# Frequency and Phase Modulation

To generate an angle modulated signal, the amplitude of the modulated carrier is held constant while either the phase or the time derivative of the phase is varied linearly with the message signal m(t).

The expression for an angle modulated signal is:

$$s(t) = A_c \cos\big(\omega_c t + \theta(t)\big)\ ,\ \omega_c \text{ is the carrier frequency.}$$

The instantaneous frequency of s(t) is :

$$f_i(t) = \frac{1}{2\pi}\frac{d}{dt}\big(\omega_c t + \theta(t)\big) = f_c + \frac{1}{2\pi}\frac{d\theta(t)}{dt}$$

For **phase modulation**, the phase is directly proportional to the modulating signal :

$$\theta(t) = k_p m(t), \qquad k_p \text{ is the phase sensitivity measured in rad/volt.}$$

The peak phase deviation is

$$\Delta\theta \quad = k_p \times \max\ \big(m(t)\big).$$

For **frequency modulation**, the frequency deviation of the carrier is proportional to the modulating signal:

$$\frac{1}{2\pi}\frac{d\theta(t)}{dt} = k_f m(t)\ \Rightarrow f_i = f_c + k_f m(t).$$

The frequency deviation from the un-modulated carrier is

$$f_i(t) - f_c = \frac{1}{2\pi}\frac{d\theta}{dt}$$

The peak frequency deviation is

$$\Delta f = \max\ \{\frac{1}{2\pi}\frac{d\theta}{dt}\}.$$

The time domain representation of a phase modulated signal is :

$$s(t) = A_c \cos\Big(\omega_c t + k_p m(t)\Big).$$

The time domain representation of a frequency modulated signal is

$$s(t) = A_c \cos\Big(\omega_c t + 2\pi k_f \int_{-\infty}^{t} m(\alpha)d\alpha\Big).$$

where $\theta(t) = 2\pi k_f \int_{-\infty}^{t} m(\alpha)d\alpha$

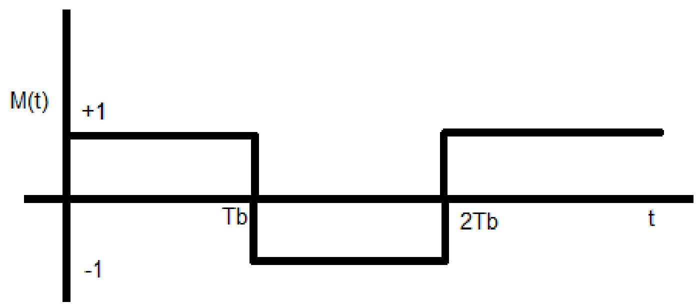
The average power in $s(t)$, for frequency modulation (FM) or phase modulation (PM) is:

$$p_{ava} = \frac{(A_c)^2}{2} = constant.$$

**Example: Binary Frequency Shift Keying.**

The periodic square signal m(t), shown below, frequency modulates the carrier $c(t) = A_c \cos(2\pi 100t)$ to produce the signal $s(t) = A_c \cos\left((2\pi 100t) + 2\pi k_f \int m(\alpha)d\alpha\right)$ where $k_f$ =10 HZ/V.

a. Find and plot the instantaneous frequency $f_i(t)$.
b. Find s(t).




**Solution**:

a) The instantaneous frequency is

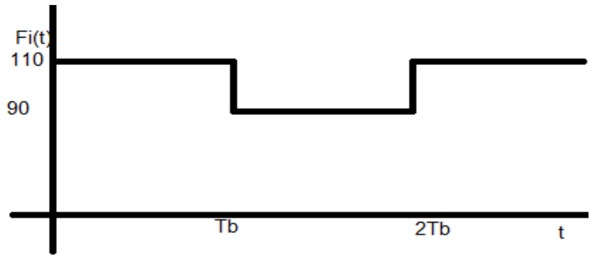
$f_i = f_c + k_f \times m(t)$

$f_i =$ 100+10 =110 Hz when $m(t) = +1$

$f_i =$ 100 -10 = 90Hz when $m(t) = -1$

For $0 < t \le T_b$ , $f_i = 110$ Hz
For $T_b \le t \le 2T_b$, $f_i = 90$ Hz

The instantaneous frequency hops between the two values 110 Hz and 90 Hz as shown below



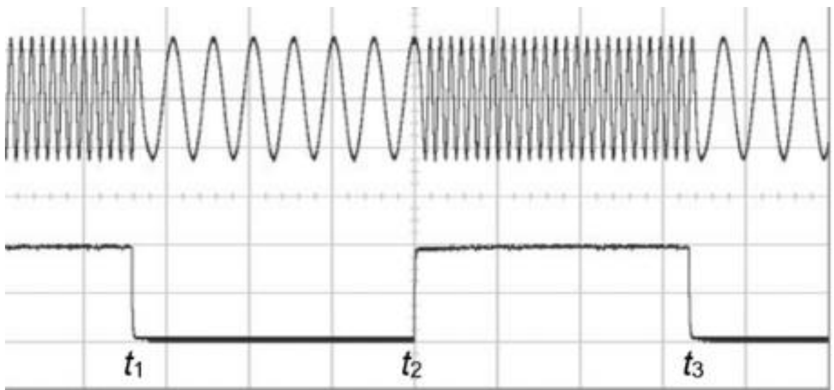
In digital transmission, we will see that a binary (1) may be represent by a signal of frequency $f_1$ for $0 \le t \le T_b$ and a binary (0) by a signal of frequency $f_2$ for $0 \le t \le T_b$ .

b) The two signals that represent the binary data are:

$s_1(t) = A\cos(2\pi(110)t\,),\qquad when\; m(t) = +1$

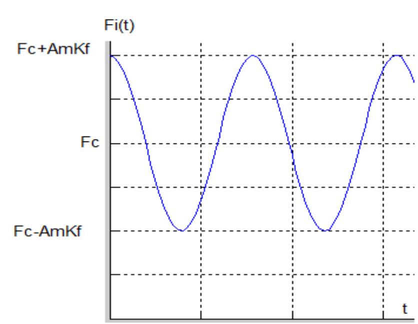
$s_2(t) = A\cos(2\pi(90)t\;\;),\qquad when\; m(t) = -1$

**Exercise**: Plot the transmitted signal $s(t)$ for $0 \le t \le 4T_b$ assuming $T_b = 10T_c$. You should obtain a figure similar to this figure




**Single Tone Frequency Modulation:**

Assume that the message $m(t) = A_m \cos\omega_m t$.




The instantaneous frequency is:

$$f_i = f_c + k_f m(t) = f_c + A_m k_f \cos 2\pi f_m t.$$

This frequency is plotted in the figure.

The peak frequency deviation (from the un-modulated carrier) is :

$$\Delta f = k_f A_m.$$

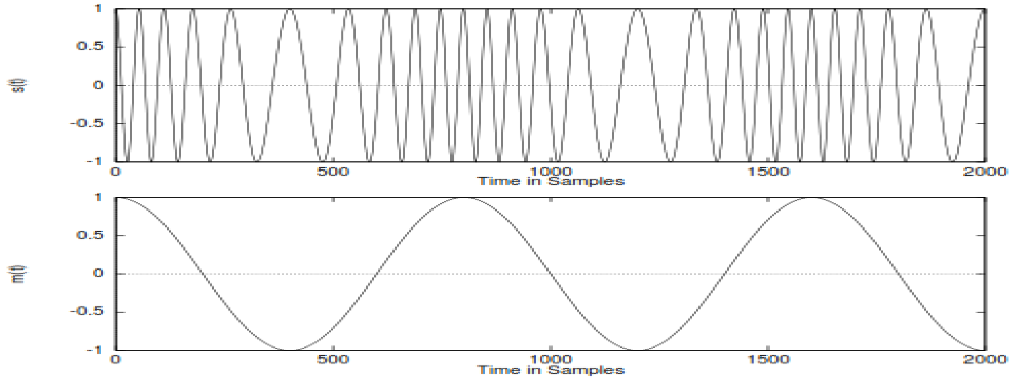
The FM signal is:

$$s(t) = A_c \cos\,(\omega_c t + \beta \sin 2\pi f_m t).$$

Where $\beta$ is the FM modulation index:

$$\beta = \frac{k_f A_m}{f_m} = \frac{peak\ frequency\ deviation}{message\ bandwidth} = \frac{\Delta f}{f_m}$$

In the figure below, we plot an FM signal when fm=2;fc=20;Ac=1;beta=5;




**Spectrum of a Single-Tone FM Signal**

The objective is to find a meaningful definition of the bandwidth of an FM signal:

Let $m(t) = A_m \cos 2\pi f_m t$ be the message signal, then the FM signal is:

$$s(t) = A_c \cos(\, 2\pi f_c t + \beta \sin 2\pi f_m t\,)$$

$s(t)$ can be rewritten as:

$$s(t) = Re\left\{e^{j(2\pi f_c t + \beta \sin 2\pi f_m t}\right\}$$

$$= Re\{e^{j(2\pi f_c t)} \times e^{j(\beta \sin 2\pi f_m t)}\}$$

Remember that: $e^{j\theta} = cos\theta + jsin\,\theta$ and that $cos\theta = Re\{e^{j\theta}\}$

The function [ $\beta \sin 2\pi f_m t$ ] is "sinusoidal" and periodic with $T_m = \frac{1}{f_m}$ . Therefore, $e^{j(\beta \sin 2\pi f_m t)}$ is also periodic with $T_m = \frac{1}{f_m}$ (but not sinusoidal)

As we know, a periodic function g(t) can be expanded into a complex Fourier series as:

$$g(t) = \sum_{-\infty}^{\infty} C_n\, e^{jn\omega_m t} \quad .$$

where,

$$C_n = \frac{1}{T_m}\int_0^{T_m} g(t)\, e^{-j\omega_m nt}\, dt$$

$\Longrightarrow$ If we let $g(t) = e^{j(\beta \sin 2\pi f_m t)}$

then, $C_n = \frac{1}{T_m} \int_0^{T_m} e^{j(\beta \sin 2\pi f_m t)} \times e^{-j\omega_m n t}\, dt$

It turns out that $\Longrightarrow \quad C_n = J_n(\beta)$.

where $J_n(\beta)$ is the Bessel function of the first kind of order n.

Hence, $g(t) = \sum_{-\infty}^{\infty} J_n(\beta)\, e^{jn\omega_m t}$

Substituting into $s(t)$, we get:

$$\Longrightarrow s(t) = A_c\, Re\{e^{j(2\pi f_c t)} \times \sum_{-\infty}^{\infty} J_n(\beta)\, e^{jn\omega_m t}\}$$

$$= A_c\, Re\{\sum_{-\infty}^{\infty} J_n(\beta) \times e^{j2\pi(f_c + n f_m)t}\}$$

$$= A_c \sum_{-\infty}^{\infty} J_n(\beta) \times \cos(2\pi(f_c + n f_m)t)$$

Finally, the FM signal can be represented as

$$s(t) = A_c \sum_{-\infty}^{\infty} J_n(\beta) \times \cos(2\pi(f_c + n f_m)t)$$

Bessel Functions:

The Bessel equation of order n is:

$$x^2 \frac{dy^2}{dx_2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0$$

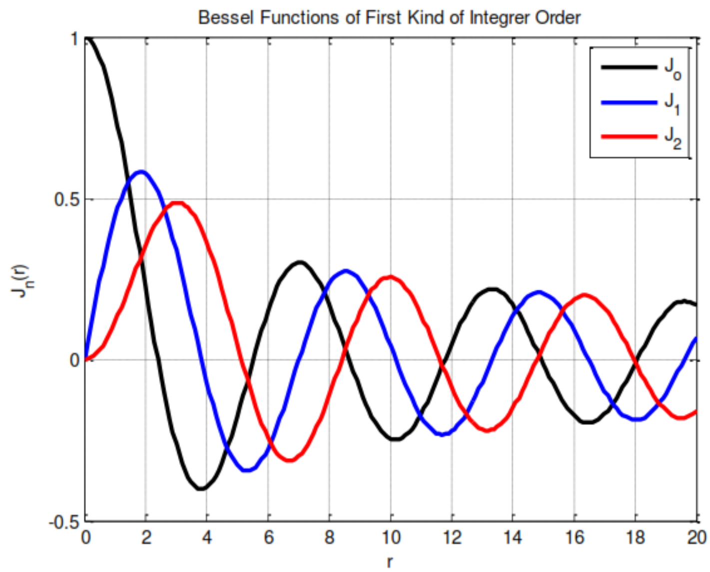
This is a second order differential equation with variable coefficient. We can solve it by the power series method, for example:

Let $y = \sum_{n=0}^{\infty} C_n\, x^n$ , $\frac{dy}{dx} = \sum_{n=1}^{\infty} n\, C_n\, x^{n-1}$ , $\frac{dy^2}{dx_2} = \sum_{n=2}^{\infty} n(n-1) C_n\, x^{n-2}$ .

Substituting $y, \frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ into the differential equation and equating terms of equal power results in:

$$y = \sum_{m=0}^{\infty} \frac{(-1)^m \times (\frac{1}{2}x)^{n+2m}}{m!\,(n+m)!}$$

The solution for each value of n (see the D.E where n appears) is $J_n(x)$, the Bessel function of the first kind of order n. The figure, below, shows the first three Bessel functions.




**Some Properties of $J_n(x)$:**

1- $J_n(x) = (-1)^n J_{-n}(x)$.

2- $J_n(x) = (-1)^n J_n(-x)$.

3- Recurrence formula

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x) .$$

4- For small values of x: $\Longrightarrow$ $J_n(x) \cong \frac{x^n}{2^n\ n!}$

Therefore, $J_0(x) \cong 1$

$J_1(x) \cong \frac{x}{2}$.

$J_n(x) \cong 0 \ \ for\ n > 1$

5- For large value of $x$:

$J_n(x) \cong \sqrt{\frac{2}{\pi x}} \cos(x - \frac{\pi}{4} - \frac{n\pi}{2})$ , $J_n(x)$ behaves like a sine function with progressively decreasing amplitude.

6- For real $x$ and fixed, $J_n(x) \longrightarrow 0\ as\ n \longrightarrow \infty$ .

7- $\sum_{-\infty}^{\infty} (J_n(x))^2 = 1 ,\ \ for\ all\ x.$

### *Table of Bessel Functions*

| $\beta$ | $J_0(\beta)$ | $J_1(\beta)$ | $J_2(\beta)$ | $J_3(\beta)$ | $J_4(\beta)$ | $J_5(\beta)$ | $J_6(\beta)$ | $J_7(\beta)$ | $J_8(\beta)$ | $J_9(\beta)$ | $J_{10}(\beta)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **0** | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **0.1** | 0.9975 | 0.0499 | 0.0012 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| **0.2** | 0.9900 | 0.0995 | 0.0050 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| **0.3** | 0.9776 | 0.1483 | 0.0112 | 0.0006 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| **0.4** | 0.9604 | 0.1960 | 0.0197 | 0.0013 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| **0.5** | 0.9385 | 0.2423 | 0.0306 | 0.0026 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| **0.6** | 0.9120 | 0.2867 | 0.0437 | 0.0044 | 0.0003 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| **0.7** | 0.8812 | 0.3290 | 0.0588 | 0.0069 | 0.0006 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| **0.8** | 0.8463 | 0.3688 | 0.0758 | 0.0102 | 0.0010 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| **0.9** | 0.8075 | 0.4059 | 0.0946 | 0.0144 | 0.0016 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| **1** | 0.7652 | 0.4401 | 0.1149 | 0.0196 | 0.0025 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| **1.1** | 0.7196 | 0.4709 | 0.1366 | 0.0257 | 0.0036 | 0.0004 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| **1.2** | 0.6711 | 0.4983 | 0.1593 | 0.0329 | 0.0050 | 0.0006 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| **1.3** | 0.6201 | 0.5220 | 0.1830 | 0.0411 | 0.0068 | 0.0009 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| **1.4** | 0.5669 | 0.5419 | 0.2074 | 0.0505 | 0.0091 | 0.0013 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| **1.5** | 0.5118 | 0.5579 | 0.2321 | 0.0610 | 0.0118 | 0.0018 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| **1.6** | 0.4554 | 0.5699 | 0.2570 | 0.0725 | 0.0150 | 0.0025 | 0.0003 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| **1.7** | 0.3980 | 0.5778 | 0.2817 | 0.0851 | 0.0188 | 0.0033 | 0.0005 | 0.0001 | 0.0000 | 0.0000 | 0.0000 |
| **1.8** | 0.3400 | 0.5815 | 0.3061 | 0.0988 | 0.0232 | 0.0043 | 0.0007 | 0.0001 | 0.0000 | 0.0000 | 0.0000 |
| **1.9** | 0.2818 | 0.5812 | 0.3299 | 0.1134 | 0.0283 | 0.0055 | 0.0009 | 0.0001 | 0.0000 | 0.0000 | 0.0000 |
| **2** | 0.2239 | 0.5767 | 0.3528 | 0.1289 | 0.0340 | 0.0070 | 0.0012 | 0.0002 | 0.0000 | 0.0000 | 0.0000 |
| **2.1** | 0.1666 | 0.5683 | 0.3746 | 0.1453 | 0.0405 | 0.0088 | 0.0016 | 0.0002 | 0.0000 | 0.0000 | 0.0000 |
| **2.2** | 0.1104 | 0.5560 | 0.3951 | 0.1623 | 0.0476 | 0.0109 | 0.0021 | 0.0003 | 0.0000 | 0.0000 | 0.0000 |
| **2.3** | 0.0555 | 0.5399 | 0.4139 | 0.1800 | 0.0556 | 0.0134 | 0.0027 | 0.0004 | 0.0001 | 0.0000 | 0.0000 |
| **2.4** | 0.0025 | 0.5202 | 0.4310 | 0.1981 | 0.0643 | 0.0162 | 0.0034 | 0.0006 | 0.0001 | 0.0000 | 0.0000 |
| **2.5** | -0.0484 | 0.4971 | 0.4461 | 0.2166 | 0.0738 | 0.0195 | 0.0042 | 0.0008 | 0.0001 | 0.0000 | 0.0000 |
| **2.6** | -0.0968 | 0.4708 | 0.4590 | 0.2353 | 0.0840 | 0.0232 | 0.0052 | 0.0010 | 0.0002 | 0.0000 | 0.0000 |
| **2.7** | -0.1424 | 0.4416 | 0.4696 | 0.2540 | 0.0950 | 0.0274 | 0.0065 | 0.0013 | 0.0002 | 0.0000 | 0.0000 |
| **2.8** | -0.1850 | 0.4097 | 0.4777 | 0.2727 | 0.1067 | 0.0321 | 0.0079 | 0.0016 | 0.0003 | 0.0000 | 0.0000 |
| **2.9** | -0.2243 | 0.3754 | 0.4832 | 0.2911 | 0.1190 | 0.0373 | 0.0095 | 0.0020 | 0.0004 | 0.0001 | 0.0000 |
| **3** | -0.2601 | 0.3391 | 0.4861 | 0.3091 | 0.1320 | 0.0430 | 0.0114 | 0.0025 | 0.0005 | 0.0001 | 0.0000 |
| **3.1** | -0.2921 | 0.3009 | 0.4862 | 0.3264 | 0.1456 | 0.0493 | 0.0136 | 0.0031 | 0.0006 | 0.0001 | 0.0000 |
| **3.2** | -0.3202 | 0.2613 | 0.4835 | 0.3431 | 0.1597 | 0.0562 | 0.0160 | 0.0038 | 0.0008 | 0.0001 | 0.0000 |
| **3.3** | -0.3443 | 0.2207 | 0.4780 | 0.3588 | 0.1743 | 0.0637 | 0.0188 | 0.0047 | 0.0010 | 0.0002 | 0.0000 |
| **3.4** | -0.3643 | 0.1792 | 0.4697 | 0.3734 | 0.1892 | 0.0718 | 0.0219 | 0.0056 | 0.0012 | 0.0002 | 0.0000 |
| **3.5** | -0.3801 | 0.1374 | 0.4586 | 0.3868 | 0.2044 | 0.0804 | 0.0254 | 0.0067 | 0.0015 | 0.0003 | 0.0001 |
| **3.6** | -0.3918 | 0.0955 | 0.4448 | 0.3988 | 0.2198 | 0.0897 | 0.0293 | 0.0080 | 0.0019 | 0.0004 | 0.0001 |
| **3.7** | -0.3992 | 0.0538 | 0.4283 | 0.4092 | 0.2353 | 0.0995 | 0.0336 | 0.0095 | 0.0023 | 0.0005 | 0.0001 |
| **3.8** | -0.4026 | 0.0128 | 0.4093 | 0.4180 | 0.2507 | 0.1098 | 0.0383 | 0.0112 | 0.0028 | 0.0006 | 0.0001 |
| **3.9** | -0.4018 | -0.0272 | 0.3879 | 0.4250 | 0.2661 | 0.1207 | 0.0435 | 0.0130 | 0.0034 | 0.0008 | 0.0002 |
| **4** | -0.3971 | -0.0660 | 0.3641 | 0.4302 | 0.2811 | 0.1321 | 0.0491 | 0.0152 | 0.0040 | 0.0009 | 0.0002 |
| **4.1** | -0.3887 | -0.1033 | 0.3383 | 0.4333 | 0.2958 | 0.1439 | 0.0552 | 0.0176 | 0.0048 | 0.0011 | 0.0002 |
| **4.2** | -0.3766 | -0.1386 | 0.3105 | 0.4344 | 0.3100 | 0.1561 | 0.0617 | 0.0202 | 0.0057 | 0.0014 | 0.0003 |
| **4.3** | -0.3610 | -0.1719 | 0.2811 | 0.4333 | 0.3236 | 0.1687 | 0.0688 | 0.0232 | 0.0067 | 0.0017 | 0.0004 |
| **4.4** | -0.3423 | -0.2028 | 0.2501 | 0.4301 | 0.3365 | 0.1816 | 0.0763 | 0.0264 | 0.0078 | 0.0020 | 0.0005 |
| **4.5** | -0.3205 | -0.2311 | 0.2178 | 0.4247 | 0.3484 | 0.1947 | 0.0843 | 0.0300 | 0.0091 | 0.0024 | 0.0006 |
| **4.6** | -0.2961 | -0.2566 | 0.1846 | 0.4171 | 0.3594 | 0.2080 | 0.0927 | 0.0340 | 0.0106 | 0.0029 | 0.0007 |
| **4.7** | -0.2693 | -0.2791 | 0.1506 | 0.4072 | 0.3693 | 0.2214 | 0.1017 | 0.0382 | 0.0122 | 0.0034 | 0.0008 |
| **4.8** | -0.2404 | -0.2985 | 0.1161 | 0.3952 | 0.3780 | 0.2347 | 0.1111 | 0.0429 | 0.0141 | 0.0040 | 0.0010 |
| **4.9** | -0.2097 | -0.3147 | 0.0813 | 0.3811 | 0.3853 | 0.2480 | 0.1209 | 0.0479 | 0.0161 | 0.0047 | 0.0012 |
| **5** | -0.1776 | -0.3276 | 0.0466 | 0.3648 | 0.3912 | 0.2611 | 0.1310 | 0.0534 | 0.0184 | 0.0055 | 0.0015 |

**The Fourier Series Representation of the FM Signal**

We saw earlier that a single tone FM signal can be represented in a Fourier series as :

$$s(t) = A_c \sum_{-\infty}^{\infty} J_n(\beta)\cos(2\pi(f_c + nf_m)t)$$

The first few terms in this expansion are:

$$s(t) = A_c \{J_0(\beta)\cos(2\pi f_c t) + J_1(\beta)\cos 2\pi(f_c + f_m)t + J_{-1}(\beta)\cos 2\pi(f_c - f_m)t \ + \ J_2(\beta)\cos 2\pi(f_c + 2f_m)t \ + J_{-2}(\beta)\cos 2\pi(f_c - 2f_m)t + \cdots..$$

The FM signal consists of an infinite number of spectral components concentrated around $f_c$. Therefore, the theoretical bandwidth of the signal is infinity. That is to say, if we need to recover the FM signal without any distortion, all spectral components must be accommodated. This means that a channel with infinite bandwidth is needed. This is, of course, not practical since the frequency spectrum is shared by many users.

In the following discussion we need to truncate the series so that say 99% of the total average power is contained within a certain bandwidth. But first let us find the total average power using the series approach.

**<u>Power in the spectral components of s(t)</u>**

Note that s(t) consists of an infinite number of Fourier terms, and the power in s(t) will be equal the power in the respective Fourier components .

Any term in s (t) takes the form: $A_c J_n(\beta)\cos(2\pi(f_c + nf_m)t)$

The average power in this term is: $\frac{(A_c)^2 (J_n(\beta))^2}{2}$

Hence the total power in s(t) is:

$$< S^2(t) > = \frac{{A_C}^2 {J_0}^2(\beta)}{2} + \frac{{A_C}^2 {J_1}^2(\beta)}{2} + \frac{{A_C}^2 {J_{-1}}^2(\beta)}{2} + \frac{{A_C}^2 {J_2}^2(\beta)}{2} + \frac{{A_C}^2 {J_{-2}}^2(\beta)}{2} + \cdots$$

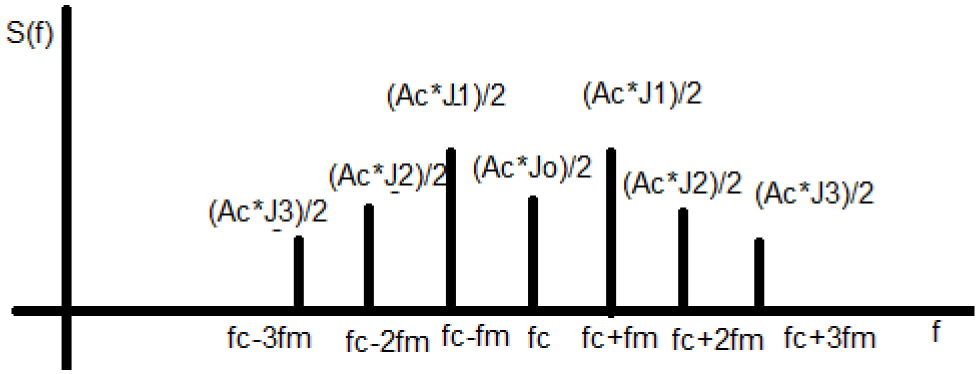
$$= \frac{{A_C}^2}{2}\{{J_0}^2(\beta) + {J_1}^2(\beta) + {J_{-1}}^2(\beta) + {J_2}^2(\beta) + {J_{-2}}^2(\beta) + \cdots..\}$$

$= \frac{{A_C}^2}{2}\{\sum_{n=-\infty}^{\infty} {J_n}^2(\beta)\}$, where $\sum_{n=-\infty}^{\infty} {J_n}^2(\beta) = 1$, ( A property of Bessel Functions).

The average power becomes

$< S^2(t) > = \frac{Ac^2}{2}$.

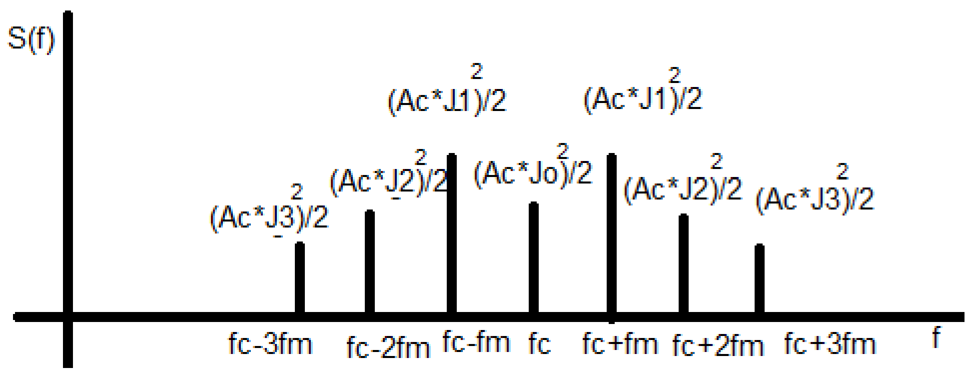
**Spectrum of an Fm Signal**




Fourier transform of $s(t) = A_c \cos\,(\omega_c t + \beta \sin 2\pi f_m t)$ (only + ve frequencies shown)

Note that in the figure above as $f_m$ *decreases*, the spectral lines become closely clustered about $f_c$.

The power spectral density, which is a plot of $|C_n|^2$ versus $f$, is shown below:




**Example: 99% power bandwidth of an FM signal**

Plot the FM spectrum and find the 99% power bandwidth when $\beta = 1$ and $\beta = 0.2$

**Solution**:

$s(t) = A_c \sum_{-\infty}^{\infty} J_n(\beta) \cos(2\pi(f_c + nf_m)t)$

**Case a: $\boldsymbol{\beta = 1}$ (wideband FM)**

The first five terms corresponding to $\beta = 1$, are
$J_0(1)$ = 0.7652, $J_1(1)$ = 0.4401, $J_2(1)$ = 0.1149, $J_3(1)$ = 0.01956, $J_4(1)$ = 0.002477

The power in s(t) is $< S^2(t) > = \frac{A_C{}^2}{2}$

Let us try to find the average power in the terms at $f_c$ , $f_c + f_m$, $f_c - f_m$, $f_c + 2f_m$, $f_c - 2f_m$

The average power in these five components can be calculated as:

1. $f_c$ : $\frac{A_C{}^2\, J_0{}^2(\beta)}{2}$
2. $f_c + f_m$: $\frac{A_C{}^2 J_1{}^2(\beta)}{2}$
3. $f_c - f_m$: $\frac{A_C{}^2 J_{-1}{}^2(\beta)}{2}$
4. $f_c + 2f_m$: $\frac{A_C{}^2 J_2{}^2(\beta)}{2}$
5. $f_c - 2f_m$: $\frac{A_C{}^2 J_{-2}{}^2(\beta)}{2}$

The average power in the five spectral components is the sum:

$$P_{av} = \frac{A^2{}_c}{2}\left[J_0{}^2(1) + 2J_1{}^2(1) + 2J_2{}^2(1)\right]$$

$$P_{av} = \frac{A^2{}_c}{2}\ [(0.7652)^2 + 2 * (0.4401)^2 + (0.1149)^2] = \ 0.9993\ \frac{A^2{}_c}{2}$$

So, these terms have 99.9 % of the total power.

Therefore, the 99.9 % power bandwidth is

$$BW = (f_c + 2f_m) - (f_c - 2f_m) = 4f_m$$

**Case b: $\boldsymbol{\beta = 0.2}$ (Narrowband FM)**

For $\beta = 0.2$, $J_0(0.2) = 0.99$, $J_1(0.2) = 0.0995$, $J_2(0.2) = 0.00498335$

The power in the carrier and the two sidebands (at $f_c$ , $f_c + f_m$, $f_c - f_m$) is

$$P = \frac{A^2{}_c}{2}\left[J_0{}^2(0.2) + 2J_1{}^2(0.2)\right]$$

$$P = \frac{A^2{}_c}{2}[0.9999]$$

Therefore, 99.99% of the total power is found in the carrier and the two sidebands. The 99% bandwidth is

$$\text{B.W}= (f_c + f_m) - (f_c - f_m) = 2f_m$$

**Remark:**

We observe that the spectrum of an FM signal when $\beta \ll 1$ ( called narrow band FM) is "similar" to the spectrum of a normal AM signal, in the sense that it consists of a carrier and two sidebands. The bandwidth of both signals is $2f_m$.

**Carson's Rule**

A 98% power B.W of an FM signal can be estimated using Carson's rule:

$$B_T = 2(\beta + 1)f_m$$

**Generation of an FM Signal**

**First: Generation of a Narrowband FM Signal**

Consider an angel modulated signal:

$$s(t) = A_c \cos(2\pi f_c t + \theta(t))$$

When $s(t)$ is an FM signal, $\theta(t) = 2\pi k_f \int m(t)dt$

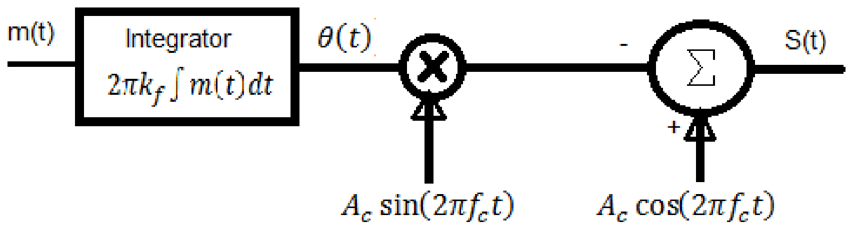
$s(t)$ can be expanded as:

$$s(t) = A_c \cos(2\pi f_c t)\cos\big(\theta(t)\big) - A_c \sin(2\pi f_c t)\sin\big(\theta(t)\big)$$

When $|\theta(t)| \ll 1$ , $cos\ \theta \cong 1$ , $\sin(\theta) \cong \theta$ and $s(t)$ , termed narrowband, can be approximated as:

$$s(t) \cong A_c \cos(2\pi f_c t) - A_c\theta \sin(2\pi f_c t)$$

Using this expression, one can generate a narrowband FM or PM signals. This is illustrated in the block diagram below:



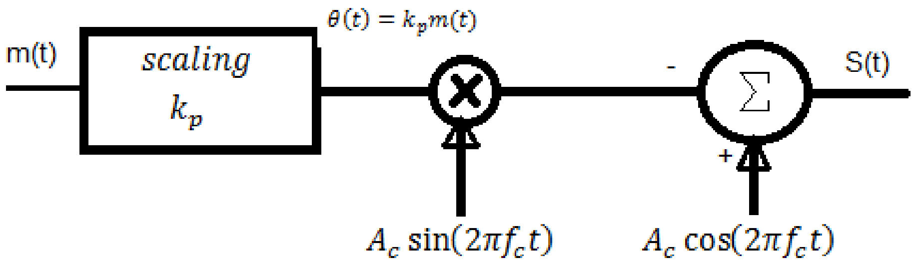

When $\quad m(t) = A_m \cos(2\pi f_m t)$

$$\theta(t) = \beta \sin(2\pi f_m t)$$

the modulated signal takes the form

$$s(t) = A_c \cos(2\pi f_c t) - A_c\beta \sin(2\pi f_m t) \sin(2\pi f_c t)$$

To generate a narrow band PM signal, we can use the scheme:



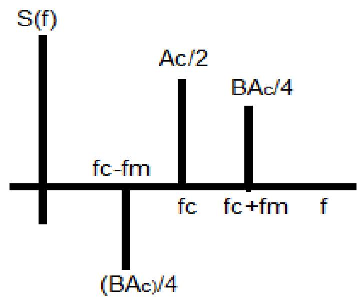

**Spectrum of a single- tone NBFM**:

For an FM signal, $\theta(t) = \beta \sin(2\pi f_m t)$

$$s(t) = A_c \cos(2\pi f_c t) - A_c\beta \sin(2\pi f_m t) \sin(2\pi f_c t)$$

$$s(t) = A_c \cos(2\pi f_c t) - \frac{A_c\beta}{2}[\cos(2\pi(f_c - f_m)t) - \cos(2\pi(f_c + f_m)t)]$$

The spectrum of $s(t)$ is shown below:




The spectrum consists of a component at the carrier frequency $f_c$ , two components at $(f_c + f_m)$ and $f_c - f_m)$. Note the negative sign at the lower sideband.

The bandwidth of this signal is $2f_m$.

Now, consider the normal AM signal with sinusoidal modulation.

$$s(t)_{AM} = A_c \cos(2\pi f_c t) + A_c A_m \cos\,(2\pi f_m t) \cos(2\pi f_c t)$$

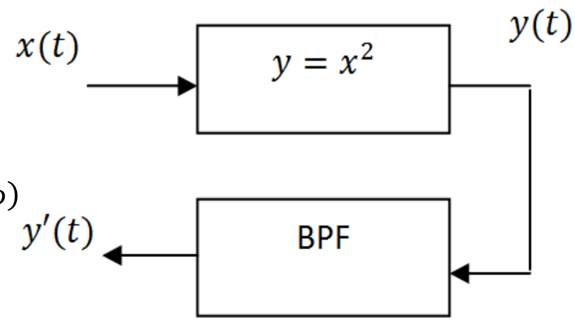
It can be represented as

$$s(t) = A_c \cos(2\pi f_c t) - \frac{A_c A_m}{2} [\cos(2\pi (f_c - f_m) t) + \cos(2\pi (f_c + f_m) t)]$$

As we recall this signal consists of a term at the carrier and two terms at $f_c + f_m$ and $f_c - f_m$.

## Frequency multiplier

It is a device for which the frequency of the output signal is an integer multiple of the frequency of the input signal. It is primarily a nonlinear characteristic followed by a band pass filter. Now we illustrate the operation of this device.




### The Square law device:

Let the input be an FM signal of the form:

$$x(t) = A_c \cos(2\pi f'_c + \beta' \sin 2\pi f_m t) = A_c \cos(\phi)$$

The output of the square law characteristic is:

$$y(t) = x(t)^2 = A_c^2 cos^2(\phi) = \frac{A_c^2}{2}[1 + \cos(2\phi)] = \frac{A_c^2}{2} + \frac{A_c^2}{2}\cos(2\phi)$$

$$= \frac{A_c^2}{2} + \frac{A_c^2}{2} cos\left[2\pi\left(2f'_c\right) + 2\beta' \sin(2\pi f_m t)\right]$$

### The bandpass filter

If $y(t)$ is passed through a BPF of center frequency $2f_c$ , then the DC term will be suppressed and the filter output is:

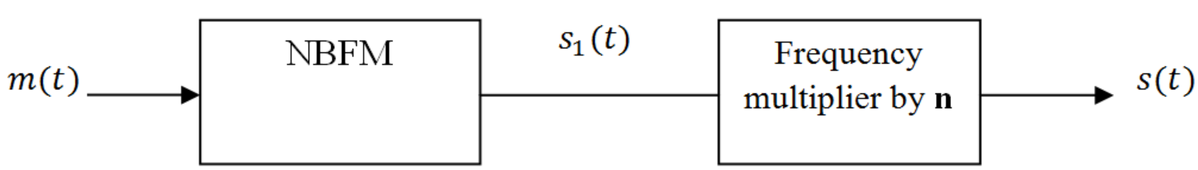
$$y'(t) = \frac{A_c^2}{2} cos\left[2\pi\left(2f'_c\right) + 2\beta' \sin(2\pi f_m t)\right]$$

$$y'(t) = \frac{A_c^2}{2} cos[2\pi(f_c) + \beta \sin(2\pi f_m t)]$$

As can be seen from this result, the output is a signal with twice the frequency of the input signal and a modulation index twice that of the input. To get frequency multiplication higher than two, a cascade of units, similar to what was described above, can be formed with the number of stages that achieve the desired frequency.

## Indirect Method for Generating a Wideband FM:

A wideband FM can be generated indirectly using the block diagram below (Armstrong Method). First, a narrowband FM is generated, and then the wideband FM is obtained by using frequency multiplication. Next, we analyze the operation of this modulator.



Let $m(t) = A_m \cos 2\pi f_m t$ be the baseband signal, then

$$s_1(t) = A_c \cos(2\pi f_c' t + \beta' \sin 2\pi f_m t) \;\; ; \;\; \beta' = \frac{k_f A_m}{f_m}$$

is a narrowband FM with $\beta' \ll 1$. The frequency of $s_1(t)$ is

$$f'_i = f_c' + k_f A_m \cos 2\pi f_m t$$

Multiplying $f_i$ by $n$ (through frequency multiplication), we get the frequency of $s(t)$ as

$$f_i = nf_c' + nk_f A_m cos 2\pi f_m t$$

This result is

$$s(t) = A_c \cos[2\pi(nf_c')t + n\beta' \sin 2\pi f_m t]$$

$$= A_c \cos[2\pi f_c t + \beta \sin 2\pi f_m t]$$

Where $\beta = n\beta'$ is the desired modulation index of WBFM

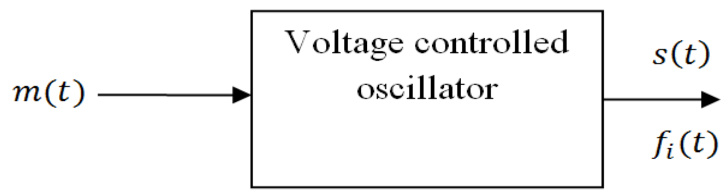
$f_c = nf_c'$ is the desired carrier frequency of WBFM

## Direct method for generating an FM signal:

In a direct FM system, the instantaneous frequency of the carrier is varied in accordance with a message signal by means of a voltage controlled oscillator (VCO). The voltage – frequency characteristic of a VCO is given by

$$f_i = f_c + k_f \, m(t)$$

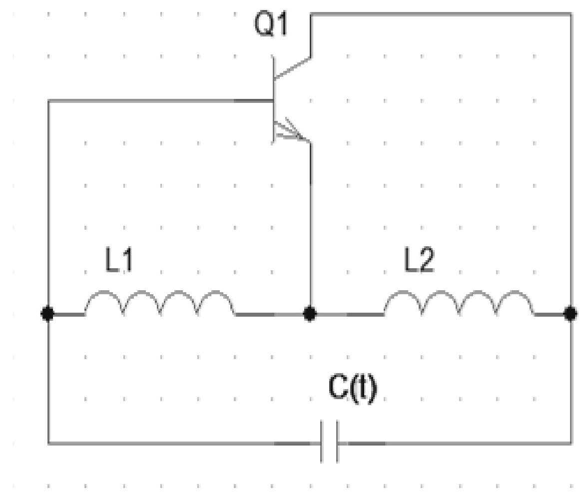
A schematic diagram of a VCO is shown in the figue




A realization of the CVO may be obtained by considering an oscillator (like the Hartley oscillator) shown below in which a varactor ((voltage variable capacitor) is used. The capacitance of the varactor varies in response to variations in the message signal. The variation is linear when the variation in the message is too small.

The frequency of the oscillator is

$$f_i(t) = \frac{1}{2\pi\sqrt{(L_1 + L_2)C(t)}}$$




Hartley Oscillator

Let $C(t) = C_0 - k\, m(t)$ (A diode operating in the reverse bias region can act like a variable capacitor)

k: is a constant,

When m(t) = 0, $C(t) = C_0$, and the unmodulated frequency of oscillation is

$$f_c = \frac{1}{\sqrt{(L_1+L_2)C_0}}$$

When m(t) has a finite value, the frequency of oscillation is

$$f_i(t) = \frac{1}{2\pi\sqrt{(L_1 + L_2)\left(C_0 - k\, m(t)\right)}}$$

$$= f_c\left(1 - \frac{k\, m(t)}{C_0}\right)^{-1/2} \qquad , \qquad [(1+x)^n \cong 1 + nx]$$

When $\frac{k\, m(t)}{C_0} \ll 1$ , we can make the approximation

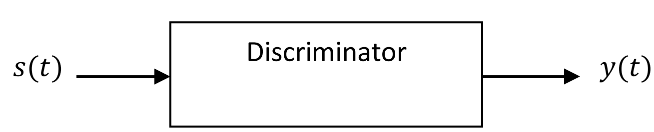
$$f_i(t) = f_c\left(1 + \frac{k\, m(t)}{2C_0}\right) = f_c + k_f\, m(t)$$

Here it is clear that the instantaneous frequency varies linearly with the message signal.

**<u>Demodulation of the FM signal:</u>**

An FM signal may be demodulated by means of what is called a *discriminator*.

Let $s(t) = A_c \cos\left(\omega_c t + \theta(t)\right)$ be an angle modulated signal. The output of an ideal discriminator is defined as:

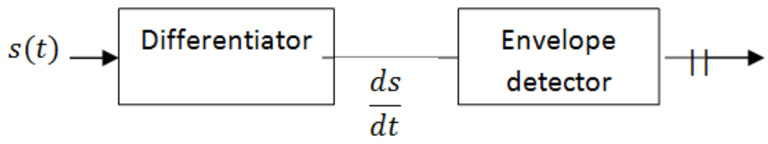



$$y(t) = \frac{1}{2\pi}\, k_D\, \frac{d\theta}{dt}$$

When $\theta = 2\pi k_f \int_{-\infty}^{t} m(\alpha)d\alpha$ , then $\frac{d\theta}{dt} = 2\pi k_f\, m(t)$ and y(t) becomes

$$y(t) = k_D k_f\, m(t)$$

One practical realization of a discriminator is a differentiator followed by an envelope detector.




The operation of this discriminator can be explained as follows:

Let $s(t) = A_c \cos\left(\omega_c t + \theta(t)\right)$

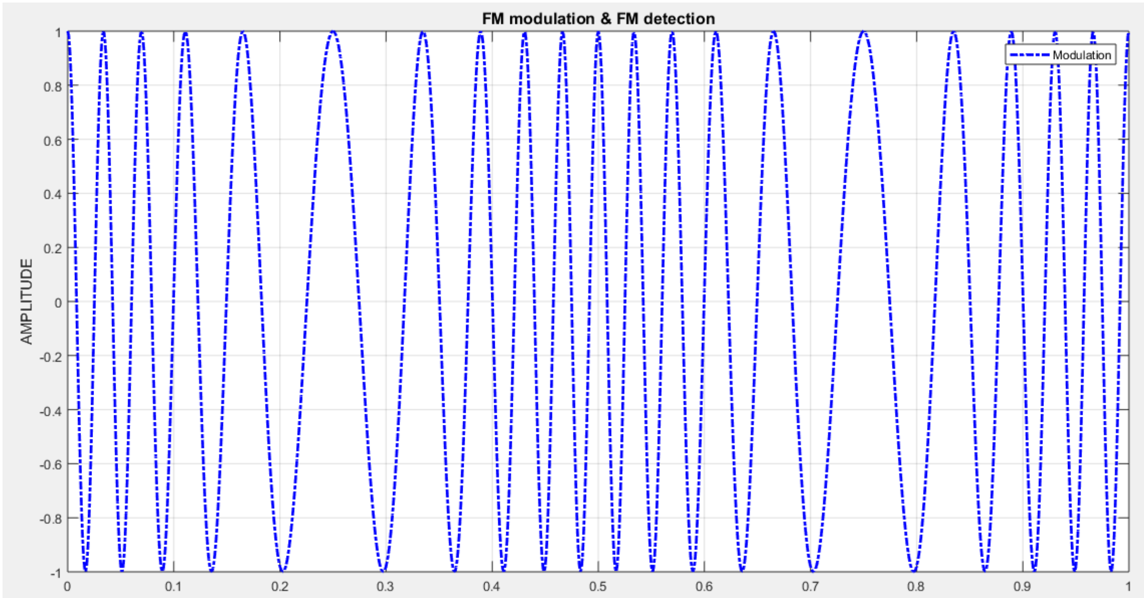
$$\frac{ds(t)}{dt} = -A_c\left(\omega_c + \frac{d\theta}{dt}\right)\sin\left(\omega_c t + \theta(t)\right)$$

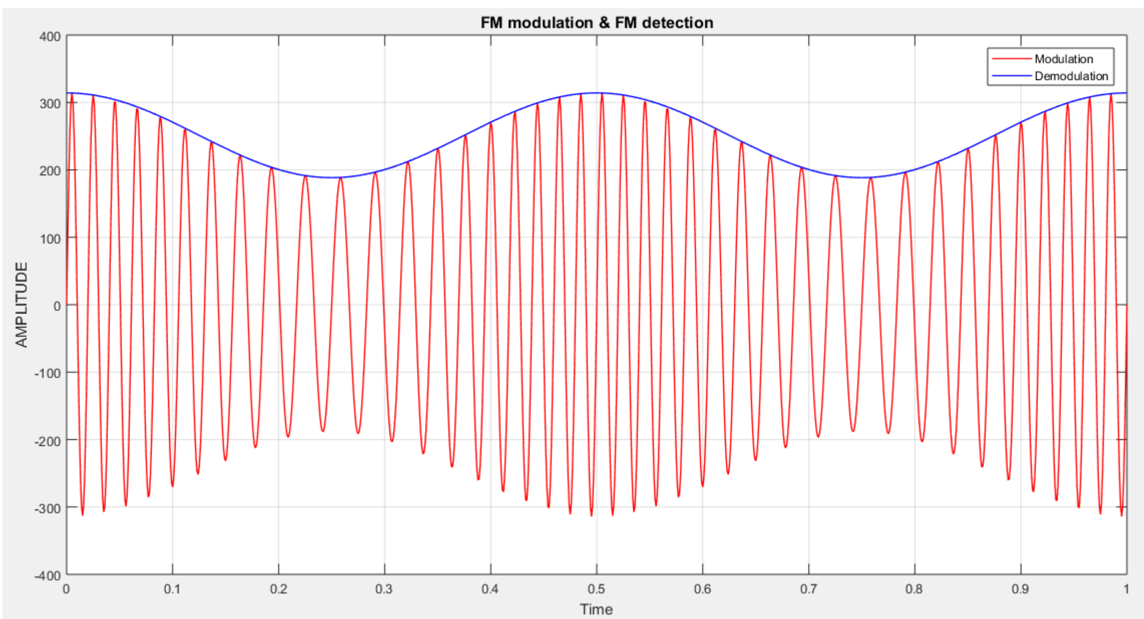
The output of the envelope detector is $A_c\left|\left(\omega_c + \frac{d\theta}{dt}\right)\right|$

The capacitor blocks the DC term and so output is:

$$V_0 = A_c\frac{d\theta}{dt} = 2\pi k_f\, A_c m(t)$$

A typical FM signal and its derivative are shown in the figure below.



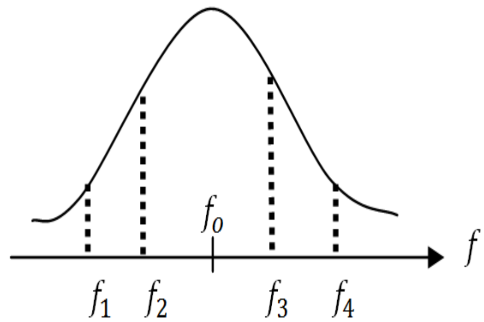




We already know what an envelope detector (recall the material on the demodulation of a normal AM signal). Now we explain how differentiation is accomplished.

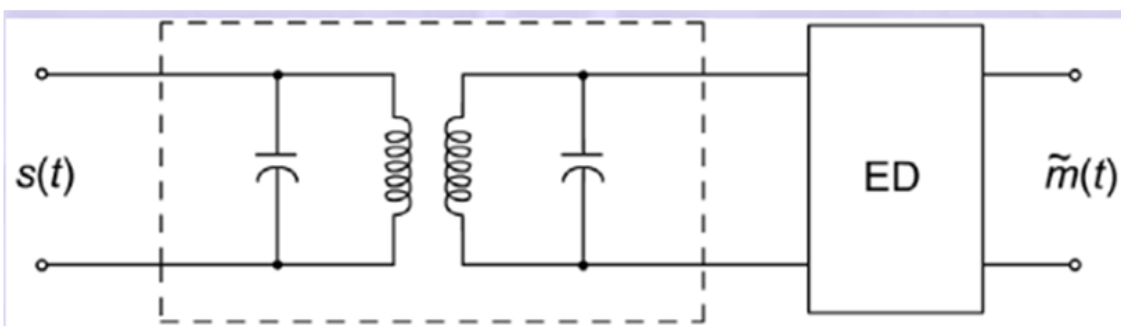
From the properties of Fourier transform we know that if $F\{g(t)\} = G(f)$ , then

$$F\left\{\frac{dg(t)}{dt}\right\} = j2\pi f \; G(f)$$

This means that multiplication by $j2\pi f$ in the frequency domain amounts to differentiating the signal in the time-domain. Hence, we need a circuit whose frequency response is linear in $f$ to perform time differentiation. A circuit that performs this task is a tuned circuit, provided that the signal frequency falls within the linear part of the characteristic, i.e., between either $(f_1, f_2)$ *or* $(f_3, f_4)$.




A balanced FM detector called *balanced discriminator* is such a circuit.




Primary circuit tuned to $f_c$     Secondary circuit tuned to $f_0 > f_c$

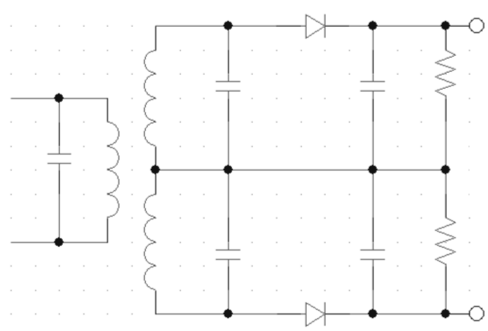
## Tuned circuit demodulator

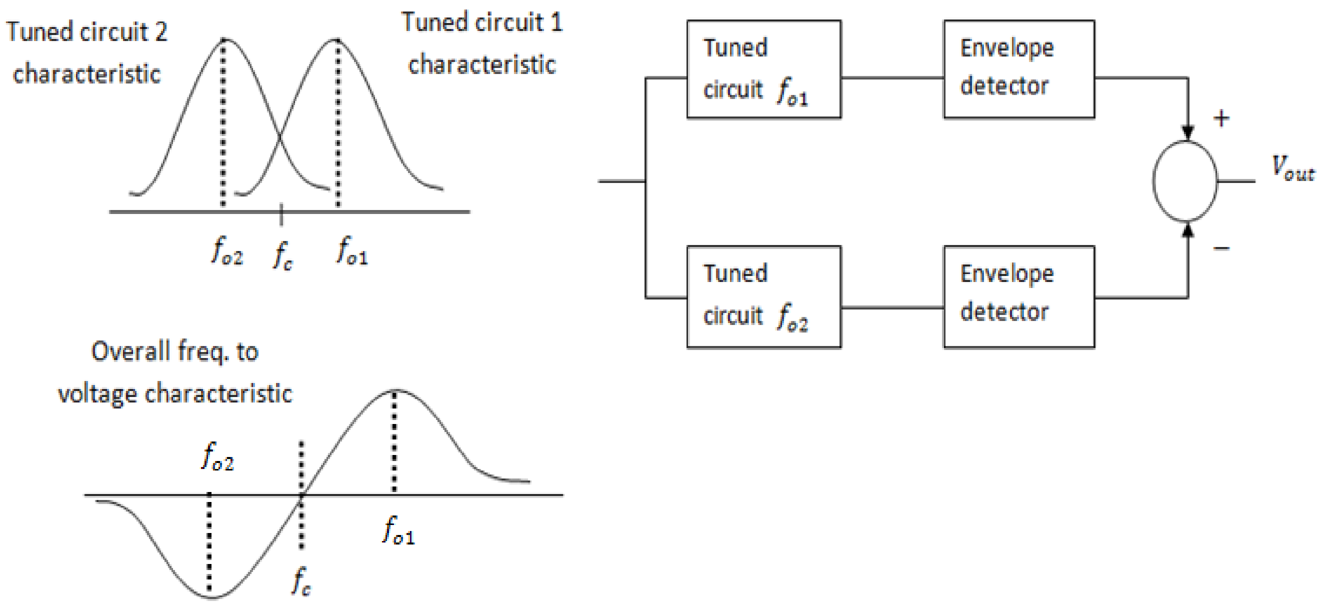
To extend the dynamic range of the differentiating circuit, two tuned circuits with center frequencies $f_{o1}$ *and* $f_{o2}$ are used as shown in the figure

## Balanced slope detector:

Two tuned circuits are tuned to two different frequencies $f_{o_1} > f_{o_2}$ . The primary circuit is tuned to $f_c$ .

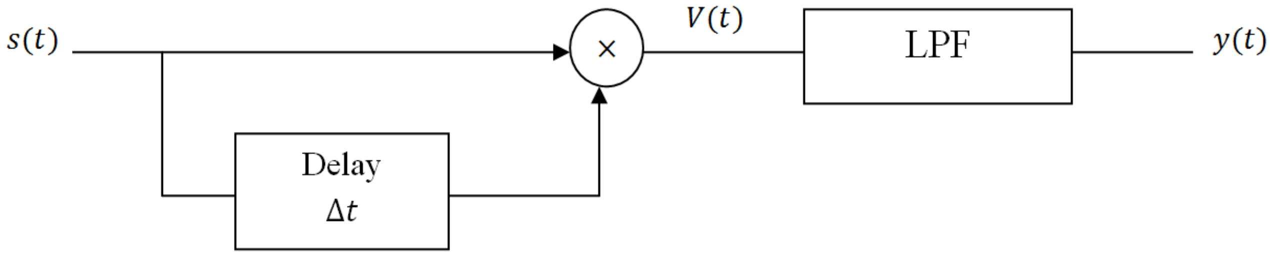
- This circuit has a wider range of linear frequency response.
- No DC blocking is necessary.




Tuned circuit 2
characteristic
Tuned circuit 1
characteristic
$f_{o2}$
$f_c$
$f_{o1}$
Overall freq. to
voltage characteristic
$f_{o2}$
$f_{o1}$
$f_c$
Tuned
circuit $f_{o1}$
Envelope
detector
Tuned
circuit $f_{o2}$
Envelope
detector
+
−
$V_{out}$

## Phase shift discriminator (not required to ENCS students)

**The quadrature detector**: This demodulator converts frequency variations into phase variation and detects the phase changes. The block diagram of the demodulator is shown below




Let $s(t) = A_c \cos\big(2\pi f_c t + \varphi(t)\big)$ ; $\varphi(t) = 2\pi K_f \int_0^t m(\alpha)\, d\alpha$

$$s(t-\Delta t) = A_c \cos[2\pi f_c(t-\Delta t) + \varphi(t-\Delta t)]$$

$$= A_c \cos[2\pi f_c t - 2\pi f_c \Delta t + \varphi(t-\Delta t)]$$

The delay $\Delta t$ is chosen such that $2\pi f_c \Delta t = {}^{\pi}/_{2}$

Hence,

$$s(t-\Delta t) = A_c \cos\left[2\pi f_c t - \frac{\pi}{2} + \varphi(t-\Delta t)\right]$$

$$= A_c \sin[2\pi f_c t + \varphi(t-\Delta t)]$$

$$V(t) = s(t)s(t-\Delta t) \quad = A_c^2 \sin[2\pi f_c t + \varphi(t-\Delta t)]\, cos[2\pi f_c t + \varphi(t)]$$

$$= \frac{A_c^2}{2} \sin[2\pi(2f_c)t + \varphi(t) + \varphi(t-\Delta t)] + \frac{A_c^2}{2} \sin[\varphi(t) - \varphi(t-\Delta t)]$$

The high frequency component is suppressed by the LPF. What remains is the second term

$$\frac{A_c^2}{2} \sin[\varphi(t) - \varphi(t-\Delta t)] \cong \frac{A_c^2}{2} [\varphi(t) - \varphi(t-\Delta t)]$$

where $\Delta t$ is small to justify the approximation $\sin(x) \cong x$. Hence,

$$y(t) = \frac{A_c^2}{2} [\varphi(t) - \varphi(t-\Delta t)]$$

$$y(t) = \frac{A_c^2}{2}\ \Delta t \frac{\varphi(t) - \varphi(t-\Delta t)}{\Delta t}$$

The second term is the derivative $\frac{d\varphi(t)}{dt}$. The output then becomes

$$y(t) = \frac{A_c^2}{2}\Delta t\frac{d\varphi}{dt}$$

But $\varphi(t) = 2\pi k_f \int_0^t m(\alpha)\, d\alpha$ and $\frac{d}{dt}\varphi(t) = 2\pi k_f m(t)$

$$y(t) = \frac{A_c^2}{2}\Delta t 2\pi k_f m(t)$$

$$y(t) = K\, m(t)$$

Therefore, $m(t)$ has been demodulated.

**<u>Transfer function of the delay:</u>**

From the Fourier transform properties

$$g(t) \to G(f)$$

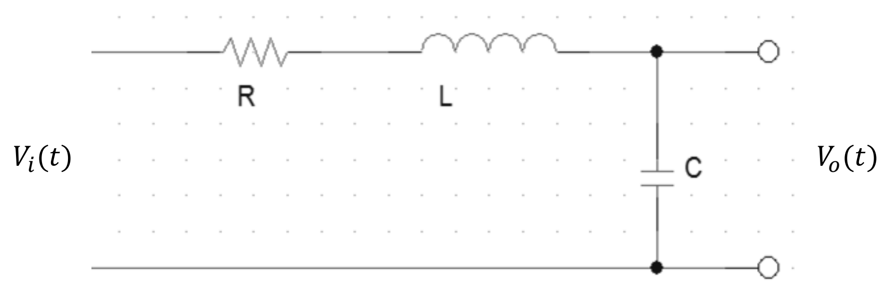
$$g(t-\Delta t) \to G(f)e^{-j2\pi f\Delta t}$$

The transfer function of the time delay is

$$H(f) = e^{-j2\pi f\Delta t}$$

Therefore, a circuit whose phase characteristic is linear in $f$ can provide time delay of the type that we need.

A circuit with linear phase characteristic is the network shown




If $f_o = \frac{1}{2\pi\sqrt{LC}}$, $f_b = \frac{R}{2\pi L}$ then it can be shown that arg $(H(f))$ for this circuit is

$$\arg(H(f)) = -\frac{\pi}{2} - \frac{2Q}{f_o}(f - f_c) \;,\; Q = \frac{f_o}{f_b}$$

$\Theta(\mathrm{f}) = a - bf$

**Remarks**:

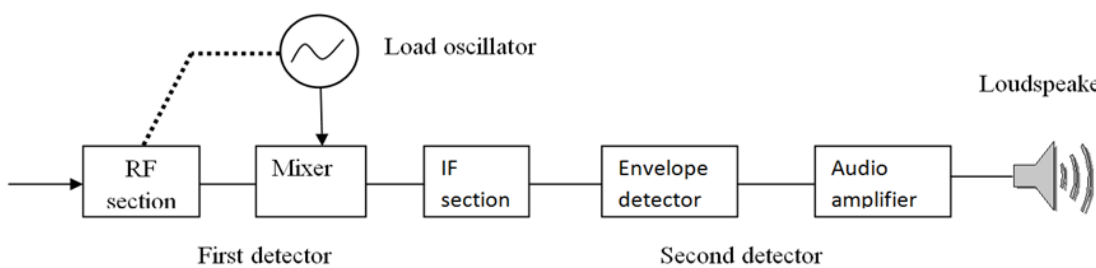
1. To perform time differentiation, we searched for a circuit whose amplitude spectrum varies linearly with frequency
2. To perform time delay, we searched for a circuit with a linear phase spectrum.

## <u>The Super heterodyne Receiver</u>:

Practically, all radio and TV receivers are made of the super heterodyne type. The receiver performs the following functions :

- <u>Carrier frequency tuning</u>: The purpose of which is to select the desired signal.
- <u>Filtering:</u> the desired signal is to be separated from other modulated signals.
- <u>Amplification</u>: to compensate for the loss of signal power incurred in the course of transmission.

The description of the receiver is summarized as follows:




- The incoming signal is picked up by the antenna and amplified in the RF section that is tuned to the carrier frequency of the incoming signal.
- The incoming RF section is down converted to a fixed intermediate frequency (IF). $f_{IF} = f_{lO} - f_{RF}$
- The IF section provides most of the amplification and selectivity in the receiver. The IF bandwidth corresponds to that required for the particular type of modulation.
- The IF output is applied to a demodulator, the purpose of which is to recover the baseband signal.

- The final operation in the receiver is the power amplification of the recovered signal.
- The basic difference between AM and FM super heterodyne lies in the use of an FM demodulator such as a discriminator (differentiator followed envelope detector)

## Quadrature Carrier Multiplexing (QAM) **(will be covered)**

### Quadrature Carrier Multiplexing: Modulation

This scheme enables two DSB-SC modulated signals to occupy the same transmission B.W and yet allows for the separation of the message signals at the receiver.

$m_1(t)$ and $m_2(t)$ are low pass signals each with a B.W = W Hz .

The composite signal is:

$$s(t) = A_c m_1(t) cos2\pi f_c t + A_c m_2(t) sin2\pi f_c t$$

$$s(t) = s_1(t) + s_2(t)$$

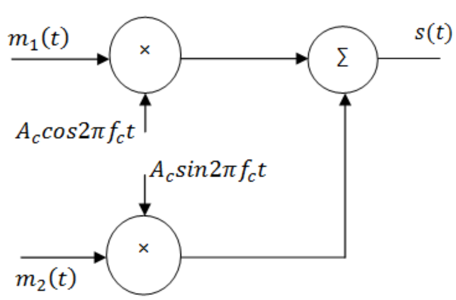
where $s_1(t)$ and $s_2(t)$ are both DSB-SC signals.

B.W of $s_1(t)$ = 2W

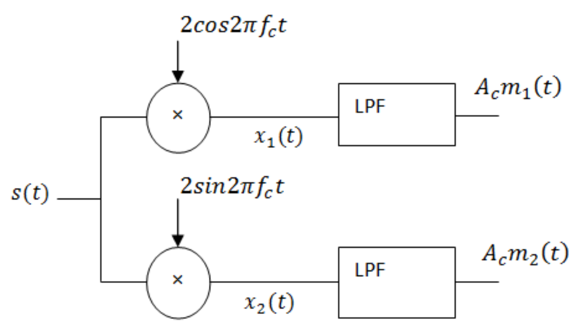
B.W of $s_2(t)$ = 2W

B.W of $s(t)$ = 2W

This method provides bandwidth conservation. That is, two DSB-SC signals are transmitted within the bandwidth of one DSB-SC signal. Therefore, this multiplexing technique provides bandwidth reduction by one half.




Modulation




Demodulation

## Quadrature Carrier Multiplexing: Demodulation

Given $s(t)$, the objective is to recover $m_1(t)$ and $m_2(t)$ from $s(t)$. Consider first the in-phase channel

$$\begin{aligned} x_1(t) &= 2cos2\pi f_c t \; s(t) \\ &= 2\cos 2\pi f_c t \; (A_c m_1(t) cos2\pi f_c t + A_c m_2(t) sin2\pi f_c t) \\ &= 2A_c m_1(t) cos^2 2\pi f_c t + 2A_c m_2(t) sin\omega_c t \; cos\omega_c t \\ &= 2A_c m_1(t) \left(\frac{1+cos2\omega_c t}{2}\right) + A_c m_2(t) sin2\omega_c t \\ &= A_c m_1(t) + A_c m_1(t) cos2\omega_c t + A_c m_2(t) sin2\omega_c t \end{aligned}$$

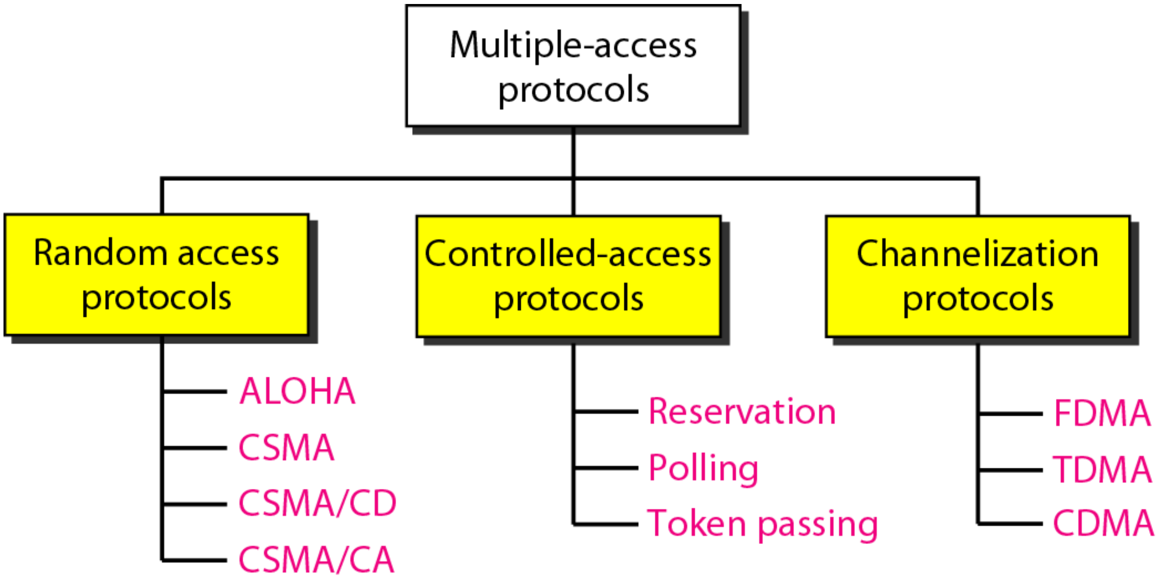
After low pass filtering, the output of the in-phase channel is

$$y_1(t) = A_c m_1(t).$$

Likewise, it can be shown that

$$y_2(t) = A_c m_2(t) \; .$$

**Note:** Synchronization is a problem. That is to recover the message signals it is important that the two carrier signals (the sine and the cosine functions) at the receiver should have the same phase and frequency as the signals at the transmitting side. A phase error or a frequency error will result is an interference type of distortion. That is, A component of $m_2(t)$ will appear in the in-phase channel in addition to the desired signal $m_1(t)$ and a component of $m_1(t)$ will appear at the quadrature output.



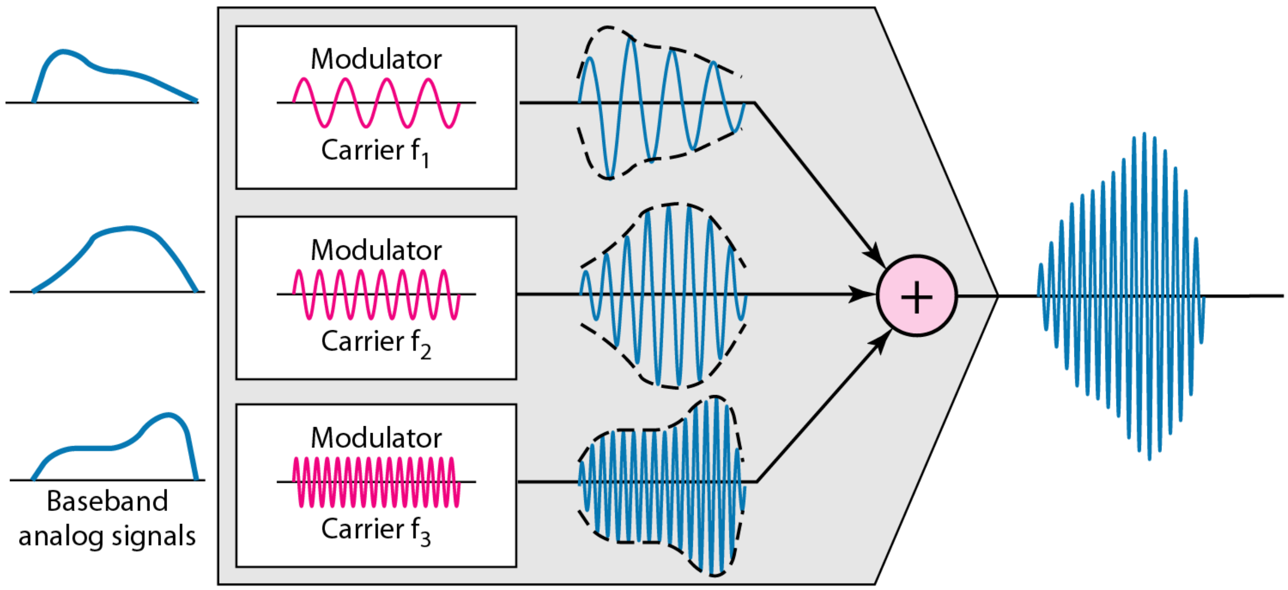

- A **multiple access channel** is one where a set of users at one end want to communicate with another set of users at the other end.
- **Channel Allocation**: The coordination of the usage of a single channel among multiple source – destination pairs.
- The algorithm which implements the channel allocation are called **medium access control** (MAC) or multiple access protocols.
- MAC protocols can be classified into
  - Conflict free protocols:
  - Random access protocols:

- A **Conflict free protocols**: Collisions are completely avoided by allocating the channel access to sources in a predetermined manner. Examples are TDMA, FDMA and CDMA. This is equivalent to circuit switching and is inefficient for bursty type of loads.
- **Random access protocols**: These are classified as contention systems where the stations compete to access the channel. The contention could be completely random or controlled.
- Collisions can occur between transmitted packets of different users trying to access the channel.
- A collided packet has to be transmitted until it is received properly at the destination.

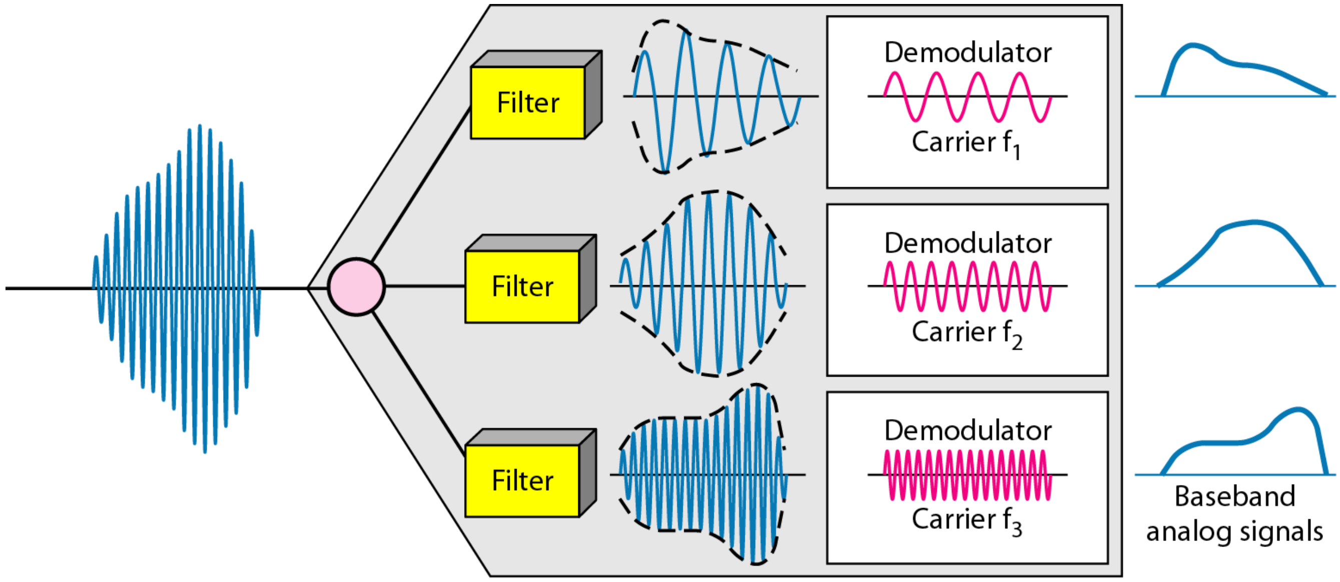
## **Frequency Division Multiplexing: (will be covered)**

A number of independent signals can be combined into a composite signal suitable for transmission over a common channel. The signals must be kept apart so that they do not interfere with each other and thus they can be separated at the receiving end.

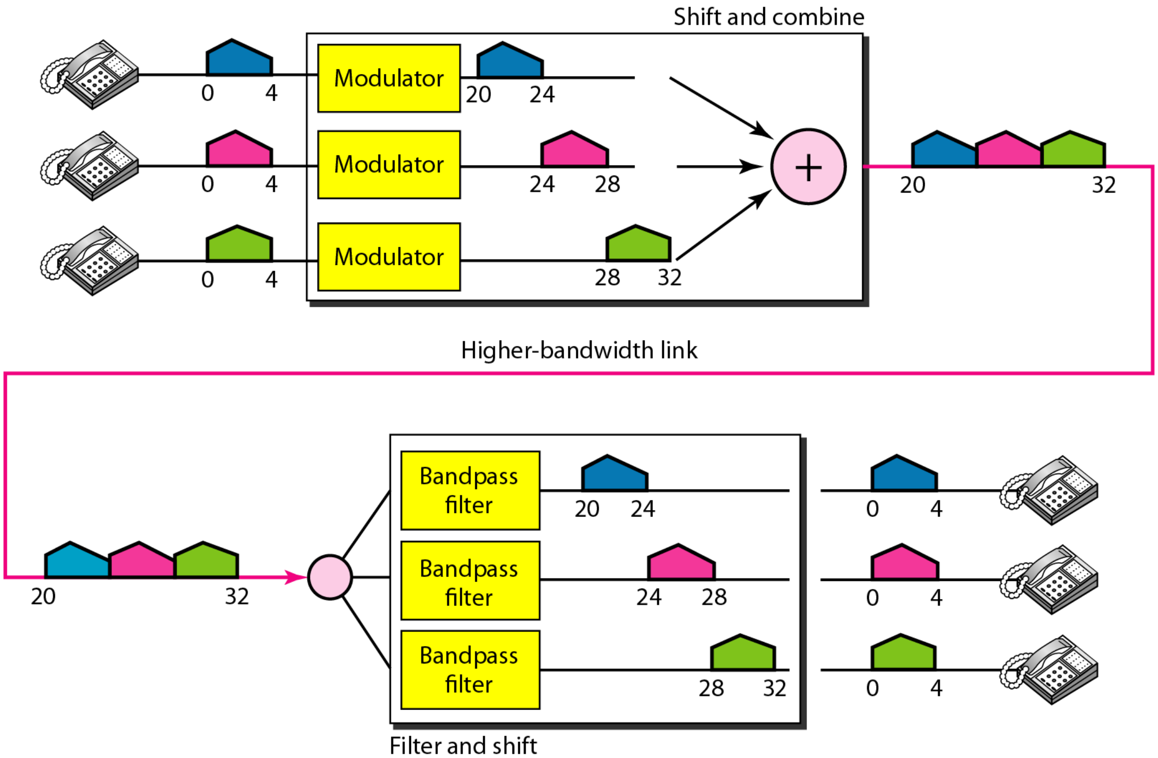
**Transmitter:**




**Receiver:**



**Example: Single Sideband Frequency Division Multiplexed Signals**



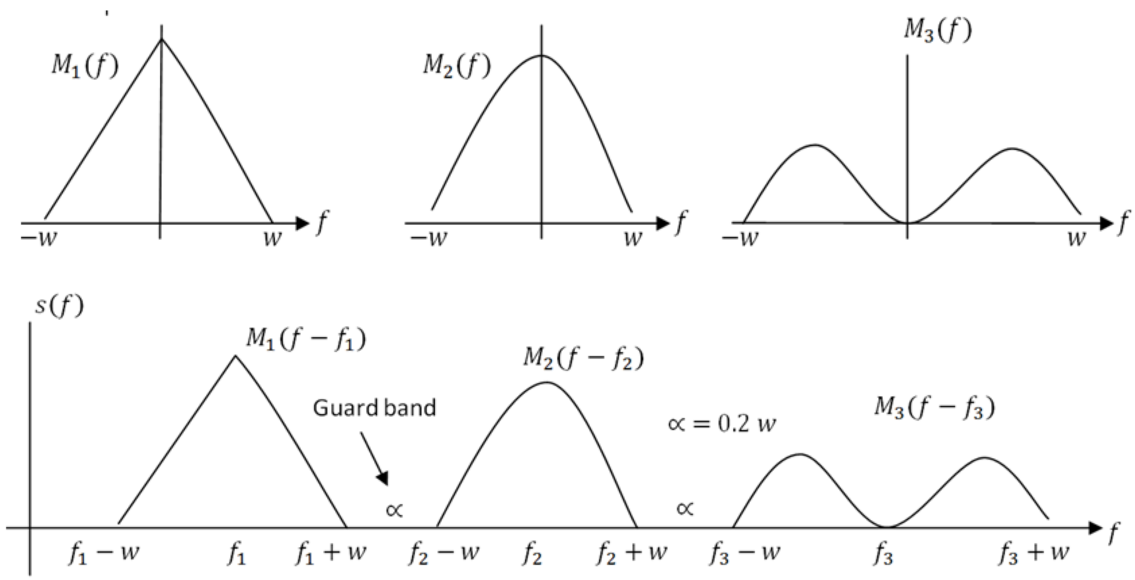

**Example: Double Sideband Frequency Division Multiplexed Signals**

Let $m_1, m_2$ *and* $m_3$ be three baseband message signals each with a B.W = w.

The composite modulated signal $s(t)$ is

$$s(t) = A_{c_1} m_1(t) cos2\pi f_1 t + A_{c_2} m_2(t) cos2\pi f_2 t + A_{c_3} m_3(t) cos2\pi f_3 t$$
$$= s_1(t) + s_2(t) + s_3(t)$$

$s_1, s_2$ *and* $s_3$ are DSB-SC signals with carrier frequencies $f_1$, $f_2$ *and* $f_3$ , respectively. If the spectrum of $m_1(t)$, $m_2(t)$ *and* $m_3(t)$ are as shown, the spectrum of $s(t)$ can be found as shown below.




To prevent interference we demand that

$$f_2 - w \geq f_1 + w \quad or \quad f_2 - f_1 \geq 2w$$

$$f_3 - w \geq f_2 + w \quad or \quad f_3 - f_2 \geq 2w$$

The structure of the receiver is as follows: