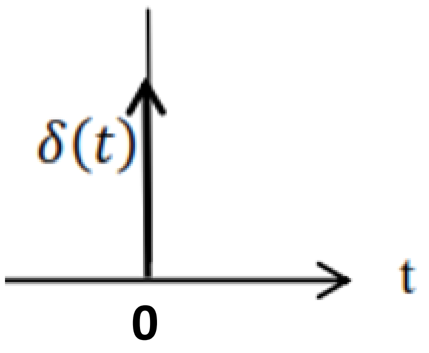
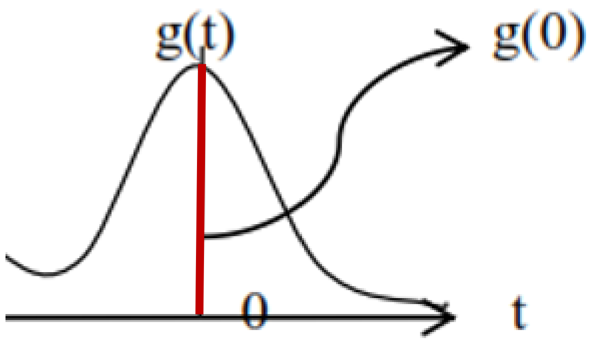
Fourier Transform of Power Signals

- For a **non-periodic (energy) signal** g(t), the Fourier transform exists when
- $E = \int_{-\infty}^{\infty} |g(t)|^2 \, dt < \infty$ (sufficient condition for existence)
- so that $G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t} dt$ **exists**
- **For power signals**, the integral $\int_{-\infty}^{\infty} g(t) e^{-j2\pi f t} dt$ **does not exist**.
- However, one can still finds the Fourier transform of power signals by employing the delta function. This function is defined next.
- **Dirac – Delta Function (Impulse Function)**





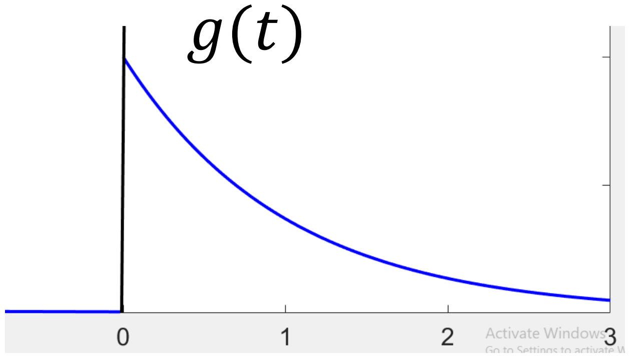
This function is defined as

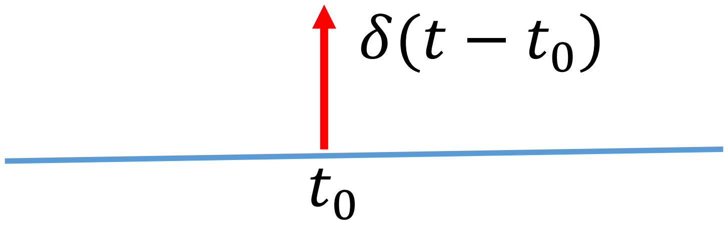
$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$

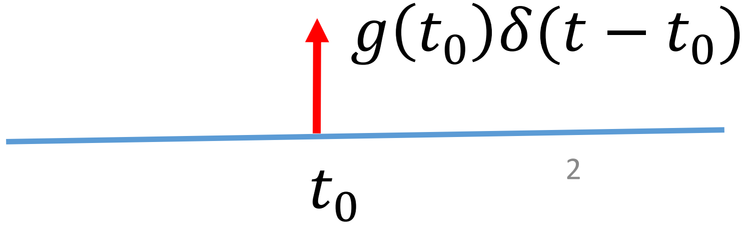
- such that: $\int_{-\infty}^{\infty} \delta(t) dt = 1$ and $\int_{-\infty}^{\infty} g(t)\delta(t) dt = g(0)$
- Here, g(t) is a continuous function of time. The second property, known as the sifting property, shows that the delta function samples the function g(t) at the time of its occurrence.

Some Properties of the Delta Function

- $g(t)\delta(t-t_0) = g(t_0)\delta(t-t_0)$; (Multiplication)
- $\int_{-\infty}^{\infty} g(t)\delta(t-t_0)dt = g(t_0)$; (Sifting or sampling property)
- $\delta(\alpha t) = \frac{1}{|\alpha|}\delta(t)$
- $\delta(t) * g(t) = g(t)$
- $\delta(t) = \frac{du(t)}{dt} \quad \Rightarrow u(t) = \int_{-\infty}^{t} \delta(t)dt$
- $\delta(t) = \delta(-t)$; an even function of its argument.
- Fourier transform: $\Im\{\delta(t)\} = 1$
- $\Im\{\delta(t-t_0)\} = e^{-j2\pi f t_0}$







Applications of the Delta Function

- **Fourier transform of the delta function**
- $\Im\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)e^{-j2\pi ft}dt = 1$. This follows from the sifting property

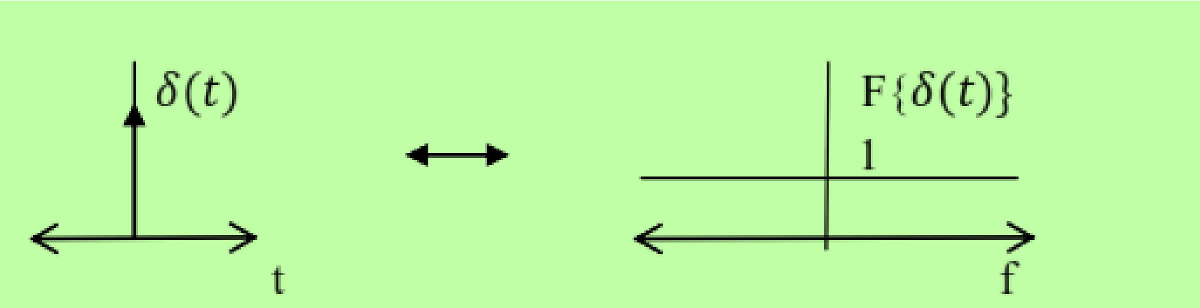
$$\int_{-\infty}^{\infty} g(t)\delta(t)dt = g(0) = 1$$

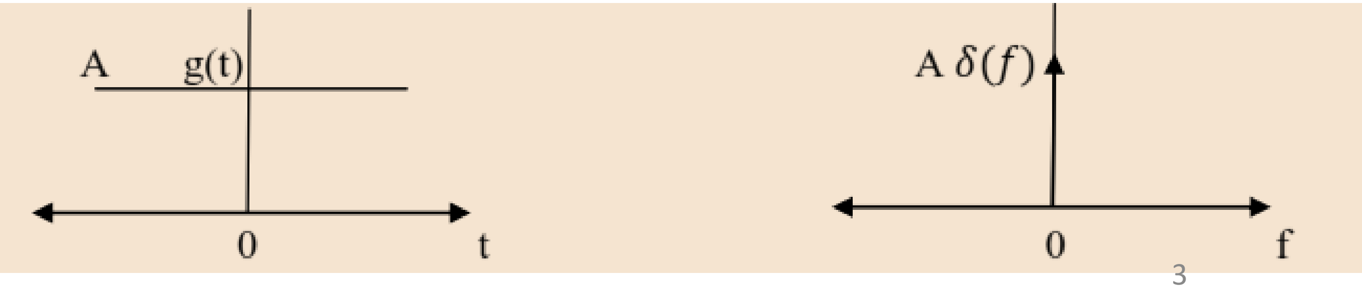
- $\Im\{\delta(t - t_0)\} = e^{-j2\pi f t_0}$; (using the time delay property $\Im\{g(t - t_0)\} = G(f)e^{-j2\pi f t_0}$
- **DC or a Constant Signal**
- Since $\Im\{\delta(t)\} = 1$, then by the duality property $\Im\{1\} = \delta(f)$

$g(t) \leftrightarrow G(f)$

$G(t) \leftrightarrow g(-f)$

- Note how the time-bandwidth relationship holds for this pair. A narrow pulse in time extends over a large frequency spectrum).
- Also, the transform of a dc signal is an impulse at f =0.





Applications of the Delta Function

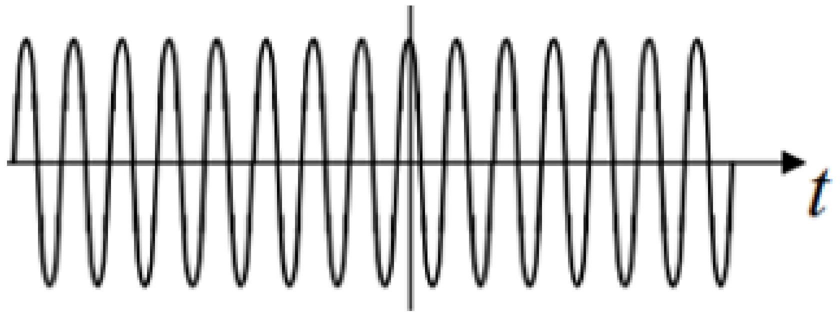
- **Complex Exponential Function**
- $\Im\{Ae^{j2\pi f_c t}\} = A\delta(f - f_c)$;
- follows from the duality property, since $\Im\{\delta(t - t_0)\} = e^{-j2\pi f t_0}$

$\boldsymbol{g(t) \leftrightarrow G(f)}$

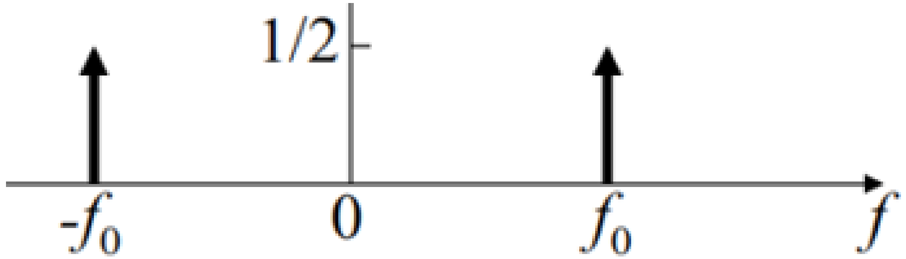
$\boldsymbol{G(t) \leftrightarrow g(-f)}$

- **Sinusoidal Functions**
- $\Im\{\cos 2\pi f_0 t\} = \Im\frac{1}{2}\{Ae^{j2\pi f_c t} + Ae^{-j2\pi f_c t}\} = \frac{1}{2}\{\delta(f - f_0) + \delta(f + f_0)\}$
- $\Im\{\sin 2\pi f_0 t\} = \Im\frac{1}{j2}\{Ae^{j2\pi f_c t} - Ae^{-j2\pi f_c t}\} = \frac{1}{2j}\{\delta(f - f_0) - \delta(f + f_0)\}$

$g(t) = \cos(2\pi f_0 t)$



G(f)



Applications of the Delta Function

- **Signum Function**
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$$sgn(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$

$$\Im\{sgn(t)\} = \frac{1}{j\pi f}$$

$$v(t) = \begin{cases} e^{-bt} & t > 0 \\ -e^{bt} & t < 0 \end{cases}$$

$$G(f) = \frac{1}{b+j2\pi f} - \frac{1}{b-j2\pi f} = \frac{-j(2)2\pi f}{b^2+(2\pi f)^2}$$

$$\log_{b\to 0} G(f) = \frac{1}{j\pi f}$$

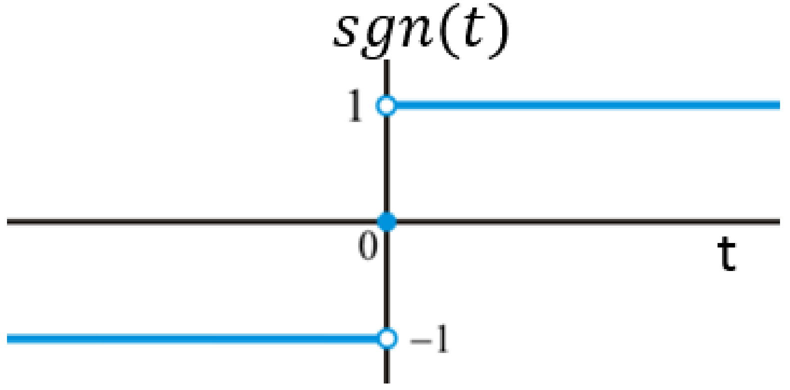
- **Unit Step Function**

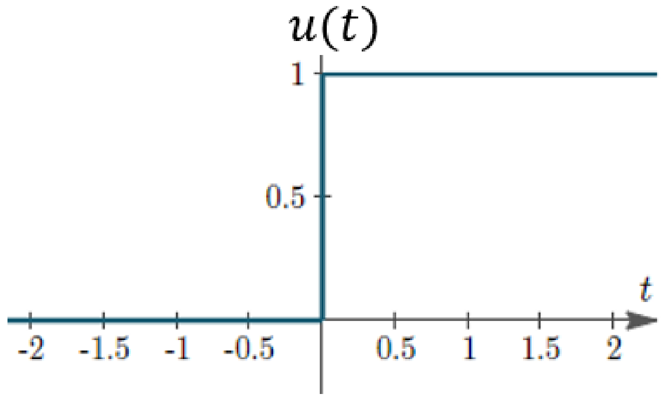
$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases}$$

$$sgn(t) = 2u(t) - 1$$

$$u(t) = \frac{1}{2}\{sgn(t) + 1\}$$

$$\Im\{u(t)\} = \frac{1}{j2\pi f} + \frac{1}{2}\,\delta(f)$$





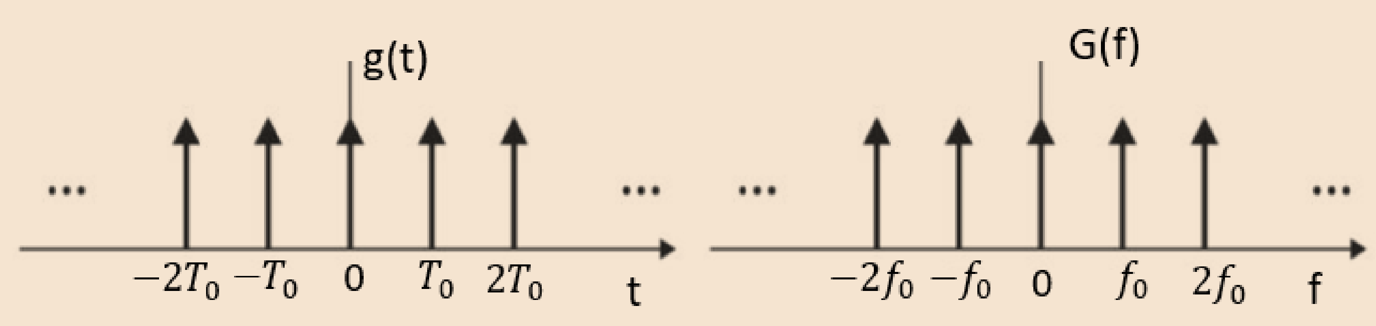
Applications of the Delta Function

- **Periodic Signals:** A periodic signal g(t) is expanded in the complex Fourier Series form as:
- $\boldsymbol{g(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \Rightarrow \Im\{g(t) = \sum_{n=-\infty}^{\infty} C_n\, \delta(f - nf_0)}$

Example: Consider the following train of impulses $g(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT_0)$

Solution: The Fourier coefficients are obtained by integrating over one period of $g(t)$.

- $C_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\, e^{-jn\omega_0 t} dt = \frac{1}{T_0} = f_0$; Note that the sifting property has been used.
- Therefore, the complex Fourier series of $g(t)$ is
- $g(t) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}; \Rightarrow \Im\{g(t)\} = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} \Im\{e^{jn\omega_0 t}\} = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} \delta(f - nf_0)$
- $\boldsymbol{\Im \sum_{m=-\infty}^{\infty} \delta(t - mT_0) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} \delta(f - nf_0)}$.

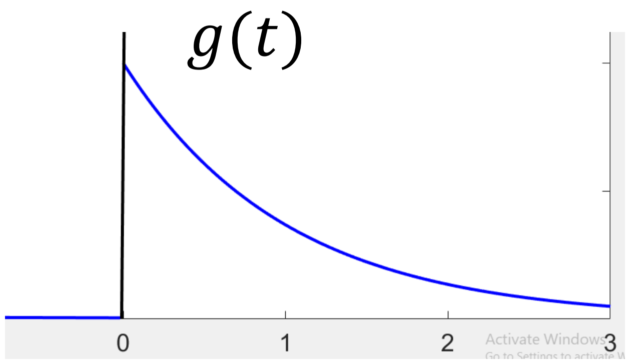


Remark 1: Note that the signal is periodic in the time domain and its Fourier transform is periodic in the frequency domain.

Remark 2: This sequence will be found useful when the sampling theorem is considered later in the course.

Examples

- Let g(t) be given as: $\mathbf{g}(t) = \begin{cases} A\,e^{-bt} & t > 0 \\ 0 & t < 0 \end{cases}.$



- The Fourier transform of g(t) is: $G(f) = \left(\frac{A}{b+j2\pi f}\right)$
- **Evaluate the following**

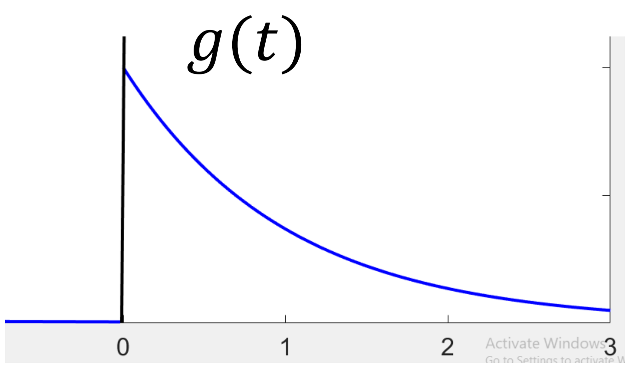
➢ $\mathbf{g}(t)\delta(t-0.5) = \mathbf{g}(t=0.5)\delta(t-0.5) = A\,e^{-0.5b}\delta(t-0.5).$

➢ $\mathbf{g}(t)\delta(t+1) = \mathbf{g}(t=-1)\delta(t+1) = (0)\delta(t+1) = 0.$

➢ $\mathbf{g}(t) * \delta(t-1) = g(t-1) = \begin{cases} A\,e^{-b(t-1)} & t > 1 \\ 0 & t < 1 \end{cases}$

➢ $\Im\{\mathbf{g}(t) * \mathbf{g}(t)\} = G(f)G(f) = \left(\frac{A}{b+j2\pi f}\right)\left(\frac{A}{b+j2\pi f}\right) = \left(\frac{A}{b+j2\pi f}\right)^2$

Examples



- Let g(t) be given as: $\mathbf{g}(t) = \begin{cases} A\, e^{-bt} & t > 0 \\ 0 & t < 0 \end{cases}.$
- The Fourier transform of g(t) is: $G(f) = \left(\frac{A}{b+j2\pi f}\right)$
- **Evaluate the following**

➢ $\int_{-\infty}^{\infty} g(t)\delta(t-1)dt = g(t=1) = A\, e^{-b}$; (sifting property)

➢ $\mathfrak{I}\{\mathbf{g}(t) - \mathbf{g}(t-1)\} = \boldsymbol{G}(\boldsymbol{f}) - \boldsymbol{G}(\boldsymbol{f})\boldsymbol{e}^{-j2\pi f} = \frac{A}{b+j2\pi f}(1 - \boldsymbol{e}^{-j2\pi f})$

➢ $\mathfrak{I}\{\mathbf{tg}(t)\} = \begin{cases} At\, e^{-bt} & t > 0 \\ 0 & t < 0 \end{cases}$

➢ $\mathfrak{I}\{\mathbf{tg}(t)\} = \frac{\boldsymbol{j}}{\boldsymbol{2\pi}}\frac{\boldsymbol{dG(f)}}{\boldsymbol{df}} = \frac{\boldsymbol{j}}{\boldsymbol{2\pi}}\frac{(-)j2\pi}{(b+j2\pi f)^2} = \frac{1}{(b+j2\pi f)^2}$

➢ Note: Prove that $\mathfrak{I}\{\mathrm{tg}(t)\} = \left(\frac{j}{2\pi}\right)\frac{dG(f)}{df}$ and $\mathfrak{I}\left\{\frac{dg(t)}{dt}\right\} = (j2\pi f)G(f)$