# Bandwidth of Signals and Systems: Lecture Outline

- Bandwidth Definitions
  - Absolute Bandwidth
  - 3-dB (half power points) Bandwidth
  - The 95 %  (energy or power) Bandwidth
  - Equivalent Rectangular Bandwidth
  - Null – to – Null Bandwidth
  - Bounded Spectrum Bandwidth
  - RMS Bandwidth
- The Definition of Decibel
- Bandwidth of Periodic Signals
- Time-Bandwidth Product

# Bandwidth of Signals and Systems

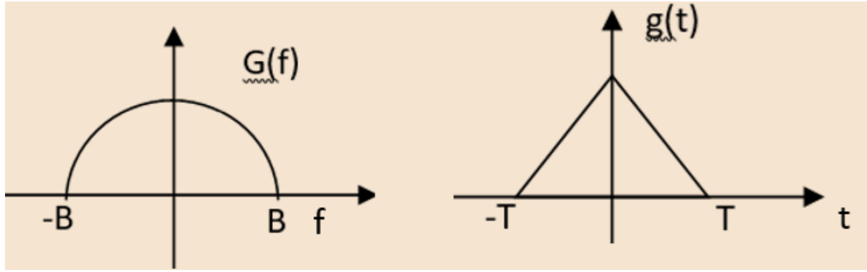
- **Definition:** The amount of **positive** frequency band that a signal $g(t)$ occupies is called the bandwidth of the signal. It provides a **measure of the extent of significant frequency content** of the signal.

- **Definition**: A signal g(t) is said to be (absolutely) band-limited to B Hz if

  G(f) = 0 for |f| >B

- **Definition**: A signal g(t) is said to be (absolutely) time-limited if

  g(t) = 0 for |t| >T




- **Theorem**: An absolutely band-limited waveform **cannot** be absolutely time-limited and vice versa, i.e., a signal g(t) **cannot** be both time-limited and bandlimited.

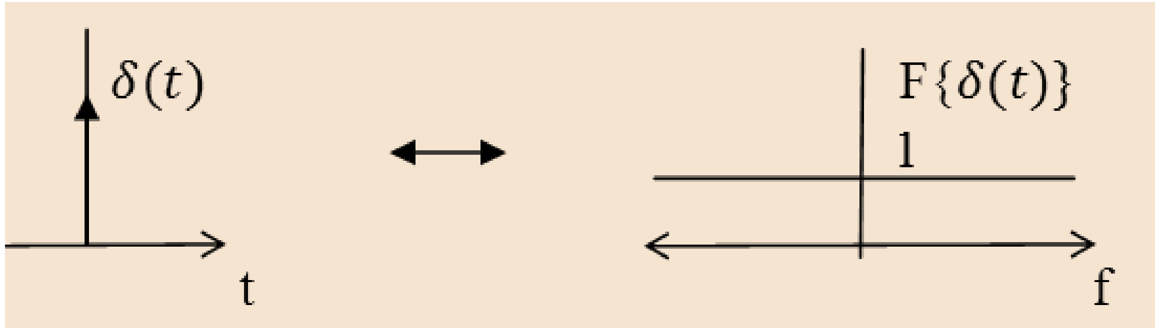
- In general, there is an inverse relationship between the signal bandwidth and the time duration. The bandwidth and the time duration are related through a relation, called the *time bandwidth product*, of the form (will investigate this more in the next lecture)

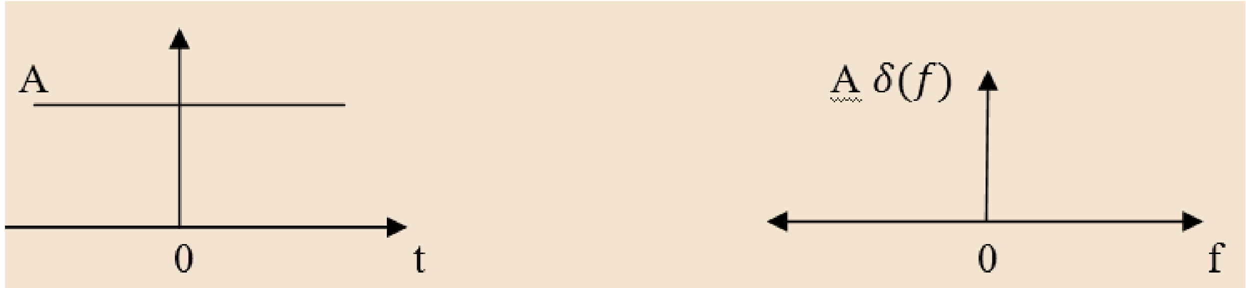
$$(\boldsymbol{Bandwidth})(\boldsymbol{Time\ Duration}) \geq \boldsymbol{Constant}$$

- The value of the constant depends on the way we define the bandwidth and the time duration. Two possible values of the constant, that we will encounter in this chapter, are ½ (for the equivalent rectangular bandwidth) and $\frac{1}{4\pi}$ (for the root mean square bandwidth).

# Bandwidth of Signals and Systems

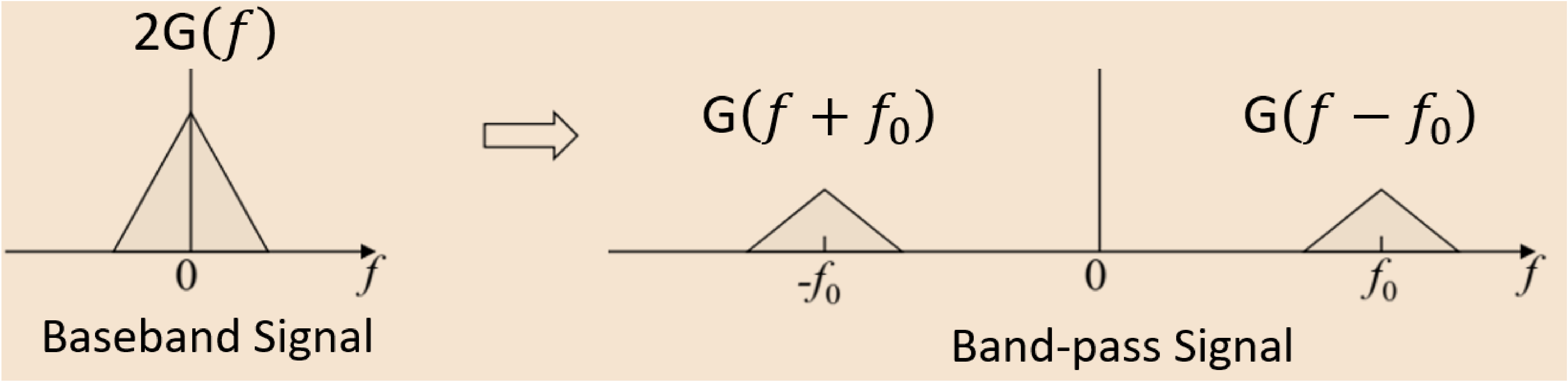
- **Theorem**: An absolutely band-limited waveform **cannot** be absolutely time-limited and vice versa, i.e., a signal g(t) **cannot** be both time-limited and bandlimited.
- We have earlier seen examples that support this theorem. For example, the delta function, which has an almost zero time duration, has a Fourier transform which extends uniformly over all frequencies (infinite bandwidth).
- Also, a constant value in the time domain (a dc) has a Fourier transform, which is an impulse in the frequency domain at the origin. These are shown below.



# Bandwidth of Signals and Systems

- **Definition of a baseband signal**: A baseband signal is one for which most of the energy is contained within a band centered around the zero frequency and negligible elsewhere. Another term synonymous with baseband is ***low-pass***. In the communication systems, the message to be transmitted is a baseband signal.
- **Definition of a band-pass signal**: A band-pass signal is one for which the energy is concentrated around some high frequency carrier $f_0$ and negligible elsewhere. This type of signal will arise in this course when the baseband message signal $m(t)$ modulates a high frequency carrier $c(t)$ to produce the modulated signal $s(t)$.



# The Definition of Decibel

- Consider a system with input voltage $v_i$ and output voltage $v_o$
- The power gain of the system is defined as:

$$G = \left(\frac{P_o}{P_i}\right)$$

- In a logarithmic scale, the gain is defined as

$$G = 10log_{10}\left(\frac{P_o}{P_i}\right) \text{ dB.}$$

- If $P_o > P_i$, G > 0. Hence, the output signal possesses more power than the input. However, if $P_o < P_i$, the system introduces attenuation or loss. In this case G < 0.
- If the input and output powers are taken relative to the same reference resistance R, then

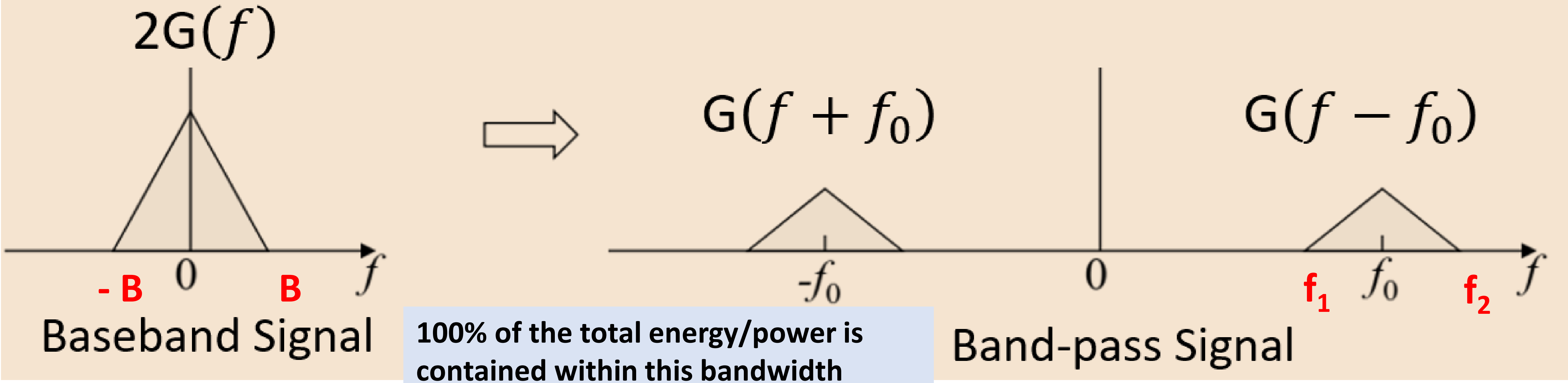
$$\mathrm{G} = 10\log\left(\frac{\frac{{V_o}^2}{R}}{\frac{{V_i}^2}{R}}\right) = 20log_{10}\left(\frac{V_o}{V_i}\right) \text{ dB}$$

- For a transfer function H(f), G becomes

$$\mathrm{G} = 20log_{10}(H(f)) \text{ dB}$$

# Definitions of Bandwidth: Absolute Bandwidth

- Here, the Fourier transform of a signal is non-zero only within a certain frequency band.
- Low-pass Signals: If G(f) = 0 for |f| > B, then g(t) is absolutely band-limited to B Hz and **B.W = B**
- Bandpass Signals: When G(f) ≠ 0 for $f_1 < |f| < f_2$, then the absolute bandwidth is **B.W= $f_2 - f_1$.**

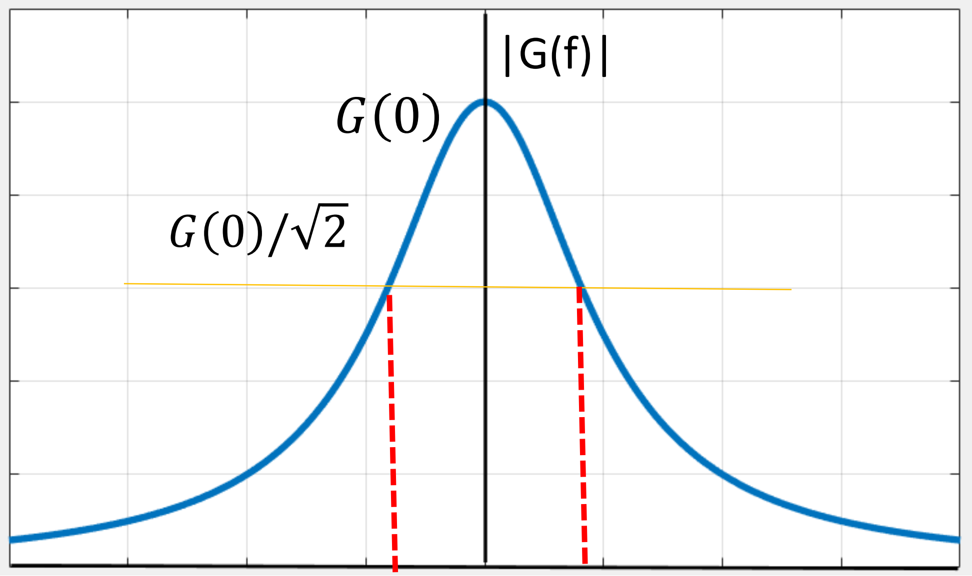



**100% of the total energy/power is contained within this bandwidth**

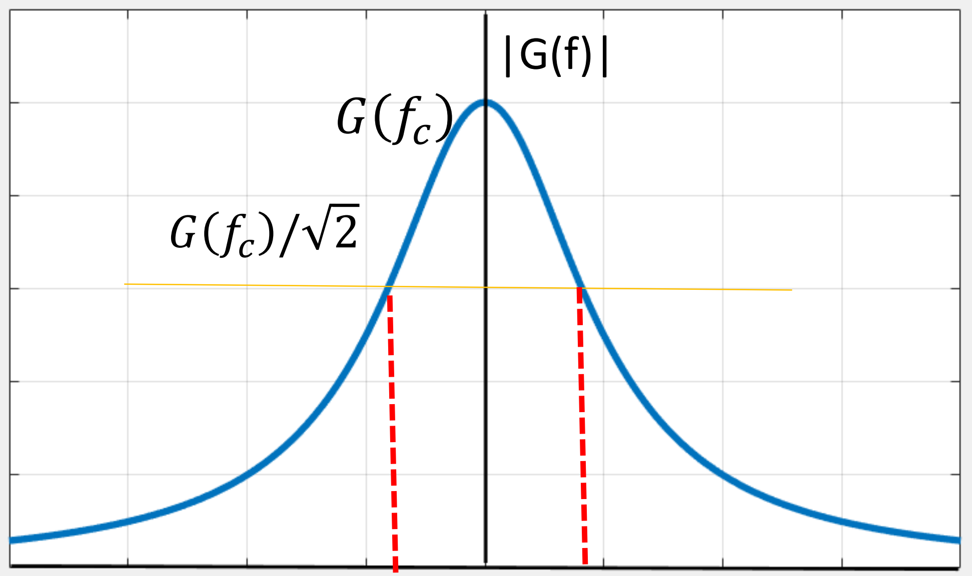
# Definitions of Bandwidth: 3-dB (half power points) Bandwidth

- The range of frequencies from 0 to some frequency B at which |G(f)| drops to $\frac{1}{\sqrt{2}}$ of its maximum value (for a low pass signal).
- As for a band pass signal, the B.W = $f_2 - f_1$.

$$G = 20\,log_{10}\left(\frac{G(B)}{G(0)}\right) = 20\,log_{10}\left(\frac{G(0)/\sqrt{2}}{G(0)}\right) = -20log_{10}(\sqrt{2}) = -3\,dB$$



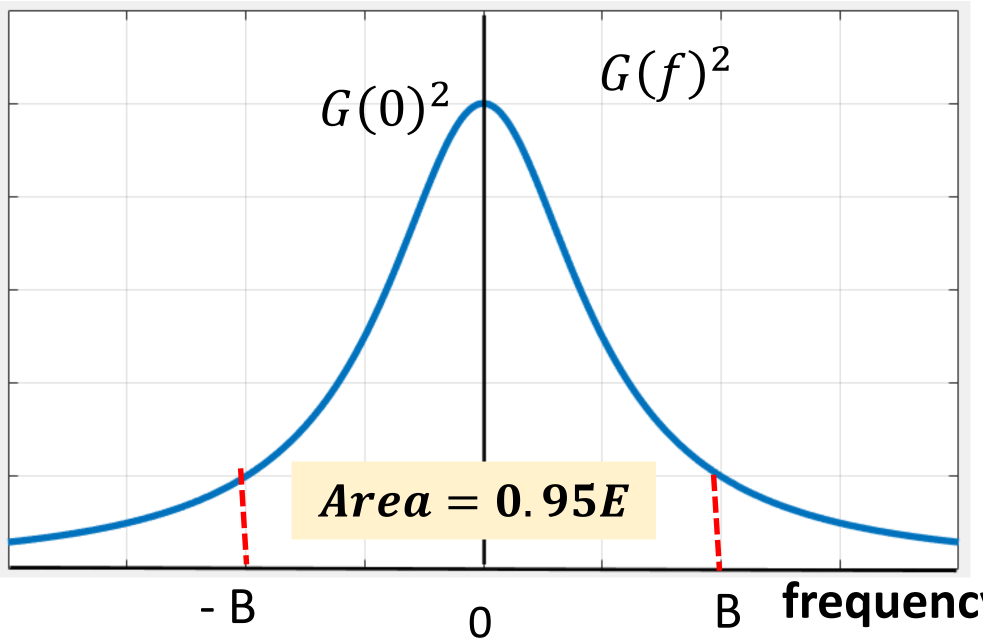

- B 0 B **frequency**




$f_1$ $f_c$ $f_2$ **frequency**

# Definitions of Bandwidth: The 95 % (energy or power) Bandwidth

- Here, the B.W is defined as the band of frequencies where the area under the energy spectral density (or power spectral density) is at least 95% (or 99%) of the total area.
- Total Signal Energy $E = \int_{-\infty}^{\infty} |G(f)|^2\, df = 2\int_{0}^{\infty} |G(f)|^2 df = \int_{-\infty}^{\infty} |g(t)|^2\, dt$
- The 95% energy bandwidth B should satisfy the relationship
- $\int_{-B}^{B} |G(f)|^2 df = 0.95 \int_{-\infty}^{\infty} |G(f)|^2\, df = 0.95\, E$
- $\boldsymbol{\int_{-B}^{B} |G(f)|^2 df = 0.95\, E}$



# Definitions of Bandwidth: Equivalent Rectangular Bandwidth

- It is the width of a fictitious rectangular spectrum such that the power in that rectangular band is equal to the energy associated with the actual spectrum. Let $B_{eq}$ be the equivalent rectangular bandwidth. To find $B_{eq}$ we set

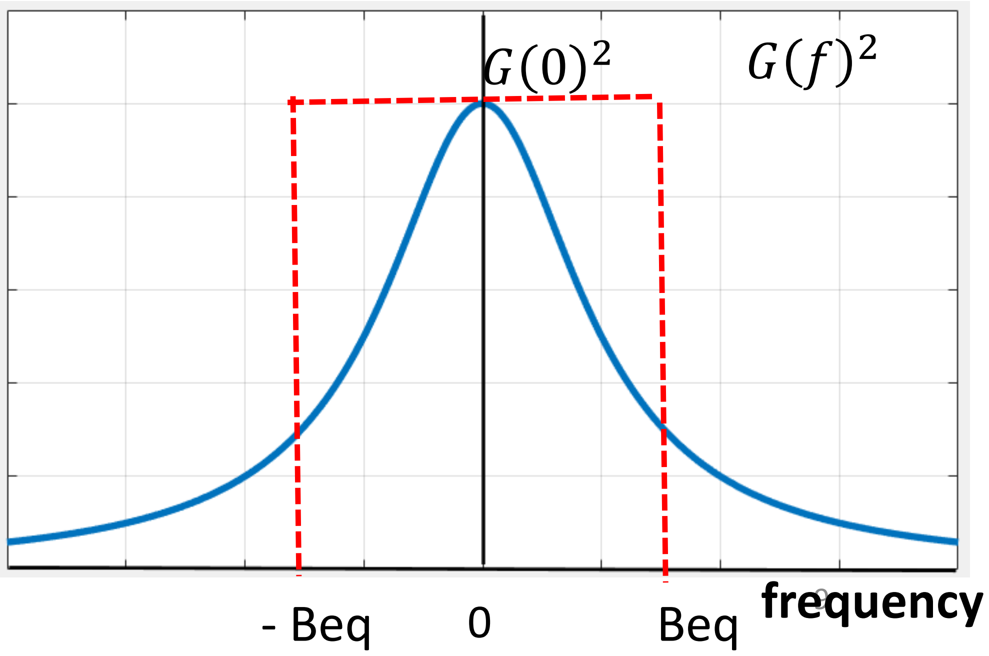
Area under fictitious rectangle = Total Signal Energy E

**Area of red rectangle = Area under the blue curve**

$$|G(0)|^2 * 2B_{eq} = \int_{-\infty}^{\infty} |G(f)|^2 df = E$$

$$|G(0)|^2 * 2B_{eq} = 2\int_{0}^{\infty} |G(f)|^2 df$$

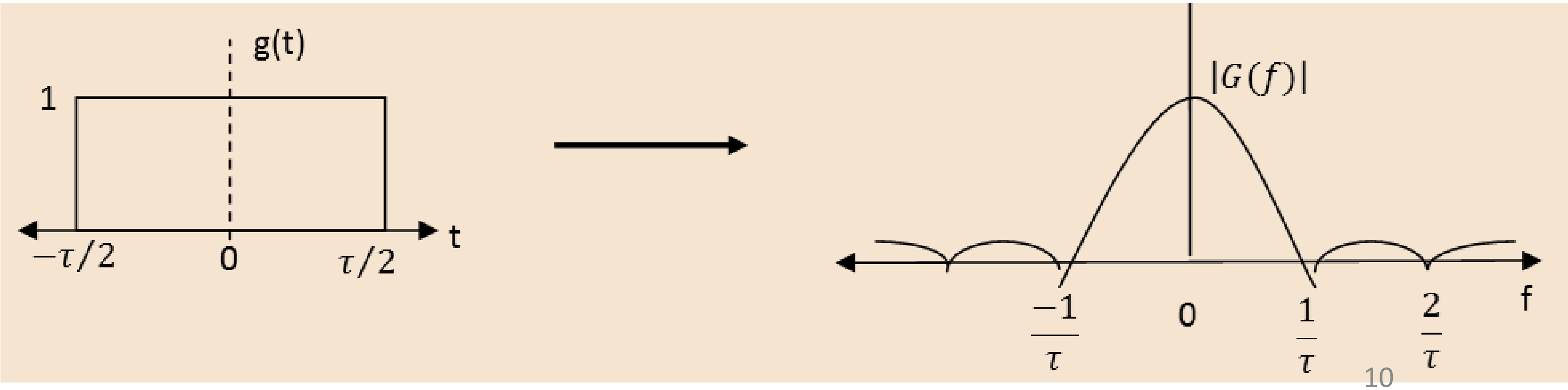
$$\boldsymbol{B_{eq} = \frac{1}{|G(0)|^2} \int_0^{\infty} |G(f)|^2 df}$$



# Definitions of Bandwidth: Null – to – Null Bandwidth

- For baseband signals, the null bandwidth is taken to be the band from zero to the first null in the envelope of the magnitude spectrum.
- For example, consider the rectangular pulse g(t), for which the Fourier transform is G(f). Note that
- $rect\left(\frac{t}{\tau}\right) \rightarrow \tau\ \mathrm{sinc} f\tau = \tau \frac{\sin\pi f\tau}{\pi f\tau}.$
- The zero crossings occur when $\sin(\pi f\tau) = 0$
- $\pi f\tau = n\pi \rightarrow f = \frac{n}{\tau}$ ; n = 1,2, … The smallest value of $n = 1$, gives
- ***Null Bandwidth*** $= \frac{1}{\tau}.$
- For a band pass signal,

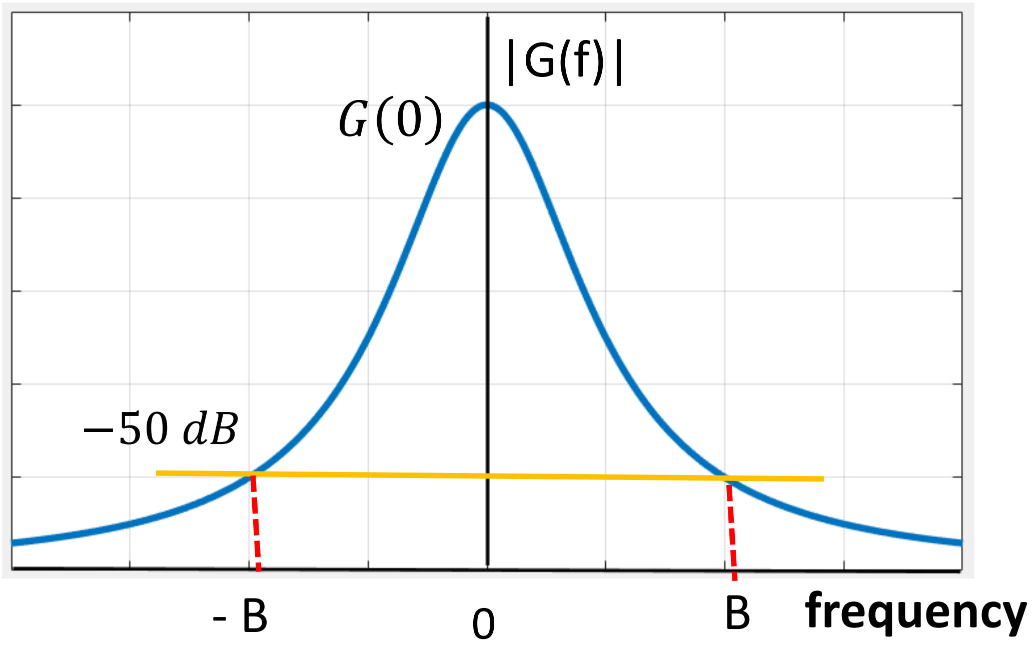
  B.W= $f_2 - f_1$



# Definitions of Bandwidth: Bounded Spectrum Bandwidth

- The range of frequencies from 0 to some frequency B at which |G(f)| drops to , say, $-50dB$ relative to its maximum value (for a low pass signal).

$$-50\ dB = 20\ log_{10}\left(\frac{G(B)}{G(0)}\right)$$



# Definitions of Bandwidth: RMS bandwidth

- The RMS bandwidth of a signal g(t) is defined as
- $$B_{rms} = \sqrt{\left(\frac{\int_{-\infty}^{\infty} f^2 |G(f)|^2 \mathrm{df}}{\int_{-\infty}^{\infty} |G(f)|^2 \mathrm{df}}\right)} = \sqrt{\left(\frac{\int_{-\infty}^{\infty} f^2 |G(f)|^2 \mathrm{df}}{E_g}\right)}$$
- In an analogous way, the corresponding RMS duration of g(t) is
- $$T_{rms} = \sqrt{\left(\frac{\int_{-\infty}^{\infty} t^2 |g(t)|^2 \mathrm{dt}}{\int_{-\infty}^{\infty} |g(t)|^2 \mathrm{dt}}\right)} = \sqrt{\left(\frac{\int_{-\infty}^{\infty} t^2 |g(t)|^2 \mathrm{dt}}{E_g}\right)}$$
- (here g(t) is assumed to be centered around the origin).
- **Remark**: The time bandwidth product is $(T_{rms})(B_{rms}) \geq \frac{1}{4\pi}$ (the proof is beyond the scope of this presentation).

# Example: 95% Energy Bandwidth of the Exponential Pulse

- Find the 95% energy bandwidth for the exponential pulse $g(t) = Ae^{-\alpha t}\ u(t)$.
- **Solution**: The Fourier transform of g(t) is

$$G(f) = \frac{A}{\alpha + j2\pi f}$$

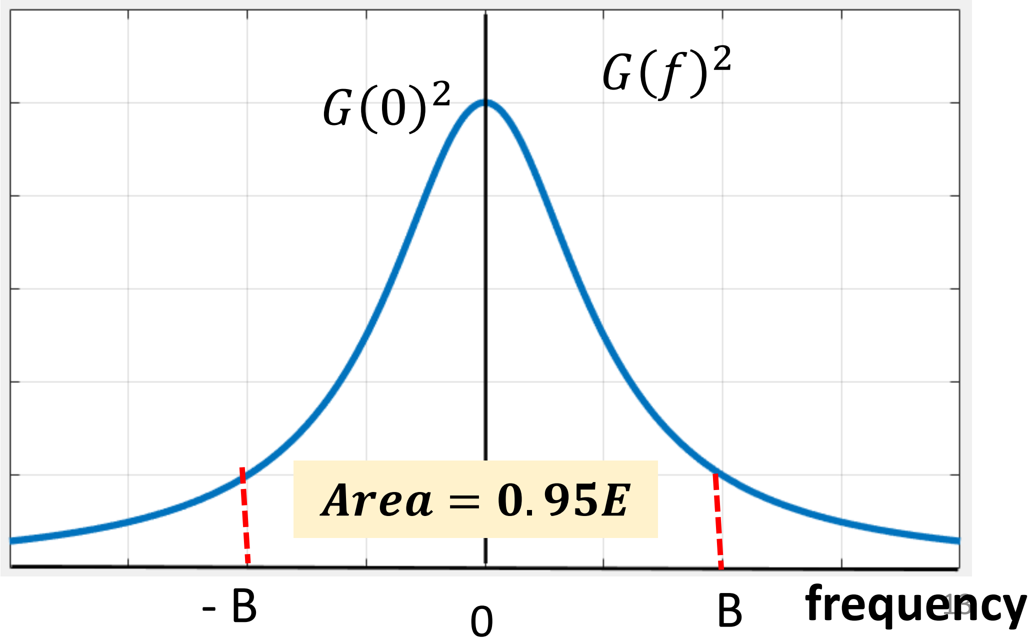
- The total energy in g(t) (calculated in the time domain) is
- $E_g = \int_0^\infty |g(t)|^2\, dt = \int_0^\infty A^2 e^{-2\alpha t}\, dt = \frac{A^2}{2\alpha}$
- Let B be the 95% energy bandwidth, then the energy contained within B is
- $E_B = \int_{-B}^{B} |G(f)|^2\, df = \int_{-B}^{B} \frac{A^2}{(\alpha^2 + (2\pi f)^2)}\, df$
- $E_B = \frac{2A^2}{2\pi\alpha} tan^{-1} \frac{2\pi B}{\alpha}$

- B should be chosen such that it satisfies the condition
- $E_B = 0.95 E_g$

$$\mathbf{\frac{2A^2}{2\pi\alpha} tan^{-1} \frac{2\pi B}{\alpha} = 0.95 \left(\frac{A^2}{2\alpha}\right)}$$

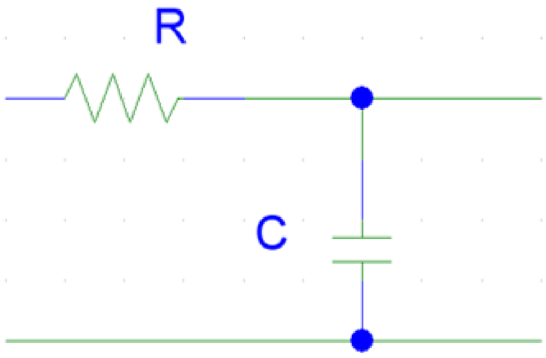
- The 95% energy bandwidth is, therefore

$$\mathbf{B_{95\%} = 2\alpha}$$



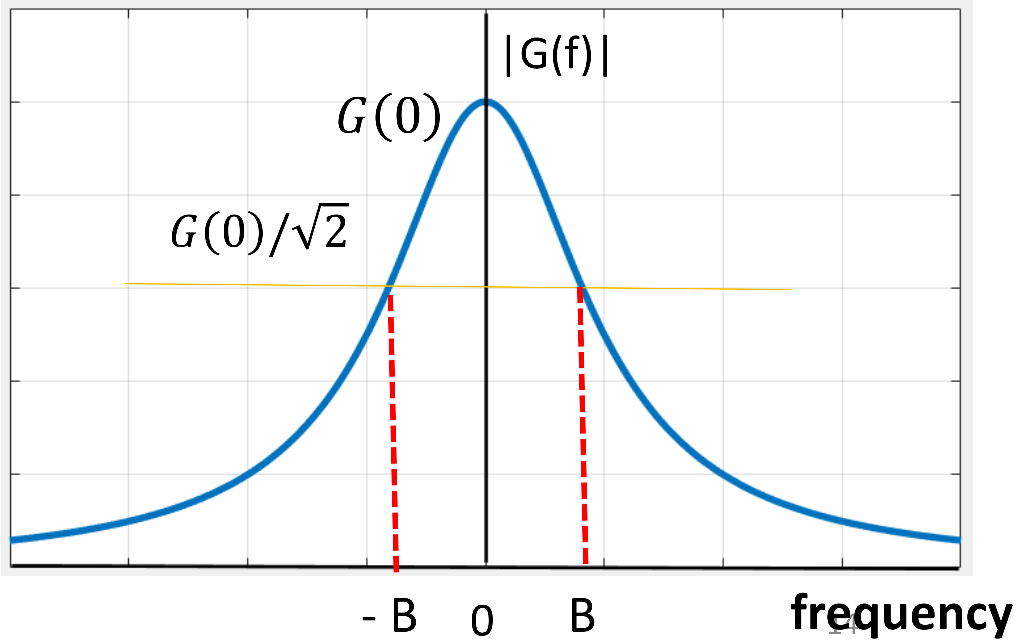
# Example: 3-dB Bandwidth of the First Order RC Circuit

- **Example**: Find the 3-dB bandwidth of a first order RC low pass filter
- **Solution**: The transfer function of the circuit is
- $H(f) = \frac{\frac{1}{j2\pi fC}}{R+\frac{1}{j2\pi fC}} = \frac{1}{1+j2\pi fRC}$




- The magnitude of $H(f)$ is
- $|H(f)| = \frac{1}{\sqrt{1+(2\pi fRC)^2}}$
- The 3-dB bandwidth is some frequency $f = B$ at which $|H(f)|$ drops to $1/\sqrt{2}$ of its maximum value. Note that the maximum value of $|H(f)|$ is 1 and occurs at $f = 0$. Therefore, B should satisfy
- $|H(B)| = \frac{1}{\sqrt{1+(2\pi BRC)^2}} = \frac{1}{\sqrt{2}}$

- From this relationship, we notice that at the 3-dB point, $2\pi BRC = 1$
- Therefore, $\boldsymbol{B = \frac{1}{2\pi RC}}$



# Example: Bandwidth of a Periodic Rectangular Signal

- **Example**: Find the 93% power bandwidth for the periodic square function define over one period as $g(t) = \begin{cases} 2A, & \frac{-T_0}{4} \le t \le \frac{T_0}{4} \\ -A, & o.w \end{cases}$
- **Solution**: The average power, computed using the time average, is
- $P_{av} = \frac{1}{T_0}\int_0^{T_0} |g(t)|^2 \ dt = \frac{1}{T_0}\left[4A^2\frac{T_0}{2} + \ A^2\frac{T_0}{2}\right] = \frac{5A^2T_0}{2T_0} = \frac{5A^2}{2} \Rightarrow \boldsymbol{P_{av} = 2.5A^2}$
- Also, by using the Parseval's theorem, the average power can be computed as:
- $P_{av} = \ |C_0|^2 + \ 2\sum_{n=1}^{\infty} |C_n|^2$
- We recall that the Fourier coefficients for this signal were found in the lecture on Fourier series. Using these values, we get
- $P_{av} \ = \left(\frac{A}{2}\right)^2 + 2\sum_{n=1}^{\infty}\frac{(3A)^2}{(n\pi)^2} \Rightarrow \boldsymbol{P_{av} = \frac{A^2}{4} + \ 2A^2\sum_{n=1}^{\infty}\frac{(3)^2}{(n\pi)^2}}$

# Example: Bandwidth of a Periodic Rectangular Signal

- $P_{av} = \left(\frac{A}{2}\right)^2 + 2\sum_{n=1}^{\infty}\frac{(3A)^2}{(n\pi)^2} \Rightarrow \boldsymbol{P_{av} = \frac{A^2}{4} + 2A^2\sum_{n=1}^{\infty}\frac{(3)^2}{(n\pi)^2} = 2.5A^2}$
- Let us take n = 1, then the power in the DC and the fundamental frequency is
- $P_1 = A^2\left\{0.25 + 2\left(\frac{9}{\pi^2}\right)\right\} = 2.073A^2 \Rightarrow \frac{P_1}{P_{av}} = \frac{2.073A^2}{2.5A^2} = 82.95\%$
- The fraction of power in these two terms relative to the total average power is only 82.95%. The 93% power limit is not yet reached. So, let us add one more term.
- When n = 3, the power in the DC, the fundamental term, and the third harmonic is
- $P_3 = A^2\left\{0.25 + 2\left(\frac{3^2}{\pi^2} + \frac{3^2}{3^2\pi^2}\right)\right\} = 2.276A^2 \Rightarrow \frac{P_3}{P_{av}} = \frac{2.276A^2}{2.5A^2} = 91.05\%.$
- The fraction of power in these three terms relative to the total average power is now 91.05%. Still, the 93% power limit is not reached yet. So, let us add one more term.
- For n = 5, the power in the DC, the fundamental term, the third harmonic, and the fifth harmonic is
- $P_5 = A^2\left\{0.25 + 2\left(\left(\frac{3}{\pi}\right)^2 + \left(\frac{3}{3\pi}\right)^2 + \left(\frac{3}{5\pi}\right)^2\right)\right\} = 2.349A^2 \Rightarrow \frac{P_5}{P_{av}} = \frac{2.349A^2}{2.5A^2} = 93.97\%.$
- With n=5, the 93% power limit has been reached. Therefore, the 93% power B.W is $\boldsymbol{B_{93\%} = 5f_0}$.

# Time-Bandwidth Product

- One more time, to illustrate the time – bandwidth product ($(Bandwidth)(Time\ Duration) \geq Constant$ ), consider the equivalent rectangular bandwidth defined earlier as
- $B_{eq} = \dfrac{\int_{-\infty}^{\infty} |G(f)|^2 \mathrm{df}}{2|G(0)|^2}$
- Analogous to this definition, we define an equivalent rectangular time duration as
- $T_{eq} = \dfrac{\left(\int_{-\infty}^{\infty} |g(t)| dt\right)^2}{\int_{-\infty}^{\infty} |g(t)|^2 \mathrm{dt}}$
- The time bandwidth product is
- $B_{eq}T_{eq} = \left(\dfrac{\int_{-\infty}^{\infty} |G(f)|^2 \mathrm{df}}{2|G(0)|^2}\right)\left(\dfrac{\left(\int_{-\infty}^{\infty} |g(t)| dt\right)^2}{\int_{-\infty}^{\infty} |g(t)|^2 \mathrm{dt}}\right)$
- Note that $\int_{-\infty}^{\infty} |g(t)|^2$ dt = $\int_{-\infty}^{\infty} |G(f)|^2$ df;
- Rayleigh energy theorem.
- Note also that $G(0) = \int_{-\infty}^{\infty} g(t) dt.$
- Using these two relations, we get
- $\boldsymbol{B_{eq}T_{eq} = \dfrac{1}{2} \dfrac{\left(\int_{-\infty}^{\infty} |g(t)| dt\right)^2}{|\int_{-\infty}^{\infty} g(t) \mathrm{dt}|^2}}$
- Case 1: When g(t) is positive for all time t, then |g(t)| = g(t) and $B_{eq}T_{eq}$ becomes
- $B_{eq}T_{eq} = \dfrac{1}{2}$
- Case 2 : For a general g(t) that can take on positive as well as negative values, $B_{eq}T_{eq}$ satisfies the inequality
- $B_{eq}T_{eq} \geq \dfrac{1}{2}$
- Note : For $B_{rms}$ and $T_{rms}$ , the time – bandwidth product satisfies the inequality $B_{rms}T_{rms} \geq \dfrac{1}{4\pi}$

# Example: Bandwidth of a Trapezoidal Signal

- **Example**: Find the equivalent rectangular bandwidth, $B_{eq}$, for the trapezoidal pulse shown.
- **Solution:**
- $T_{eq} = \frac{\left(\int_{-\infty}^{\infty} |g(t)| dt\right)^2}{\int_{-\infty}^{\infty} |g(t)|^2 \mathrm{dt}}$
- $\int_{-\infty}^{\infty} |g(t)| dt \;= A\,(\,t_a + t_b)$
- $\int_{-\infty}^{\infty} |g(t)|^2 dt \;= \frac{2A^2}{3}\,(\,2t_a + t_b)$
- $T_{eq} = \frac{3}{2}\frac{(t_a+t_b)^2}{(2t_a+t_b)}$
- $B_{eq} = \frac{0.5}{T_{eq}} = \frac{2t_a+t_b}{3(t_a+t_b)^2}.$

- **Remark**: Note that using this method we were able to determine the signal bandwidth without the need to go through the Fourier transform.