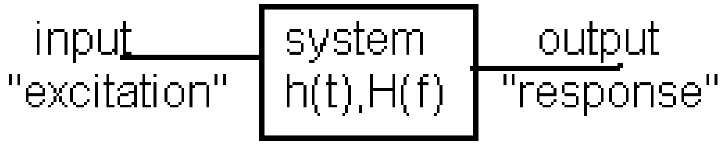
Transmission of Signals through Linear Systems

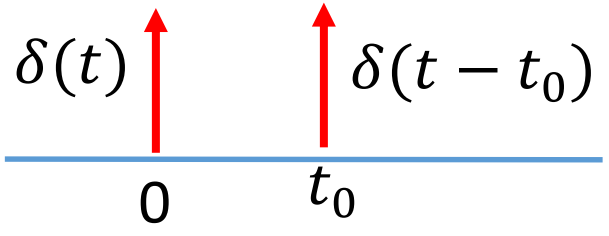
- ***Definition:*** A **system** refers to any physical device that produces an output signal in response to an input signal.
- ***Definition:*** A system is **linear** if the principle of superposition applies.
- If $x_1(t)$ produces output $y_1(t)$
- $x_2(t)$ produces output $y_2(t)$
- then $a_1x_1(t)+ a_2x_2(t)$ produces an output $a_1y_1(t)+ a_2y_2(t)$
- Also, a zero input should produce a zero output.

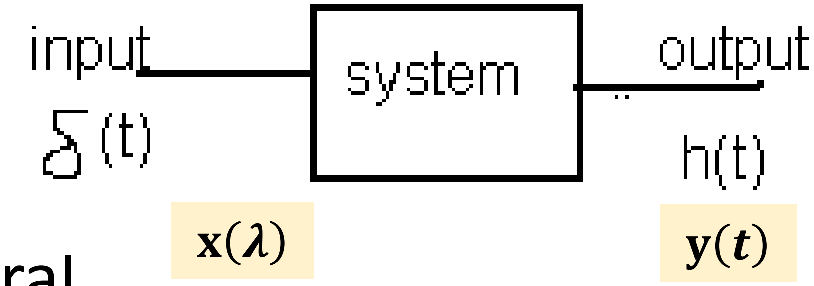


- Examples of linear systems include **filters and communication channels**.
- ***Definition:*** A **filter** refers to a frequency selective device that is used to limit the spectrum of a signal to some band of frequencies (will be discussed in detail in a later lecture)
- ***Definition:*** A **channel** refers to a transmission medium that connects the transmitter and receiver of a communication system.
- Time domain and frequency domain may be used to evaluate system performance.

Basic Time-domain Definitions

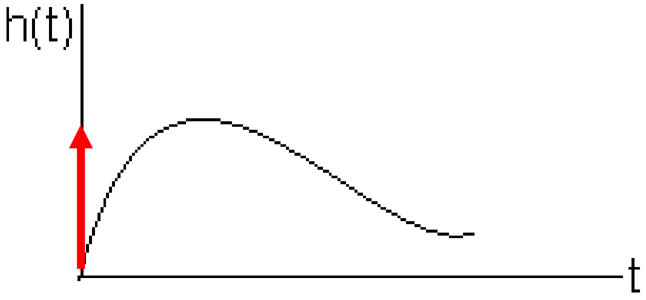
- ***Definition***: The **impulse response h(t)** is defined as the response of a system to an impulse $\delta(t)$ applied to the input at t=0 .
- ***Definition***: A system is **time-invariant** when the shape of the impulse response is the same no matter when the impulse is applied to the system.
- $\delta(t) \rightarrow h(t)$, then $\delta(t - t_0) \rightarrow h(t - t_0)$
- When the input to a linear time-invariant system in a signal x(t) , then the output is given by

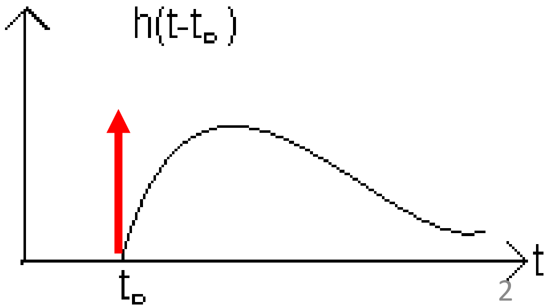




- $\mathbf{y}(t) = \int_{-\infty}^{\infty} \mathbf{x}(\lambda)\mathbf{h}(t - \lambda)\mathbf{d}\lambda$

  $= \int_{-\infty}^{\infty} \mathbf{h}(\lambda)\mathbf{x}(t - \lambda)d\lambda$; convolution integral



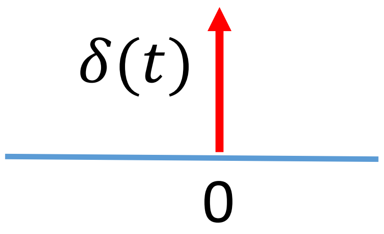


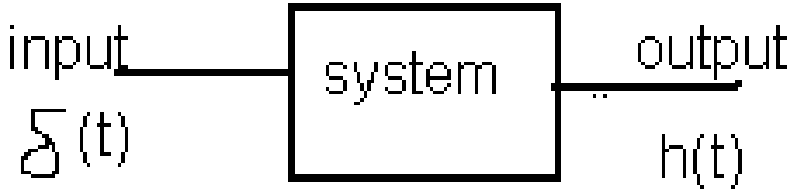
Basic Time-domain Definitions

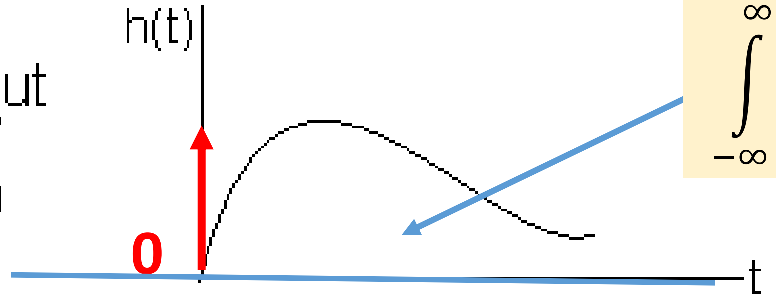
- ***Definition***: A system is said to be **causal** if it does not respond before the excitation is applied, i.e.,
- h(t) = 0 for t < 0 ; the causal system is physically realizable.
- ***Definition***: A system is said to be **stable** if the output signal is bounded for all bounded input signals.

$$\mathbf{y}(t) = \int_{-\infty}^{\infty} \mathbf{h}(\tau)\mathbf{x}(t-\tau)\ \mathbf{d\tau}$$

- If | x(t) | $\leq$ M ; M is the maximum value of the input
- then | y(t) | $\leq \int_{-\infty}^{\infty} |h(\tau)||x(t-\tau)|\, d\tau$ = M $\int_{-\infty}^{\infty} |h(\tau)|\, d\tau$
- Therefore, a necessary and sufficient condition for stability (a bounded output) is
- $\int_{-\infty}^{\infty} |\mathbf{h}(t)|\ \mathbf{dt} < \infty$; h(t) is absolutely integrable (zero initial conditions assumed)







$$\int_{-\infty}^{\infty} |\mathbf{h}(t)|\ \mathbf{dt} < \infty$$

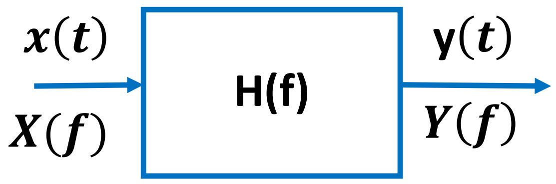
Basic Frequency-domain Definitions

- ***Definition***: The **transfer function** of a linear time invariant system is defined as the Fourier transform of the impulse response h(t)

$$\boldsymbol{H(f) = \mathfrak{I}\{h(t)\}}$$

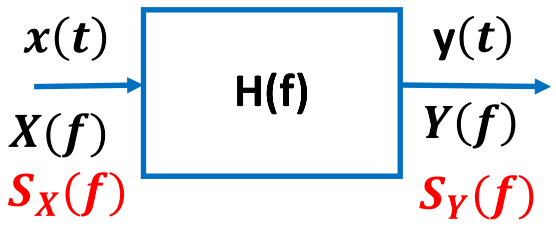
- Since $\boldsymbol{y(t) = x(t) * h(t)}$, then $\boldsymbol{Y(f) = H(f)X(f)}$.
- The system transfer function is thus the ratio of the Fourier transform of the output to that of the input $\boldsymbol{H(f) = \frac{Y(f)}{X(f)}}$
- The transfer function $H(f)$ is a complex function of frequency, which can be expressed as
- $H(f) = |H(f)|e^{j\,\theta(f)}$
- where, $|H(f)|$: Amplitude spectrum

  $\theta(f)$: Phase spectrum.



System input–output energy spectral density

- Let $x(t)$ be applied to a LTI system, then the Fourier transform of the output is related to the Fourier transform of the input through the relation
- $Y(f) = H(f)X(f)$.
- Taking the absolute value and squaring both sides, we get
- $|Y(f)|^2 = |H(f)|^2 |X(f)|^2$



$$\boldsymbol{S_Y(f) = |H(f)|^2 S_X(f)}$$

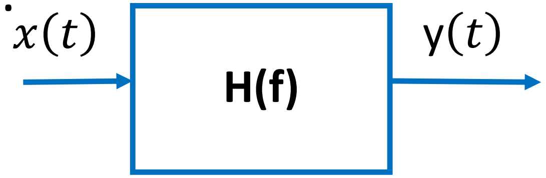
- $S_X(f)$, $S_Y(f)$: Input and output Energy Spectral Density

$$output\ energy\ spectral\ densit = |H(f|^2\ (input\ energy\ spectral\ density)$$

- Total input and output energies
- $E_x = \int_{-\infty}^{+\infty} S_x(f)df = \int_{-\infty}^{+\infty} |X(f)|^2 df$; **Recall Rayleigh Energy Theorem**
- $E_y = \int_{-\infty}^{+\infty} S_Y(f)df = \int_{-\infty}^{+\infty} |\mathrm{H(f)}|^2 S_X(f)df$

Example: Response of a LPF filter to a sinusoidal input

- **Example**: The signal $x(t) = \cos(2\pi f_0 t), -\infty < t < \infty$, is applied to a filter described by the transfer function $H(f) = \frac{1}{1+jf/B}$, B is the 3-dB bandwidth. Find the filter output $y(t)$.
- **Solution**: Here, we will find the output using the frequency domain approach.



- $Y(f) = H(f)X(f), \;\; H(f) = \frac{1}{\sqrt{1+(\frac{f}{B})^2}} e^{-j\theta}; \;\; \theta = \tan^{-1}\frac{f}{B}; \;\; \theta_0 = \tan^{-1}\frac{f_0}{B}$
- $Y(f) = H(f)[\frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0), \; \Rightarrow Y(f) = \frac{1}{2}H(f_0)\delta(f - f_0) + \frac{1}{2}H(-f_0)\delta(f + f_0)$
- $Y(f) = \frac{1}{2}\frac{1}{\sqrt{1+(\frac{f_0}{B})^2}} e^{-j\theta_0}\delta(f - f_0) + \frac{1}{2}\frac{1}{\sqrt{1+(\frac{f_0}{B})^2}} e^{j\theta_0}\delta(f + f_0)$

$\boldsymbol{g(t)\delta(t-t_0) = g(t_0)\delta(t-t_0);}$

- Taking the inverse Fourier transform, we get
- $y(t) = \frac{1}{\sqrt{1+(\frac{f_0}{B})^2}}\frac{1}{2}[e^{j(2\pi f_0 t - \theta_0)} + e^{-j(2\pi f_0 t - \theta_0)}]$, $\quad \boldsymbol{y(t) = \frac{1}{\sqrt{1+(\frac{f_0}{B})^2}} cos(2\pi f_0 t - tan^{-1}\frac{f_0}{B})}$
- Note that in the last step we have made use of the Fourier transform pair $e^{j2\pi f_0 t} \leftrightarrow \delta(f - f_0)$
- **Remark**: Note that the amplitude of the output as well as its phase depend on the frequency of the input, f_0, and the bandwidth of the filter, B.

Response of a LPF to a sum of two sinusoidal signals

- **Example**: The signal $x(t) = \ cos\ w_0 t - \frac{1}{\pi} cos\ 3w_0 t$ is applied to a filter described by the transfer function $H(f) = \frac{1}{1+jf/B}$. Use the result of the previous example to find the filter output $y(t)$.
- **Solution**: From the previous example, we have
- $\cos(2\pi f_0 t) \ \rightarrow \frac{1}{\sqrt{1+(\frac{f_0}{B})^2}} \cos(2\pi f_0 t - \tan^{-1}\frac{f_0}{B})$
- Therefore, using linearity property
- $cos\ w_0 t - \frac{1}{\pi} cos\ 3w_0 t \rightarrow$
- $\frac{1}{\sqrt{1+\left(\frac{f_0}{B}\right)^2}} \cos\left(2\pi f_0 t - \tan^{-1}\frac{f_0}{B}\right) \quad - \frac{1}{\pi}\frac{1}{\sqrt{1+\left(\frac{3f_0}{B}\right)^2}} \cos\left(2\pi 3 f_0 t - \tan^{-1}\frac{3f_0}{B}\right)$

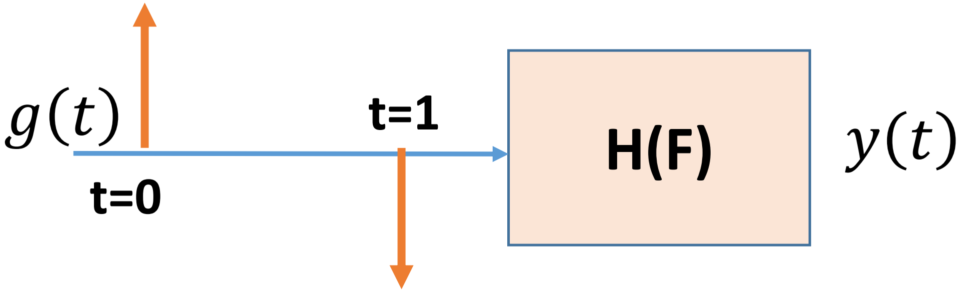
Example: Response of a LPF to a periodic square pulse

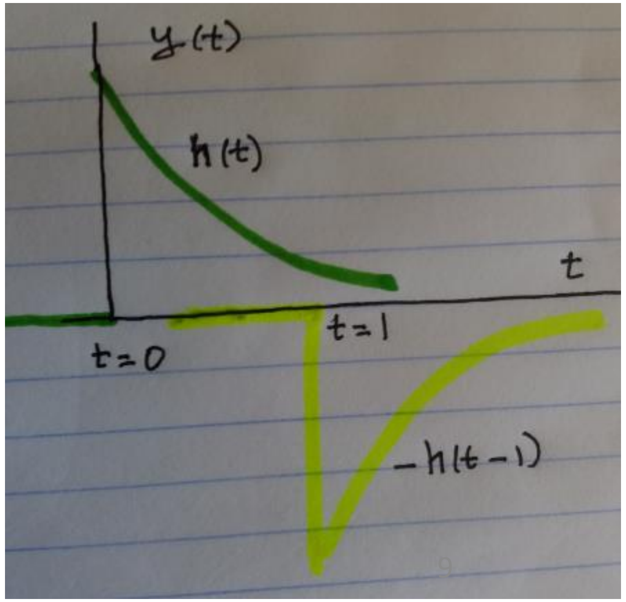
- **Example**: Consider the periodic rectangular signal $g(t)$ defined over one period T_0 as
$$g(t) = \begin{cases} +A, & -T_0/4 \le t \le T_0/4 \\ 0, & otherwise \end{cases}.$$
- If $g(t)$ is applied to a filter described by the transfer function $H(f) = \frac{1}{1+jf/B}$. use the result of the previous example to find the filter output $y(t)$.
- **Solution**: The Fourier series of g(t) is:
- $g(t) = \frac{A}{2} + \frac{2A}{\pi}\{\cos(2\pi f_0 t) - \frac{1}{3}\cos(2\pi 3 f_0 t) + \frac{1}{5}\cos(2\pi 5 f_0 t) - \frac{1}{7}\cos(2\pi 7 f_0 t)$
- Using the result of the previous example:
- $y(t) = \frac{A}{2} + \frac{2A}{\pi}\frac{1}{\sqrt{1+\left(\frac{f_0}{B}\right)^2}}\cos\left(2\pi f_0 t - \tan^{-1}\frac{f_0}{B}\right)$

$$-\frac{2A}{\pi}\frac{1}{3}\frac{1}{\sqrt{1+\left(\frac{3f_0}{B}\right)^2}}\cos\left(2\pi 3 f_0 t - \tan^{-1}\frac{3f_0}{B}\right) + \ldots$$

Transmission of Signals through Linear Systems: A Convolution Example

- **Example:** The signal $g(t) = \delta(t) - \delta(t-1)$ is applied to a channel described by the transfer function $H(f) = \frac{1}{1+jf/B}$. Use the convolution integral to find the channel output.
- **Solution:** The impulse response of the channel is obtained by taking the inverse Fourier transform of $H(f)$, which is $h(t) = 2\pi B e^{-2\pi B t} u(t)$
- Using the linearity and time invariance property, the output can be obtained as
- $y(t) = h(t) * [\delta(t) - \delta(t-1)];\quad y(t) = h(t) - h(t-1)$
- $y(t) = 2\pi B[e^{-2\pi B t}u(t) - e^{-2\pi B(t-1)}u(t-1)]$



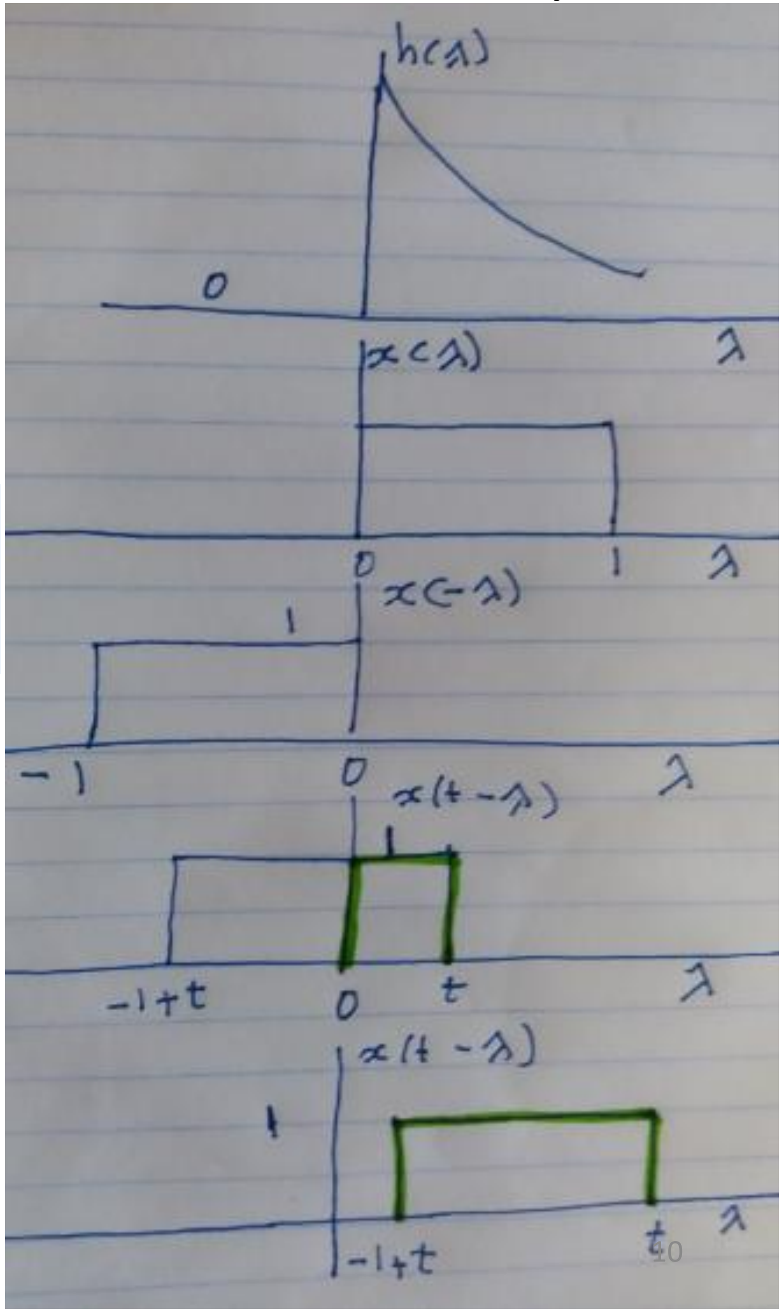


Transmission of Signals through Linear Systems: A Convolution Example

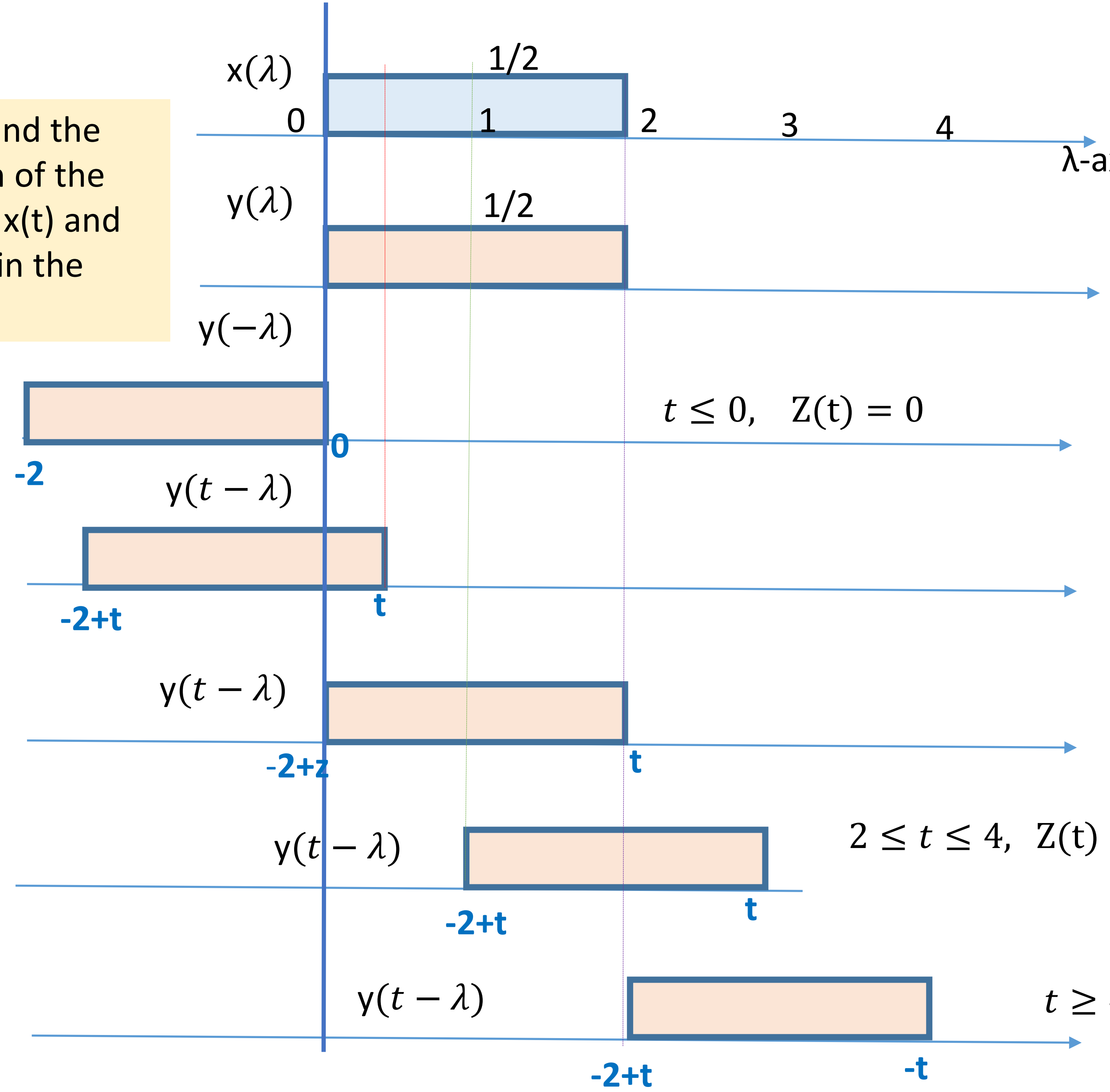
- **Example: channel response due to a rectangular pulse**
- The signal $g(t) = u(t) - u(t-1)$ is applied to a channel described by the transfer function $H(f) = \frac{1}{1+jf/B}$. Find the channel output $y(t)$.
- **Solution:** The impulse response of the channel is:
- $h(t) = 2\pi B e^{-2\pi B t} u(t)$
- The output is the convolution

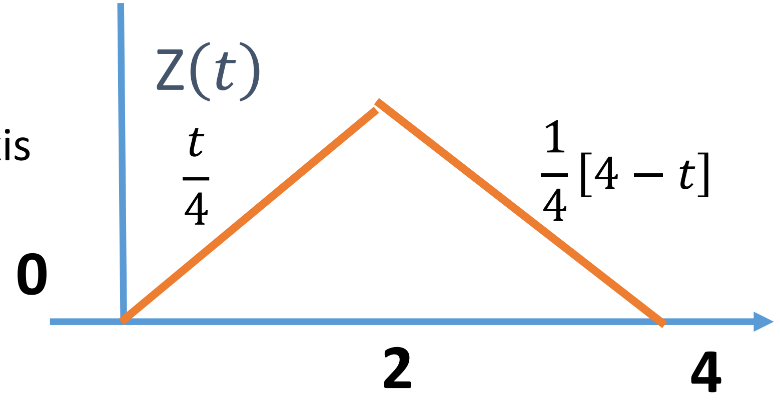
$$\mathbf{y}(t) = \int_{-\infty}^{\infty} \mathbf{h}(\lambda)\mathbf{x}(t-\lambda)\ \mathbf{d\tau}$$

- $y(t) = h(t) * [u(t) - u(t-1)]$. The answer is
- $\mathrm{y(t)} = \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)\ \mathrm{d}\lambda$
- $y(t) = 0\ for\ t < 0$
- $\mathrm{y(t)} = \int_0^t 2\pi B e^{-2\pi B\lambda}\ \mathrm{d}\lambda = 1 - e^{-2\pi B t}$, for $0 \le t < 1$
- $\mathrm{y(t)} = \int_{-1+t}^{t} 2\pi B e^{-2\pi B\lambda}\ \mathrm{d}\lambda = (e^{2\pi B} - 1)e^{-2\pi B t}$, for $t \ge 1$



Example: Find the convolution of the two signals x(t) and y(t) shown in the figure.





$$\mathbf{Z(t)} = \int_{-\infty}^{\infty} \boldsymbol{x}\,(\boldsymbol{\lambda})\,\mathbf{y}(\mathbf{t}-\boldsymbol{\lambda})\,\boldsymbol{d\lambda}$$

$t \le 0, \quad \mathrm{Z(t)} = 0$

$$0 \le t \le 2, \qquad \mathrm{Z(t)} = \int_0^t \frac{1}{2} \times \frac{1}{2}\; d\lambda = \frac{t}{4}$$

$$t = 2, \quad \mathrm{Z(t)} = \int_0^2 \frac{1}{2} \times \frac{1}{2}\; d\lambda = \frac{1}{2}$$

$$2 \le t \le 4, \quad \mathrm{Z(t)} = \int_{-2+t}^{2} \frac{1}{2} \times \frac{1}{2}\; d\lambda = \frac{\lambda}{4}\Big|_{-2+t}^{2} = \frac{1}{4}[4-t]$$

$t \ge 4, \quad \mathrm{Z(t)} = 0$