Fourier Transform

- Let $g(t)$ be a function of time t. The Fourier transform maps the function g(t) into another function G(f) defined into the frequency domain. The Fourier transform is defined as:

$$\boldsymbol{G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t} \, dt}$$

- The inverse Fourier transform is defined as

$$\boldsymbol{g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi f t} \, df}$$

- Conditions for existence (Dirichlet conditions, which are the same as those for the FS)
  - The function g(t) is absolutely integrable, i.e.,, $\int_{0}^{T_0} |g(t)| \, dt < \infty$.
  - Any discontinuities in g(t) are finite
  - g(t) has only a finite number of discontinuities and only a finite number of maxima and minima in any finite interval.
- **Remarks**:
  - These conditions are sufficient but not necessary
  - A **weaker sufficient condition** for existence is $\boldsymbol{\int_{-\infty}^{\infty} |g(t)|^2 \, dt < \infty}$ (g(t) is an energy signal). This is the finite-energy condition that is satisfied by all physically realizable waveforms.
  - Generally, physical waveforms encountered in engineering practice are Fourier transformable.
  - The Fourier transform can be derived from the Fourier series by allowing the period T_0 to go to infinity, but this will not be covered in this presentation.

Fourier Transform: Amplitude and Phase Spectrum

Observations: $\boldsymbol{G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}\, dt}$

- G(f) is a complex function of frequency f, which can be expressed as:

  $$G(f) = |G(f)|j^{\theta(f)}$$

- The function $G(f)$ is often referred to as the spectrum of $g(t)$.
  - $|G(f)|$: is the ***continuous amplitude spectrum*** of $g(t)$, (an even function of f).
  - $\theta(f)$: is the ***continuous phase spectrum*** of $g(t)$, (an odd function of f).

- **Notation**:
  - To denote that $G(f)$ is the Fourier transform of $g(t)$, we write $G(f) = \mathfrak{I}(g(t))$
  - To denote that $g(t)$, *is the inverse Fourier transform of* $G(f)$, we write $g(t) = \mathfrak{I}^{-1}(G(f))$
  - Sometimes, the following notation is used for a Fourier transform pair g$(t) \leftrightarrow G(f)$.

Rayleigh Energy Theorem

Rayleigh Energy Theorem: The energy in a signal g(t) is given by:

$$\boldsymbol{E = \int_{-\infty}^{\infty} |g(t)|^2\, dt = \int_{-\infty}^{\infty} |G(f)|^2 df}$$

- The proof of this result is the same as that for Parseval's power theorem
- The function $|G(f)|^2$ is called the ***energy spectral density***. It depicts the range of frequencies over which the signal energy extends and the frequency bands which are significant in terms of their energy contents.
- For a non-period signal energy signal, the **energy spectral density is a continuous function of f.**

A General Form of the Rayleigh Energy Theorem

- For two energy functions $g(t)$ and $v(t)$, the following result holds:

$$\boldsymbol{\int_{-\infty}^{\infty} g(t)v(t)^*\, dt = \int_{-\infty}^{\infty} G(f)V(f)^*\, df}$$

Example: Exponential Pulse

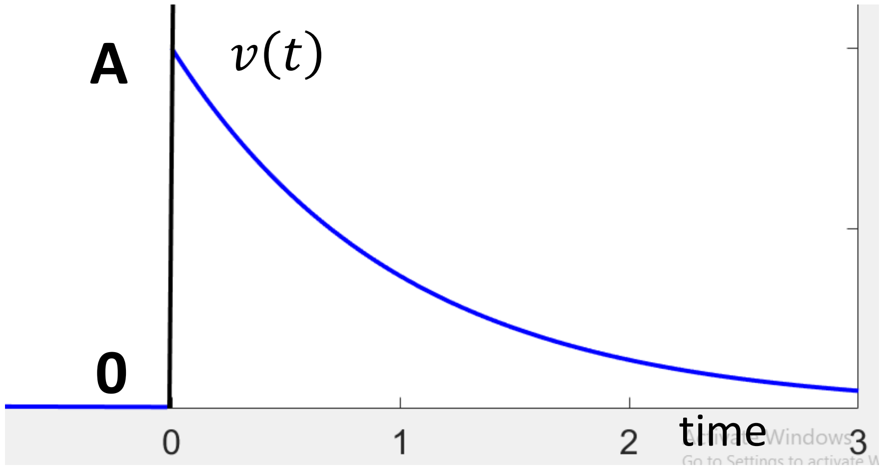
- $v(t) = \begin{cases} A\, e^{-bt} & t > 0 \\ 0 & t < 0 \end{cases}$
- $\boldsymbol{E = \int_{-\infty}^{\infty} |g(t)|^2\, dt = \int_0^{\infty} A^2\, e^{-2bt} dt = (A^2/\,2b)}$ **, F.T exists**
- $V(f) = \int_0^{\infty} v(t) e^{-j2\pi f t}\; dt = \int_0^{\infty} A e^{-bt}\, e^{-j2\pi f t} dt$

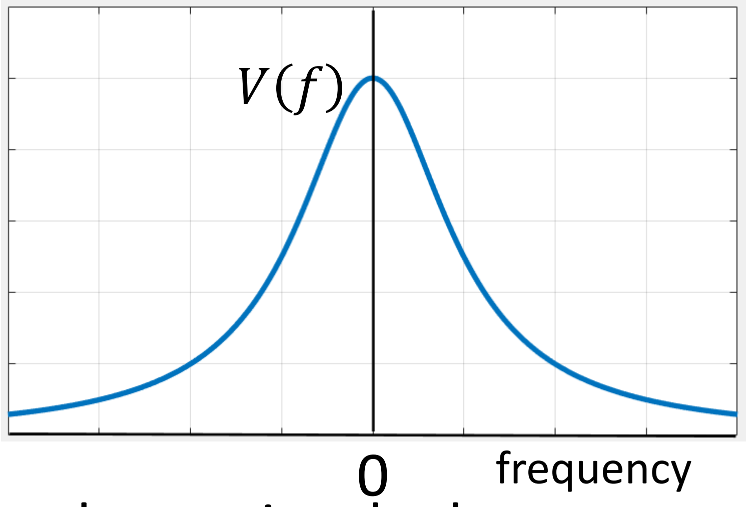
$$V(f) = A \int_0^{\infty} e^{-(b+j2\pi f)t}\; dt = A \frac{e^{-(b+j2\pi f)t}}{-(b+j2\pi f)} \Big|_0^{\infty} = \frac{A}{b+j2\pi f}.$$

- $V(f) = \frac{A}{b+j2\pi f},$
- $|V(f)| = \frac{A}{(b^2+(2\pi f)^2)^{1/2}}$
- The energy spectral density is: $S_v(f) = |V(f)|^2 = \frac{A^2}{b^2+(2\pi f)^2}$

Exercise: For the given v(t), verify Rayleigh Energy Theorem:

$$\boldsymbol{E = \int_{-\infty}^{\infty} |v(t)|^2\, dt = \int_{-\infty}^{\infty} |V(f)|^2 df}$$

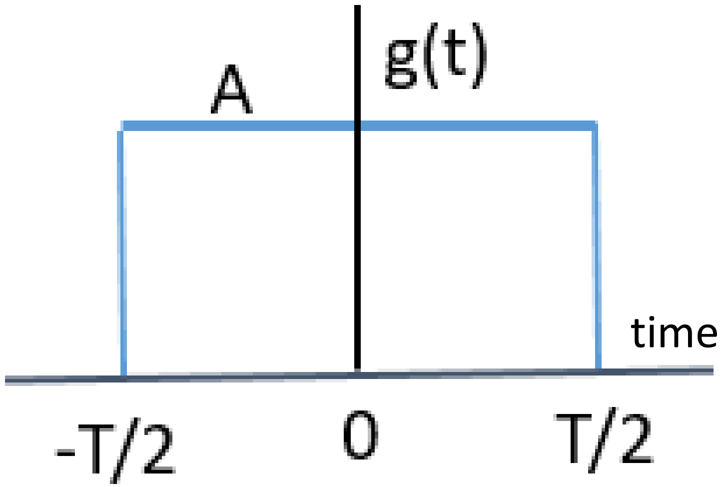


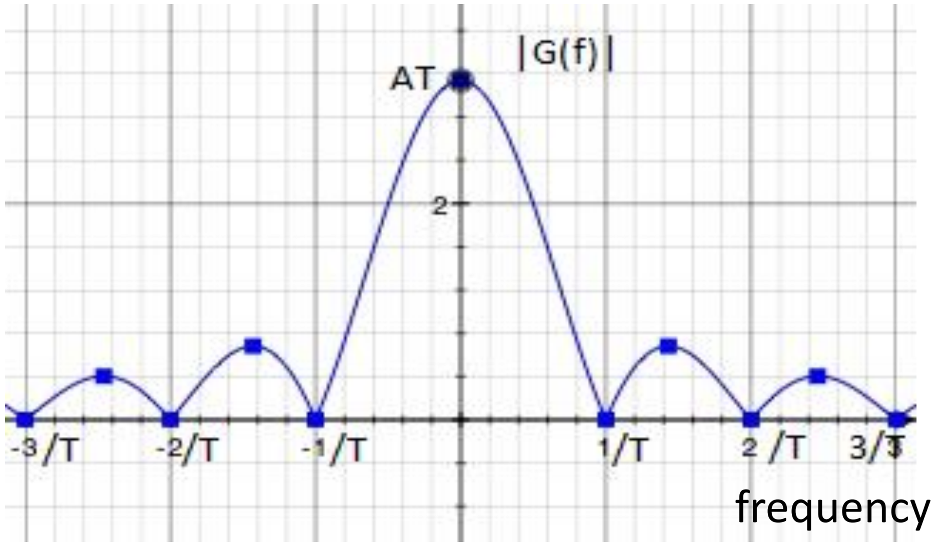


- **Remark**: The signal $v(t)$ is called a ***baseband signal*** since the signal occupies the low frequency part of the spectrum. That is, the energy in the signal is found around the zero frequency. When the signal is multiplied by a high frequency carrier, the spectrum becomes centered around the carrier and the modulated signal is called a ***bandpass signal***.

Example: The Rectangular Pulse $g(t) = A\,rect(\frac{t}{T})$

- $G(f) = \int_{-T/2}^{T/2} Ae^{-j2\pi ft}dt = \frac{A}{\pi f}\sin \pi fT$, $AT\frac{\sin \pi fT}{\pi fT} \triangleq AT\, sinc fT$
- $|\boldsymbol{G(f)}| = \boldsymbol{AT}\ |\boldsymbol{sinc fT}|$
- The maximum of $|G(f)|$ occurs at $f = 0$ since $\lim_{x\to 0}\frac{\sin x}{x} = 1$. Also, $G(f) = 0$ when $sin(\pi fT) = 0$, which occurs at the points that satisfy $\pi fT = n\pi, \Rightarrow fT = n, or\ f = \frac{n}{T}, n = \pm 1, \pm 2, \pm 3, \ldots$





Properties of the Fourier Transform

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}\, dt$$

- **Linearity (superposition)**

  Let $g_1(t) \leftrightarrow G_1(\mathrm{f})$ and $g_2(t) \leftrightarrow G_2(\mathrm{f})$, then

  $c_1 g_1(t)+c_2 g_2(t) \leftrightarrow c_1 G_1(f)+c_2 G_2(f)$; c_1, c_2 are constants

- **Time Scaling**

$g(t) \leftrightarrow G(f)$	$g(at) \leftrightarrow \frac{1}{\lvert a\rvert} G(f/a)$

- **Duality**

$g(t) \leftrightarrow G(f)$	$G(t) \leftrightarrow g(-f)$

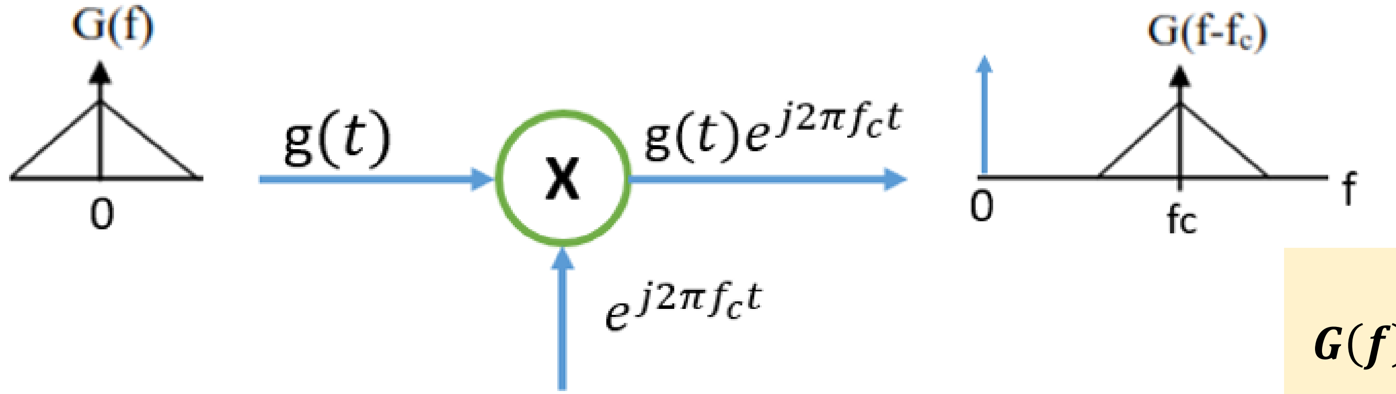
- **Time Shifting**

$g(t) \leftrightarrow G(f)$	$g(t-t_0) \leftrightarrow G(f)e^{-j2\pi f t_0}$
Delay in time domain corresponds to a phase shift in frequency domain	

Properties of the Fourier Transform

- **Frequency Shifting**

$g(t) \leftrightarrow G(f)$	$g(t)e^{j2\pi f_c t} \leftrightarrow G(f - f_c)$; f_c is constant



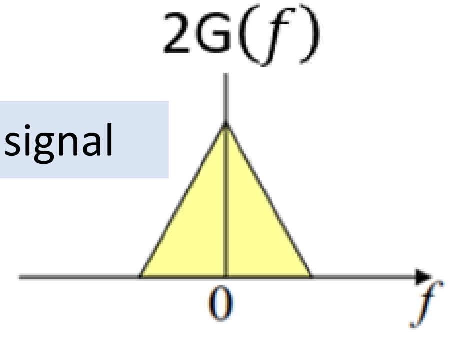
$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi f t}\, dt$$

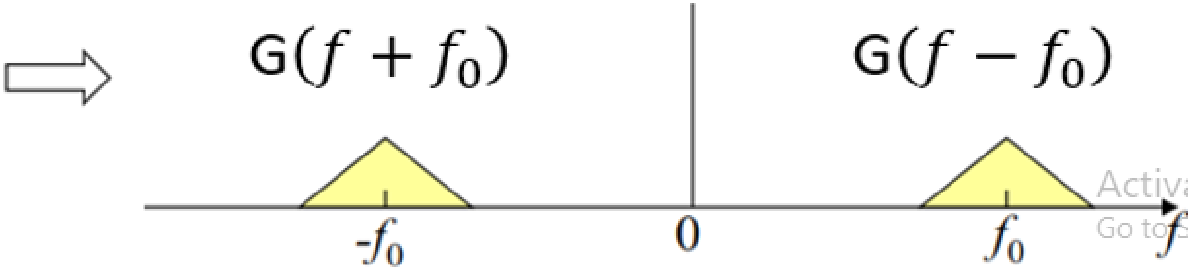
Frequency Shifting Property of the Fourier Transfer

- **Modulation Property**

$g(t) \leftrightarrow G(f)$	$2g(t)\cos(2\pi f_0 t) \leftrightarrow G(f - f_0) + G(f + f_0)$; f_0 is a constant

Low-pass signal





Band-pass signal

Properties of the Fourier Transform

- **Area under $G(f)$**

$g(t) \leftrightarrow G(f)$	$g(t=0) = \int_{-\infty}^{\infty} G(f)df$
The value $g(t=0)$ is equal to the area under its Fourier transform function	

- **Area under $g(t)$**

$g(t) \leftrightarrow G(f)$	$G(0) = \int_{-\infty}^{\infty} g(t)dt$
The area under a function g(t) is equal to the value of its Fourier transform G(f) at f = 0, where $G(0)$ implies the presence of a dc component.	

$$\boldsymbol{g(t) = \int_{-\infty}^{\infty} G(f)e^{j2\pi ft}\, df}$$

$$\boldsymbol{G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}}$$

Properties of the Fourier Transform

- **Differentiation in the Time Domain**

If $g(t)$ and its derivative $g'(t)$ are Fourier transformable, then,

$$g'(t) \leftrightarrow (j2\pi f)G(f)$$

i.e., differentiation in the time domain $\Longrightarrow$ multiplication by j$2\pi f$ in the frequency domain.

(Differentiation in the time domain enhances high frequency components of a signal)

Also, $$\frac{d^n g(t)}{dt^n} \leftrightarrow (j2\pi f)^n G(f)$$

$$\boldsymbol{g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi f t}\, df}$$

- **Integration in the Time Domain**

$$\int_{-\infty}^{t} g(\tau)d\tau \leftrightarrow \frac{1}{j2\pi f}G(f);\ \text{assuming } G(0) = 0$$

i.e., integration in the time domain corresponds to division by (j$2\pi f$) in the frequency domain. This amounts to low pass filtering, where high frequency components are attenuated due to filtering.

When $G(0) \neq 0$, the above result becomes:

$$\int_{-\infty}^{t} g(\tau)d\tau \leftrightarrow \frac{1}{j2\pi f}G(f) + \frac{1}{2}\, G(0)\delta(f).$$

Properties of the Fourier Transform

- **Multiplication of two signals in the time domain**

$$g_1(t)\, g_2(t) \leftrightarrow \int_{-\infty}^{\infty} G_1(\lambda)\, G_2(f-\lambda) d\lambda = G_1(f) * G_2(f)$$

Multiplication of two signals in the time domain is transformed into the convolution of their Fourier transforms in the frequency domain.

- **Convolution of two signals in the time domain**

$$g_1(t) * \; g_2(t) \leftrightarrow G_1(f) G_2(f)$$

Convolution of two signals in the time domain is transformed into a multiplication of their Fourier transforms in the frequency domain

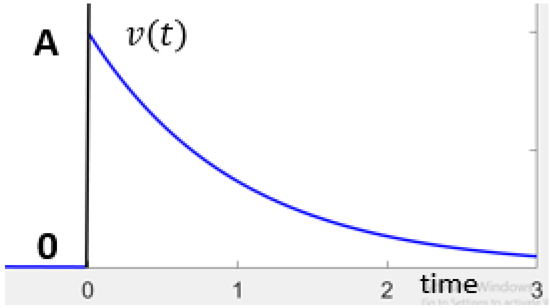
- **Multiplication by t in the time domain corresponds to differentiation in the** frequency domain

$$\mathfrak{I}\{\mathbf{t}\mathbf{g}(t)\} = \frac{j}{2\pi}\frac{dG(f)}{df}$$

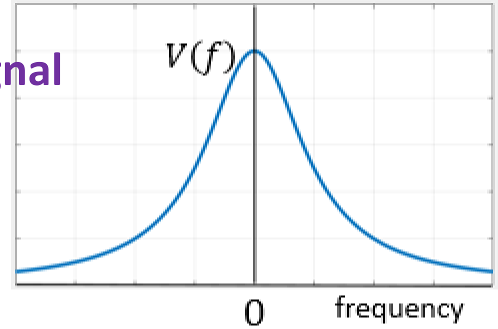
$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t}$$

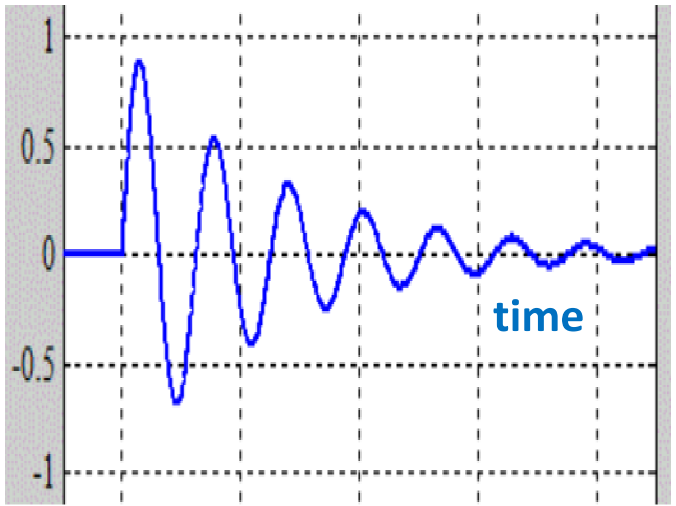
Examples: The RF Negative Exponential Pulse

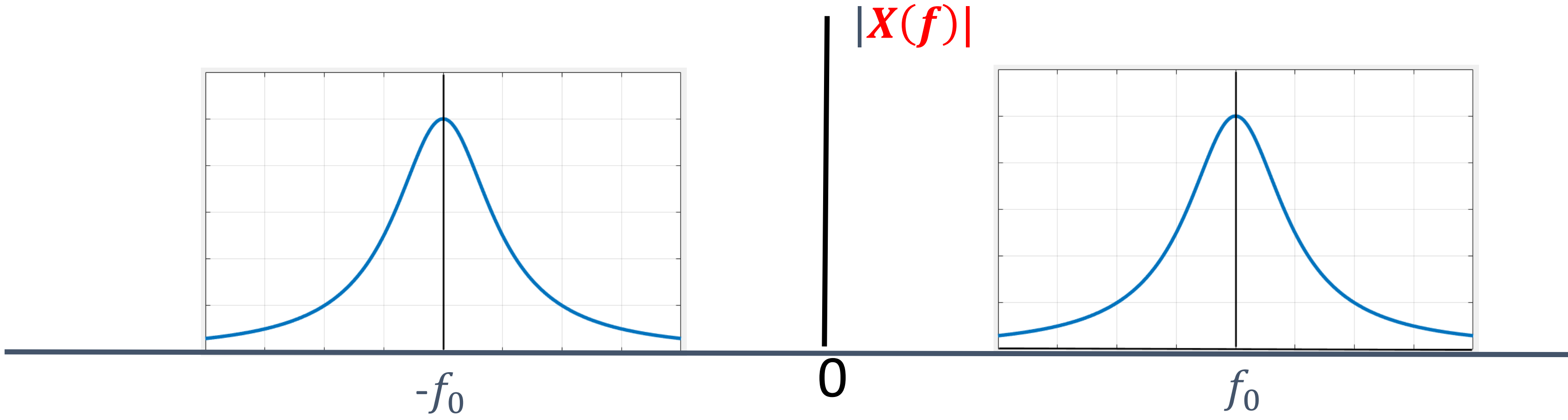
- **Example**: Find the Fourier transform of $x(t) = A\, e^{-bt} cos(2\pi f_0 t),\ t > 0$
- **Solution**: Note that x(t) can be expressed as
- $\mathbf{x(t)} = \mathbf{g}(t) cos(2\pi f_0 t),\ g(t) = \begin{cases} A\, e^{-bt} & t > 0 \\ 0 & t < 0 \end{cases}$
- $G(f) = \left(\frac{A}{b + j2\pi f}\right)$
- Use the modulation property
- $X(f) = \frac{1}{2}\{G(f - f_0) + G(f + f_0)\}$
- $X(f) = \frac{1}{2}\left\{\frac{A}{b + j2\pi(f - f_0)} + \frac{A}{b + j2\pi(f + f_0)}\right\}$; **Band-pass signal**



Baseband signal







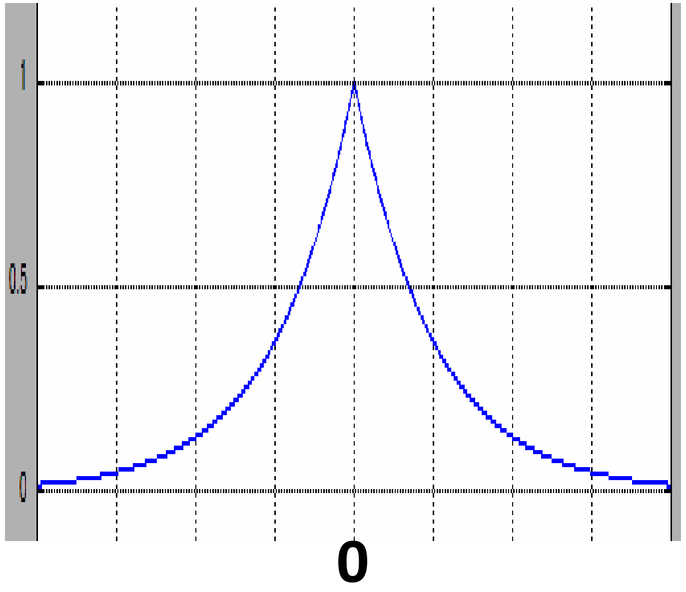
Example: double-sided exponential pulse

- **Example**: Find the Fourier transform of the double-sided exponential pulse

$$\mathbf{g}(t) = Ae^{-b|t|}, -\infty < t < \infty$$

- **Solution**: You can easily find that the energy in g(t) is finite, and hence the F.T. exists.

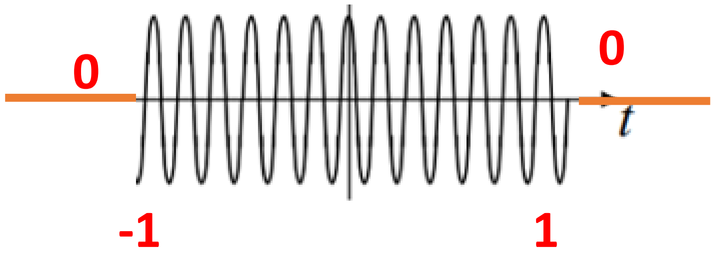
- $$G(f) = \int_{-\infty}^{0} Ae^{bt} e^{-j2\pi ft}\, dt + \int_{0}^{\infty} Ae^{-bt}\, e^{-j2\pi ft} dt$$
- $$G(f) = \frac{A}{b-j2\pi f} + \frac{A}{b+j2\pi f} = \frac{2bA}{b^2+(2\pi f)^2}$$



Examples: Fourier Transform of an RF Pulse

Find the Fourier transform of the RF pulse $x(t) = \cos(2\pi f_0 t)\,; 1 \leq t \leq 1,$

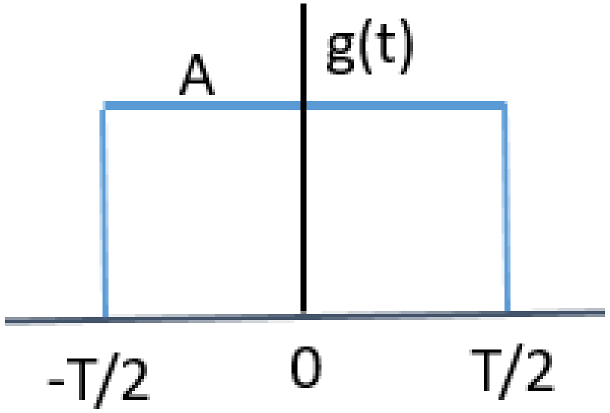
Solution: x(t) can be viewed as a product of the rectangular pulse and the cosine function $x(t) = \mathrm{g(t)}\cos(2\pi f_0 t)$, where

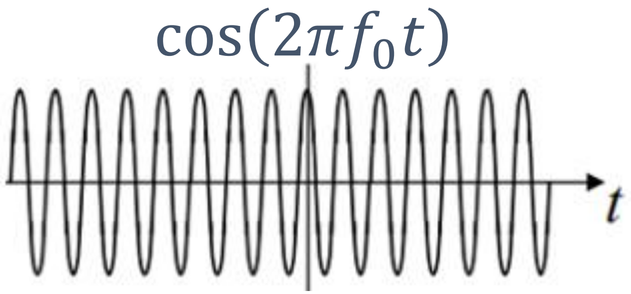


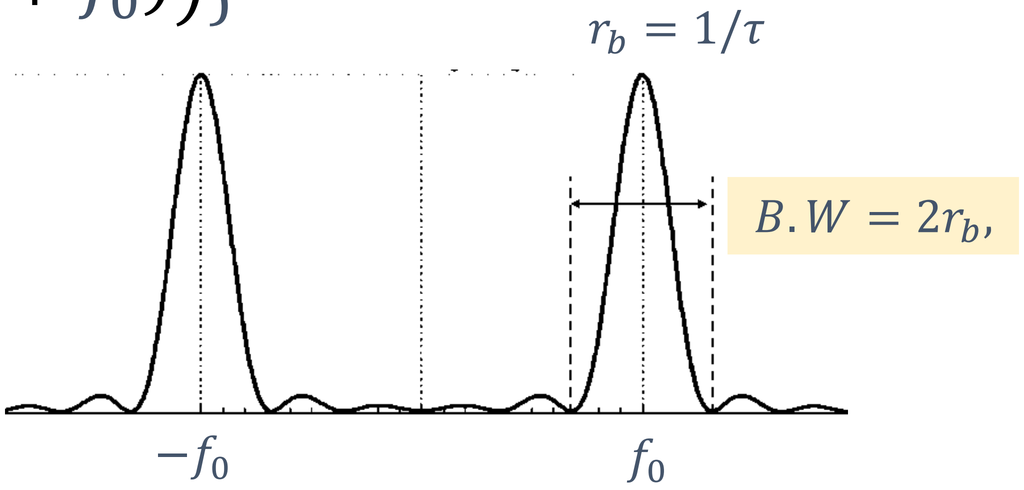
$$g(t) = u(t+1) - u(t+1) = rect(\frac{t}{2})$$

$$\mathrm{G}(f) = AT\ sinc fT = 2sinc(2f)$$

- $X(f) = \frac{1}{2}\{X(f - f_0) + X(f + f_0)\}$
- $X(f) = \frac{1}{2}\{2sinc\big(2(f - f_0)\big) + 2sinc\big(2(f + f_0)\big)\}$







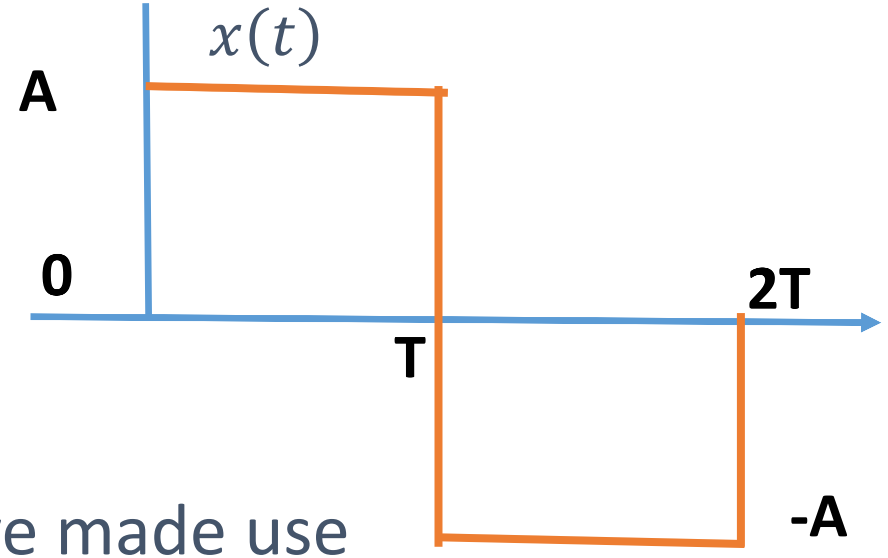
Examples: Fourier Transform of the doublet pulse

Find the Fourier transform of the pulse $x(t)$ shown in the figure

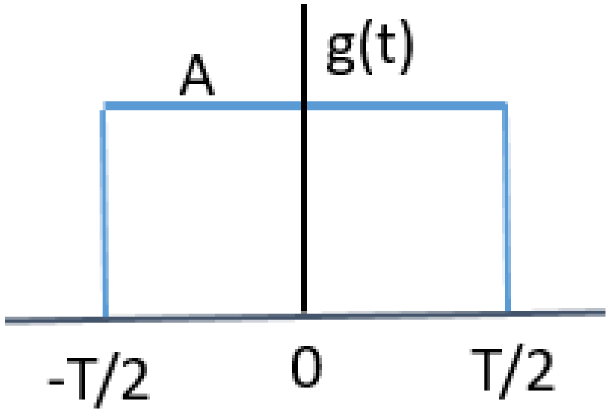
Solution: x(t) can be expressed in terms of the rectangular pulse g(t) as

$$x(t) = g(t - T/2) - g(t - 3T/2)$$

$$X(f) = G(f)e^{-\frac{j2\pi fT}{2}} - G(f)e^{-\frac{j2\pi f3T}{2}}$$



- $X(f) = G(f)e^{-j2\pi fT}(e^{\frac{j2\pi fT}{2}} - e^{-\frac{j2\pi fT}{2}})$
- $X(f) = G(f)e^{-j2\pi fT}(j2)sin(\frac{2\pi fT}{2})$
- Remark: Note that in this example, we have made use of the linearity and time shifting properties.



Examples: Fourier Transform of the triangular pulse

Find the Fourier transform of the pulse $y(t)$ shown in the figure

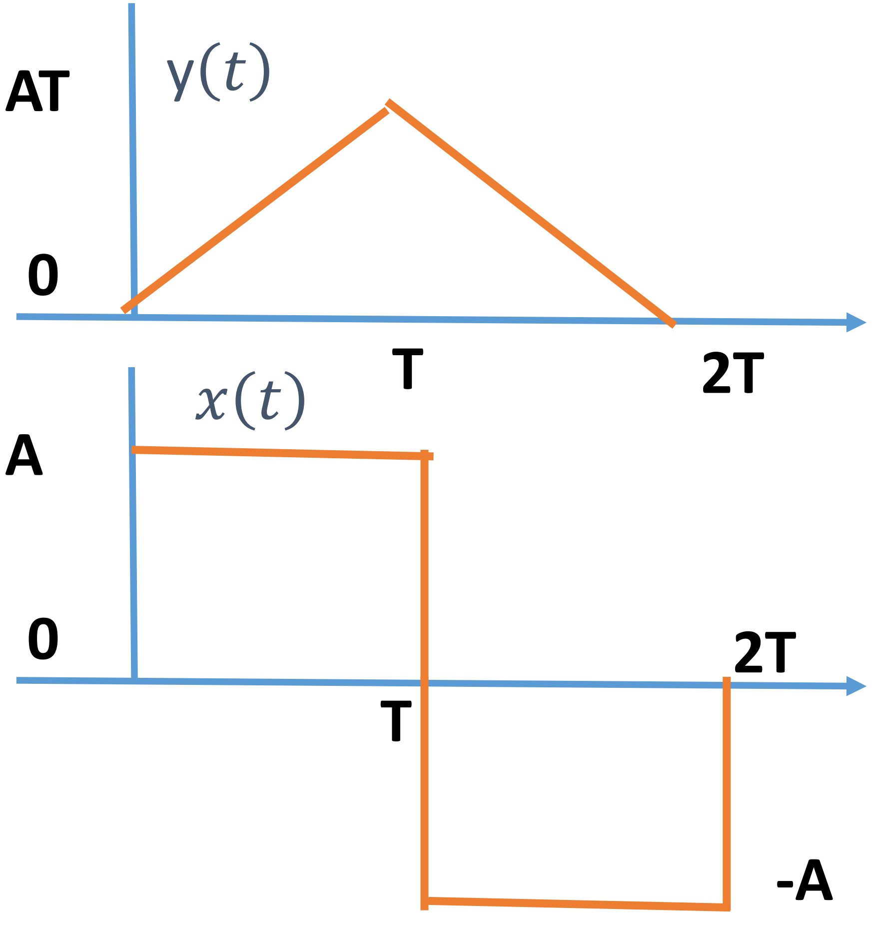
Solution: If we differentiate y(t), we get x(t) of the previous example

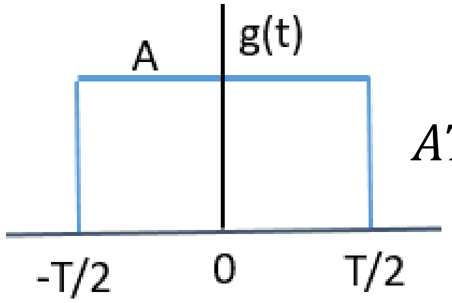
$$\frac{dy(t)}{dt} = x(t). \quad \text{Taking the F.T of both sides,}$$

$$j2\pi f Y(f) = X(f)$$

$$Y(f) = \frac{X(f)}{j2\pi f} = \frac{G(f)e^{-j2\pi fT}(j2)sin(\frac{2\pi fT}{2})}{j2\pi f}$$

$$G(f) = \frac{TG(f)e^{-j2\pi fT}sin(2\pi fT)}{\pi fT} = AT^2(sincfT)^2e^{-j2\pi fT}$$





$$AT\frac{\sin \pi fT}{\pi fT} \triangleq AT\ sincfT$$

Same result can be obtained by realizing that y(t)=g(t)*g(t) and using the convolution property Y(f)=G(f).G(f) and then using the time shifting property