi f_c\tau + 2\theta) + cos2\pi f_c\tau\; dt$$

$$= \frac{A^2}{2}cos2\pi f_c\tau$$

These are the same values found in the previous example.

$\Rightarrow$ process is ergodic.

Power spectral Density and Autocorrelation Function:

Consider a stationary random process $X(t)$ that is ergodic. Consider a truncated segment of $x(t)$ defined over the observation interval $-T < t < T$. Let X_{2T} be the truncated signal:

$$x_{2T}(t) = \begin{cases} x(t) & -T < t < T \\ 0 & o.w \end{cases}$$

The Fourier transform of x_{2T} is:

$$X_{2T}(f) = \int_{-T}^{T} x(t) e^{-j2\pi f t} dt$$

The energy spectral density of $X_{2T}(t)$ is $|X_{2T}(f)|^2$. Since $x(t)_{2T}$ is only one realization of a random process, then we need to find its mean value $E\{|X_{2T}(f)|^2\}$. Dividing this by the observation interval 2T, and letting T becomes very large, we get the power spectral density of the whole process, averaged over all sample functions and over all time.

The power spectral density, of a stationary process, may then be defined as:

$$S_X(f) = \lim_{T \to \infty} \frac{1}{2T} E\{|X_{2T}(f)|^2\}$$

The Wiener –Khintchine Thorem:

The power spectral density $S_X(f)$ and the autocorrelation function $R_X(\tau)$ of a stationary random process $X(t)$ form a Fourier transform pairs:

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f \tau} d\tau$$

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) e^{j2\pi f \tau} df$$

Properties of the power spectral Density:

1. The zero frequency value of the power spectral density of a stationary process equals the total area under the graph of the autocorrelation function ;

$$S_X(0) = \int_{-\infty}^{\infty} R_X(\tau) d\tau$$

2. The mean squared value (the total signal power) of a stationary process equals the area under the power spectral density curve.

$$E\{X(t)^2\} = R_X(0) = \int_{-\infty}^{\infty} S_X(f)df$$

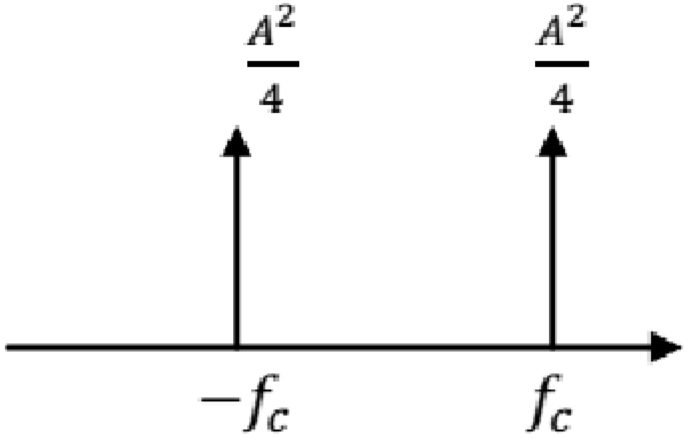
3. The power spectral density of a stationary process is always nonnegative ; i.e., $S_X(f) \geq 0$ for all f.

4. The power spectral density of a real-valued random process is an even function of f.

$$S_X(f) = S_X(-f)$$

Find the power spectral density of the random process $X(t) = A\cos(2\pi f_c t + \theta)$;

θ is a uniform r.v over $(-\pi, \pi)$.



For this process, we found earlier that $R_X(\tau) = \frac{A^2}{2} cos 2\pi f_c \tau$

Since $S_X(f) = F\{ R_X(\tau) \}$, then

$$S_X(f) = \frac{A^2}{4}\{ \delta(f - f_c) + \delta(f + f_c) \}$$

The total average power is obtained by integrating $S_X(f)$ over all frequencies.

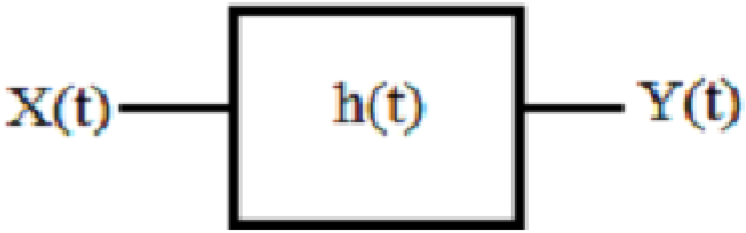
$$\int_{-\infty}^{\infty} S_X(f) df = \int_{-\infty}^{\infty} \frac{A^2}{4}[\delta(f - f_c) + \delta(f + f_c)] df$$

$$= \frac{A^2}{4} + \frac{A^2}{4} = \frac{A^2}{2}$$

$$= R_X(0)$$

Transmission of a Random Process Through a LTI Filter

Suppose that a stationary process X(t) is applied to a LTI filter of impulse response h(t) producing a new process Y(t) at the filter output.



The input Signal X (t) is a stationary process characterized by an auto-correlation function $R_x(\tau)$ and a power spectral density S_x (f).

Mean Value of Y(t)

Y (t) is related to X(t) through the convolution integral

$$Y(t) = \int_{-\infty}^{\infty} h(\lambda) X(t-\lambda) d\lambda$$

The mean value of Y(t) is

$$E\{Y(t)\} = \int_{-\infty}^{\infty} h(\lambda) E\{X(t-\lambda)\} d\lambda$$

Since X (t) is a stationary process, then $E\{X(t-\lambda)\} = \mu_x$, a constant, so

$$E\{Y(t)\} = \int_{-\infty}^{\infty} h(\lambda)\ \mu_x\ d\lambda = \mu_x \int_{-\infty}^{\infty} h(\lambda) d\lambda$$

$$\boxed{\mu_y = \mu_x \int_{-\infty}^{\infty} h(\lambda) d\lambda = \mu_x H(0)}$$

where, H(0) is the value of the transfer function evaluated at f = 0.

Autocorrelation Function of Y(t)

The autocorrelation function of Y(t) can be evaluated as:

$$R_Y(t,u) = E\{Y(t).\ Y(u)\} \qquad ;\ u = t+\tau$$

$$= E\{\int_{-\infty}^{\infty} h(\lambda_1) X(t-\lambda_1) d\lambda_1 . \int_{-\infty}^{\infty} h(\lambda_2) X(u-\lambda_2) d\lambda_2$$

$$= \iint_{-\infty}^{\infty} h(\lambda_1) h(\lambda_2) E\{X(t-\lambda_1) X(u-\lambda_2)\} d\lambda_1 d\lambda_2$$

$$= \iint_{-\infty}^{\infty} h(\lambda_1) h(\lambda_2)\, R_X[(t-\lambda_1) - (u-\lambda_2]\, d\lambda_1 \lambda_2$$

$$R_X\,[(t-\lambda_1)-(u-\lambda_2] = R_X\,[(t-\lambda_1-u+\lambda_2] = R_X\,[(\tau-\lambda_1-\lambda_2]$$

Where, $\tau = t - u$. With this, $R_X(t,u)$ becomes

$$R_Y(\tau) = \iint_{-\infty}^{\infty} h\,(\lambda_1) h\,(\lambda_2)\, R_X\;(\tau-\lambda_1+\lambda_2) d\lambda_1\, d\lambda_2$$

Which can be expressed in a compact form as:

$$R_y\,(\tau) = h\,(\tau) * h(-\tau) * R_x(\tau)$$

Mean Square Value of Y(t)

Setting $\tau = 0$ in the expressions for $R_X(\tau)$, we get

$$E\{Y(t)^2\} = R_y(0) = \iint_{-\infty}^{\infty} h(\lambda_1)h(\lambda_2)\, R_x(\lambda_2-\lambda_1)\, d\lambda_1\, d\lambda_2$$

Power Spectral density $S_y(f)$ of Y(t)

The power spectral density of Y(t) is related to the autocorrelation function through the relations

$$S_y(f) = F\{R_y(\tau)\} = F\{h(\tau) * h(-\tau) * R_x(\tau)\}$$

$$= H(f) \,.\, H^*(f) \,.\, S_x(f)$$

$$S_Y(f) = \int_{-\infty}^{\infty} |H(f)|^2 S_X(f) df$$

Total Input and Output Power

The total input and output powers can be found as the total area under the power spectral density curve.

$$E\{X(t)^2\} = \int_{-\infty}^{\infty} S_x(f)df = R_x(0)$$

$$E\{Y(t)^2\} = \int_{-\infty}^{\infty} S_y(f)df = R_y(0)$$

The Gaussian Random Process

A random variable X is said to be Gaussian if its probability density function is :

$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-(x-\mu_x)^2/2\sigma_x^2}$$

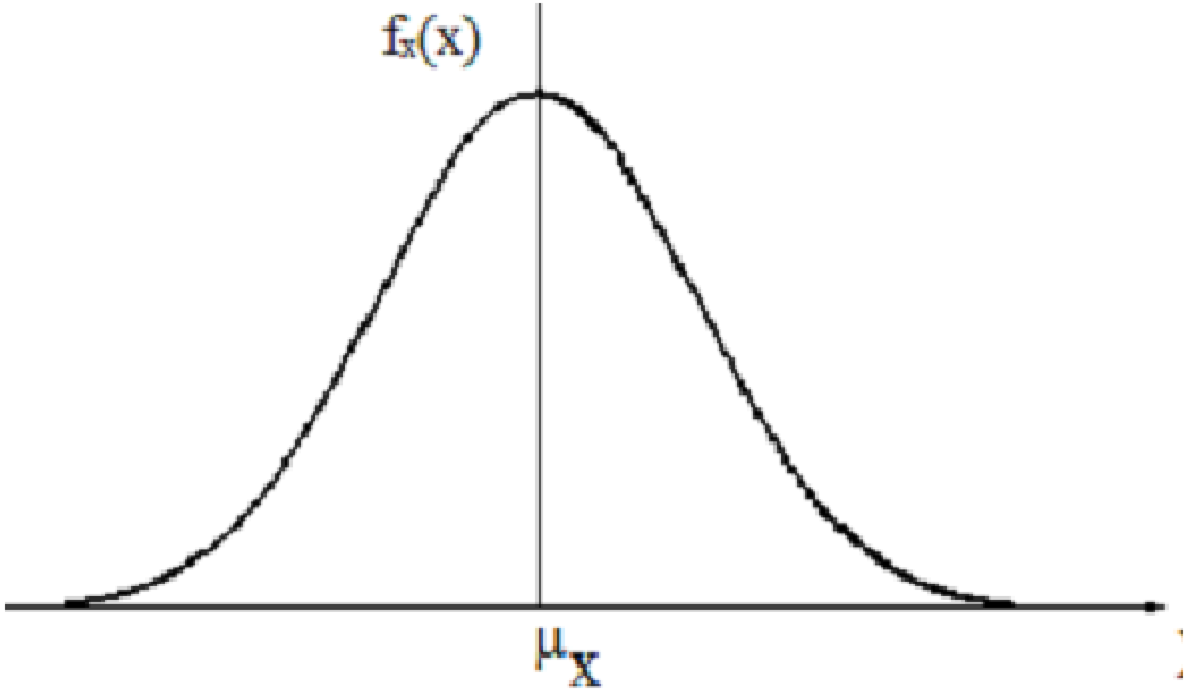
where,

$\mu_X = E(x)$ is the mean value of X

$\sigma_x^2 = E\{(X-\mu_x)^2\}$ is the variance of X.

A random process $X(t)$ is said to Gaussian if the random variables X_1, X_2,., X_n (obtained by observing the process at times $t_1, t_2, \ldots, t_n$) have a jointly Gaussian probability density function for all possible values of n and all times $t_1, t_2, \ldots, t_n$.

Two Virtues of the Gaussian Process



First : The process has many properties that make analytic results possible (easy to handle mathematically).

Second: The random process produced by physical phenomena is often such that a Gaussian model is appropriate . The use of a Gaussian model to describe the physical phenomena is usually confirmed by experiments.

The Central Limit Theorem

Let $X_1, X_2, .., X_n$ be a set of independent and identically distributed (iid) random variables such that $E(X_i) = \mu_x$ and Var $(X_i) = \sigma_x^2$. Define the random variable

$$U = \frac{\sum_1^n X_i}{n}$$

The probability distribution of U approaches a Gaussian distribution with mean μ_x and variance σ_x^2/n in the limit as n → ∞ .

The theorem provides a justification for using a Gaussian process as a model for a large number of physical phenomena in which the observed random variable at a particular instant of time, is a result of a large number of individual events.

Properties of the Gaussian Process

1- If a Gaussian process X(t) is applied to a stable linear filter, then the random process Y(t) at the output of the filter is also Gaussian .

To see that we consider the convolution integral relating Y(t) to X(t)

$$Y(t) = \int_{-\infty}^{\infty} X(\lambda)h\,(t - \lambda)d\lambda$$

Which comes from the approximation

$$Y(t) = \sum X(\lambda_i)h\,(t - \lambda_i)$$

Note that Y(t) is a linear combination of Gaussian random variables, and so Y(t) is Gaussian for any value of t (any linear operation on X(t) produces another Gaussian process).

2- Consider the set of random variables $X(t_1)$, $X(t_2)$,…., $X(t_n)$, obtained by observing a random process X(t) at times t_1 , t_2 , … , t_n . If the process X(t) is Gaussian , then this set of random variables is jointly Gaussian for any n. The joint pdf is completely determined by specifying

the mean vector

$$\mu = [\ \mu_1 , \mu_2 , \ldots \mu_n]^T$$

and the covariance matrix

$$\Sigma = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix}, \quad c_{ij} = E\{(X_i - \mu_i)(X_j - \mu_j)\}, \quad i, j = 1, \ldots, n$$

The joint pdf of the n random variables is

$$f(X_1, \ldots, X_n) = \frac{1}{(2\pi)^{n/2}\sqrt{\det \Sigma}} e^{-0.5\,(X-\mu)^T \Sigma^{-1} (X-\mu)}$$

3- If a Gaussian process is stationary in the wide sense, then it is also stationary in the strict sense (this follows from property 2 above).

4- If the random variables $X(t_1), X(t_2), \ldots, X(t_n)$ obtained by sampling a Gaussian process $X(t)$ at times $t_1, t_2, \ldots, t_n$ are uncorrelated, that is

$$E\{(X_i - \mu_i)(X_j - \mu_j)\} = 0; \qquad i \neq j$$

then these random variables are statistically independent. Here the covariance matrix is diagonal.

$$\Sigma = \begin{bmatrix} \sigma_1^2 & 0 \ldots & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & \cdots & \sigma_n^2 \end{bmatrix}$$

The joint pdf becomes a product of the marginal pdf's.

$$f_x(x) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-(x_i-\mu_i)^2/2\sigma_i^2}$$

A diagonal covariance matrix is a necessary and sufficient condition for statistical independence .

Noise in Communication Systems

The term noise is used to designate unwanted signals that tend to disturb the transmission and processing of signals in communication system and over which we have no control .

The noise may be external (man-made noise, galactic noise) or internal (arising from spontaneous fluctuation of current or voltage in electronic devices), example of which are shot noise and thermal noise .

Shot Noise

The noise arises in electronic devices such as diodes, transistors and photo-detector circuits, due to the discrete nature of current flow in these devices. Remember that current is a result of the flow of electrons, which have a discrete nature. The Poisson distribution is often used to model this type of noise. Using this model the number of electrons emitted in an interval of length T is a random variable with the pdf

$$P(X=x) = e^{-\lambda T} \frac{(\lambda T)^x}{x!} \quad , x = 0, 1, 2, \ldots$$

λ: average number of electrons emitted /unit time (rate of emission).

Thermal Noise :-

Voltages and currents that exist in a network due to the random motion of electrons in conductors is referred to as thermal noise (Johnson's noise).

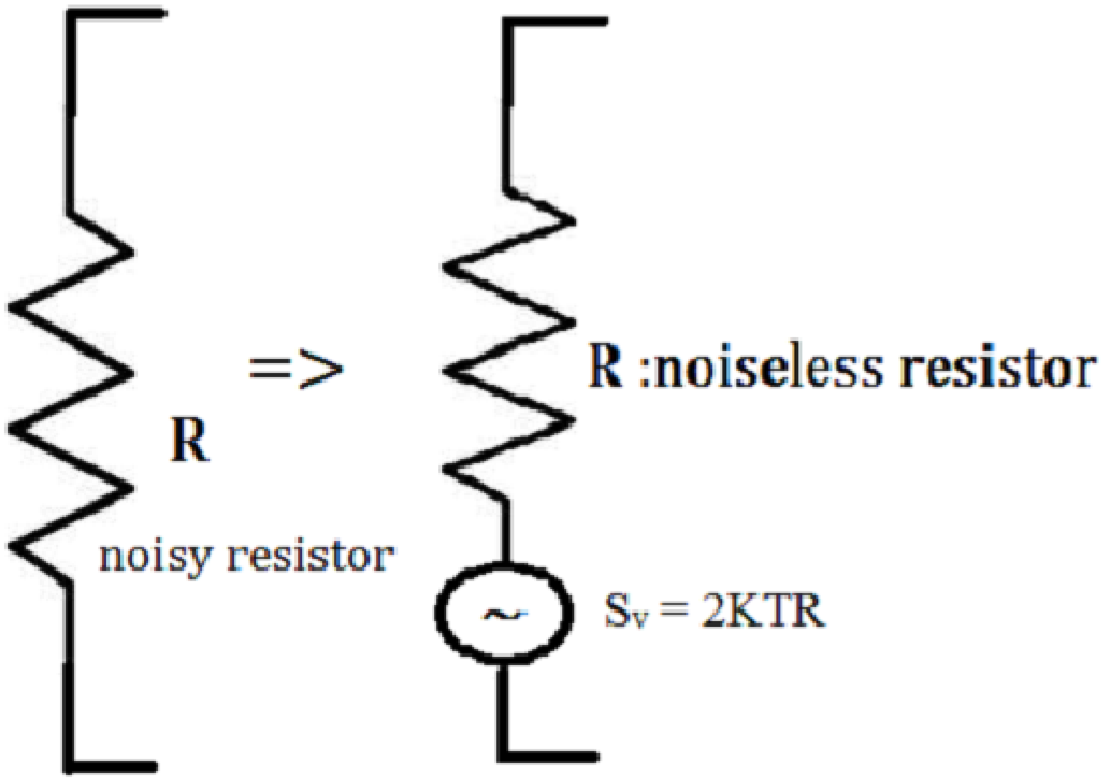
Quantum mechanics shows that the power spectral density of the thermal noise associated with a resistor with a resistance R is given by

$$S_v(f) = \frac{2Rh|f|}{e^{\frac{h|f|}{kT}} - 1} \qquad V^2/Hz$$

K: Boltsman constant ($1.38 * 10^{-23}$ J/degree)

h : Planck constant ($6.62 * 10^{-34}$ Joules-sec)

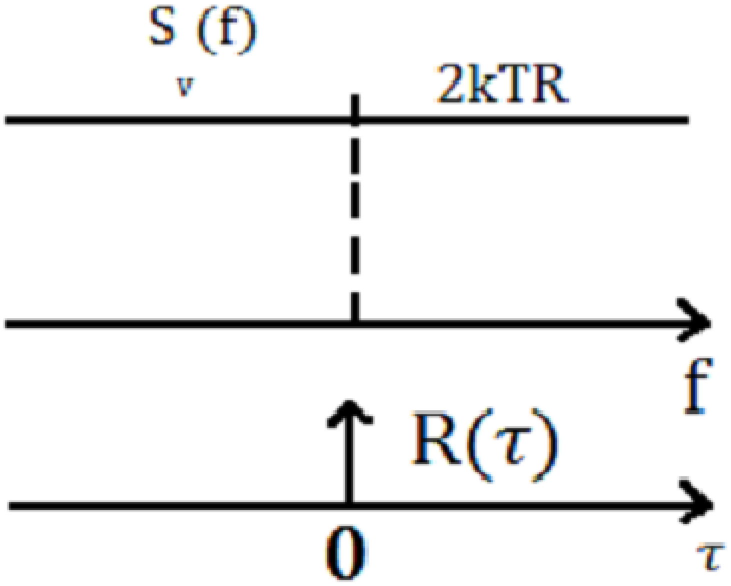
T: degree in Kelvin



For frequencies up to 10^{12} =1000 GHz, this power spectral density is almost constant having the value

$$S_v(f) = 2kTR \quad V^2/Hz .$$

The thermal noise voltage in a zero –mean Gaussian random process.



$E\{V(t)\}=0;$ zero mean.

$S_v(f) = 2kTR;$ constant power spectral density.

The autocorrelation function corresponding to this constant power spectral density is:

$$R_v(\tau) = 2kTR\,\delta(\tau);$$

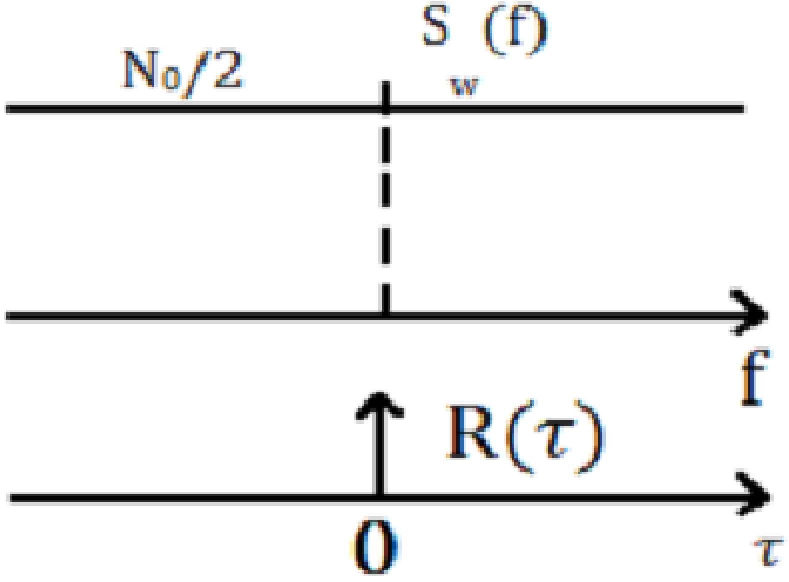
This result shows that two random variables V_{t1} , V_{t2} taken at times t_1 and t_2 are statistically independent for any value of $\tau = t_2 - t_1$ $\tau > 0$.

White Noise

White noise is one whose power spectral density is constant over all frequencies. The power spectral density and autocorrelation function for this type of noise are:

$$S_w(f) = N_0/2$$

$$R_w(\tau) = N_0/2\ \delta(\tau)$$



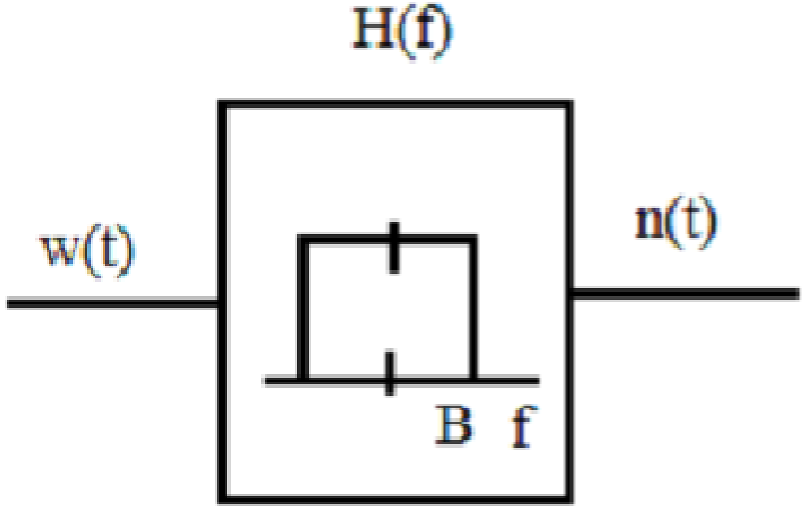
This is the type of noise (model) that we will use in the analysis of communication systems.

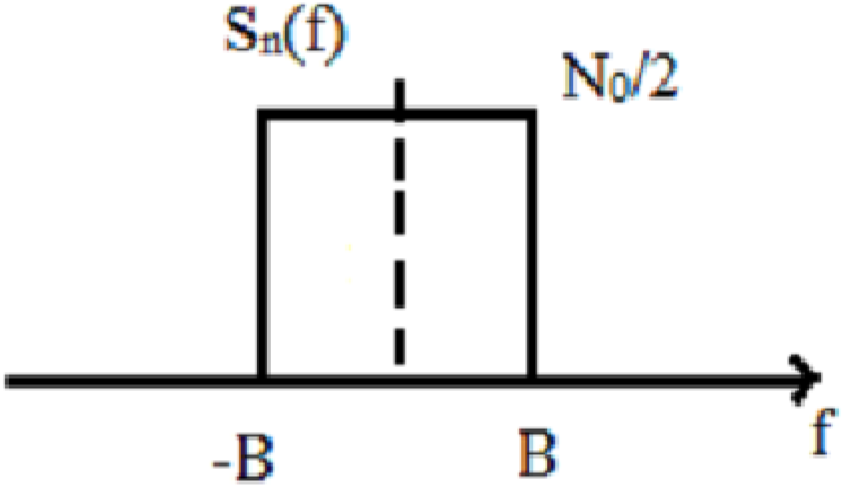
The assumption made is that this noise is additive white Gaussian (AWGN). If s(t) is the transmitted signal and r(t) is the received signal, then

$$r(t) = s(t) + w(t); \quad \text{additive channel noise .}$$

Filtered White Noise

Assume that a white Gaussian noise w(t) of zero mean and psd = $N_0/2$ is applied to an ideal LPF of B.W = B. Let n(t) denote the filtered noise, then





$S_n(f) = |H(f)|^2 \, S_w(f)$

$S_n(f) = \begin{cases} N_0/2 & , -B < f < B \\ 0 & , o.w \end{cases}$; Output psd

$R_n(\tau) = \int_{-B}^{B} \frac{N_0}{2} \; e^{(j2\pi f\tau)} \; df$

$R_n(\tau) = N_0 B \text{ sinc } 2B\tau$; Output autocorrelation function

$E\{n(t)\} = 0$; zero mean noise

$E\{n(t)^2\} = \int_{-B}^{B} S_n(f)df = \frac{N_0}{2}(2B) = N_0 B$; Total output noise power

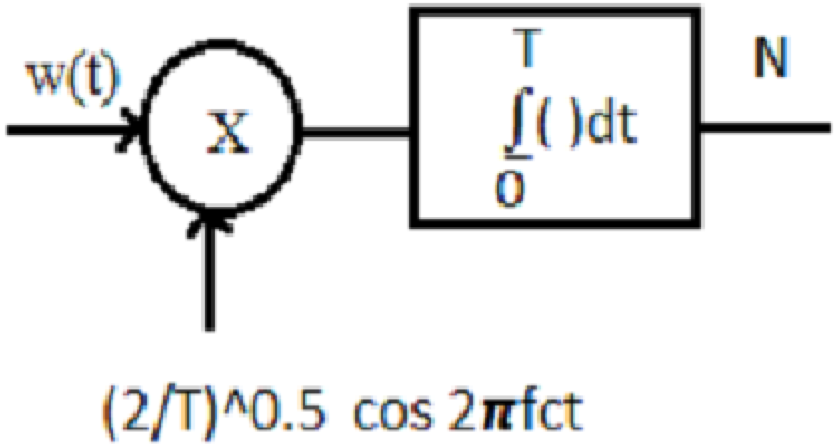
The pdf of the filtered noise at any particular time t is

$$f_n(n) = \frac{1}{\sqrt{2\pi(N_o B)}} \, e^{\frac{-n^2}{2(N_0 B)}} \quad , \; -\infty < n < \infty$$

Remark: note that the pdf is not a function of time indicating that this filtered Gaussian process in stationary in the strict sense.

Correlation of White Noise with a Sinusoidal Signal

Let w(t) be a white Gaussian noise with zero mean. This noise is multiplied by a sinusoidal basis function and integrated over an interval of duration T to produce the scalar N. This scheme is repeatedly used in the coherent demodulation of digital signals. The interval T corresponds to one symbol interval and f_c is the frequency of the carrier. The carrier period and the symbol period are related by $T=nT_c$ where n is an integer. We wish to study the properties of N.



Mathematically, the correlation process is represented as:

$$N = \int_0^T w(t)\sqrt{\frac{2}{T}}\cos 2\pi f_c t \ dt$$

The mean value of N is:

$$E\{N\} = \int_0^T E\{w(t)\}\sqrt{\frac{2}{T}}\cos 2\pi f_c t \ dt = 0.$$

The variance of N is:

$$E\{N^2\} = \frac{2}{T}\iint_0^T E\{w(t_1)w(t_2)\}cos2\pi f_c t_1 \ \cos 2\pi f_c t_2 \ dt_1\, dt_2$$

$$E\{w(t_1)w(t_2)\} = R_w(t_2 - t_1) = N_0/2\ \delta(t_2 - t_1)$$

$$\Rightarrow E\{N^2\} = \frac{2}{T}\frac{No}{2}\int_0^T\left(\int_0^T \delta(t_2 - t_1)\cos 2\pi f_c\, t_1\, dt_1\right)\cos 2\pi f_c t_2 dt_2$$

$$= \frac{2}{T}\frac{No}{2}\int_0^T \cos^2 2\pi f_c t_2\, dt_2; \quad T = nT_c$$

The last step comes by virtue of the sifting property of the delta function. By performing the integration, we get

$$E\{N^2\} = \frac{No}{2} = \sigma^2$$

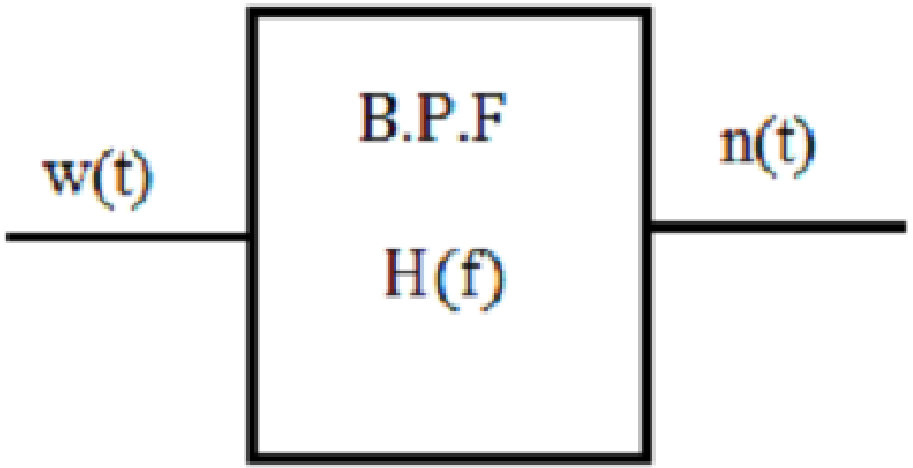
=> N is a zero mean Gaussian r.v with variance $\sigma^2 = \frac{No}{2}$. Its pdf can be written as

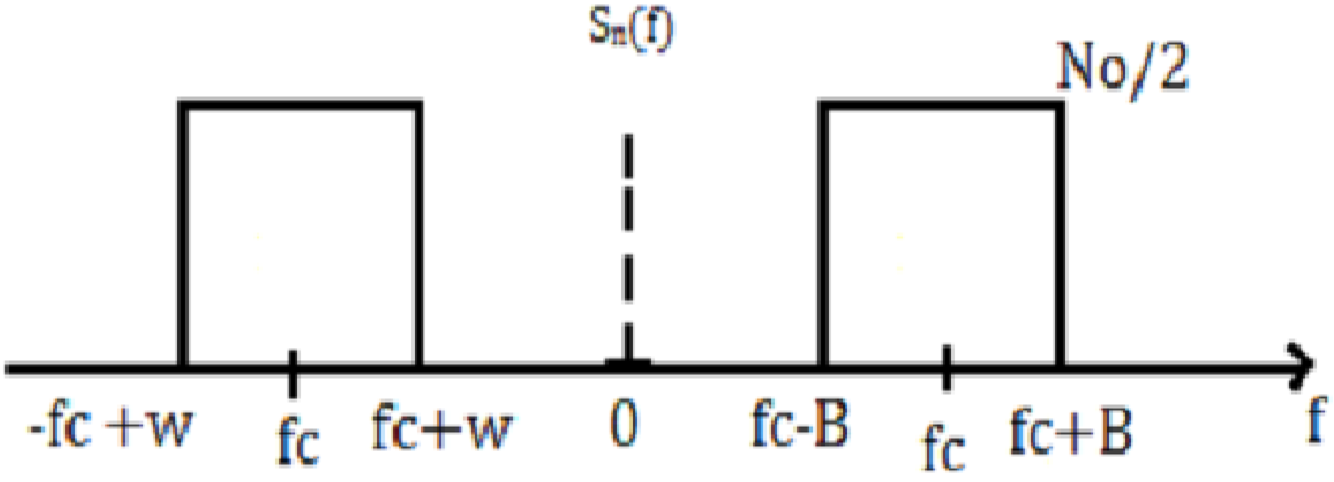
$$f_N(n) = \frac{1}{\sqrt{2\pi(\frac{N_o}{2})}} \ e^{\frac{-n^2}{2.(N_0/2)}}$$

$$f_N(n) = \frac{1}{\sqrt{\pi N_o}} \ e^{\frac{-n^2}{No}}$$

Narrow-band Noise

Now let the white Gaussian noise w(t) of psd $S_w(f) = N_0 / 2$ be applied to an ideal band pass filter with center frequency f_c and bandwidth 2B.





The noise is described as narrow band when $2B \ll f_c$. The analysis is similar to that done for the LPF and the results are summarized as follows:

$S_n(f) = N_0/2$ for $f_c - B < |f| < f_c + B$; Output psd

$E\{n(t)\} = 0$; zero mean noise

$E\{n(t)^2\} = 2 \cdot \frac{N_o}{2} \cdot 2B = N_o(2B) = \sigma^2$; Total output power

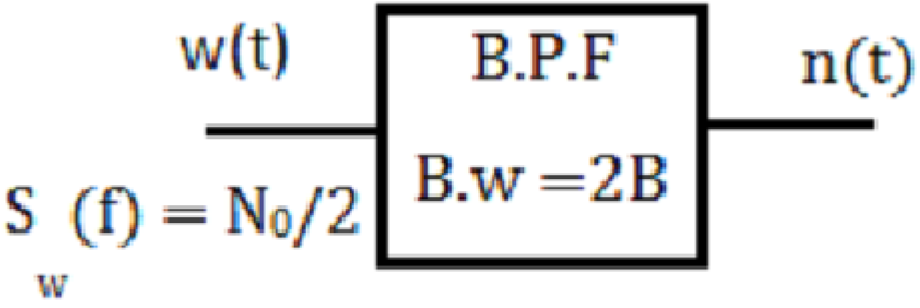
$f_N(n) = \frac{1}{\sqrt{2\pi(2BN_0)}} \, e^{\frac{-n^2}{2(2BN_0)}}$, $-\infty \le n \le \infty$; output noise pdf.

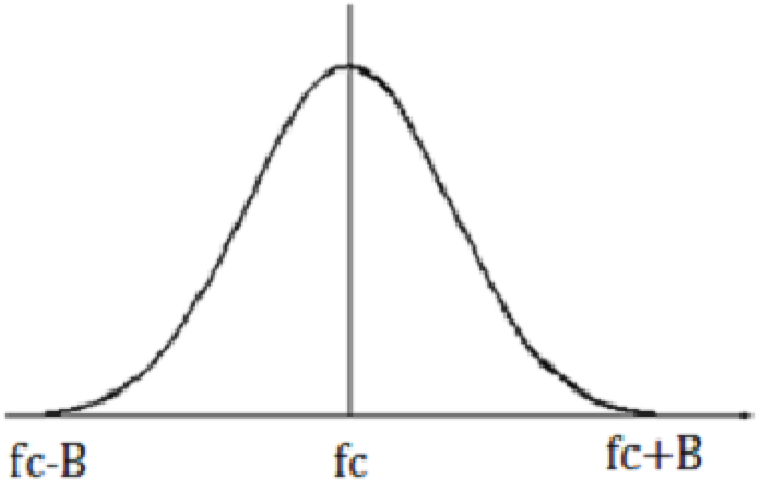
$R_n(0) = E\{n(t)^2\} = \int_{-\infty}^{\infty} S_n(f)df$; Mean square value.

Narrow-band Noise: In-phase and Quadrature Representation

Let $w(t)$ be applied to a bandpass filter of B.w = 2B centered at f_c to produce a narrow band noise $n(t)$.

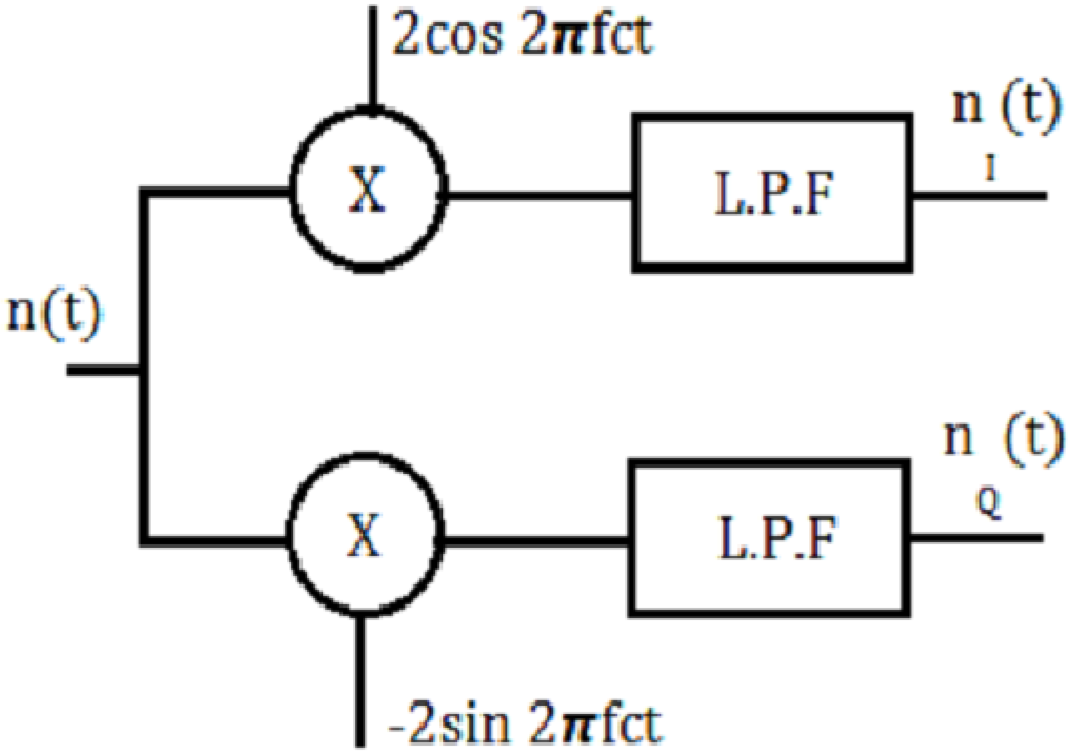
The narrow band noise $n(t)$ can be represented in terms of an in-phase $n_I(t)$ and a quadrature component $n_Q(t)$ as:





$$n(t) = n_I(t) \cos 2 \pi f_c t - n_Q(t) \sin 2 \pi f_c t$$

The in-phase and quadrature components $n_I(t)$ and $n_Q(t)$ can be recovered from n(t) as demonstrated in the block diagram.



$n_I(t) = Lp\{2n(t)\cos 2\pi f_c t\}$; in-phase noise component

$S_{NI}(f) = Lp\{S_n(f-f_c)+S_n(f+f_c)\}$; in-phase noise psd.

$n_Q(t) = -\, Lp\{2n(t)\sin 2\pi f_c t\}$; quadrature noise component

$S_{NI}(f) = S_{NQ}(f)$; both components have the same psd

Finally, $n_I(t)$ and $n_Q(t)$ can be retrieved from n(t) as:

$$n_I(t) = n(t)\cos 2\pi f_c t + \widehat{n(t)} sin 2\pi f_c t$$

$$n_I(t) = \widehat{n(t)}\cos 2\pi f_c t - n(t) sin 2\pi f_c t$$

Properties of the Noise Components

- The in-phase component $n_I(t)$ and the quadrature component $n_Q(t)$ of narrow band noise n(t) have zero mean .
- If the narrow band noise n(t) is Gaussian, then $n_I(t)$ and $n_Q(t)$ are jointly Gaussian .
- If n(t) is wide sense stationary, then $n_I(t)$ and $n_Q(t)$ are jointly wide sense stationary .
- Both $n_I(t)$ and $n_Q(t)$ have the same power spectral density

$$S_{NI}(f) = S_{NQ}(f) = \begin{cases} S_N(f - f_c) + S_N(f + f_c), & -B < f < B \\ 0 & , o.w \end{cases}$$

- $n_I(t)$, $n_Q(t)$ and n(t) have the same variance

$$E\{n(t)^2\} = E\{n_I(t)^2\} = E\{n_Q(t)^2\} = \sigma^2$$

- The cross-spectral densities of $n_I(t)$ and $n_Q(t)$ are imaginary

$$S_{NINQ}(f) = - S_{NQNI}(f) = \begin{cases} j[S_N(f + f_c) - S_N(f - f_c)], & -B < f < B \\ 0 & , o.w \end{cases}$$

- If n(t) is Gaussian with zero mean and a power spectral density $S_n(f)$ that is symmetric about f_c, then $n_I(t)$ and $n_Q(t)$ are statistically independent. The joint pdf of $n_I(t)$ and $n_Q(t)$ is the product of the marginal pdf's

- The cross-spectral densities of $n_I(t)$ and $n_Q(t)$ are imaginary

$$S_{NINQ}(f) = -S_{NQNI}(f) = \begin{cases} j[S_N(f+f_c) - S_N(f-f_c)], & -B < f < B \\ 0 & , o.w \end{cases}$$

- If n(t) is Gaussian with zero mean and a power spectral density $S_n(f)$ that is symmetric about f_c, then $n_I(t)$ and $n_Q(t)$ are statistically independent. The joint pdf of $n_I(t)$ and $n_Q(t)$ is the product of the marginal pdf's

$$f(n_I, n_Q) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-n_I^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-n_Q^2}{2\sigma^2}}$$

(i.e, when the cross spectral density = 0 $\forall$ f, then n_I and n_Q are independent)

Polar Representation of Narrow-band Noise

Let n(t) be a narrow band zero-mean, white Gaussian noise with a symmetric psd about some center frequency f_c .

$$n(t) = n_I(t) \cos 2\pi f_c t - n_Q(t) \sin 2\pi f_c t.$$

Because $S_n(f)$ is symmetric, it follows that $n_I(t)$ and $n_Q(t)$, observed at a fixed time t, are independent Gaussian r.v with zero mean and variance σ^2 .

n(t) can also be represented as

$$n(t) = R(t) \cos (2\pi f_c t + \emptyset(t))$$

where the envelope R(t) and the phase Ø(t) are given as:

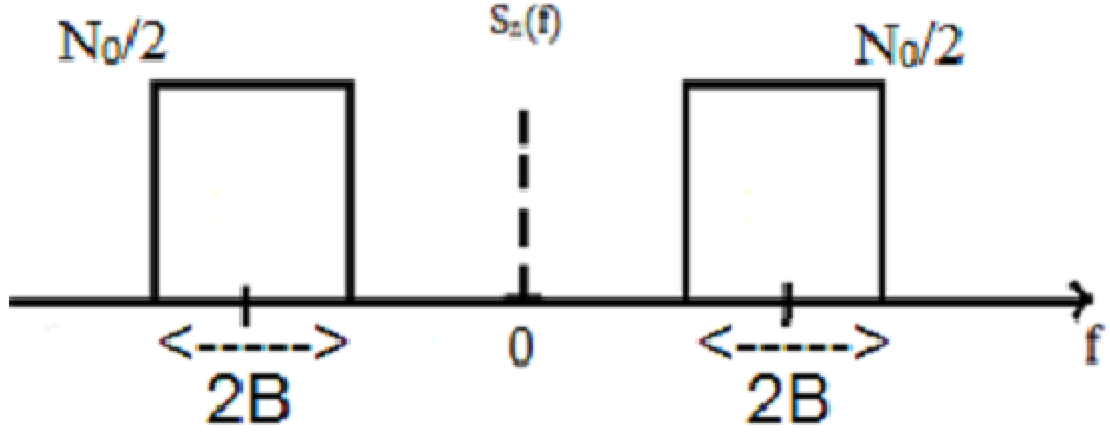
$$R(t) = [n_I(t)^2 + n_Q(t)^2]^{1/2}$$

$$\emptyset(t) = \tan^{-1} (n_Q(t) / n_I(t))$$

It can be shown (Go back to your ENEE 331 lecture notes and go over the proof) that R and Ø are independent random variables with pdf 's

$$f_{\Phi}(\Phi) = \begin{cases} \frac{1}{2\pi} & ,0 \le \emptyset \le 2\pi \\ 0 & ,\text{o.w} \end{cases} ; \qquad \text{(Uniform pdf)}$$

$$f_R(r) = \begin{cases} \frac{r}{\sigma^2} \exp[-(r^2/2\sigma^2)] & ,\ r \ge 0 \\ 0 & ,\text{o.w} \end{cases} ; \qquad \text{(Rayleigh distribution)}$$



If $S_n(f)$ has the psd shown then , $\sigma^2 = 2\,(N_0/2)(2B) = 2N_0\,B$ and the pdf of R is as given above .

NOISE IN ANALOG COMMUNICATIONS

Lessons to learn about Noise:

Lesson 1: Minimizing the effects of noise is a prime concern in analog communications, and consequently the ratio of signal power to noise power is an important metric for assessing analog communication quality.

Lesson 2: Amplitude modulation may be detected either coherently requiring the use of a synchronized oscillator or non-coherently by means of a simple envelope detector. However, there is a performance penalty to be paid for non-coherent detection.

Lesson 3: Frequency modulation is nonlinear and the output noise spectrum is parabolic when the input noise spectrum is flat. Frequency modulation has the advantage that it allows us to trade bandwidth for improved performance.

Lesson 4: Pre- and de-emphasis filtering is a method of reducing the output noise of an FM demodulator without distorting the signal. This technique may be used to significantly improve the performance of frequency modulation systems.

Properties of Noise

- *The mean of the random process*. For noise, the mean value corresponds to the *dc offset*. In most communication systems, dc offsets are removed by design since they require power and carry little information. Consequently, both noise and signal are generally assumed to have zero mean.

- *The autocorrelation of the random process.*

$$R_w(\tau) = \frac{N_0}{2}\delta(\tau)$$

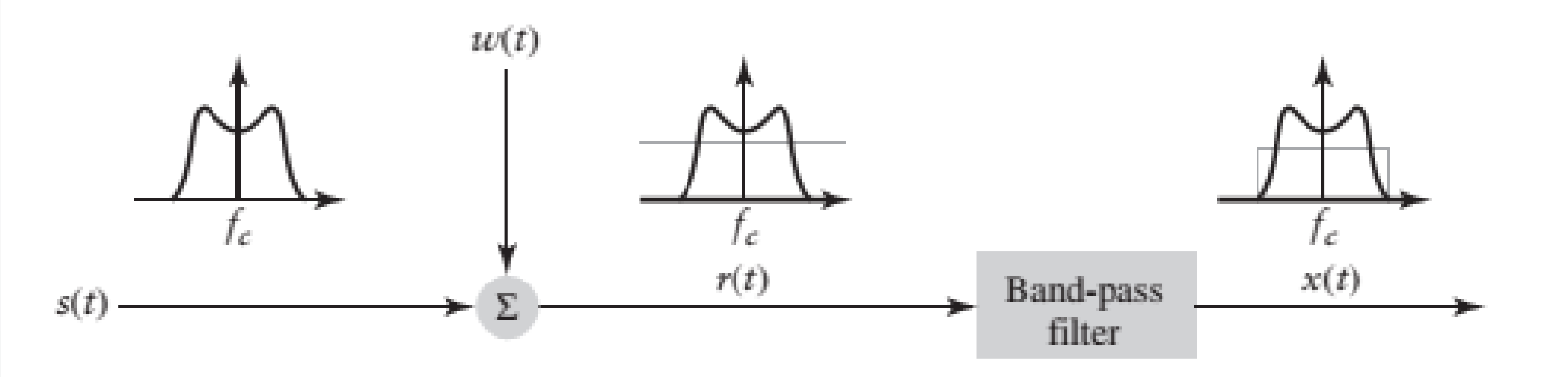
- *The spectrum of the random process*. For additive white Gaussian noise the spectrum is flat and defined as

$$S_w(f) = \frac{N_0}{2}$$

- The noise power at the output of a filter of equivalent-noise bandwidth is:

$$N = N_0 B_T$$

Block diagram of signal plus noise before and after filtering showing spectra at each point.



Noise in Communication Systems

-Given that communication deals with random signals,
how do we quantify the performance of a particular communication system?
we will focus on *signal-to-noise ratio* (SNR) as the measure of quality for analog systems;
-For zero-mean processes, a simple measure of the signal quality is the ratio of the variances of the desired and undesired signals.

$$\text{SNR} = \frac{\mathbf{E}[s^2(t)]}{\mathbf{E}[n^2(t)]}$$

-The signal-to-noise ratio is often considered to be a ratio of the average signal power to the average noise power

Example: Sinusoidal Signal-to-Noise Ratio

Consider the case where the transmitted signal is $s(t) = A_c \cos(2\pi f_c t + \theta)$

where the phase is unknown at the receiver. The signal is received in the presence of additive AWGN noise, determine SNR?

$$
\begin{aligned}
\mathbf{E}[s^2(t)] &= \frac{1}{T}\int_0^T (A_c \cos(2\pi f_c t + \theta))^2 \, dt \\
&= \frac{A_c^2}{2T}\int_0^T (1 + \cos(4\pi f_c t + 2\theta)) \, dt \\
&= \frac{A_c^2}{2T}\left[t + \frac{\sin(4\pi f_c t + 2\theta)}{4\pi f_c} \right]_0^T \\
&= \frac{A_c^2}{2}
\end{aligned}
$$

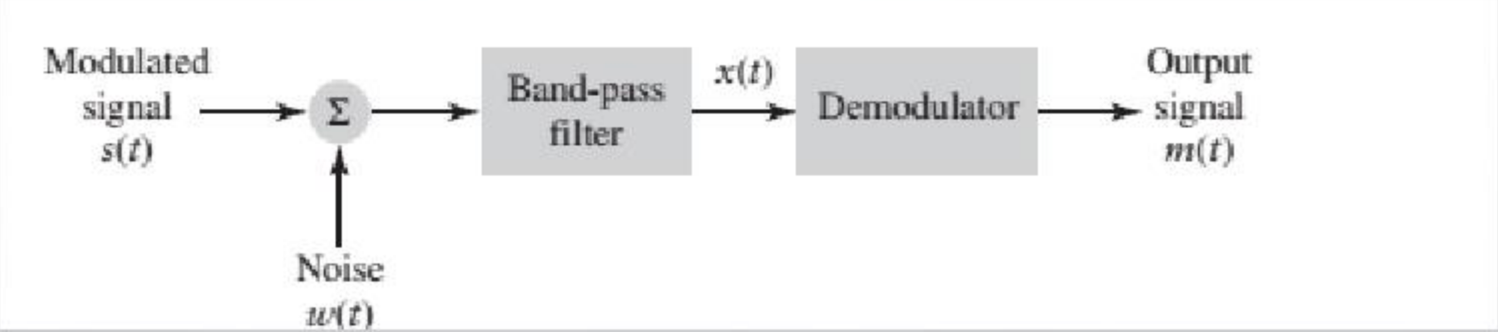
$$
\begin{aligned}
\mathbf{E}[n^2(t)] &= N \\
&= N_0 B_T
\end{aligned}
$$

$$\mathrm{SNR} = \frac{A_c^2}{2N_0 B_T}$$

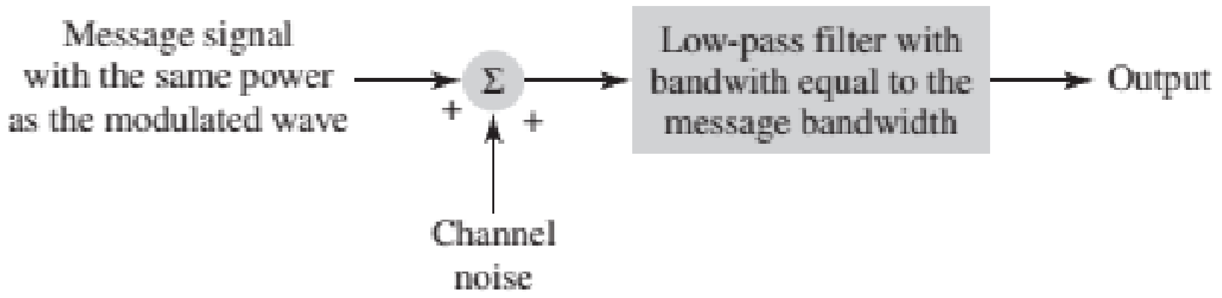
The signal-to-noise ratio is measured at the receiver, but there are several
points in the receiver where the measurement may be carried out. In fact, measurements at particul
points in the receiver have their own particular importance and value.
For instance:

pre-detection signal-to-noise ratio: If the signal-to-noise ratio is measured at the front-end of the receiv
then it is usually a measure of the quality of the transmission link and the receiver front-end.

post-detection signal-to-noise ratio: If the signal-to-noise ratio is measured at the output of the receiver
a measure of
the quality of the recovered information-bearing signal whether it be audio, video, or
otherwise.



Reference transmission model for analog communications

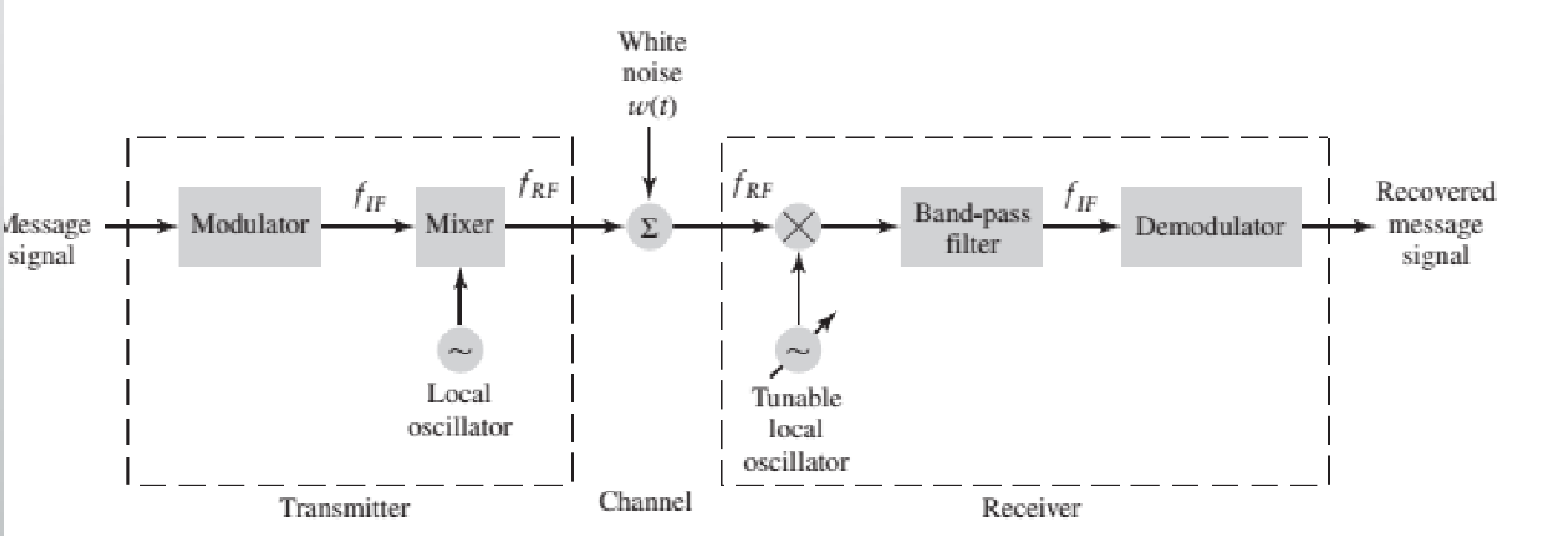


$$\text{Figure of merit} = \frac{\text{post-detection SNR}}{\text{reference SNR}}$$

$$\text{SNR}_{\text{ref}} = \frac{\text{average power of the modulated message signal}}{\text{average power of noise measured in the message bandwidth}}$$

- The pre-detection SNR is measured before the signal is demodulated.
- The post-detection SNR is measured after the signal is demodulated.
- The reference SNR is defined on the basis of a baseband transmission model.
- The figure of merit is a dimensionless metric for comparing different analog modulation–demodulation schemes and is defined as the ratio of the post-detection and reference SNRs.

Bandpass Receiver Structure:
Block diagram of band-pass transmission showing a superheterodyne receiver



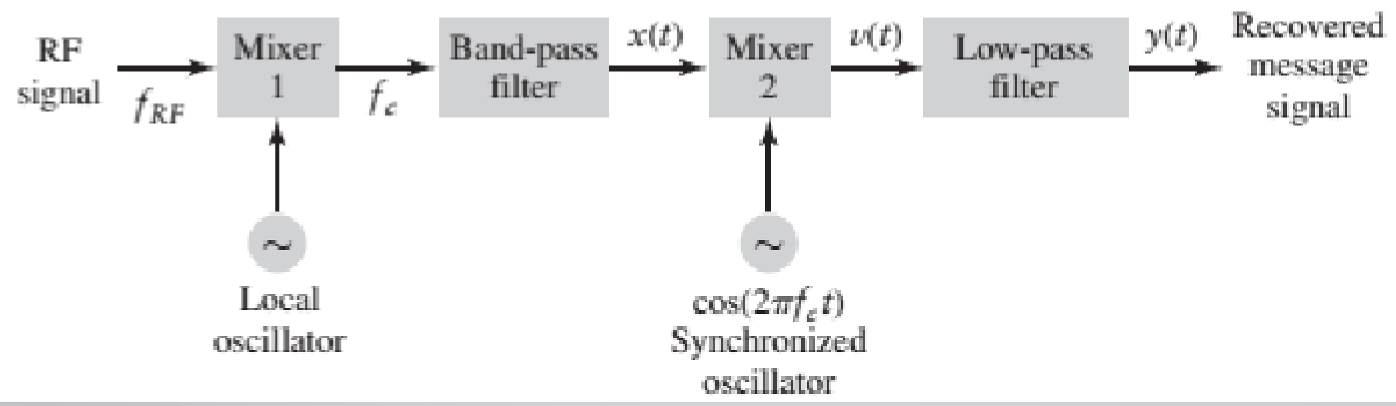
Common examples are AM radio transmissions, where the RF channels' frequencies lie in the range between 510 and 1600 kHz, and a common IF is 455 kHz; another example is FM radio, where the RF channels are in the range from 88 to 108 MHz and the
IF is typically 10.7 MHz

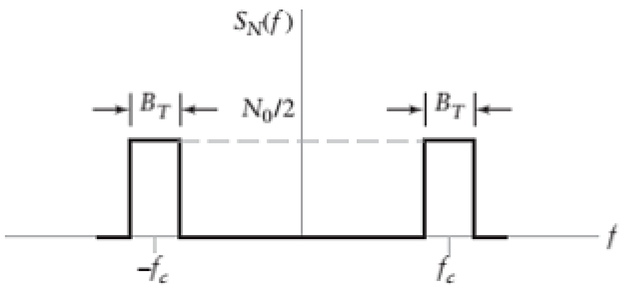
Noise in Linear Receivers Using Coherent Detection

Double sideband suppressed-carrier (DSB-SC) modulation, the modulated signal is represented as

$$s(t) = A_c m(t) \cos(2\pi f_c t + \theta)$$

$$x(t) = s(t) + n(t)$$





Power spectral density of band-pass noise

Pre-detection SNR for DSBSC

$$E[s^2(t)] = E[(A_c \cos(2\pi f_c t + \theta))^2]E[m^2(t)]$$

If we let

$$P = E[m^2(t)]$$

$$E[s^2(t)] = \frac{A_c^2 P}{2}$$

$$\text{SNR}_{\text{pre}}^{\text{DSB}} = \frac{A_c^2 P}{2N_0 B_T}$$

POST-DETECTION SNR for DSBSC

$$x(t) = s(t) + n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)$$

$$\begin{aligned} v(t) &= x(t)\cos(2\pi f_c t) \\ &= \tfrac{1}{2}(A_c m(t) + n_I(t)) \\ &+ \tfrac{1}{2}(A_c m(t) + n_I(t))\cos(4\pi f_c t) - \tfrac{1}{2}n_Q(t)\sin(4\pi f_c t) \end{aligned}$$

$$y(t) = \tfrac{1}{2}(A_c m(t) + n_I(t))$$

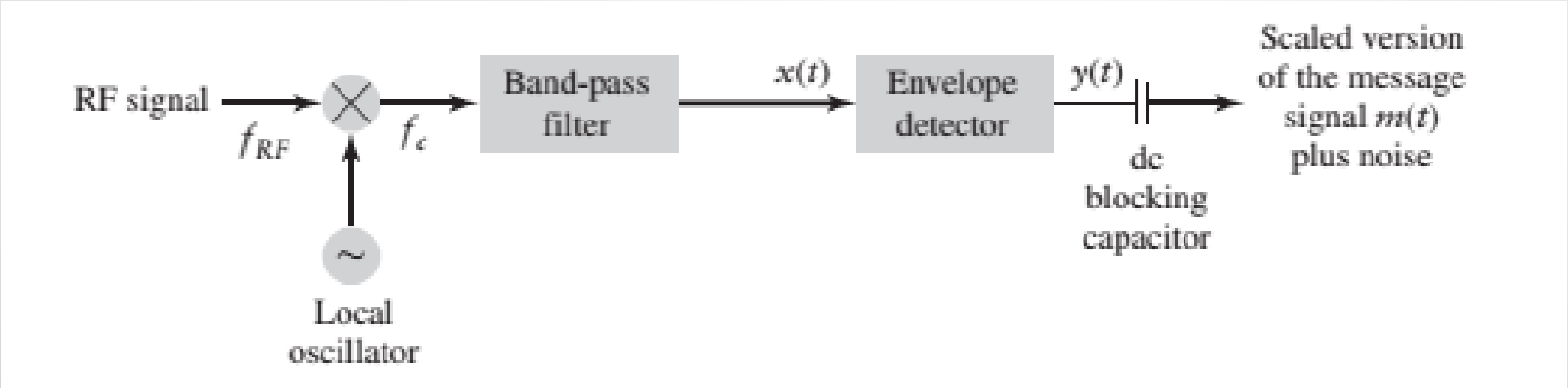
$$\begin{aligned} \mathrm{E}[n_I^2(t)] &= \int_{-W}^{W} N_0\, df \\ &= 2N_0 W \end{aligned}$$

$$\mathrm{SNR}_{\mathrm{ref}} = A_c^2 P/(2N_0 W)$$

$$\begin{aligned} \mathrm{SNR}_{\mathrm{post}}^{\mathrm{DSB}} &= \frac{\frac{1}{4}(A_c^2)P}{\frac{1}{4}(2N_0 W)} \\ &= \frac{A_c^2 P}{2N_0 W} \end{aligned}$$

$$\text{Figure of merit} = \frac{\mathrm{SNR}_{\mathrm{post}}^{\mathrm{DSB}}}{\mathrm{SNR}_{\mathrm{ref}}} = 1$$

Noise In AM Receivers Using Envelope Detection



$$s(t) = A_c(1 + k_a m(t)) \cos(2\pi f_c t)$$

PRE-DETECTION SNR

$$\mathbf{E}[(1 + k_a m(t))^2] = \mathbf{E}[1 + 2k_a m(t) + k_a^2 m^2(t)]$$
$$= 1 + 2k_a \mathbf{E}[m(t)] + k_a^2 \mathbf{E}[m^2(t)]$$
$$= 1 + k_a^2 P$$

$$\mathrm{SNR}_{\mathrm{pre}}^{\mathrm{AM}} = \frac{A_c^2(1 + k_a^2 P)}{2N_0 B_T}$$

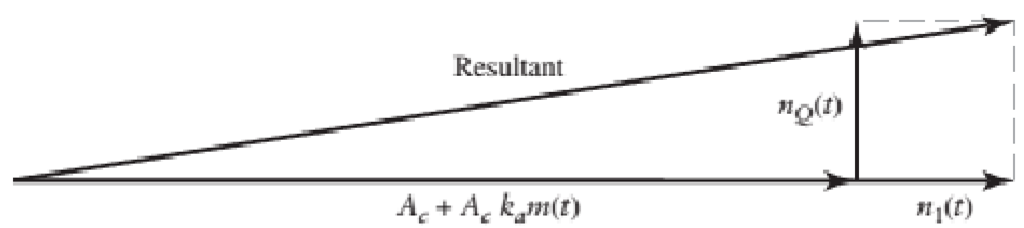
Post-detection SNR

$$x(t) = s(t) + n(t)$$
$$= [A_c + A_c k_a m(t) + n_I(t)] \cos(2\pi f_c t) - n_Q(t) \sin(2\pi f_c t)$$

$$y(t) = \text{envelope of } x(t)$$
$$= \{[A_c(1 + k_a m(t)) + n_I(t)]^2 + n_Q^2(t)\}^{1/2}$$

$$y(t) \approx A_c + A_c k_a m(t) + n_I(t)$$

under high SNR conditions



$$\text{SNR}_{\text{post}}^{\text{AM}} = \frac{A_c^2 k_a^2 P}{2N_0 W}$$

Conditions:
- The SNR is high.
- *ka* is adjusted for 100% modulation or less, so there is no distortion of the signal envelope.

Figure of Merit

SNR is $A_c^2(1 + k_a^2 P)/(2N_0 W)$

$$\text{Figure of merit} = \frac{\text{SNR}_{\text{post}}^{\text{AM}}}{\text{SNR}_{\text{ref}}} = \frac{k_a^2 P}{1 + k_a^2 P}$$

The figure of merit for this system is always less than 0.5. Hence, the noise performance of an envelope-detector receiver is always inferior to a DSB-SC receiver, the reason
is that at least half of the power is wasted transmitting the carrier as a component of the modulated (transmitted) signal

Noise in SSB
Do it…

$$s(t) = \frac{A_c}{2} m(t)\cos(2\pi f_c t) + \frac{A_c}{2} \hat{m}(t)\sin(2\pi f_c t)$$

the reference SNR is $A_c^2 P/(4N_0 W)$

$$\mathrm{SNR}_{\mathrm{pre}}^{\mathrm{SSB}} = \frac{A_c^2 P}{4N_0 W}$$

$$y(t) = \frac{1}{2}\left(\frac{A_c}{2} m(t) + n_I(t)\right)$$

$$\text{Figure of merit} = \frac{\mathrm{SNR}_{\mathrm{post}}^{\mathrm{SSB}}}{\mathrm{SNR}_{\mathrm{ref}}} = 1$$

$$\mathrm{SNR}_{\mathrm{post}}^{\mathrm{SSB}} = \frac{A_c^2 P}{4N_o W}$$

Summery:

Comparing the results for the different amplitude modulation schemes, we find that there are a number of design tradeoffs. Double-sideband suppressed carrier modulation provides the same SNR performance as the baseband reference model but requires synchronization circuitry to perform coherent detection. Non-suppressed-carrier AM simplifies the receiver design significantly as it is implemented with an envelope detector.

However, non-suppressed-carrier AM requires significantly more transmitter power to obtain the same SNR performance as the baseband reference model. Single-sideband modulation achieves the same SNR performance as the baseband reference model but only requires half the transmission bandwidth of the DSC-SC system. On the other hand, SSB requires more transmitter processing. *These observations are our first indication that communication system design involves a tradeoff between power, bandwidth, and processing complexity.*

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$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
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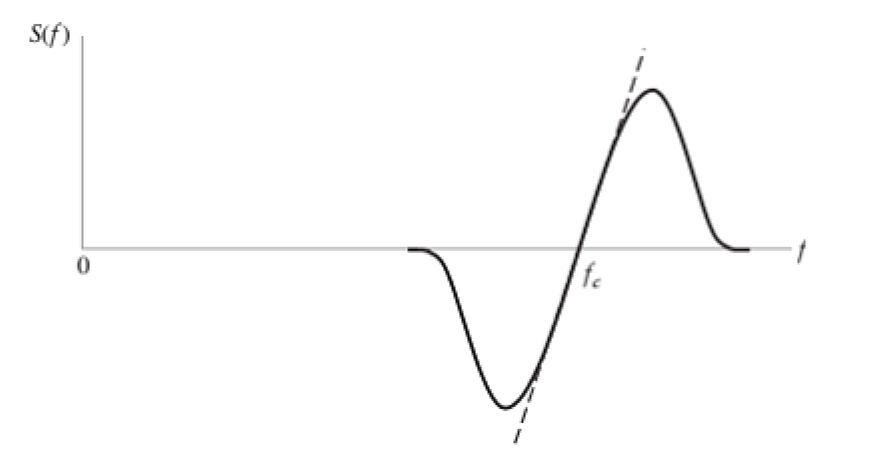
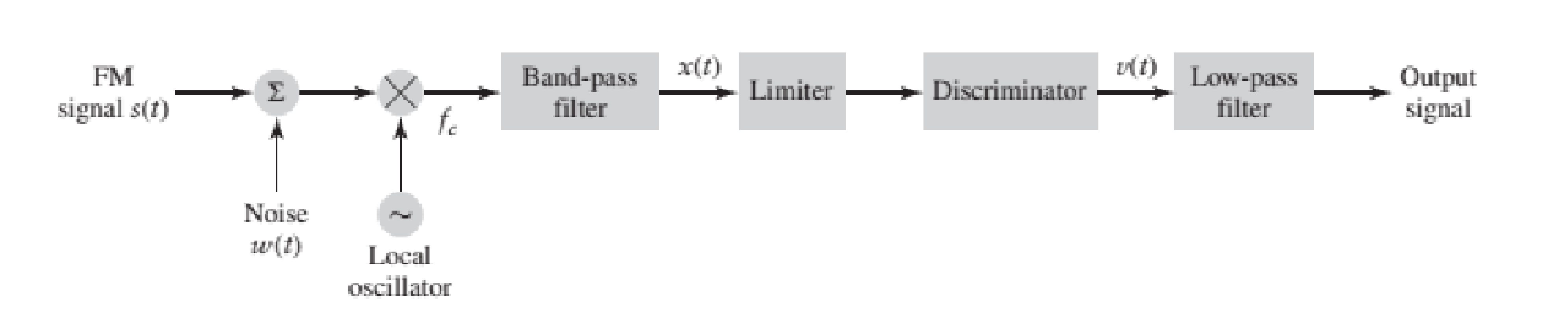
Detection of Frequency Modulation (FM)

$$s(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau)\, d\tau\right]$$

$$\text{SNR}_{\text{pre}}^{\text{FM}} = \frac{A_c^2}{2N_0 B_T}$$





Post Detection SNR

$$x(t) = s(t) + n(t)$$

$$n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)$$

$$n(t) = r(t)\cos[2\pi f_c t + \phi_n(t)]$$

$$r(t) = [n_I^2(t) + n_Q^2(t)]^{1/2}$$

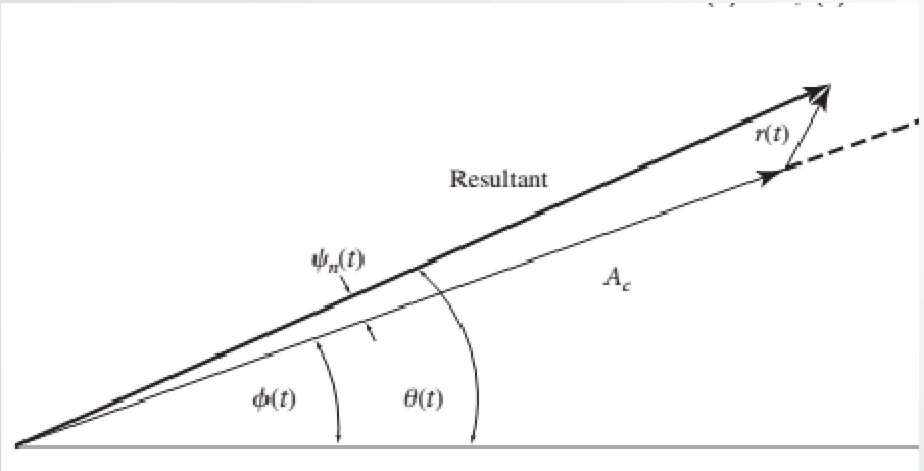
$$\phi_n(t) = \tan^{-1}\left(\frac{n_Q(t)}{n_I(t)}\right)$$

$$\phi(t) = 2\pi k_f \int_0^t m(\tau)\,d\tau$$

$$\begin{aligned} x(t) &= s(t) + n(t) \\ &= A_c\cos[2\pi f_c t + \phi(t)] + r(t)\cos[2\pi f_c t + \phi_n(t)] \end{aligned}$$

$$\theta(t) = \phi(t) + \tan^{-1}\left\{\frac{r(t)\sin(\psi_n(t))}{A_c + r(t)\cos(\psi_n(t))}\right\}$$

$$\psi_n(t) = \phi_n(t) - \phi(t)$$



$$\theta(t) = \phi(t) + \frac{r(t)}{A_c}\sin[\psi_n(t)]$$

$$\tan^{-1}\xi \approx \xi \text{ since } \xi \ll 1$$

$$\theta(t) = \phi(t) + \frac{n_Q(t)}{A_c}$$

$$n_Q(t) = r(t)\sin[\phi_n(t)]$$

$$\theta(t) \approx 2\pi k_f \int_0^t m(\tau)\, d\tau + \frac{n_Q(t)}{A_c}$$

$$v(t) = \frac{1}{2\pi}\frac{d\theta(t)}{dt} = k_f m(t) + n_d(t)$$

$$n_d(t) = \frac{1}{2\pi A_c}\frac{dn_Q(t)}{dt}$$

$$G(f) = \frac{j2\pi f}{2\pi A_c} = \frac{jf}{A_c}$$

$$S_{N_d}(f) = |G(f)|^2 S_{N_Q}(f) = \frac{f^2}{A_c^2} S_{N_Q}(f)$$

$$S_{N_d}(f) = \begin{cases} \frac{N_0 f^2}{A_c^2}, & |f| < \frac{B_T}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$S_{N_o}(f) = \begin{cases} \frac{N_0 f^2}{A_c^2}, & |f| < W \\ 0, & \text{otherwise} \end{cases}$$

$$\text{Average post-detection noise power} = \frac{N_0}{A_c^2}\int_{-W}^{W} f^2\, df = \frac{2N_0W^3}{3A_c^2}$$

$$\text{SNR}_{\text{post}}^{\text{FM}} = \frac{3A_c^2 k_f^2 P}{2N_0 W^3}$$

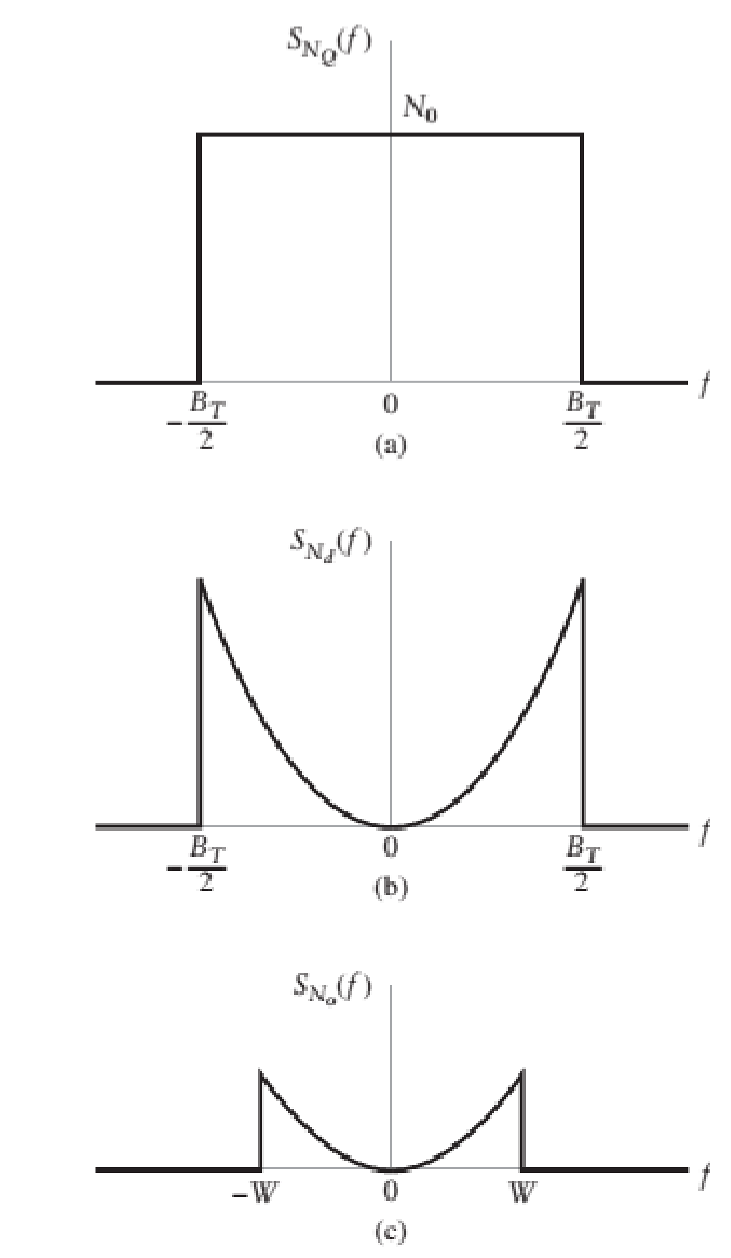


Figure of Merit

With FM modulation, the modulated signal power is simply $A_c^2/2$, hence the reference SNR is $A_c^2/(2N_0W)$. Consequently, the figure of merit for this system is given by

$$\text{Figure of merit} = \frac{\text{SNR}_{\text{post}}^{\text{FM}}}{\text{SNR}_{\text{ref}}} = \frac{\dfrac{3A_c^2k_f^2P}{2N_0W^3}}{\dfrac{A_c^2}{2N_0W}}$$

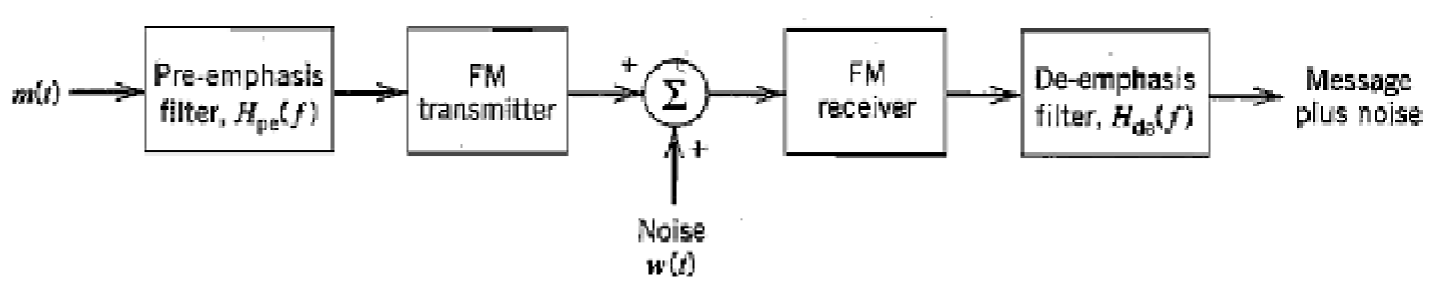
$$= 3\left(\frac{k_f^2P}{W^2}\right)$$

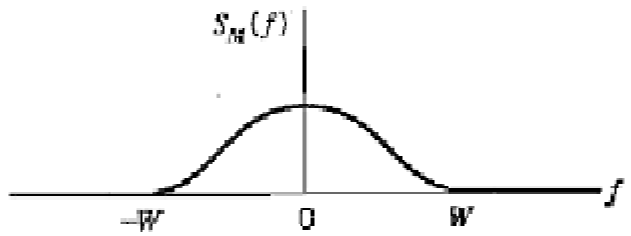
$$= 3D^2$$

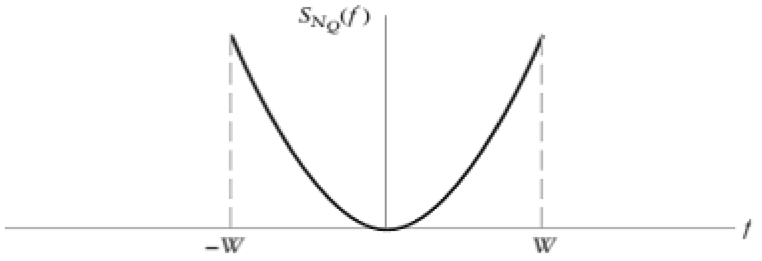
where, we have introduced the definition D as the deviation ratio . Recall from generalized Carson rule yields the transmission bandwidth $B_T = 2(k_fP^{1/2} + W) \approx 2k_fP^{1/2}$ for an FM signal. So, substituting $B_T/2$ for $k_fP^{1/2}$ for in the definition of *D*, the figure of merit for an FM system is approximately given by

$$\text{Figure of merit} \approx \frac{3}{4}\left(\frac{B_T}{W}\right)^2$$

FM Pre-emphasis and De-emphasis







$$H_{pre}(f) = \frac{1}{H_{de}(f)} \qquad |f| < W$$

$$H_{de}(f) = \frac{1}{1 + j\dfrac{f}{f_{3dB}}}$$

$$H_{pre}(f) = 1 + j\frac{f}{f_{3dB}}$$

$$|H_{de}(f)|^2 S_{N_d}(f) = \begin{cases} \dfrac{N_0 f^2}{A_c^2}|H_{de}(f)|^2, & |f| \le \dfrac{B_T}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$I = \frac{\text{average output noise power without pre-emphasis and de-emphasis}}{\text{average output noise power with pre-emphasis and de-emphasis}}$$

$$\begin{pmatrix}\text{Average output noise} \\ \text{power with de-emphasis}\end{pmatrix} = \frac{N_0}{A_c^2}\int_{-W}^{W} f^2 |H_{de}(f)|^2 \, df$$

$$I = \frac{2W^3}{3\int_{-W}^{W} f^2 |H_{de}(f)|^2 \, df}$$

In commercial FM broadcasting, we typically have $f_{3dB} = 2.1$ kHz, and we may reasonably assume $W = 15$ kHz. This set of values yields $I = 22$, which corresponds to an improvement of 13 dB in the post-detection signal-to-noise ratio of the receiver. This example illustrates that a significant improvement in the noise performance of an FM system may be achieved by using pre-emphasis and de-emphasis filters made up of simple RC circuits.