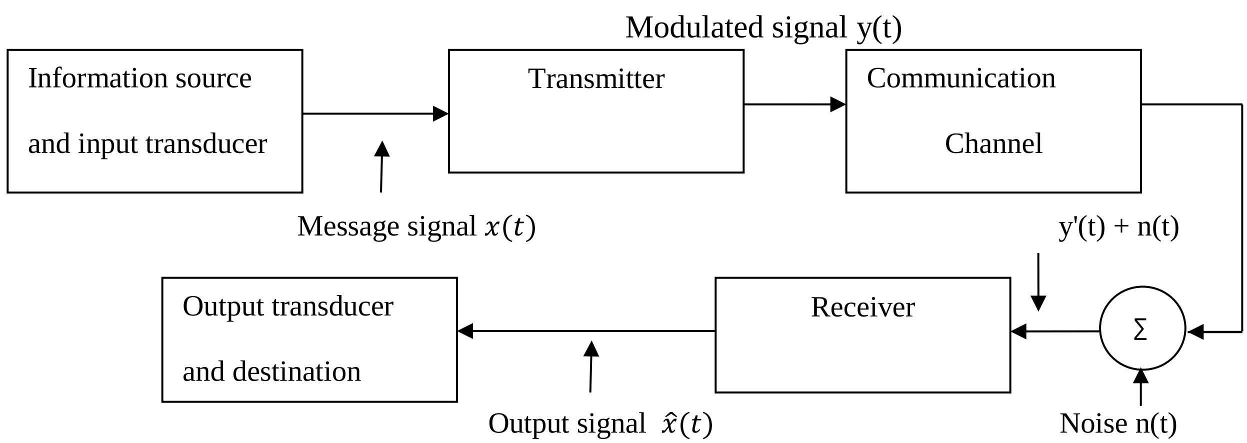
# Model of a Communication System

- *Communication* is defined as "exchange of information".
- *Telecommunication* refers to communication over a distance greater than would normally be possible without artificial aids.
- Telephony is an example of point-to-point communication and normally involves a two – way flow of information.
- Broadcast radio and television : Information is transmitted from one location but is received at many locations using different receivers (point to multi-point communication)
- Model of a communication system :




- The purpose of a communication system is to transmit information – bearing signals from a source located at one point to a user located at another end.
- The input transducer is used to convert the physical message generated by the source into a time-varying electrical signal called the *message signal*.
- The original message is recreated at the destination using an output transducer.
- The *transmitter* modifies the message signal into a form suitable for transmission over the channel. *Here modulation takes place*.
- The *channel* is the medium over which signal is transmitted, (like free space, an optical fiber, transmission lines, twisted pair of wires…). Here signal is distorted due to

   A. Nonlinearities and/or imperfections in the frequency response of the channel.
   B. Noise and interference are added to the signal during the course of transmission.

- The purpose of the *receiver* is to recreate the original signal x(t) from the degraded version x(t) + n(t) of the transmitted signal after propagating through channel .
  *Here, demodulation takes place*.

## Classification of Signals

**<u>Definition</u>:** A signal may be defined as a single valued function of time that conveys information.

Depending on the feature of interest, we may distinguish four different classes of signals:

**<u>1. Periodic Signals, Non-periodic Signals:</u>**

A *periodic signal* g(t) is a function of time that satisfies the condition $g(t) = g(t+T_0), \forall t$.

The smallest value of $T_0$ that satisfies this condition is called the period of g(t).

**Example: A Periodic Signal**

The saw-tooth function shown below is an example of a periodic signal.

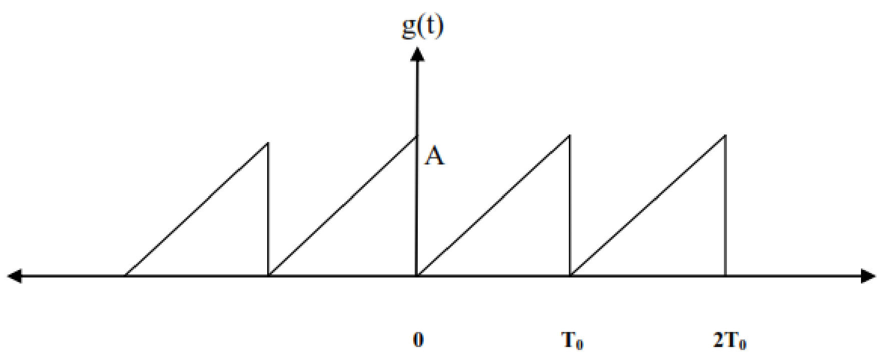



Fig. 1.1: A periodic signal with period $T_0$

**Example: A Non-periodic Signal**

The signal

$$g(t) = \begin{cases} A, & 0 \le t \le \tau \\ 0, & otherwise \end{cases}$$

is non-periodic, since there does not exist a $T_0$ for which the condition $g(t) = g(t+T_0)$ is satisfied.

**<u>2. Deterministic Signals, Random Signals:</u>**

A *deterministic signal* is one about which there is no uncertainty with respect to its value at any time. It is a completely specified function of time .

**Example: A Deterministic Signal**

$x(t) = Ae^{-at}u(t)$ ; $A$ and α are constants.

A *random signal* is one about which there is some degree of uncertainty before it actually occurs. (It is a function of a random variable )

**Example : A Random Signal**

x(t) = A $e^{-at}$u(t) ; α is a constant and A is a random variable with the following probability density function (pdf).

$F_A(a) = \begin{cases} 1 & 0 \le a \le 1 \\ 0 & otherwise \end{cases}$

**<u>3. Energy Signals, Power Signals:</u>**

The *instantaneous power* in a signal g(t) is defined as that power dissipated in a 1-Ω resistor, i.e.,

$$p(t) = |g(t)|^2$$

The *average power* is defined as:

$$P_{av} \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |g(t)|^2 \, dt$$

The total energy of a signal g(t) is

$$E \triangleq \lim_{T\to\infty} \int_{-T}^{T} |g(t)|^2 \, dt$$

A signal g(t) is classified as *energy signal* if it has a finite energy,i.e, $0 < E < \infty$

A signal g(t) is classified as *power signal* if it has a finite power,i.e, $0 < P_{av} < \infty$

The average power in a periodic signal is

$$P_{av} = \frac{1}{T_0} \int_{0}^{T_0} |g(t)|^2 \, dt$$ ; $T_0$ is the period;

$f_0 = 1/T_0$ is referred to as the fundamental frequency

Usually, periodic signals and random signals are power signals. Both deterministic and non periodic signals are energy signals (We will show a counterexample shortly).

**<u>4. Analog Signals, Digital Signals :</u>**

An *analog signal* is a continuous time - continuous amplitude function of time .

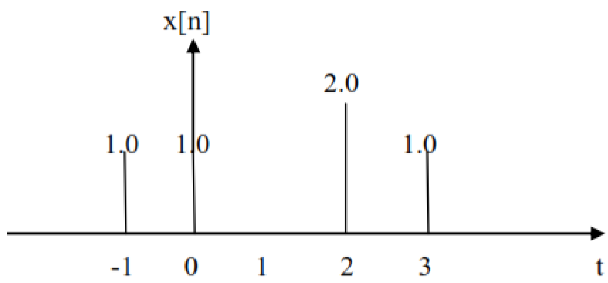
**Example**:

The sinusoidal signal$x(t) = A\cos 2\pi f t$ , $-\infty < t < \infty$, is an example of an analog signal.

A *discrete time- discrete amplitude*(digital) signal is defined only at discrete times. Here, the independent variable takes on only discrete values.

**Example**:

The sequence x[n] shown below is an examples of a digital signal. The amplitudes are drawn from the finite set {1,0,2}.




**More Examples**

**Example: The Exponential Pulse**

Find the energy in the signal g(t) = A $e^{-\propto t}$ u(t).

E= $\int_0^{\infty} A^2\, e^{-2\propto t} dt = A^2 \frac{-e^{-2\propto t}}{2\propto} \Big|_0^{\infty} = \frac{A^2}{2\propto}$ . Since E is finite, then g(t) is an energy signal.

**Example: The Rectangular Pulse**

Find the energy in the signal:

$$g(t) = \begin{cases} A, & 0 < t < \tau \\ 0, & o.w \end{cases}$$

E= $\int_0^{\tau} A^2\, dt = A^2\tau$. This signal is an energy since E is finite.

**Example: The Periodic Sinusoidal Signal**

Find the average power in the signal :

g(t) = A $\cos \omega t$ , $-\infty < t < \infty$

Since g(t) is periodic, then :

$$P_{av} = \frac{1}{T_0}\int_0^{T_0} A^2 \cos^2 \omega t \, dt = \frac{A^2}{T_0}\int_0^{T_0}\left(\frac{1+\cos 2\omega t}{2}\right) = \frac{A^2}{T_0} \cdot \frac{T_0}{2} = \frac{A^2}{2} \; .$$

Here, $P_{av}$ is finite and, therefore, g(t) is a power signal.

**Example: The Periodic Saw-tooth Signal**

Find the average power in the saw-tooth signal g(t) plotted in Fig.1.

g(t) = $\frac{A}{T_0}$ t , $0 \le t \le T_0$

$$P_{av} = \frac{1}{T_0}\int_0^{T_0} \frac{A^2}{{T_0}^2} t^2 dt = \frac{1}{T_0}\frac{A^2}{{T_0}^2}\frac{t^3}{3}\Big|_0^{T_0} = \frac{A^2 {T_0}^3}{3\,{T_0}^3} = \frac{A^2}{3} \; .$$

**Example: The Unit Step Function**

Consider the signal: g(t) = A u(t).

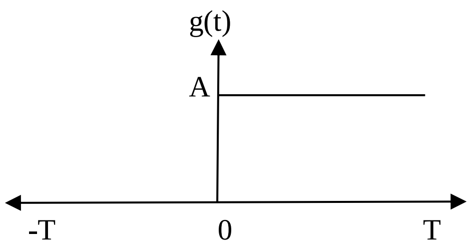



Fig. 1.2

This is a non periodic signal. So let us first try to find its energy:

$$E = \int_0^{\infty} A^2 \, dt = \infty \; .$$

Sine E is not finite, then g(t) is not an energy signal.

To find the average power, we employ the definition :

$$P_{av} \triangleq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |g(t)|^2 \, dt,$$

where 2T is chosen to be a symmetrical interval about the origin, as in Fig. 1.2 above.

$$P_{av} = \lim_{T\to\infty} \frac{1}{2T}\int_0^{T} A^2 \, dt = \lim_{T\to\infty} \frac{A^2\,T}{2T} = \frac{A^2}{2} \; .$$

So, even-though g(t) is non-periodic, it turns out that it is a power signal.

**Remark**: This is an example where the general rule (periodic signals are power signals and energy signals are non periodic signals) fails to hold.

# Fourier Series

Let g(t) be a periodic signal with period $T_0 = \frac{1}{f_0}$ such that it absolutely integrable over one period,

i.e., $\int_0^{T_0} |g(t)|\, dt < \infty$.

The signal g(t), satisfying the above integrability condition, may be expanded in one of three possible Fourier series forms (We will not address the question of series convergence in this discussion):

**The complex form**:

$$g(t) = \sum_{n=-\infty}^{\infty} \mathrm{C_n} \mathrm{e}^{\mathrm{j}n\omega_0 \mathrm{t}}$$

where, $\mathrm{C_n} = \frac{1}{T_0}\int_0^{T_0} g(t)\, e^{-jn\omega_0 t} dt$ ;

$C_n$ : is a complex valued quantity that can be written as:

$C_n = | C_n |\, e^{j\theta n}$

*Discrete Amplitude Spectrum*: A plot of $|C_n|$ vs. frequency

*Discrete Phase Spectrum*: A plot of $\theta_n$ vs. frequency

The term at $f_0$ is referred to as the fundamental frequency. The term at $2f_0$ is referred to as the second order harmonic, and so on.

**The trigonometric form**:

$$g(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

Where : $a_0 = \frac{1}{T_0}\int_0^{T_0} g(t)\, dt$ (dc or average value)

$$a_n = \frac{2}{T_0}\int_0^{T_0} g(t) \cos n\omega_0 t\, dt$$

$$b_n = \frac{2}{T_0}\int_0^{T_0} g(t) \sin n\omega_0 t\, dt$$

**The polar form** :

$$g(t) = c_0 + \sum_{n=1}^{\infty} 2|\mathrm{C_n}| \cos(n\omega_0 t + \theta_n)$$

where $C_n$ and $\theta_n$ are those terms defined in the complex form.

**Remark**: The above three forms are equivalent and are representations of the same waveform. If you know one representation, you can easily deduce the other.

**Example**: Find the trigonometric Fourier series of the periodic rectangular signal defined over one period $T_0$ as:

$$g(t) = \begin{cases} +A, & -T_0/4 \le t \le T_0/4 \\ 0, & otherwise \end{cases}$$

**Solution**:

$$a_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\, dt = \frac{1}{T_0}\int_{-T_0/4}^{T_0/4} A\, dt = A/2$$

$$b_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\sin(\frac{2\pi n}{T_0}t)\, dt = \frac{2}{T_0}\int_{-T_0/4}^{T_0/4} A\sin(\frac{2\pi n}{T_0}t)\, dt = 0$$

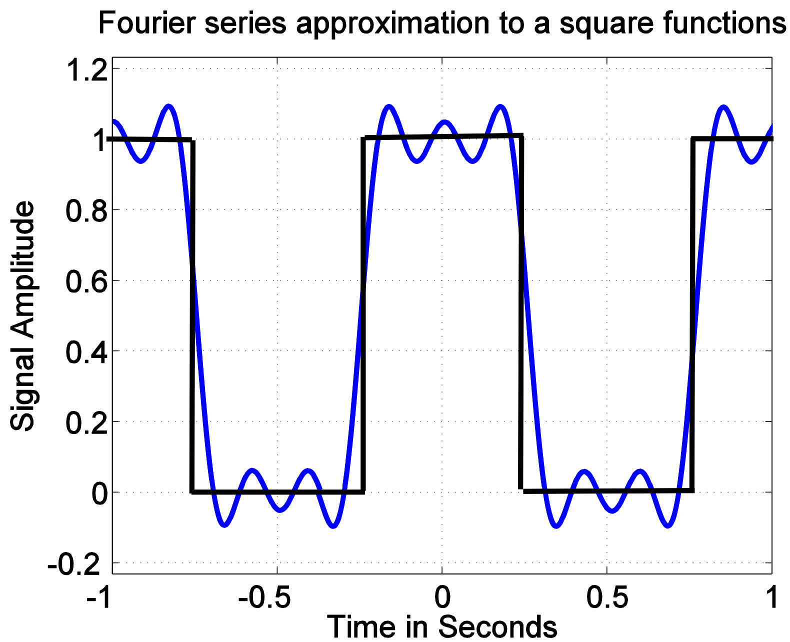
$$a_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\cos(\frac{2\pi n}{T_0}t)\, dt = \frac{2}{T_0}\int_{-T_0/4}^{T_0/4} A\cos(\frac{2\pi n}{T_0}t)\, dt$$

$$a_n = \begin{cases} \frac{2A}{n\pi}, & n = 1, 5, 9, \dots \\ \frac{-2A}{n\pi}, & n = 3, 7, 11, \dots \\ 0, & n = 2, 4, 6 \dots \end{cases}$$

The first four terms in the expansion of $g(t)$ are:

$$\tilde{g}(t) = \frac{A}{2} + \frac{2A}{\pi}\{\cos(2\pi f_0)\, t - \frac{1}{3}\cos(2\pi 3 f_0)\, t + \frac{1}{5}\cos(2\pi 5 f_0)\, t\}$$

The function $\tilde{g}(t)$ along with $g(t)$ are plotted in the figure for $-1 \le t \le 1$ assuming $A = 1$ and $f_0 = 1$




**Remark**: As more terms are added to $\tilde{g}(t)$, $\tilde{g}(t)$ becomes closer to $g(t)$ and in the limit as $n \to \infty$, $\tilde{g}(t)$ becomes equal to $g(t)$ at all points except at the points of discontinuity.

**Parseval's Power Theorem**

The average power of a periodic signal g(t) is given by:

$$\mathrm{P_{av}} = \frac{1}{T_0}\int_0^{T_0} |g(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |\mathrm{C_n}|^2 \;\; = |\mathrm{C_n}|^2 + 2\sum_{n=1}^{\infty} |\mathrm{C_n}|^2$$

$$= |a_0|^2 + \frac{1}{2}\sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2)$$

**Power Spectral Density**

The plot of $|\mathrm{C_n}|^2$ versus frequency is called the *power spectral density* (PSD). It displays the power content of each frequency (spectral) component of a signal. For a periodic signal, the PSD consists of discrete terms at multiples of the fundamental frequency.

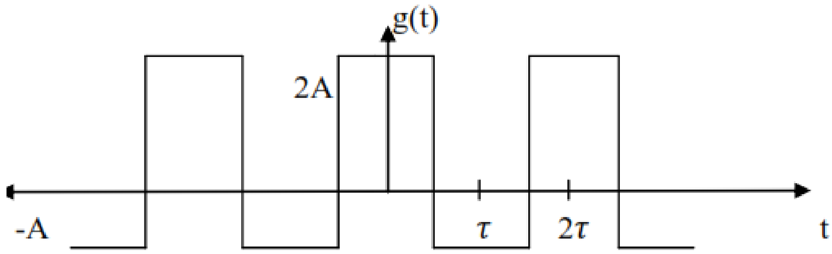
**Exercise**: Consider again the saw-tooth function defined over one period as $g(t) = t, 0 \le t \le 1$

a. Use matlab to find the dc terms and the first three harmonics(i.e., let n = 3) in the Fourier series expansion

$$\tilde{g}(t) = a_0 + \sum_{n=1}^{3} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

b. Plot $\tilde{g}(t)$ and $g(t)$ versus time for $-1 \le t \le 1$ on the same graph.
c. Find the fraction of the power contained in $\tilde{g}(t)$ to that in $g(t)$.
d. Sketch the power spectral density.

**Example :** Find the power spectral density of the periodic function g(t) shown in the figure :

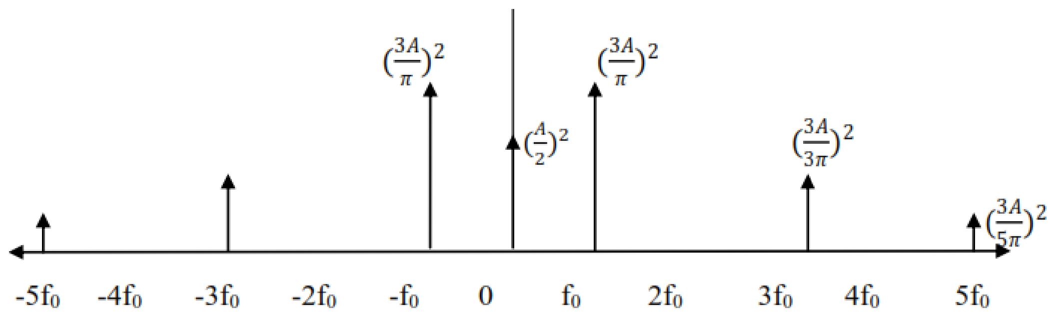



**Solution:** Here, we need to find the complex Fourier series expansion, where the period $T_0 = 2\tau$

$$g(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}; \qquad C_n = \frac{1}{T_0}\int_0^{T_0} g(t)\, e^{-jn\omega_0 t} dt$$

$$C_n = \begin{cases} \frac{A}{2}, & n = 0 \\ \frac{3A}{|n|\pi}, & n = \pm 1, \pm 5, \pm 9, \ldots \\ \frac{-3A}{|n|\pi}, & n = \pm 3, \pm 7, \pm 11, \ldots \\ 0, & n = \pm 2, \pm 4, \ldots \end{cases} \Longrightarrow |C_n|^2 = \begin{cases} (\frac{A}{2})^2, & n = 0 \\ (\frac{3A}{n\pi})^2, & n: odd \\ 0, & n: even \end{cases}$$

$$S_g(f) = \sum_{n=-\infty}^{\infty} |C_n|^2\, \delta(f - nf_0)$$




As can be seen, the power spectral density of this periodic signal is a discrete function in frequency.

**Exercise:** Verify Parseval's power theorem for this signal, i.e., show that

$$P_{av} = \frac{1}{T_0}\int_0^{T_0} |g(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |C_n|^2 = 2.5A^2$$

# Fourier Transform

Let g(t) be a non periodic square integrable function of time. That is one for which

$\int_{-\infty}^{\infty} |g(t)|^2 \, dt < \infty$

The Fourier transform of g(t) exists and is defined as:

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t} \, dt$$

The time function g(t) can be recovered from $G(f)$ using the inverse Fourier Transform:

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi f t} \, df$$

**Remarks**:

- All energy signals for which $E = \int_{-\infty}^{\infty} |g(t)|^2 \, dt < \infty$ are Fourier transformable.
- G(f) is a complex function of frequency f, which can be expressed as:

  G(f) = | G(f) | $e^{j\theta(f)}$

where, | G(f) | : is the *continuous amplitude spectrum* of g(t), (even function of f).

$\theta(f)$ : is the *continuous phase spectrum* of g(t), (odd function of f).

**Rayleigh Energy Theorem :**

The energy in a signal g(t) is given by :

E = $\int_{-\infty}^{\infty} |g(t)|^2 \, dt$ = $\int_{-\infty}^{\infty} |G(f)|^2 df$

The function $|G(f)|^2$ is called the *energy spectral density*. It illustrates the range of frequencies over which the signal energy extends and the frequency bands which are significant in terms of their energy contents. For a non-period signal energy signal, the energy spectral density is a continuous function of f.

**A General Form of the Rayleigh Energy Theorem**

For two energy functions g(t) and $v(t)$, the following result holds:

$$\int_{-\infty}^{\infty} g(t)v(t)^* \, dt = \int_{-\infty}^{\infty} G(f)V(f)^* \, df$$

**Example**: **Energy spectral density of the exponential signal**

$$v(t) = \begin{cases} A\, e^{-bt} & t > 0 \\ 0 & t < 0 \end{cases}$$

$$V(f) = \int_0^{\infty} v(t)e^{-j2\pi ft} \, dt = \int_0^{\infty} Ae^{-bt}\, e^{-j2\pi ft} dt$$

$V(f) = A\int_0^{\infty} e^{-(b+j2\pi f)t} \, dt$ = A $\frac{e^{-(b+j2\pi f)t}}{-(b+j2\pi f)}\Big|_0^{\infty} = \frac{A}{b+j2\pi f}$ .

$V(f) = \frac{A}{b+j2\pi f}$ ⇨ $|V(f)| = \frac{A}{(b^2+(2\pi f)^2)^{1/2}}$

The energy spectral density is: $S_v(f) = |V(f)|^2 = \frac{A^2}{b^2+\omega^2}$

**Remark**: The signal $v(t)$ is called a *baseband signal* since the signal occupies the low frequency part of the spectrum. That is, the energy in the signal is found around the zero frequency. When the signal is multiplied by a high frequency carrier, the spectrum becomes centered around the carrier and the modulated signal is called a *bandpass signal*.

**Exercise** : For the exponential pulse, verify Rayleigh energy theorem, i.e., show that

$\int_0^{\infty} |v(t)|^2 dt = 2\int_0^{\infty} |V(f)|^2 df = \frac{A^2}{2b}$.

**Example: The Rectangular Pulse** $g(t) = Arect(\frac{t}{T})$

$G(f) = \int_{-T/2}^{T/2} Ae^{-j2\pi ft} dt = \frac{A}{\pi f}\sin \pi fT$

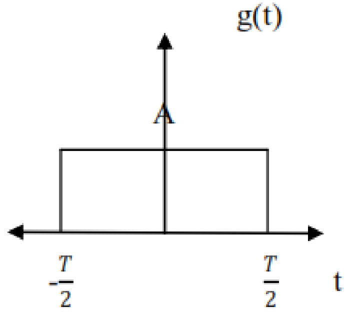
$= AT\frac{\sin \pi fT}{\pi fT} \triangleq AT\; sinc\; Tf$

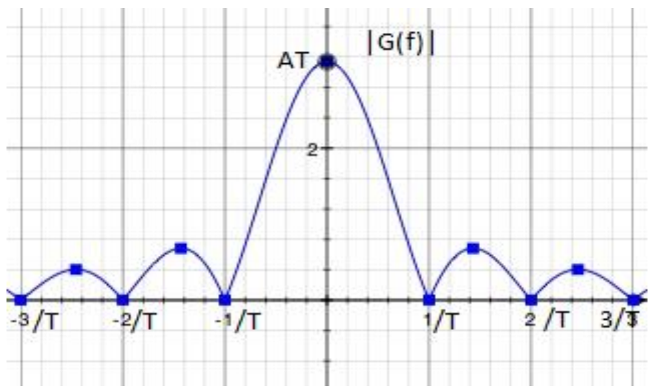
$$|G(f)| = AT\; |sinc\; Tf|$$

$\lim_{x\to 0}\frac{\sin x}{x} = 1$ ⇨ max. of function

$G(f) = 0$ $when\; sin\; (\pi fT) = 0$ or $when\; \pi fT = n\pi$ , n=$\pm 1, \pm 2, \pm 3, \ldots$

$$fT = n, \therefore f = \frac{n}{T}$$




**<u>Remark:</u> Time duration and bandwidth :**

Note that as the signal time duration $T$ *increases*, the first zero crossing at $f = \frac{1}{T}$ decreases, implying that the bandwidth of the signal decreases. More on this will be said later when we discuss the time bandwidth product.

**Exercise** : For the rectangular pulse $g(t) = Arect(\frac{t}{T})$, verify Rayleigh energy theorem, i.e., show that

$$\int_0^\infty |g(t)|^2 dt = 2\int_0^\infty |G(f)|^2 df = A^2 T.$$

**Properties of the Fourier Transform:**

1. Linearity (superposition)
   Let $g_1(t) \leftrightarrow G_1$(f)
   and $g_2(t) \leftrightarrow G_2$(f) , then

   $c_1 g_1(t)$+$c_2 g_2(t) \leftrightarrow c_1 G_1(f)$+$c_2 G_2(f)$ ; $c_1, c_2$ are constants

2. Time scaling
   If $g_1(t) \leftrightarrow G_1$(f),
   then

$$g(at) \leftrightarrow \frac{1}{|a|} G(f/a)$$

3. Duality
   If $g(t) \leftrightarrow G(f)$ ,
   Then, $G(t) \leftrightarrow g(-f)$

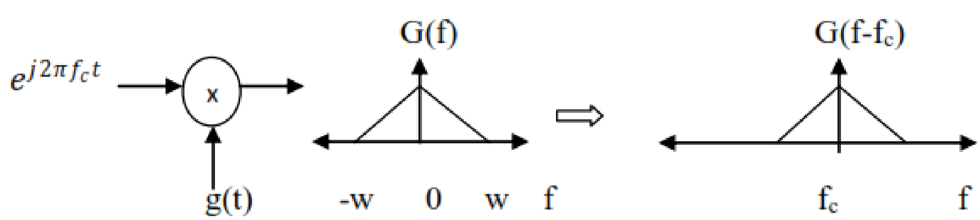
4. Time shifting

   If $g(t) \leftrightarrow G(f)$

   then $g(t - t_0) \leftrightarrow G(f)e^{-j2\pi f t_0}$

   Delay in time domain $\Longrightarrow$ phase shift in frequency domain

5. Frequency shifting
   If $g(t) \leftrightarrow G(f)$ ,
   then $g(t)e^{j2\pi f ct} \leftrightarrow G(f - fc)$ ; $f_c$ is a real constant




6. Area under $G(f)$
   If $g(t) \leftrightarrow G(f)$ ,
   then: $g(t = 0) = \int_{-\infty}^{\infty} G(f)df$

The value $g(t = 0)$ is equal to the area under its Fourier transform.

7. Area under $g(t)$
   If $g(t) \leftrightarrow G(f)$

Then, $G(0) = \int_{-\infty}^{\infty} g(t)dt$

The area under a function g(t) is equal to the value of its Fourier transform G(f) at f = 0, where $G(0)$ implies the presence of a dc component.

8. Differentiation in the time domain

If $g(t)$ and its derivative $g'(t)$ are Fourier transformable, then,

$$g'(t) \leftrightarrow (j2\pi f)G(f)$$

i.e., differentiation in the time domain $\Longrightarrow$ multiplication by $j2\pi f$ in the frequency domain.

(differentiation in the time domain enhances high frequency components of a signal)

Also, $\frac{d^n g(t)}{dt^n} \leftrightarrow (j2\pi f)^n G(f)$

9. Integration in the time domain

$\int_{-\infty}^{t} g(\tau)d\tau \leftrightarrow \frac{1}{j2\pi f} G(f)$; assuming $G(0) = 0$

i.e., integration in the time domain $\Longrightarrow$ division by ($j2\pi f$) in the frequency domain. This amounts to low pass filtering where high frequency components are attenuated.

When $G(0) \neq 0$, the above result becomes:

$$\int_{-\infty}^{t} g(\tau)d\tau \leftrightarrow \frac{1}{j2\pi f} G(f) + \frac{1}{2} G(0)\delta(f).$$

10. Conjugate Functions

For a complex – valued time signal $g(t)$, we have:

$$g^*(t) \leftrightarrow G^*(-f) \quad ;$$

Also, $g^*(-t) \leftrightarrow G^*(f) \quad ;$

Therefore, Re{g(t)} $\leftrightarrow \frac{1}{2}\{ G(f) + G^*(-f)\}$

Im{g(t)} $\leftrightarrow \frac{1}{2j}\{ G(f) - G^*(-f)\}$

11. Multiplication in the time domain

$$g_1(t)\, g_2(t) \leftrightarrow \int_{-\infty}^{\infty} G_1(\lambda)\, G_2(f-\lambda)d\lambda = G_1(f) * G_2(f)$$

Multiplication of two signals in the time domain is transformed into the convolution of their Fourier transforms in the frequency domain.

12. Convolution in the time domain

$$g_1(t) * \ g_2(t) \leftrightarrow G_1(f)G_2(f)$$

Convolution of two signals in the time domain is transformed into a multiplication of their Fourier transforms in the frequency domain.

**Fourier Transform of Power Signals**

For a non-periodic (energy) signal, the Fourier transform exists when

$$\mathrm{E}= \int_{-\infty}^{\infty} |g(t)|^2 \, dt < \infty$$

So that $(f) = \ \int_{-\infty}^{\infty} g(t)e^{-j2\pi f t}dt$ .

For power signals, the integral $\int_{-\infty}^{\infty} g(t)e^{-j2\pi f t}dt$ **does not exist**.

However, one can still find the Fourier transform of power signals by employing the delta function. This function is defined next.

**Dirac – Delta Function (impulse function)**

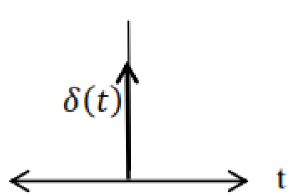
This function is defined as

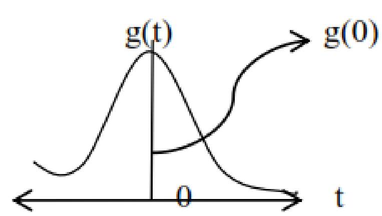
$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$

such that $\int_{-\infty}^{\infty} \delta(t)dt = 1$

and $\int_{-\infty}^{\infty} g(t)\delta(t)dt = g(0)$

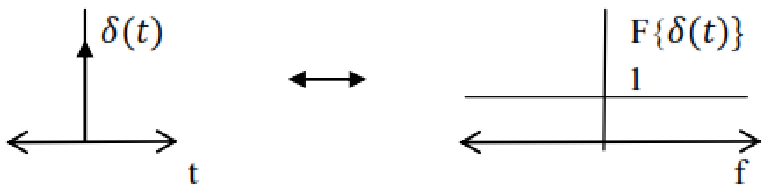
(Here, g(t) is a continuous function of time).



**Some properties of the delta function:**

1. $g(t)\delta(t-t_0) = g(t_0)\delta(t-t_0)$; (Multiplication)
2. $\int_{-\infty}^{\infty} g(t)\delta(t-t_0)dt = g(t_0)$ ; (Sifting or sampling property)
3. $\delta(\alpha t) = \frac{1}{|\alpha|}\delta(t)$
4. $\delta(t) * g(t) = g(t)$
5. $\delta(t) = \frac{du(t)}{dt}$ $\Longrightarrow$ u(t) = $\int_{-\infty}^{t} \delta(t)dt$
6. $\delta(t) = \delta(-t)$
7. Fourier transform: F{$\delta(t)$} = 1




(Note how the time-bandwidth relationship holds for this pair. A narrow pulse in time extends over a large frequency spectrum)

8. F{$\delta(t-t_0)$} $= e^{-j2\pi f t_0}$

**Applications of delta functions**

1. Dc Signal

Since F{$\delta(t)$} = 1, then by the duality property F{1} = $\delta(f)$}


(Again, note that the transform of a dc signal is an impulse at f =0)

2. Complex exponential function

F{A $e^{j2\pi f_c t}$} = A $\delta(f-f_c)$

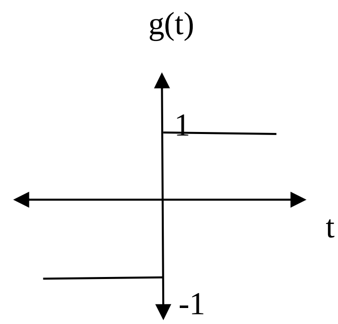
3. Sinusoidal functions

$$F\{\cos 2\pi f_c t\} = \frac{1}{2}\{\delta(f - f_c) + \delta(f + f_c)\}$$

$$F\{\sin 2\pi f_c t\} = \frac{1}{2j}\{\delta(f - f_c) - \delta(f + f_c)\}$$

4. Signum function

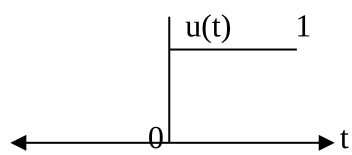
$$\text{sgn}(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$




$$F\{\text{sgn}(t)\} = \frac{1}{j\pi f}$$

4. Unit Step function :

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases}$$




sgn(t) = 2u(t) – 1

$$u(t) = \frac{1}{2}\{\text{sgn}(t) + 1\}$$

$$F\{u(t)\} = \frac{1}{j2\pi f} + \frac{1}{2}\,\delta(f)$$

7. Periodic Signals

A periodic signal g(t) is expanded in the complex form as :

$$g(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$F\{g(t)\} = \sum_{n=-\infty}^{\infty} C_n\, \delta(f - nf_0)$$

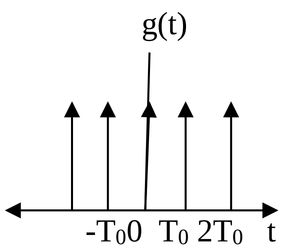
**Example**: Consider the following train of impulses

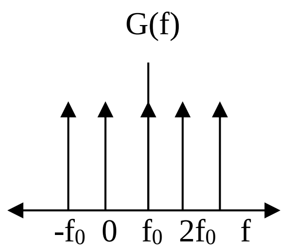
$$g(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT_0)$$

$$C_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\, e^{-jn\omega_0 t} dt = \frac{1}{T_0} = f_0$$

$\Rightarrow F\{g(t)\} = \frac{1}{T_0}\sum_{n=-\infty}^{\infty}\delta(f - nf_0)$

$$\sum_{m=-\infty}^{\infty}\delta(t - mT_0) \leftrightarrow \frac{1}{T_0}\sum_{n=-\infty}^{\infty}\delta(f - nf_0)$$




Note that the signal is periodic in the time domain and its Fourier transform is periodic in the frequency domain.

**Remark**: This sequence will be found useful when the sampling theorem is considered later in the course.