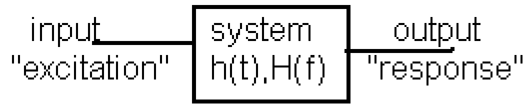
Transmission of Signals Through Linear Systems

Definition : A system refers to any physical device that produces an output signal in response to an input signal.



Definition : A system is linear if the principle of superposition applies.

If	$x_1(t)$	produces output	$y_1(t)$
	$x_2(t)$	produces output	$y_2(t)$
then	$a_1x_1(t)+ a_2x_2(t)$	produces an output	$a_1y_1(t)+ a_2y_2(t)$
Also,	a zero input	should produce	a zero output.

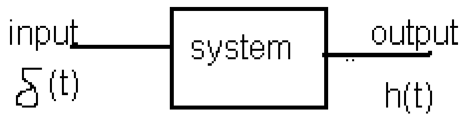
Example of linear systems include filters and communication channels.

Definition: A filter refers to a frequency selective device that is used to limit the spectrum of a signal to some band of frequencies.

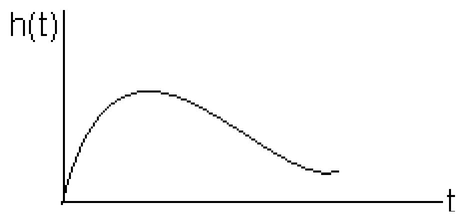
Definition: A channel refers to a transmission medium that connects the transmitter and receivers of a communication system .

Time domain and frequency domain may be used to evaluate system performance.

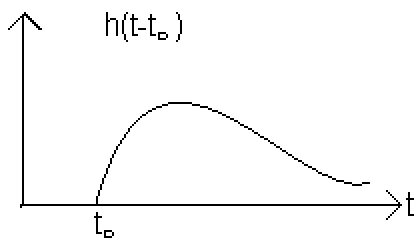
Time response :



Definition: The impulse response h(t) is defined as the response of a system to an impulse $\delta(t)$ applied to the input at t=0 .



Definition: A system is time-invariant when the shape of the impulse response is the same no matter when the impulse is applied to the system.



$\delta(t) \longrightarrow h(t)$, then $\delta(t - t_d) \longrightarrow h(t - t_d)$

When the input to a linear time-invariant system in a signal x(t) , then the output is given by

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t - \lambda)\, d\lambda$$
$$= \int_{-\infty}^{\infty} h(\lambda)x(t - \lambda)\, d\lambda \; ; \quad \text{convolution integral}$$

Definition: A system is said to be causal if it does not respond before the excitation is applied, i.e.,

$$h(t) = 0 \qquad t < 0$$

The causal system is physically realizable.

Definition: A system is said to be stable if the output signal is bounded for all bounded input signals.

If $\quad |\, x(t) \,| \le M$; $\qquad$ M is the maximum value of the input

then $\quad |\, y(t) \,| \le \int_{-\infty}^{\infty} |\, h(\tau) \,||x(t-\tau\,)\,|\, d\tau$

$$= M \int_{-\infty}^{\infty} |\, h(\tau) \,|\, d\tau$$

$\Longrightarrow$ A necessary and sufficient condition for stability (a bounded output) is

$\int_{-\infty}^{\infty} |h(t)|\, dt \;<\; \infty$; $\qquad$ h(t) is absolutely integrable.

$\therefore$ $\qquad$ zero initial conditions assumed .

Frequency Response:

Definition: The transfer function of a linear time invariant system is defined as the Fourier transform of the impulse response h(t)

$$H(f) = F\{h(t)\}$$

Since $\quad y(t) = x(t)*h(t)$, then

$$Y(f) = H(f)\, X(f)$$

or $\qquad H(f) = \frac{Y(f)}{X(f)}$

The transfer function H(f) is a complex function of frequency, which can be obtained as the ratio of the Fourier transform of the output to that of the input.

$$H(f) = |H(f)| e^{j\,\theta(f)}$$

where,

H(f) : amplitude spectrum

θ(f) : phase spectrum.

System Input–Output Energy Spectral Density

Let x(t) be applied to a LTI system , then the Fourier transform of the output is related to the Fourier transform of the input through the relation

$Y(f) = H(f)\ X(f)$

Taking the absolute value and squaring both sides, we get

$|Y(f)|^2 = |H(f)|^2\ |X(f)|^2$

$S_Y(f) = |H(f)|^2\ S_X(f)$

$S_Y(f)$: Output Energy Spectral Density
$S_X(f)$: Input Energy Spectral Density.

Output energy spectral density $= |H(f)|^2$ x Input energy spectral density

The total output energy

$$E_y = \int_{-\infty}^{+\infty} S_Y(f)df$$
$$= \int_{-\infty}^{+\infty} |H(f)|^2\ S_X(f)df.$$

The total input energy is

$$E_x = \int_{-\infty}^{+\infty} S_x(f)df\ .$$

Example: Response of a Filter to a Sinusoidal Input

The signal $x(t) = \ cos\ w_0 t$ is applied to a filter described by the transfer function $H(f) = \frac{1}{1+jf/B}$. Find the filter output $y(t)$.

Solution:

We will find the output using the frequency domain approach.

$$Y(f) = H(f)X(f)$$

$$H(f) = \frac{1}{\sqrt{1+(\frac{f}{B})^2}} e^{-j\theta}; \qquad \theta = \tan^{-1}\frac{f}{B}\,; \qquad \theta_0 = \tan^{-1}\frac{f_0}{B}$$

$$Y(f) = H(f)[\tfrac{1}{2}\delta(f - f_0) + \tfrac{1}{2}\delta(f + f_0)]$$

$$Y(f) = \tfrac{1}{2}H(f_0)\delta(f - f_0) + \tfrac{1}{2}H(-f_0)\delta(f + f_0)$$

$$Y(f) = \frac{1}{2}\frac{1}{\sqrt{1+(\frac{f_0}{B})^2}} e^{-j\theta_0}\delta(f - f_0) + \frac{1}{2}\frac{1}{\sqrt{1+(\frac{f_0}{B})^2}} e^{j\theta_0}\delta(f + f_0)$$

Taking the inverse Fourier transform, we get

$$y(t) = \frac{1}{\sqrt{1+(\frac{f_0}{B})^2}} \frac{1}{2} [e^{j(2\pi f_0 t - \theta_0)} + e^{-j(2\pi f_0 t - \theta_0)}]$$

$$y(t) = \frac{1}{\sqrt{1+(\frac{f_0}{B})^2}} \cos(2\pi f_0 t - \theta_0)$$

Note that in the last step we have made use of the Fourier transform pair

$$e^{j2\pi f_c t} \leftrightarrow \delta(f - f_c)$$

Remark: Note that the amplitude of the output as well as its phase depend on the frequency of the input and the bandwidth of the filter.

Assume, for instance, that $f_0 = B$. Then $\theta_0 = \tan^{-1}\frac{f_0}{B} = \tan^{-1} 1 = 45°$ and the output can be written as:

$$y(t) = \frac{1}{\sqrt{1+1}} \cos(2\pi f_0 t - 45°)$$

$$y(t) = \frac{1}{\sqrt{2}} \cos(2\pi f_0 t - 45°)$$

Exercise: The signal $x(t) = \ cos\ w_0 t - \frac{1}{\pi} cos\ 3w_0 t$ is applied to a filter described by the transfer function $H(f) = \frac{1}{1+jf/B}$.

a. Use the result of the previous example to find the filter output $y(t)$.
b. Is the transmission through this filter distortion-less ?

Exercise: Consider the periodic rectangular signal $g(t)$ defined over one period T_0 as:

$$g(t) = \begin{cases} +A, & -T_0/4 \le t \le T_0/4 \\ 0, & otherwise \end{cases}$$

If $g(t)$ is applied to a filter described by the transfer function $H(f) = \frac{1}{1+jf/B}$. Use the result of the previous example to find the filter output $y(t)$.

Example:

The signal $g(t) = Arect(\frac{t}{T})$ is applied to the filter $(f) = \frac{1}{1+jf/B}$. Find the output energy spectral density.

Solution:

$$S_Y(f) = |H(f)|^2 S_X(f)$$

$$S_Y(f) = \frac{1}{1+(\frac{f}{B})^2} (AT\ |sinc\ Tf|)^2$$

Example:

The signal $g(t) = \delta(t) - \delta(t-1)$ is applied to a channel described by the transfer function $H(f) = \frac{1}{1+jf/B}$. Find the channel output.

Solution:

The impulse response of the channel is obtained by taking the inverse Fourier transform of $H(f)$, which is

$$h(t) = 2\pi B e^{-2\pi B t} u(t)$$

Using the linearity and time invariance property, the output can be obtained as:

$$y(t) = h(t)u(t) - h(t-1)u(t-1)$$

$$y(t) = 2\pi B[e^{-2\pi B t} u(t) - e^{-2\pi B(t-1)} u(t-1)]$$

Exercise: The signal $g(t) = u(t) - u(t-1)$ is applied to a channel described by the transfer function $H(f) = \frac{1}{1+jf/B}$. Find the channel output $y(t)$.

Signal Distortion in Transmission

As we have said before, the objective of a communication system is to deliver to the receiver almost an exact copy of what the source generates. However, communication channels are not perfect in the sense that impairments on the channel will cause the received signal to differ from the transmitted one. During the course of transmission, the signal undergoes attenuation, phase delay, interference from other transmissions, Doppler shift in the carrier frequency, and many other effects. In this introductory discussion we will explain some of the reasons that cause the received signal to be distorted.

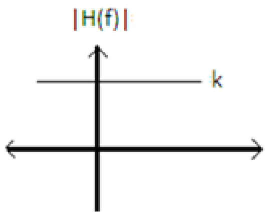
Linear Distortion

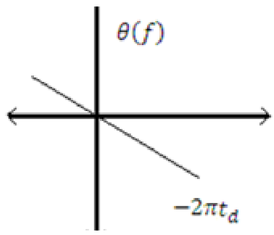
A signal transmission is said to be *distortion-less* if the output signal y(t) is an exact replica of the input signal x(t) , i.e., y(t) has the same shape as the input, except for a constant amplification (or attenuation) and a constant time delay.

Condition for a distortion-less transmission in the time domain is:

$$\boldsymbol{y(t) = kx(t - t_d)}; \textbf{ Condition for a distortion-less transmission}$$

where k: is a constant amplitude scaling
t_d: is a constant time delay





In the frequency domain, the condition for a distortion-less transmission becomes

$$Y(f) = k\, X(f)\, e^{-j2\pi f t_d}$$

or $$H(f) = \frac{Y(f)}{X(f)} = k\, e^{-j2\pi f t_d} = ke^{-j\theta(f)}$$

That is, for a distortion-less transmission, the transfer function should satisfy two conditions:

1. **$|H(f)| = k$** ; The amplitude of the transfer function is constant (gain or attenuation) over the frequency range of interest.
2. $\boldsymbol{\theta(f) = -2\pi f t_d = -(2\pi t_d) f}$; The phase function is linear in frequency with a negative slope that passes through the origin (or multiples of π).

When |H(f)| is not constant for all frequencies of interest, *amplitude distortion* results.

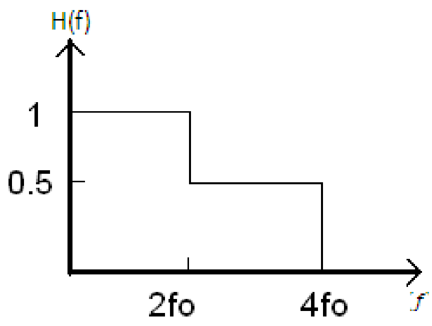
When $\theta(f) \neq -2\pi f t_d \pm 180^0$, then we have *phase distortion* (or delay distortion).

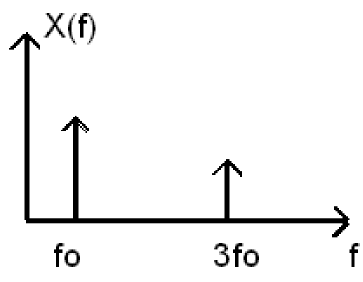
The following examples demonstrate the two types of distortion mentioned above.

Example : Amplitude Distortion

Consider the signal $x(t) = \cos w_0 t - \frac{1}{3} \cos 3w_0 t$. If this signal passes through a channel with zero time delay (i.e., $t_d = 0$) and amplitude spectrum as shown in the figure

a. Find y(t)
b. Is this a distortion-less transmission?





Solution:

x(t) consists of two frequency components, f_0 and $3f_0$. Upon passing through the channel, each one of them will be scaled by a different factor.

a. $y(t) = \cos w_0 t - \frac{1}{2} . \frac{1}{3} \cos 3w_0 t$

b. Since $y(t) \neq k\ x(t)$, this is not a distortion-less transmission .

Example: Phase Distortion

If x(t) in the previous example is passed through a channel whose amplitude spectrum is a constant k. Each component in x(t) suffers a $\frac{\pi}{2}$ phase shift

a. Find y(t).

b. Is this a distortion-less transmission ?

Solution:

$$x(t) = \cos w_o t - \frac{1}{3} \cos 3w_o t$$

$$y(t) = \mathrm{k}\ \cos(w_o t - \frac{\pi}{2}) - \frac{1}{3}\ k\ \cos\left(3w_o t - \frac{\pi}{2}\right)$$

$$y(t) = \mathrm{k}\ \cos w_o (t - \frac{\pi}{2w_o}) - \frac{1}{3}\ k\ \cos\left(3w_o (t - \frac{\pi}{2x3w_o})\right)$$

$$y(t) = \mathrm{k}\ \cos w_o (t - t_{d1}) - \frac{1}{3}\ k\ \cos(3w_o (t - t_{d2}))$$

Note that $t_{d1} \neq t_{d2}$, i.e., each component in $x(t)$ suffers from a different time delay. Hence this transmission introduces phase (delay) distortion.

Nonlinear Distortion

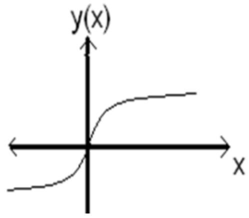
When a system contains nonlinear elements, it is not described by a transfer function H(f), but rather by a transfer characteristic of the form

$y(t) = a_1\ x(t) + a_2\ x^2(t) + a_3\ x^3(t) + \ldots$ (time domain)

In the frequency domain ,

$Y(f) = a_1\ X(f) + a_2\ X(f)*X(f) + a_3\ X(f)*X(f)*X(f) + \ldots$

Here, the output contains new frequencies not originally present in the original signal. The nonlinearity produces undesirable frequency component for $|f| \leq W$, in which W is the signal bandwidth.



Harmonic Distortion in Nonlinear Systems

Let the input to a nonlinear system be the single tone signal

$$x(t) = \cos 2\pi f_0 t$$

This signal is applied to a channel with characteristic

$$y(t) = a_1 x(t) + a_2 x(t)^2 + a_3 x(t)^3$$

upon substituting x(t) and arranging terms, we get

$$y(t) = \frac{1}{2}a_2 + \left(a_1 + \frac{3}{4}a_3\right)\cos 2\pi f_0 t + \frac{1}{2}\, a_2 \cos 4\pi f_0 t + \frac{1}{4}a_3 cos6\pi\, f_0 t$$

Note that the output contains a component proportional to x(t) which is $\left(a_1 + \frac{3}{4}a_3\right)\cos 2\pi f_0 t$, in addition to a second and a third harmonic terms (terms at twice and three times the frequency of the input). These new terms are the result of the nonlinear characteristic and are, therefore, considered as harmonic distortion. The DC term does not constitute a distortion, for it can be removed using a blocking capacitor.

Define second harmonic distortion

$$D_2 = \frac{|amplitude\ of\ second\ harmonic\ |}{|amplitude\ of\ fundamental\ term|}$$

$$D_2 = \frac{|\frac{1}{2}a_2\ |}{|\left(a_1 + \frac{3}{4}a_3\right)|}\ x\ 100\%$$

In a similar way we can define the third harmonic distortion as:

$$D_2 = \frac{|amplitude\ of\ third\ harmonic\ |}{|amplitude\ of\ fundamental\ term|}$$

Therefore,

$$D_3 = \frac{|\frac{1}{4}a_3\ |}{|\left(a_1 + \frac{3}{4}a_3\right)|}\ x\ 100\%$$

Remark: In the solution above we have made use of the following two identities:

$\mathrm{Cos}^2 x = \frac{1}{2}\{1 + cos2x\}$

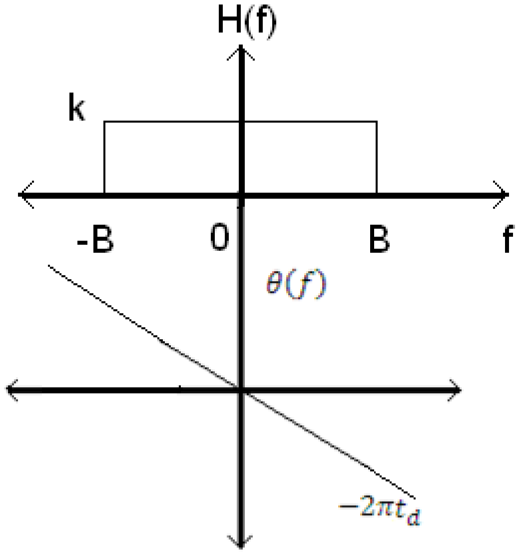
$\mathrm{Cos}^3 x = \frac{1}{4}\{3cosx + cos3x\}$.

Filters and Filtering

A filter is a frequency selective device. It allows certain frequencies to pass almost without attenuation wile it suppresses other frequencies

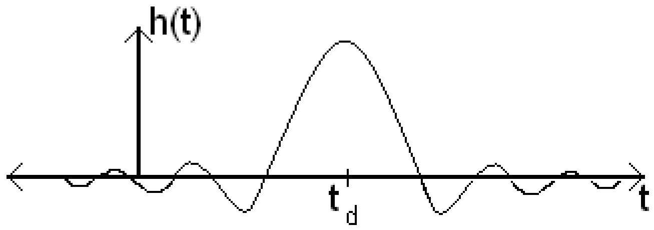
A. Ideal Filter:

Ideal low pass filter



$$H(f) = \begin{cases} k\, e^{-j2\pi f t_d} & |f| < B \\ 0 & o.w \end{cases}$$

$$h(t) = 2Bk\ sinc\ 2B(t - t_d)$$



since h(t) is the response to an impulse applied at t=0, and because h(t) has nonzero values for t<0 , the filter is *non-causal* (physically non realizable).

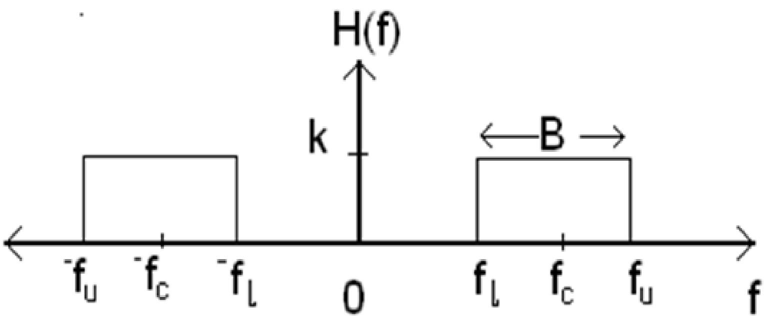
<u>Band Pass Filter</u>

$$H(f) = \begin{cases} k\, e^{-j2\pi f t_d} & f_l < |f| < f_u \\ 0 & o.w \end{cases}$$

Filer bandwidth B = $f_u - f_l$; difference between upper and lower positive frequencies

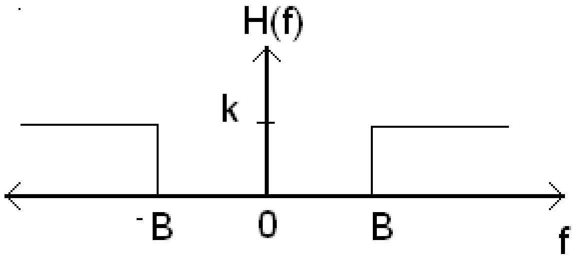
$f_c = \frac{f_u + f_l}{2}$; center frequency of the filter

$h(t) = 2Bk\ sinc\ B(t - t_d)\ \ cos\ w_c(t - t_d)$; impulse response



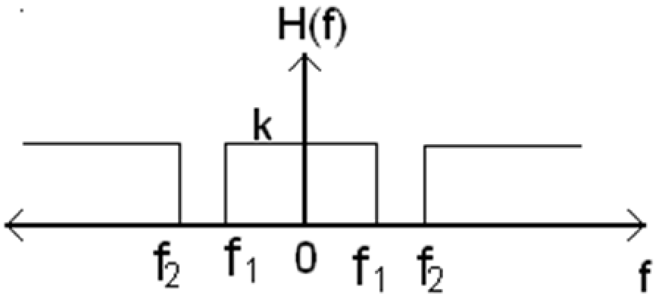
High-pass filter

$$H(f) = \begin{cases} k\, e^{-j2\pi f t_d} & |f| > B \\ 0 & o.w \end{cases}$$



Band Rejection or Notch Filter

$$H(f) = \begin{cases} k\, e^{-j2\pi f t_d} & o.w \\ 0 & f_1 < |f| < f_2 \end{cases}$$



Real Filter

Here, we only consider a Butterworth low pass filter. The transfer function of a low pass Butterworth filter is of the form

$$H(f) = \frac{1}{P_n\left(\frac{jf}{B}\right)}$$

B is the 3-dB bandwidth of the filter and $P_n(jf/B)$ is a complex polynomial of order n . The family of Butterworth polynomials is defined by the property

$$|P_n\left(\frac{jf}{B}\right)|^2 = 1 + \left(\frac{f}{B}\right)^{2n}$$

So that

$$|H(f)| = \frac{1}{\sqrt{1+\left(\frac{f}{B}\right)^{2n}}}$$

The first few polynomials are:

$P_1(x) = 1 + x$

$P_2(x) = 1 + \sqrt{2}x + x^2$

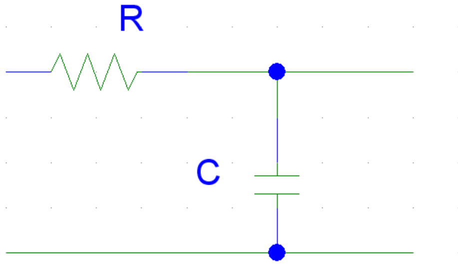
$P_3(x) = (1+x)(1+x+x^2)$

A first order LPF

$$H(f) = \frac{\frac{1}{j2\pi f_C}}{R+\frac{1}{j2\pi f_C}} = \frac{1}{1+j2\pi fRC}$$

Let $B = \frac{1}{2\pi RC}$

$$H(f) = \frac{1}{1+jf/B} = \frac{1}{P_1(jf/B)} = \frac{1}{P_1(x)}$$

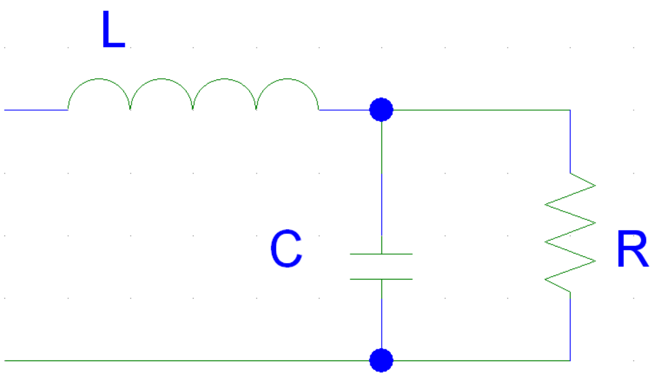


A Second order LPF

$$H(f) = \frac{1}{1+\frac{jwL}{R}-\left(2\pi\sqrt{LC}f\right)^2}$$

$$H(f) = \frac{1}{1+j\sqrt{2}f/B-(f/B)^2}$$

$where\ \ R = \sqrt{\frac{L}{2C}},\ B = \frac{1}{2\pi\sqrt{LC}}$



$$H(f) = \frac{1}{1+j\sqrt{2}f/B-(f/B)^2}$$

$$H(f) = \frac{1}{P_2(jf/B)}$$

Hilbert Transform **(Details are not required for ENEE 339)**

The quadrature filter: is an all pass filter that shifts the phase of positive frequency by (-90°) and negative frequency by (+90°). The transfer function of such a filter is

$$H(f) = \begin{cases} -j & f > 0 \\ j & f < 0 \end{cases}$$

Using the duality property of Fourier transform, the impulse response of the filter is

$$h(t) = \frac{1}{\pi t}$$

The Hilbert transform is the output of the quadrature filter to the signal g(t)

$$\hat{g}(t) = \frac{1}{\pi t} * g(t) \quad = \int_{-\infty}^{\infty} \frac{g(\lambda)}{\pi(t-\lambda)}\, d\lambda$$

Note that the Hilbert transform of a signal is a function of time. The Fourier transform of $\hat{g}(t)$ is

$$\hat{G}(f) = -j\ \mathrm{sgn}(f)\ G(f)$$

Hilbert transform can be found using either the time domain approach or the frequency domain approach depending on the given problem, that is

- Direct convolution in the time domain of g(t) and $\frac{1}{\pi t}$.
- Find the Fourier transform $\hat{G}(f)$, then find the inverse Fourier transform $\hat{g}(t) \ = \int_{-\infty}^{\infty} \hat{G}(f)\, e^{j2\pi f t}\, df$

Some properties of the Hilbert transform

1. A signal g(t) and its Hilbert transform $\hat{g}(t)$ have the same energy spectral density

$$\left|\hat{G}(f)\right|^2 = |-j\, sgn(f)|G(f)||^2 = |-j\, sgn(f)|^2 |G(f)|^2$$
$$= |G(f)|^2$$

The consequences of this property are:

- If a signal g(t) is bandlimited, then $\hat{g}(t)$ is bandlimited to the same bandwidth (note that $\left|\hat{G}(f)\right| = |G(f)|$)
- $\hat{g}(t)$ and g(t) have the same total energy (or power).
- $\hat{g}(t)$ and g(t) have the same autocorrelation function.

2. A signal g(t) and $\hat{g}(t)$ are orthogonal, i.e.,

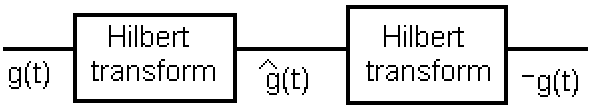
$$\int_{-\infty}^{\infty} g(t)\, \hat{g}(t) dt = 0$$

This property can be verified using the general formula of Rayleigh energy theorem

$$\int_{-\infty}^{\infty} g(t)\, \hat{g}(t) dt \quad = \int_{-\infty}^{\infty} G(f)\, \hat{G}^{*}(f) df = \int_{-\infty}^{\infty} G(f)\, \{-jsgn(f)\, G(f)\}^{*} df$$
$$= \int_{-\infty}^{\infty} jsgn(f)\, |G(f)|^{2} df = 0$$

The result above follows from the fact that $|G(f)|^2$ is an even function of f while $sgn(f)$ is an odd function of f. Their product is odd. The integration of an odd function over a symmetrical interval is zero.

3. If $\hat{g}(t)$ is a Hilbert transform of g(t) , then the Hilbert transform of $\hat{g}(t)$ is $-g(t)$.



Example on Hilbert Transform
Find the Hilbert transform of the impulse function $g(t) = \delta(t)$

Solution:
Here, we use the convolution in the time domain

$$\hat{g}(t) = \frac{1}{\pi t} * \delta(t)$$

As we know, the convolution of the delta function with a continuous function is the function itself. Therefore,

$$\hat{g}(t) = \frac{1}{\pi t}$$

Example on Hilbert Transform

Find the Hilbert transform of $g(t) = \frac{\sin t}{t}$

<u>Solution</u>

Here, we will first find the Fourier transform of $g(t)$, find $\hat{G}(f)$, and then find $\hat{g}(t)$

$$A\, rect\left(\frac{t}{\tau}\right) \quad \xleftrightarrow{transform} \quad A\tau\, sinc\, f\tau \quad ; \quad when\ \tau = \frac{1}{\pi}$$

$$A \text{ rect}\left(\frac{t}{1/\pi}\right) \quad \xleftrightarrow{transform} \quad A\ \frac{1}{\pi}\ \frac{\sin \pi f \tau}{\pi f \tau} = \frac{1}{\pi}\frac{\sin f}{f}$$

$$\pi\, rect\left(\frac{t}{1/\pi}\right) \quad \xleftrightarrow{transform} \quad \frac{\sin f}{f}$$

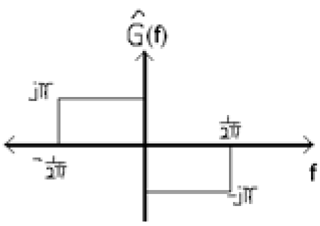
So by the duality property, we get the pair

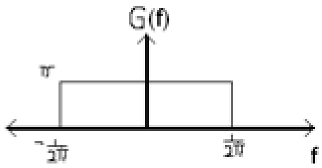
$$\pi\, rect\left(\frac{f}{1/\pi}\right) \quad \xleftrightarrow{transform} \quad \frac{\sin t}{t}$$

i.e. , $\quad G(f) = \pi\, rect(\frac{f}{1/\pi})$, (See the figure below)

$$\hat{G}(f) = -jsgn(f)G(f) = \begin{cases} -j\pi & 0 < f < 1/2\pi \\ j\pi & -1/2\pi < f < 0 \end{cases}$$

$$\begin{aligned} \hat{g}(t) &= \int_{-\infty}^{\infty} \hat{G}(f)\, e^{j2\pi f t}\, df \\ &= \int_{-1/2\pi}^{0} j\pi \quad e^{j2\pi f t}\, df \ - \ \int_{0}^{1/2\pi} j\pi \quad e^{j2\pi f t}\, df \\ &= \frac{1}{2t}\left(1 - e^{-jt}\right) - \frac{1}{2t}\left(e^{jt} - 1\right) \\ &= \frac{1}{t} - \frac{1}{t}\ \frac{\left(e^{jt} + e^{-jt}\right)}{2} \\ &= \frac{1 - \cos t}{t} \end{aligned}$$





Correlation and Spectral Density (Details are not required for ENEE 339)

Here, we consider the relationship between the autocorrelation function and the power spectral density. In this discussion we restrict our attention to real signals. First, we consider power signals and then energy signals.

Definition: The autocorrelation function of a signal g(t) is a measure of similarity between g(t) and a delayed version of g(t).

a. **Autocorrelation function of a power signal**

The autocorrelation function of a power signal g(t) is defined as:

$R_g(\tau) = \langle g(t)g(t-\tau) \rangle;$ $\quad\quad$ $\langle (.) \rangle$: represents time average.

$$R_g(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} g(t)g(t-\tau)dt$$

Exercise: Show that for a periodic signal with period T_0, the above definition becomes

$$R_g(\tau) = \frac{1}{T_0} \int_0^{T_0} g(t)g(t-\tau)dt$$

Remark: We can take this as a definition for the autocorrelation function of a periodic signal.

Exercise: Show that if $g(t)$ is periodic with period T_0, then $R_g(\tau)$ is also periodic with the same period T_0.

Hint: Expand $g(t)$ in a complex Fourier series $g(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0\tau}$. Form the delayed signal $g(t-\tau)$, and then perform the integration over a complete period T_0. You should get the following result:

$$R_g(\tau) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0\tau}; \quad\quad D_n = |C_n|^2$$

This formula bears two results

a. $R_g(\tau)$ is periodic with period T_0.
b. The Complex Fourier coefficients D_n of $R_g(\tau)$ are related to the complex Fourier coefficients C_n of $g(t)$ by the relation $D_n = |C_n|^2$.

Properties of R(τ)

- $R_g(0) = \frac{1}{T_0} \int_0^{T_0} g(t)^2 dt;$ is the total average signal power.
- $R_g(\tau)$ is an even function of τ, i.e., $R_g(\tau) = R_g(-\tau)$.
- $R_g(\tau)$ has a maximum (positive) magnitude at $\tau = 0$, i.e. $|R_g(\tau)| \le R_g(0)$.
- If g(t) is periodic with period T_0, then $R_g(\tau)$ is also periodic with the same period T_0.

- The autocorrelation function of a periodic signal and its power spectral density (represented by a discrete set of impulse functions) are Fourier transform pairs

$$S_g(f) = \mathrm{F}\{R_g(\tau)\}$$

$$S_g(f) = \sum_{n=-\infty}^{\infty} |C_n|^2 \delta(f - nf_0)$$

Cross Correlation Function
The cross correlation function of two periodic signals $g_1(t)$ and $g_2(t)$ with the same period T_0 is defined as:

$$R_{1,2}(\tau) = \frac{1}{T_0}\int_0^{T_0} g_1(t)g_2(t-\tau)dt$$

b- **Autocorrelation function of an energy signal**

When g(t) is an energy signal, $R_g(\tau)$ is defined as:

$$R_g(\tau) = \int_{-\infty}^{\infty} g(t)g(t-\tau)dt$$

Properties of R(τ)

- $R_g(0) = \int_{-\infty}^{\infty} g(t)^2 dt$; is the total signal energy.
- $R_g(\tau)$ is an even function of τ, i.e., $R_g(\tau) = R_g(-\tau)$.
- $R_g(\tau)$ has a maximum (positive) magnitude at τ = 0 , i.e. $|R_g(\tau)| \leq R_g(0)$.
- The autocorrelation function of an energy signal and its energy spectral density (a continuous function of frequency) are Fourier transform pairs, i.e.,

$$S_g(f) = \mathrm{F}\{R_g(\tau)\}$$

$$S_g(f) = \int_{-\infty}^{\infty} R_g(\tau)e^{-j2\pi f\tau}d\tau$$

$$R_g(\tau) = \int_{-\infty}^{\infty} S_g(f)e^{j2\pi f\tau}df.$$

Proof:
The autocorrelation function is defined as:

$$R_g(\tau) = \int_{-\infty}^{\infty} g(\lambda)g(\lambda-\tau)d\,\lambda$$

In this integral we have replaced t by λ (both are dummy variables of integration).
With this substitution, we can rewrite the integral as

$$R_g(\tau) = \int_{-\infty}^{\infty} g(\lambda)g(-(\tau-\lambda))d\,\lambda$$

One can realize that $R_g(\tau)$ is nothing but the convolution of $g(\tau)$ and $-g(\tau)$. That is,

$$R_g(\tau) = g(\tau) * g(-\tau)$$

Taking the Fourier transform of both sides, we get

$$F\{R_g(\tau)\} = G(f)G^*(f)$$

Therefore, $S_g(f) = \mathrm{F}\{R_g(\tau)\} = |G(f)|^2$.

Cross Correlation Function

The cross correlation function of two energy signals $g_1(t)$ and $g_2(t)$ is defined as;

$$R_{1,2}(\tau) = \int_{-\infty}^{\infty} g_1(t)g_2(t-\tau)dt$$

Example:

Find the auto-correlation function of the sine signal $g(t) = A\cos(2\pi f_0 t + \theta)$, where A and θ are constants.

Solution:

As we know, $g(t)$ is a periodic signal. So, we find $R_g(\tau)$ using the definition

$$R_g(\tau) = \frac{1}{T_0}\int_0^{T_0} g(t)g(t-\tau)dt$$

$$R_g(\tau) = \frac{1}{T_0}\int_0^{T_0} A\cos(2\pi f_0 t + \theta)A\cos(2\pi f_0 t - 2\pi f_0 \tau + \theta)dt$$

$$R_g(\tau) = \frac{A^2}{2T_0}\int_0^{T_0}[\cos(4\pi f_0 t - 2\pi f_0 \tau + 2\theta) + \cos(2\pi f_0 \tau)]dt$$

$$R_g(\tau) = \frac{A^2}{2T_0}[0 + \cos(2\pi f_0 \tau)T_0]$$

$R_g(\tau) = \frac{A^2}{2}\cos(2\pi f_0 \tau)$; periodic with period T_0.

Example:

Determine the autocorrelation function of the sinc pulse $g(t) = Asinc2Wt$.

Solution:

Using the duality property of the Fourier transform, we can deduce that

$$G(f) = \frac{A}{2W}rect(\frac{f}{2W})$$

The energy spectral density of $g(t)$ is

$$S_g(f) = |G(f)|^2 = (\frac{A}{2W})^2 rect(\frac{f}{2W})$$

Taking the inverse Fourier transform, we get the autocorrelation function

$$R_g(\tau) = \frac{A^2}{2W} sinc 2Wt$$

Exercise:

a. Find and plot the cross correlation function of the two signals

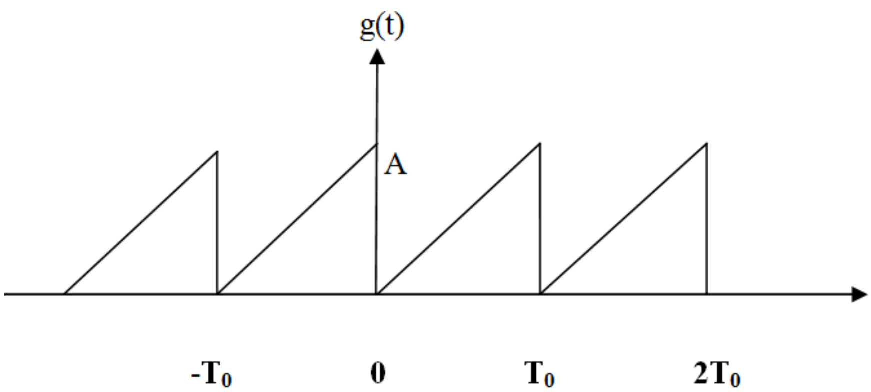
$$g_1(t) = \begin{cases} 1 & 0 \le t \le 2 \\ 0, otherwise \end{cases}$$

$$g_2(t) = \begin{cases} 1 & 0 \le t \le 1 \\ -1 & 1 < t \le 2 \end{cases}$$

b. Are $g_1(t)$ and $g_2(t)$ orthogonal?

Exercise:

Find and plot the autocorrelation function for the periodic saw-tooth signal shown below:



Example:

Find the autocorrelation function of the rectangular pulse g(t).

Solution:

As we saw earlier, this pulse is an energy signal and , therefore, we can find its $R_g(\tau)$ as:

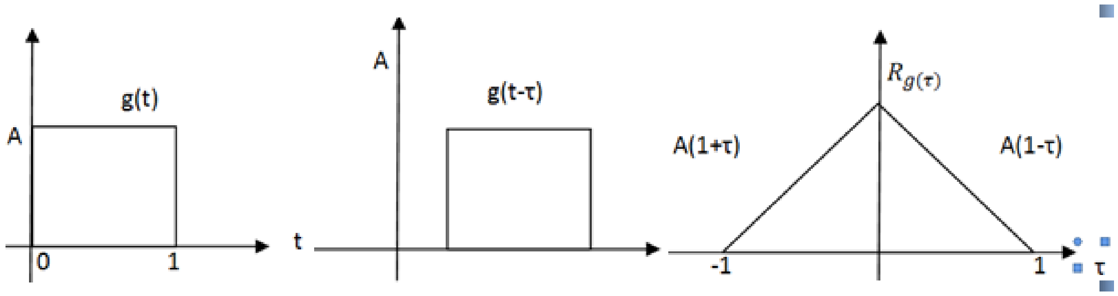
$$R_g(\tau) = \int_{\tau}^{1} (A)(A)dt = A^2\,(1-\tau)\ ;\ 0< \tau <1$$

Using the even symmetry property of the autocorrelation function, we can find $R_g(\tau)$ for – ve values of τ as:

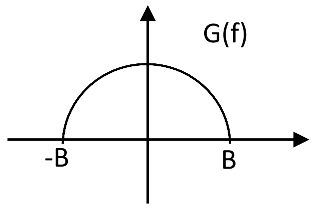
$$R_g(\tau) = = A^2\,(1+\tau)\ ;\ -1< \tau <0$$

This function is sketched below. Note that that the maximum value occurs at τ = 0 and that g(t) and g(t-τ) become uncorrelated for τ = 1 sec, which is the duration of the pulse.

The energy spectral density is $S_g(f) = F\{R_g(\tau)\} = A^2 sinc^2 f$



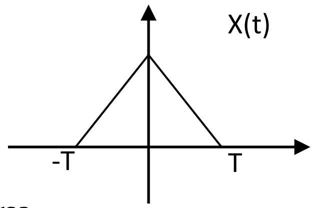
Bandwidth of Signals and Systems:



Def: A signal g(t) is said to be (absolutely) band-limited to BHz if

$G(f) = 0 \qquad \text{for } |f| > B$

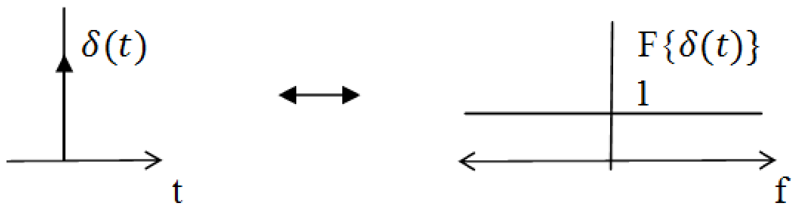
Def: A signal x(t) is said to be (absolutely) time-limited if



$x(t) = 0 \qquad \text{for } |t| > T$

Theorem: An absolutely band-limited waveform cannot be absolutely time-limited (theoretically has an infinite time duration) and vice versa.

We have earlier seen examples that support this theorem. For example, the delta function, which has an almost zero time duration, has a Fourier transform which extends uniformly over all frequencies (infinite bandwidth). Also, a constant value in the time domain (a dc) has a Fourier transform, which is an impulse in the frequency domain. This is repeated here for convenience.



In general, there is an inverse relationship between the signal bandwidth and the time duration. The bandwidth and the time duration are related through a relation, called the *time bandwidth product*, of the form

$$\textit{Bandwidth} \ * \textit{Time Duration} \ \geq \ \textit{constant}$$

The value of the constant depends on the way the bandwidth and the time duration of a signal are defined as will be illustrated later (Possible values of the constant = $\frac{1}{2}$, $\frac{1}{4\pi}$)

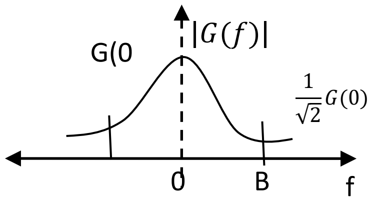
Remarks:

1. The bandwidth of a signal provides a measure of the extent of significant frequency content of the signal.
2. The bandwidth of a signal is taken to be the width of a positive frequency band.

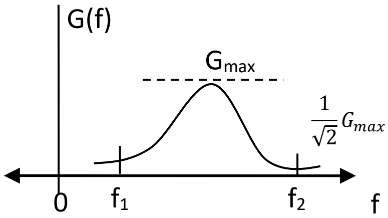
3. For baseband signals or networks , where the spectrum extends from –B to B, the bandwidth is taken to be B Hz.
4. For bandpass signals or systems where the spectrum extends between (f_1, f_2) and ($-f_1$, $-f_2$), the B.W= $f_2 - f_1$.

<u>Some Definitions of Bandwidth:</u>

1- Absolute bandwidth
Here, the Fourier transform of a signal is non zero only within a certain frequency band. If G(f) = 0 for |f| >B , then g(t) is absolutely band-limited to BHz. When $G(f) \neq 0$ for $f_1 < |f| < f_2$, then the absolute bandwidth is $f_2 - f_1$.



2- 3-dB (half power points) bandwidth
The range of frequencies from 0 to some frequency B at which |G(f)| drops to $\frac{1}{\sqrt{2}}$ of its maximum value (for a low pass signal).
As for a band pass signal, the B.W = $f_2 - f_1$



3- The 95 % (energy or power) bandwidth.
Here , the B.W is defined as the band of frequencies where the area under the energy spectral density (or power spectral density) is at least 95% (or 99%) of the total area .

Total Energy = $\int_{-\infty}^{\infty} |G(f)|^2\, df = 2\int_0^{\infty} |G(f)|^2 df$

$\int_{-B}^{B} |G(f)|^2 df = 0.95 \int_{-\infty}^{\infty} |G(f)|^2\, df$

4- Equivalent Rectangular Bandwidth.
It is the width of a fictitious rectangular spectrum such that the power in that rectangular band is equal to the power associated with the actual spectrum over positive frequency
Area under rectangle = Area under curve

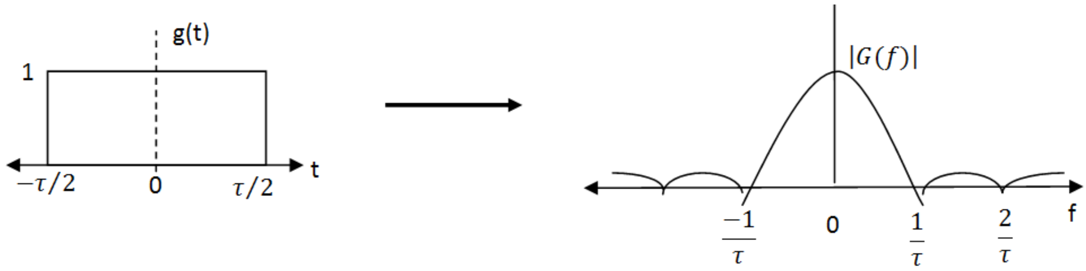
$|G(0)|^2 * 2B_{eq} = \int_{-\infty}^{\infty} |G(f)|^2 df$

$|G(0)|^2 * 2B_{eq} = 2\int_0^{\infty} |G(f)|^2 df$

$B_{eq} = \frac{1}{|G(0)|^2} \int_0^{\infty} |G(f)|^2 df$

5- <u>Null – to –null bandwidth:</u>

For baseband signals , B.W is the first null in the envelope of the magnitude spectrum above zero.



$\mathrm{rect}(\frac{t}{\tau}) \rightarrow \tau\ \mathrm{sinc} f\tau\ = \tau\, \frac{\sin\pi f\tau}{\pi f\tau}$

Zero crossing take place when $\sin(\pi f\tau) = 0$

$$\pi f\tau = n\pi \ \rightarrow\ f = \frac{n}{\tau}\ ;\ n = 1,2,\dots\dots\dots$$

B.W $= \frac{1}{\tau}$ smaller τ large bandwidth.

For a band pass signal, B.W= $f_2 - f_1$

6- <u>Bounded spectrum bandwidth:</u>

Range of frequencies as (0,B) such that outside the band , the power spectral density must be down by say 50 dB below the maximum value

$-50\ \text{dB} = 10 \log \frac{|G(B)|^2}{|G(0)|^2}$

7- <u>RMS Bandwidth:</u>

$$B_{rms} = \left(\frac{\int_{-\infty}^{\infty} f^2 |G(f)|^2 df}{\int_{-\infty}^{\infty} |G(f)|^2 df}\right)^{1\backslash 2}$$

The corresponding rms duration of g(t) is

$$T_{rms} = = \left(\frac{\int_{-\infty}^{\infty} t^2 |g(t)|^2 dt}{\int_{-\infty}^{\infty} |g(t)|^2 dt}\right)^{1\backslash 2}$$

(here g(t) is assumed to be centered around the origin).

Remark**:** The time bandwidth product is $T_{rms}\, B_{rms} \geq \frac{1}{4\pi}$

Time – Bandwidth Product

To illustrate the time – bandwidth product, consider the equivalent rectangular bandwidth defined earlier as

$$B_{eq} = \frac{\int_{-\infty}^{\infty} |G(f)|^2 df}{2|G(0)|^2}$$

Analogous to this definition, we define an equivalent rectangular time duration as :

$$T_{eq} = \frac{\left(\int_{-\infty}^{\infty} |g(t)|dt\right)^2}{\int_{-\infty}^{\infty} |g(t)|^2 \mathrm{dt}}$$

The time bandwidth product is

$$B_{eq}T_{eq} = \frac{\int_{-\infty}^{\infty} |G(f)|^2 \mathrm{df}}{2|G(0)|^2} \cdot \frac{\left(\int_{-\infty}^{\infty} |g(t)|dt\right)^2}{\int_{-\infty}^{\infty} |g(t)|^2 \mathrm{dt}}$$

Note $\int_{-\infty}^{\infty} |g(t)|^2 \mathrm{dt} = \int_{-\infty}^{\infty} |G(f)|^2 \mathrm{df}$; Rayleigh energy theorem. Note also that $G(0) = \int_{-\infty}^{\infty} g(t)dt$. Using these relations, we get

$$B_{eq}T_{eq} = \frac{1}{2} \frac{\left(\int_{-\infty}^{\infty} |g(t)|dt\right)^2}{|\int_{-\infty}^{\infty} g(t)\mathrm{dt}|^2}$$

Case 1: When g(t) is positive for all time t, then |g(t)| = g(t) and $B_{eq}T_{eq}$ becomes

$$B_{eq}T_{eq} = \frac{1}{2}$$

Case 2 : For a general g(t) that can take on positive as well as negative values, $B_{eq}T_{eq}$ satisfies the inequality

$$B_{eq}T_{eq} \geq \frac{1}{2}$$

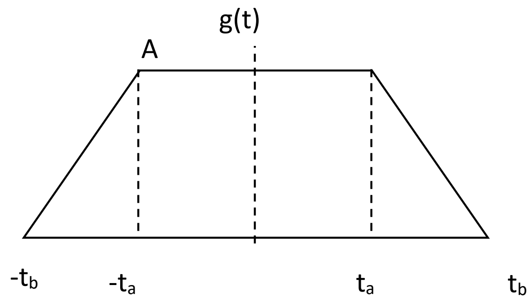
Note : For B_{rms} and T_{rms} , the time – bandwidth satisfies the inequality

$$B_{rms}\, T_{rms} \geq \frac{1}{4\pi}$$

Example : Bandwidth of a trapezoidal signal

Find the equivalent rectangular bandwidth, B_{eq}, for the trapezoidal pulse shown.

Solution :



$$T_{eq} = \frac{\left(\int_{-\infty}^{\infty} |g(t)|dt\right)^2}{\int_{-\infty}^{\infty} |g(t)|^2 \mathrm{dt}}$$

$$\int_{-\infty}^{\infty} |g(t)|dt = A\,(t_a + t_b)$$

$$\int_{-\infty}^{\infty} |g(t)|^2 dt = \frac{2A^2}{3}\,(2t_a + t_b)$$

$$T_{eq} = \frac{3}{2}\frac{(t_a+t_b)^2}{(2t_a+t_b)}$$

$$B_{eq} = \frac{0.5}{T_{eq}} = \frac{2t_a+t_b}{3(t_a+t_b)^2}.$$

Remark: Note that using this method we were able to determine the signal bandwidth without the need to go through the Fourier transform.

Exercise: Use the above method to find the equivalent rectangular bandwidth for the triangular signal $g(t) = tri(\frac{t}{T})$.

Example: Bandwidth of a periodic signal:

Find the bandwidth For the periodic square function define over one period as

$$g(t) = \begin{cases} 2A, & \frac{-T}{4} \leq t \leq \frac{T}{4} \\ -A, & o.w \end{cases}$$

Solution:

The average power, computed using the time average, is

$$P_{av} = \frac{1}{T_0}\int_0^{T_0} |g(t)|^2 \ dt$$

$$= \frac{1}{T_0}[4A^2\tau + \ A^2\tau] = \frac{5A^2\tau}{2\tau} = \frac{5A^2}{2} = 2.5A^2$$

Also, by using the Parseval's theorem, the average power can be computed as:

$$P_{av} = \ |C_0|^2 + \ 2\sum_{n=1}^{\infty}|C_n|^2$$

We recall that the Fourier coefficients for this signal were found in Chapter 1. Using these values we get

$$P_{av} \ = \ \left(\frac{A}{2}\right)^2 + 2\sum_{n=1}^{\infty}\frac{(3A)^2}{(n\pi)^2}$$

$$P_{av} = \ \frac{A^2}{4} + \ 2A^2\sum_{n=1}^{\infty}\frac{(3)^2}{(n\pi)^2}$$

Let us take n = 1

$$P_1 = \ A^2\left\{0.25 + 2\ .\ \frac{9}{\pi^2}\right\} = 2.073A^2$$

$$\frac{P_1}{P_{av}} = \frac{2.073A^2}{2.5A^2} = 82.95\%$$

(This is the percentage of the total power that lies in the dc and the fundamental frequency).

For n = 3

$$P_3 = A^2 \left\{0.25 + 2 \left(\frac{3^2}{\pi^2} + \frac{3^2}{3^2\pi^2}\right)\right\} = 2.276A^2$$

$$\frac{P_3}{P_{av}} = \frac{2.276A^2}{2.5A^2} = 91.05\%$$

(Fraction of power in the dc, fundamental and third harmonic terms)

For n = 5

$$P_5 = A^2 \left\{0.25 + 2 \left(\left(\frac{3}{\pi}\right)^2 + \left(\frac{3}{3\pi}\right)^2 + \left(\frac{3}{5\pi}\right)^2\right)\right\} = 2.349A^2$$

$$\frac{P_5}{P_{av}} = \frac{2.349A^2}{2.5A^2} = 93.97\%$$

Here , the 93% power band width is $5f_0$.

Example: Bandwidth of an energy signal .

If the signal $g(t) = Ae^{-\propto t}\ u(t)$ is passed through an ideal LPF with B.W = B Hz,

find the fraction of the signal energy contained in B.

Solution

The Fourier transform of g(t) is:

$$G(f) = \frac{A}{\propto + j2\pi f}$$

The energy in g(t), using the time domain, is

$$E_g = \int_0^\infty |g(t)|^2\, dt = \int_0^\infty A^2 e^{-2\propto t}\, dt = \frac{A^2}{2\propto}$$

Energy contained in the filter output y(t) is

$$E_y = \int_{-B}^{B} |G(f)|^2\, df = \int_{-B}^{B} \frac{A^2}{(\propto^2 + (2\pi f)^2)}\, df$$

$$E_y = \frac{2A^2}{2\pi \propto} tan^{-1}\frac{2\pi B}{\propto}$$

The ratio of E_y to the total energy is

$$\frac{E_y}{E_g} = \frac{2}{\pi} tan^{-1}\frac{2\pi B}{\propto}$$

The table below shows this ratio for various values of B .

B	$\left(E_y/E_g\right) \times \mathbf{100}$
$\frac{\propto}{4}$	63.9
$\frac{\propto}{2}$	80.38
$\propto$	89.95
$2 \propto$	94.94

Thus, the 95% energy bandwidth is $2 \propto$.

Exercise: Find the 98% energy bandwidth.

Pulse Response and Risetime

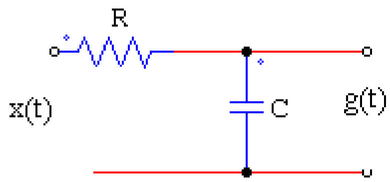
A rectangular pulse contains significant high frequency components. When that pulse is passed through a LPF, the high frequency components will be attenuated resulting in signal distortion.

We need to investigate the relationship that should exist between the pulse bandwidth and the channel bandwidth. This subject is of particular importance, especially, when we study the transmission of data over band-limited channels. In the simplest form, a binary digit 1 may be represented by a pulse $A,\ \ 0 \le t \le T_b$, while binary digit 0 may be represented by the negative pulse $-A,\ \ 0 \le t \le T_b$. So, in order to retrieve the transmitted data, the channel bandwidth must be wide enough to accommodate the transmitted data.

To convey this idea in a simple form, we first consider the response of a first order low pass filter to a unit step function and then to a pulse.

Step response of a first order LPF (channel)

Let x(t) = u(t) be applied to a first order RC circuit. This first order filter is a fair representation of a low pass communication channel.



The system differential equation is

$$x(t) = Ri(t) + g(t) = RC\frac{dg(t)}{dt} + g(t)$$

where g(t) is the channel output.

$$RC\frac{dg(t)}{dt} + g(t) = u(t)$$

The solution to this first order system is

$$g(t) = (1-e^{-t/RC})\ u(t)$$

The 3-db B.W of the channel is

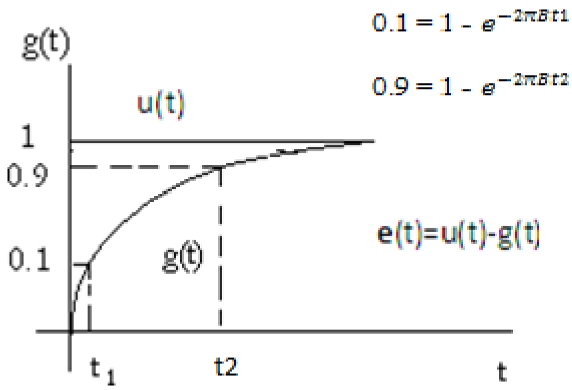
$$B= \frac{1}{2\pi RC}\ \text{(to be derived shortly)}$$

$$g(t) = (1-e^{-2\pi Bt})\ u(t)$$

Define the difference between the input and the output as:

$\mathbf{e}(t) = u(t) - g(t) = e^{-2\pi Bt}$

Note that e(t) decreases as B increases. Meaning that as the channel bandwidth increases, the output becomes closer and closer to the input. In the ideal case, when the channel bandwidth becomes infinity, the output becomes a step function. In essence, to reproduce a step function (or a rectangular pulse), a channel with infinite bandwidth is needed.



The Risetime

The Rise time is a measure of the speed of a step response. One common measure is the 10-90 % rise time defined as the time it takes for the output to rise between 10% to 90% of the final (steady state) value (1) when a step function is applied to a LIT system. For the step response g(t) and the first order RC circuit considered above, the rise time can be easily calculated as:

$t_r = t_2 - t_1 \approx \frac{0.35}{B}$

From this result, we conclude that: **increasing the bandwidth of the channel will decrease the rise time** (a faster response).

Exercise: For the system above, verify that the rise time is given as $t_r = \frac{0.35}{B}$

Exercise: Find the 10-90% rise time for a second order low pass filter with 3-dB bandwidth B and transfer function

$$H(f) = \frac{1}{P_2(\frac{jf}{B})}$$

Where, $P_2(x) = 1 + \sqrt{2}x + x^2$.

(Hints: You may let B=10, for example, use matlab to find the step response, and then find the rise time).

Pulse response

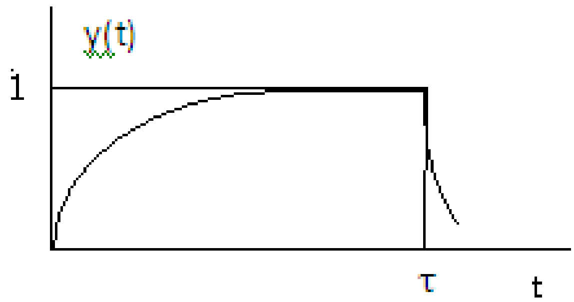
It is the response of the circuit to a pulse of duration τ. For the same circuit let us apply the pulse

x(t) = u(t) - u(t- τ)

Using the linearity and time invariance properties, the output can be obtained from the step response as:

$$y(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-t/RC} & 0 < t < \tau \\ \left(1 - e^{-\frac{\tau}{RC}}\right) . e^{-\frac{t-\tau}{RC}} & t > \tau \end{cases}$$

This is sketched in the figure below.



Bandwidth Considerations

The transfer function of the RC circuit is

$$H(f) = \frac{1/j2\pi fc}{R+1/j2\pi fc} = \frac{1}{1+j2\pi fRc}$$

$$|H(f)| = \frac{1}{\sqrt{1+(2\pi fRc)^2}}$$

Let B = $\frac{1}{2\pi Rc}$; 3-db bandwidth ; $2\pi fRc = 1$; f = $\frac{1}{2\pi Rc}$

Then , $H(f) = \frac{1}{1+jf/B}$

$$|H(f)| = \frac{1}{\sqrt{1+\left(\frac{f}{B}\right)^2}}$$

For the rectangular pulse x(t), we have

$$X(f) = sincf\tau$$

The first null frequency of X(f) is an estimate of the bandwidth B_x of x(t), which is of the order of $\approx \frac{1}{\tau}$.

1. When τ is large, such that signal bandwidth $B_x = \frac{1}{\tau} << B$ (channel B.W)

$$Y(f) = X(f)H(f) \approx X(f)$$

and the output resembles the input. There is enough time for x(t) to reach the maximum value .

2. When τ is small, such that signal $B_x = \frac{1}{\tau} >> B$ (channel B.W)

$$Y(f) = X(f)H(f) \approx H(f)$$

The signal suffers a considerable amount of distortion and $Y(f)$ is no longer proportional to X(f).

Band-pass Signals and Systems

(Details are not required for ENEE 339)

A signal g(t) is called a *band pass signal* if its Fourier transform G(f) is non-negligible only in a band of frequencies of total extent 2W centered about f_c .

A signal is called *narrowband* if 2W is small compared with f_c.

A band pass signal g(t) represented in the canonical form:

$$g(t) = g_I(t) \cos \omega_c t - g_Q(t) \sin\omega_c t.$$

$g_I(t)$ is a low pass signal of B.W = W Hz called the *in phase component* of g(t) .

$g_Q(t)$ is a low pass signal of B.W = W Hz called the *quadrature component* .

g(t) appears as a modulated signal in which $g_I(t)$ and $g_Q(t)$ are the low pass signals and f_c is the carrier frequency. Recall the modulation property of the Fourier transform :

$$x(t) \cos \omega_c t \rightarrow \frac{1}{2}(X(f- f_c)+ X(f+ f_c))$$
$$x(t) \sin \omega_c t \rightarrow \frac{1}{j2}(X(f- fc)- X(f+ fc))$$

Define the *complex envelope* of a signal g(t) as:

$$\tilde{g}(t) = g_I(t) + j\, g_Q(t)$$

$\tilde{g}(t)$ is a low pass signal of B.W =W. The signals g(t) and $\tilde{g}(t)$ are related by :

$$g(t) = \mathrm{Re}\{\tilde{g}(t)\, e^{j\omega ct}\}$$

How to get $g_I(t)$ and $g_Q(t)$ from g(t)

If we multiply g(t) by cos $\omega_c t$, we get

$$g(t) \cos\omega_c t = g_I(t) \cos^2 \omega_c t - g_Q(t) \sin \omega_c t \cos \omega_c t$$

$$= \frac{1}{2} g_I(t) + \frac{1}{2} g_I(t) \cos 2\omega_c t - \frac{1}{2} g_Q(t) \sin 2\omega_c t .$$

The first term is the desired low pass signal. The second and third terms are high frequency components centered about 2 f_c.

$$g_I(t) = \mathrm{lowpass}\{2g(t) \cos\omega_c t\}$$

Or, in the frequency domain

$$G_I(f)= \begin{cases} G(f - fc) + G(f + fc) & -w \le f \le w \\ 0 & otherwise \end{cases}$$

Now if we multiply g(t) by sinω_ct, we get

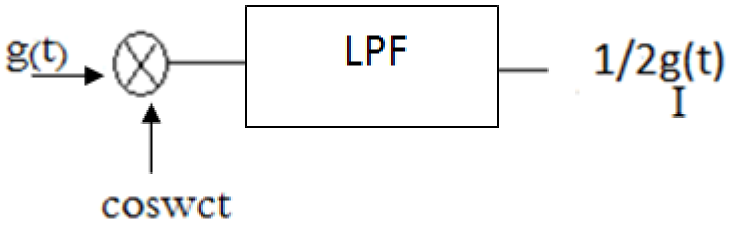
$$g(t)\sin\omega_c t = g_I(t)\sin\omega_c t\cos\omega_C t - g_Q(t)\sin^2\omega_c t$$

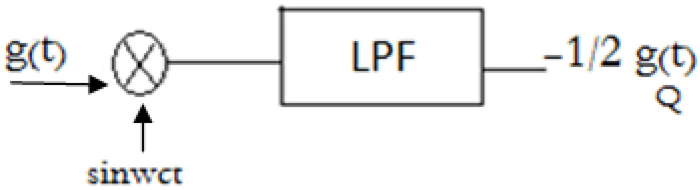
$$= -\frac{1}{2}g_Q(t) + \frac{1}{2}g_I(t)\sin 2\omega_c t + \frac{1}{2}g_Q(t)\cos 2\omega_c t$$

Again, the first term is a low pass signal, while the second and third are high frequency terms centered about 2 f_c.

$g_Q(t)$ = - low pass{2g(t) sinω_ct}In the frequency domain, this is equivalent to

$$G_Q(t) = \begin{cases} j[G(f-fc) - G(f+fc) & -w \le f \le w \\ 0 & otherwise \end{cases}$$





Band pass systems:

The analysis of band pass systems can be simplified by using the complex envelope concept. Here, results and techniques from low pass systems can be easily applied to band pass systems .

The problem to be addressed is :

The input x(t) is a band pass signal

$$x(t) = x_I(t)\cos\omega_c t - x_Q(t)\sin\omega_c t$$

x(t) is applied to a band pass filter represented as:

$$h(t) = h_I(t)\cos\omega_c t - h_Q(t)\sin\omega_c t$$

The objective is to find the filter output y(t). The output is, of course, the convolution of x(t) and h(t) (y(t) = x(t)*h(t)), which can also be expressed as:

$$y(t) = y_I(t)\cos\omega_c t - y_Q(t)\sin\omega_c t$$

Due to the band-pass nature of the problem, carrying out the direct convolution will be a tedious task due to the presence of the sin and cos functions in all terms. The complex envelope concept simplifies the problem to a very great extent. The procedure is summarized as follows:

a. Form the complex envelope for both the input and the channel:

$\tilde{x}$(t)= x_I(t) + jx_Q(t)

$\tilde{h}$(t)= h_I(t) + jh_Q(t)

b. Carry out the convolution between $\tilde{x}$(t) and $\tilde{h}$(t). Note that both signals are low pass signals and so $\tilde{y}$(t) is also low pass.

$$2\,\tilde{y}(t)= \tilde{h}(t) * \tilde{x}(t)$$

$$\tilde{y}(t) = y_I(t) + jy_Q(t)$$

c. The band-pass filter output is obtained from the low pass signal $\tilde{y}$(t) through the relation

$$y(t) =\mathrm{Re}\{\tilde{y}(t)\, e^{jw_c t}\}$$

or through the relation

$$y(t) = y_I(t)\cos\omega_c t - y_Q(t)\sin\omega_c t$$

Example :

The rectangular radio frequency (RF) pulse

$$x(t) = \begin{cases} A\ cos2\pi f_c t & 0 \le t \le T \\ 0 & otherwise \end{cases}$$

is applied to a linear filter with impulse response (We will see later that this is a filter matched to x(t), called the *matched filter*).

$$h(t) = x(T - t)$$

Assume that T= nT_C; n is an integer, $T_C = \frac{1}{fc}$, determine the response of the filter and sketch it.

Solution: We follow the three steps outlined above.

$h(t) = A\ cos2\pi f_c(T - t)$

$= A\ cos2\pi f_c T\ cos2\pi f_c t + A\ sin2\pi f_c T sin2\pi f_c$ t

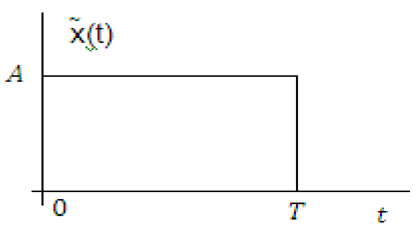
$$=\underbrace{A\ cos2\pi\left(\frac{nT_c}{T_c}\right)}_{cos2n\pi \equiv 1} cos2\pi f_c t + \underbrace{A\ sin2\pi\left(\frac{nT_c}{T_c}\right)}_{sin2n\pi \equiv 0} sin2\pi f_c t$$

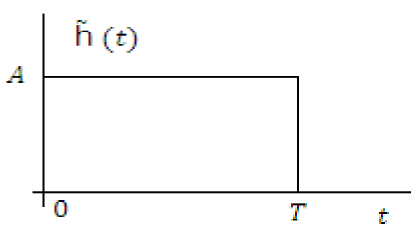
Therefore, $h(t) = \begin{cases} A\ cos2\pi f_c t & 0 \le t \le T \\ 0 & otherwise \end{cases}$

The complex envelopes of x(t) and h(t) are (step a)

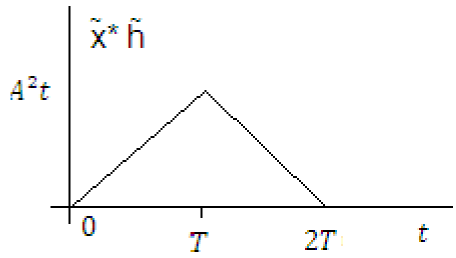
$$\tilde{x}(t) = \begin{cases} A & 0 \le t \le T \\ 0 & o.w \end{cases}$$

$$\tilde{h}(t) = \begin{cases} A & 0 \le t \le T \\ 0 & o.w \end{cases}$$





$\tilde{y}(t) = \tilde{x}(t) * \tilde{h}(t)$ is the triangular signal shown in the Figure (step b).

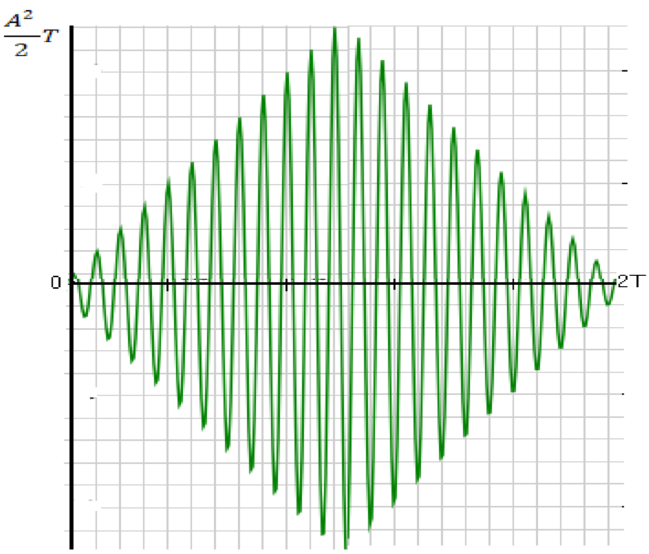


$$2\tilde{y}(t)=\begin{cases} A^2t & 0 \le t \le T \\ A^2(2T-t) & T \le t \le 2T \end{cases}$$

The bandpass signal is obtained as (step c)

$$y(t)=\begin{cases} \frac{A^2}{2}t\, cosw_c t & 0 \le t \le T \\ \frac{A^2}{2}(2T-t)\, cosw_c t & T \le t \le 2T \end{cases}$$

and is sketched as in the figure below.



Exercise

The band-pass signal $x(t) = e^{-\frac{t}{\tau}}\cos(2\pi f_c t)\, u(t)$ is applied to a band-pass filter with impulse response $h(t)$ given as:

$$h(t) = \begin{cases} A\, cos 2\pi f_c t & 0 \le t \le T \\ 0 & otherwise \end{cases}$$

Find and sketch the filter output.