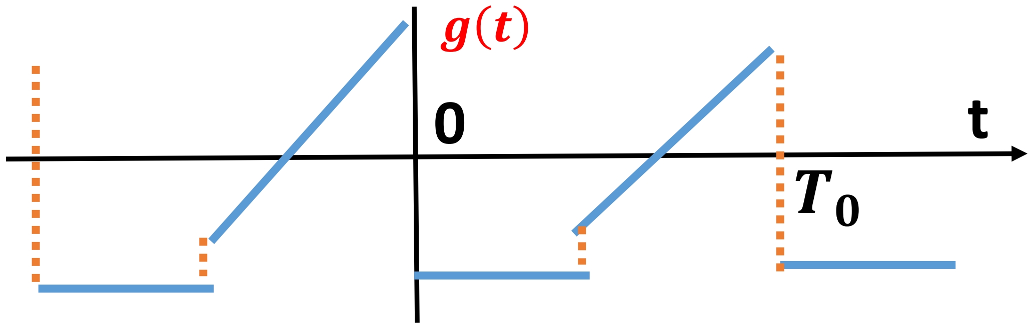
Fourier Series

- Let $g(t)$ be a periodic function of time with period $T_0 = \frac{1}{f_0}$ such that
  - The function g(t) is absolutely integrable over one period, i.e.,, $\int_0^{T_0} |g(t)|\, dt < \infty$
  - Any discontinuities in g(t) are finite (the amount of jump at points of discontinuity is finite).
  - g(t) has only a finite number of discontinuities and only a finite number of maxima and minima in the period
- When these conditions (called the **Dirichlet's conditions**) apply, g(t) may be expanded in a trigonometric Fourier series of the form
- $\boldsymbol{g(t) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)}$, where,
  - $a_0 = \frac{1}{T_0}\int_0^{T_0} g(t)\, dt$; (dc or average value)
  - $a_n = \frac{2}{T_0}\int_0^{T_0} g(t) \cos n\omega_0 t\, dt$
  - $b_n = \frac{2}{T_0}\int_0^{T_0} g(t) \sin n\omega_0 t\, dt$



- These conditions are sufficient (but not necessary)
- In this representation, we can associate with g(t) a FS. This does not mean equality.
- At points where g(t) is continuous, the FS converges to the function g(t)
- At a point of discontinuity t_0, the FS converges to $\frac{1}{2}\left(g(t_0 -) + g(t_0 +)\right)$

Coefficients of the Fourier Series

- $\boldsymbol{g(t) \sim a_0 + \sum_{n=1}^{\infty}(a_n \; cos\, n\omega_0 t + b_n \, sin\, n\omega_0 t)}$, (1)
- Orthogonality Relations: You can easily verify the following relations:
- $\int_0^{T_0} \cos n\omega_0 t \cos m\omega_0 t = \begin{cases} \frac{T_0}{2} & , n = m \\ 0 & n \neq m \end{cases}$,
- $\int_0^{T_0} \sin n\omega_0 t \sin m\omega_0 t \; = \begin{cases} \frac{T_0}{2} & , n = m \\ 0 & n \neq m \end{cases}$
- $\int_0^{T_0} \sin n\omega_0 t \cos m\omega_0 t = 0$ for all n and m.

- To get a_0, we integrate both sides of (1) with respect to t over one period.
- $\int_0^{T_0} g(t)\, dt \;\; \sim \;\; \int_0^{T_0} a_0 \, dt + \int_0^{T_0} [\sum_{n=1}^{\infty}(a_n \; cos\, n\omega_0 t + b_n \, sin\, n\omega_0 t)] \, dt$
- **Result:** $a_0 = \frac{1}{T_0} \int_0^{T_0} g(t)\, dt$; (dc or average value)

Coefficients of the Fourier Series

- $\boldsymbol{g(t) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)}$, (1)
- To get a_n, we multiply both sides of (1) by $\cos m\omega_0 t$, integrate over one period and use the orthogonality relations.
- $\int_0^{T_0} g(t)\, \boldsymbol{\cos m\omega_0 t}\, dt \sim \int_0^{T_0} a_0\, \boldsymbol{\cos m\omega_0 t}\, dt$

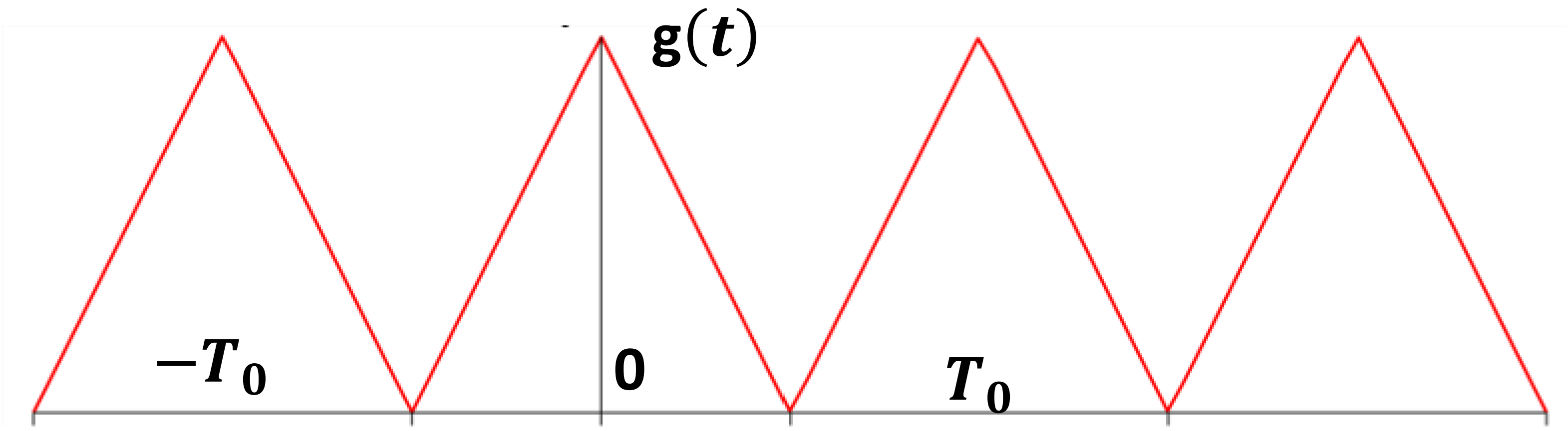
  $+ \int_0^{T_0} [\sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)\, \boldsymbol{\cos m\omega_0 t}]\, dt$
- **Result**: $a_n = \frac{2}{T_0} \int_0^{T_0} g(t) \cos n\omega_0 t\, dt$
- To get b_n, we multiply both sides of (1) by $\sin m\omega_0 t$, integrate over one period and use the orthogonality relations.
- $\int_0^{T_0} g(t)\, \boldsymbol{\sin m\omega_0 t}\, dt \sim \int_0^{T_0} a_0\, \boldsymbol{\sin m\omega_0 t}\, dt$

  $+ \int_0^{T_0} [\sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)\, \boldsymbol{\sin m\omega_0 t}]\, dt$
- **Result:** $b_n = \frac{2}{T_0} \int_0^{T_0} g(t) \sin n\omega_0 t\, dt$

Example: Existence of Fourier Series

- The Dirichlet conditions apply to the waveform given below.
- The function g(t) is absolutely integrable, i.e., $\int_0^{T_0} |g(t)|\, dt < \infty$.
- The function g(t) is continuous over the period (no discontinuities)
- Has one maximum and one minimum within one period.
- Therefore**, the FS exists**. Moreover, the **FS converges to g(t) at all points**. That is,

- $\boldsymbol{g(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)}$; Note the equality sign

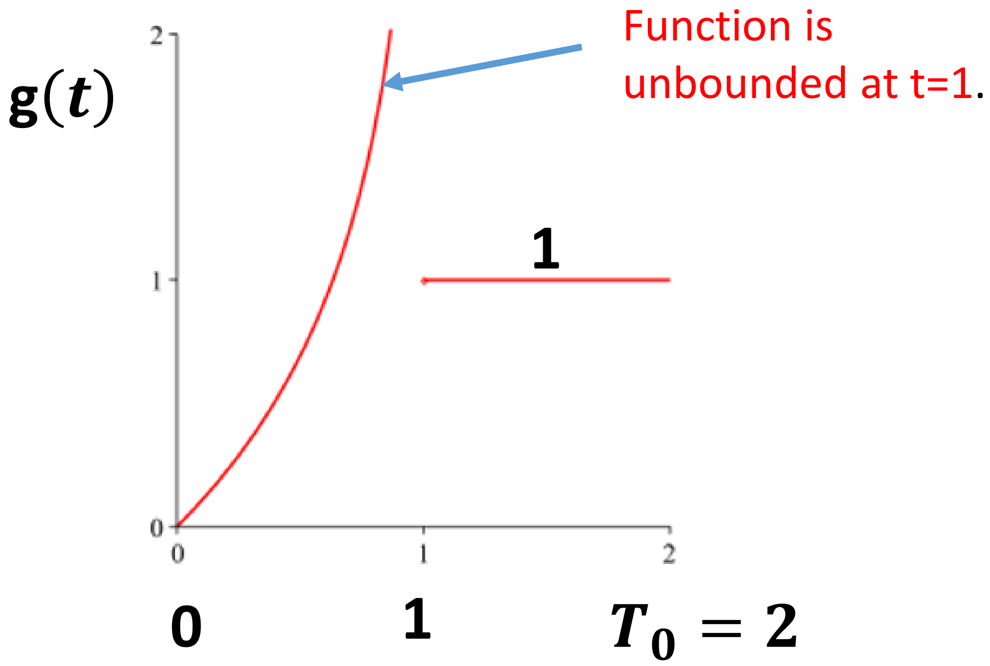


Example: Existence of Fourier Series

- Let g(t), defined over one period, be given by

$$g(t) = \begin{cases} -\ln(1-t), & 0 < t < 1 \\ 1, & 1 < t < 2 \end{cases}$$

- $\lim_{t \to 1}\big(g(t)\big) = -\ln(1-t) \to \infty$
- the function g(t) has a discontinuity. However, this discontinuity is infinite.
- **Therefore, the FS does not exists**



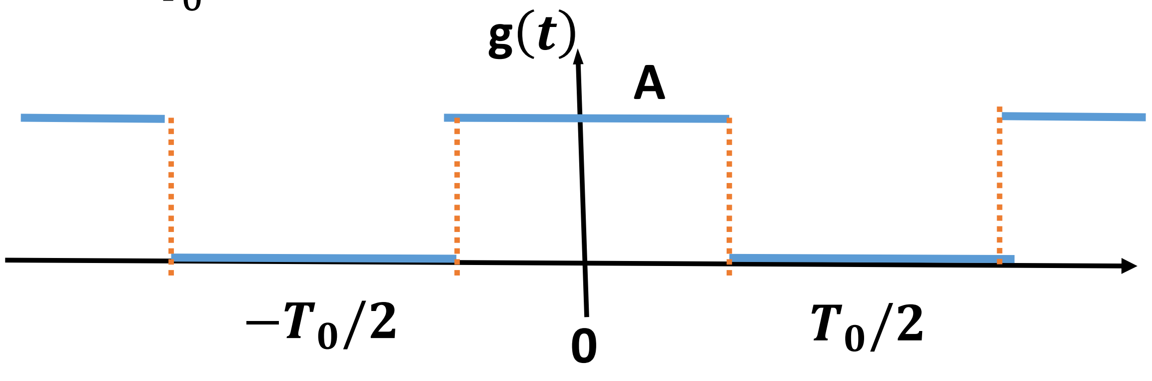
Example: Fourier Series Coefficient Evaluation

- **Example**: Find the trigonometric Fourier series of the periodic rectangular signal defined over one period T_0 as:

$$g(t) = \begin{cases} +A, & -T_0/4 \leq t \leq T_0/4 \\ 0, & otherwise \end{cases}$$

- **Solution**: The FS is given as $g(t) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$
- $a_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\, dt = \frac{1}{T_0}\int_{-T_0/4}^{T_0/4} A\, dt = A/2$
- $b_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} g(t) \sin(\frac{2\pi n}{T_0}t)\, dt = \frac{2}{T_0}\int_{-T_0/4}^{T_0/4} A \sin(\frac{2\pi n}{T_0}t)\, dt = 0$
- $a_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} g(t) \cos(\frac{2\pi n}{T_0}t)\, dt = \frac{2}{T_0}\int_{-T_0/4}^{T_0/4} A \cos(\frac{2\pi n}{T_0}t)\, dt$
- $a_n = \begin{cases} \frac{2A}{n\pi}, & n = 1, 5, 9, \ldots \\ \frac{-2A}{n\pi}, & n = 3, 7, 11, \ldots \\ 0, & n = 2, 4, 6 \ldots \end{cases}$

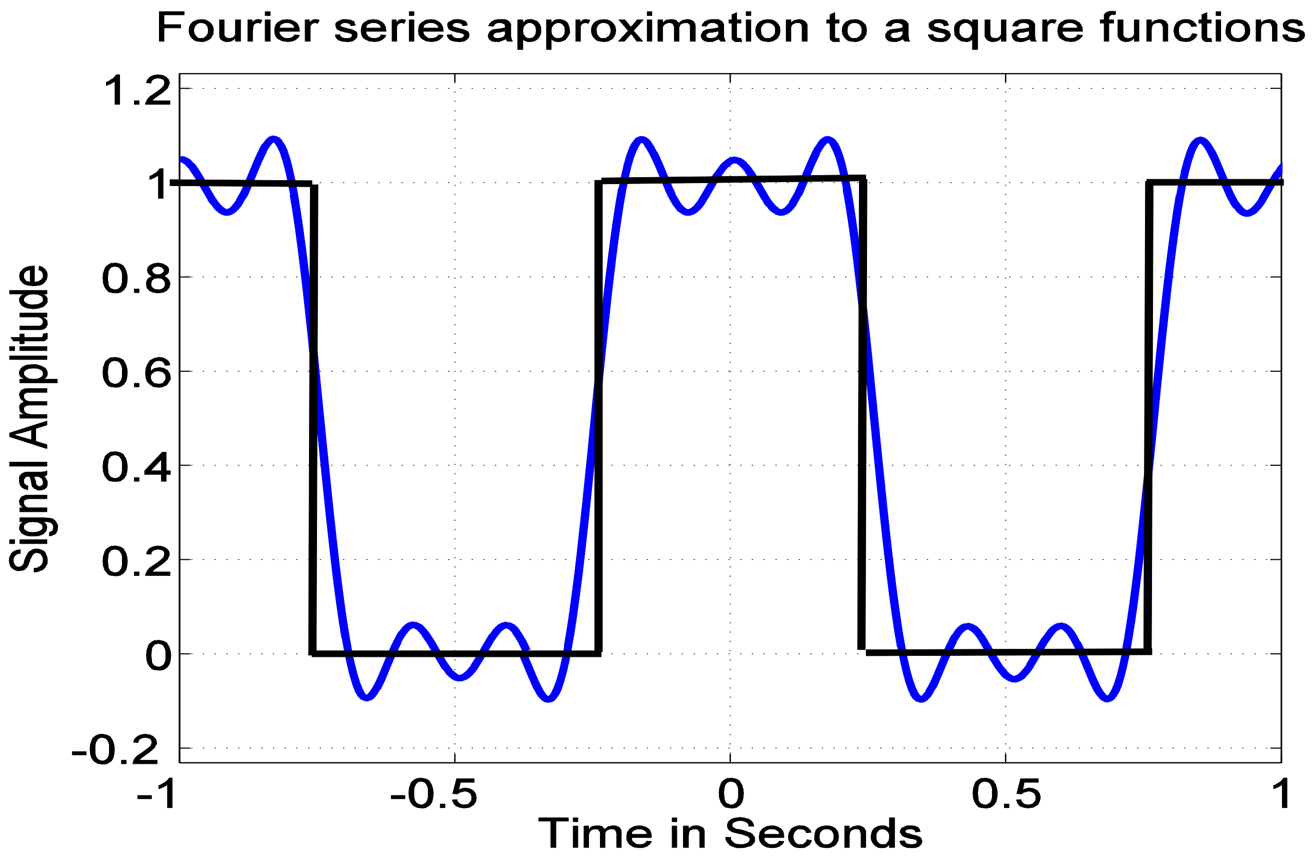
Dirichlet conditions apply. Therefore, a FS exists



Example: Convergence of Fourier Series

- The first four terms in the expansion of $g(t)$ are:
- $\tilde{g}(t) = \frac{A}{2} + \frac{2A}{\pi}\{\cos(2\pi f_0 t) - \frac{1}{3}\cos(2\pi 3 f_0 t) + \frac{1}{5}\cos(2\pi 5 f_0 t)\}$
- The function $\tilde{g}(t)$ along with $g(t)$ are plotted in the figure for $-1 \leq t \leq 1$ assuming $A = 1$ and $f_0 = 1$

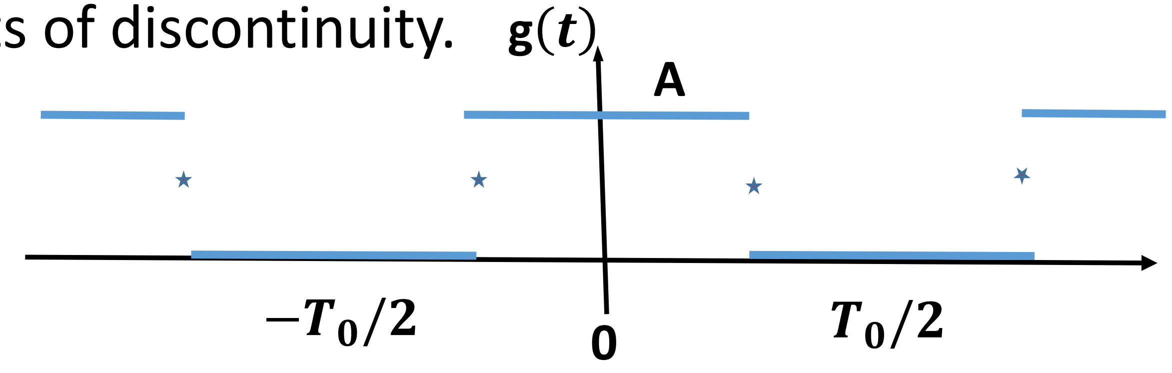
Comments: As more terms are added to $\tilde{g}(t)$, $\tilde{g}(t)$ becomes closer to $g(t)$ and in the limit as $n \rightarrow \infty$, $\tilde{g}(t)$ becomes equal to $g(t)$ at all points except at the points of discontinuity.



Convergence of the Fourier Series

The Fourier series of the signal $g(t) = \begin{cases} +A, & -T_0/4 \le t \le T_0/4 \\ 0, & otherwise \end{cases}$ is given by

- $g(t) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$. The FS is shown in the figure below
- $a_0 = A/2$, $b_n = 0$,
- $a_n = \begin{cases} \frac{2A}{n\pi}, & n = 1, 5, 9, \dots \\ \frac{-2A}{n\pi}, & n = 3, 7, 11, \dots \\ 0, & n = 2, 4, 6 \dots \end{cases}$
- The FS converges to g(t) at all points where g(t) is continuous
- Converges to A/2, the average value at points of discontinuity.



Fourier Cosine and Sine Series

- Let $g(t)$ be a periodic function of time with period $T_0 = \frac{1}{f_0}$ such that its FS exists.
- **Fourier Cosine Series**:
- Let g(t) be an even function of t, then
- $b_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\sin(\frac{2\pi n}{T_0}t)\,dt = 0$
- $a_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\cos(\frac{2\pi n}{T_0}t)\,dt = \frac{4}{T_0}\int_{0}^{T_0/2} g(t)\cos(\frac{2\pi n}{T_0}t)\,dt$
- The FS becomes a Fourier cosine series $\boldsymbol{g(t) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t}$
- **Fourier Sine Series**:
- Let g(t) be an odd function of t, then
- $a_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\,dt = 0, \;\; a_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\cos(\frac{2\pi n}{T_0}t)\,dt = 0$
- $b_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\sin(\frac{2\pi n}{T_0}t)\,dt = \frac{4}{T_0}\int_{0}^{T_0/2} g(t)\sin(\frac{2\pi n}{T_0}t)\,dt$
- The FS becomes a Fourier sine series $\boldsymbol{g(t) \sim \sum_{n=1}^{\infty} b_n \sin n\omega_0 t}$

Complex Form of the Fourier Series

The Fourier series can also be expressed in the complex form:

$$g(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

where, $$C_n = \frac{1}{T_0} \int_0^{T_0} g(t)\, e^{-jn\omega_0 t} dt.$$

- Note that C_n is a complex valued quantity, which can be written as
- $C_n = |C_n| e^{j\theta n}$
- The plot of $|C_n|$ versus frequency is called the ***Discrete Amplitude Spectrum***.
- The plot of θ_n versus frequency is called the ***Discrete Phase Spectrum***.
- The term at f_0 is referred to as the fundamental frequency. The term at $2f_0$ is referred to as the second order harmonic, the term at $3f_0$ is referred to as the third order harmonic and so on.

Parseval's Power Theorem

- The average power of a periodic signal g(t) is given by:

$$P_{av} = \frac{1}{T_0}\int_0^{T_0}|g(t)|^2\,dt = \sum_{n=-\infty}^{\infty}|C_n|^2 = |C_0|^2 + 2\sum_{n=1}^{\infty}|C_n|^2$$

- $= |a_0|^2 + \frac{1}{2}\sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2$

- **Proof:** $g(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$, where, $C_n = \frac{1}{T_0}\int_0^{T_0} g(t)\, e^{-jn\omega_0 t}dt.$
- $|g(t)|^2 = g(t)g^*(t) = \left(\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}\right)\left(\sum_{m=-\infty}^{\infty} C_m^* e^{-jm\omega_0 t}\right)$
- $\frac{1}{T_0}\int_0^{T_0}|g(t)|^2 dt = \frac{1}{T_0}\int_0^{T_0}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} C_n\, C_m^*\, e^{j(n-m)\omega_0 t}dt$
- Orthogonality: $\int_0^{T_0} e^{j(n-m)\omega_0 t}dt = \begin{cases} T_0 & , n = m \\ 0 & n \neq m \end{cases}$
- $\frac{1}{T_0}\int_0^{T_0}|g(t)|^2 dt = \sum_{n=-\infty}^{\infty}|C_n|^2 = |C_0|^2 + 2\sum_{n=1}^{\infty}|C_n|^2$

Power Spectral Density

- The plot of $|C_n|^2$ versus frequency is called the ***power spectral density*** (PSD).
- It displays the power content of each frequency (spectral) component of a signal.
- For a periodic signal, the PSD consists of discrete terms at multiples of the fundamental frequency.
- The next example demonstrate these properties

Power Spectral Density

- **Example:** Find the power spectral density of the $g(t)$ shown in the figure.
- Here, we need to find the complex Fourier series expansion, where the period $T_0 = 2\tau$
- $g(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$; $\quad C_n = \frac{1}{T_0}\int_0^{T_0} g(t)\, e^{-jn\omega_0 t} dt$
- $C_n = \begin{cases} \frac{A}{2}, & n = 0 \\ \frac{3A}{|n|\pi}, & n = \pm1, \pm5, \pm9, \ldots \\ \frac{-3A}{|n|\pi}, & n = \pm3, \pm7, \pm11, \ldots \\ 0, & n = \pm2, \pm4, \ldots \end{cases}$

$$|C_n|^2 = \begin{cases} (\frac{A}{2})^2, & n = 0 \\ (\frac{3A}{n\pi})^2, & n: odd \\ 0, & n: even \end{cases}$$

$$S_g(f) = \sum_{n=-\infty}^{\infty} |C_n|^2\, \delta(f - nf_0)$$