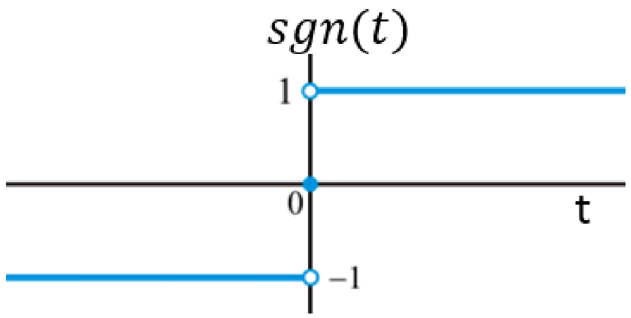
Hilbert Transform

- **The quadrature filter** is an all pass filter that shifts the phase of positive frequency by (-90°) and negative frequency by (+90°).
- **The transfer function** of such a filter is
- $H(f) = \begin{cases} -j & f > 0 \\ j & f < 0 \end{cases} = \boldsymbol{-jsgn(f)}$
- Note that $|H(f)| = 1$ for all f.
- Using the duality property of Fourier transform, the impulse response of the filter is $\boldsymbol{h(t) = \frac{1}{\pi t}}$ $(\Im\{sgn(t)\} = \frac{1}{j\pi f})$
- The Hilbert transform is the output of the quadrature filter to the signal $g(t)$
- $\boldsymbol{\hat{g}(t) = \frac{1}{\pi t} * g(t)}$ $= \int_{-\infty}^{\infty} \frac{g(\lambda)}{\pi(t-\lambda)} d\lambda$
- Note that the Hilbert transform of a signal is a function of time (not frequency as in the case of the Fourier transform). The Fourier transform of $\hat{g}(t)$
- $\boldsymbol{\hat{G}(f) = -jsgn(f)G(f)}$
- Hilbert transform can be found using either the time domain approach or the frequency domain approach depending on the given problem. That is
- **Time-domain**: Perform the convolution $\frac{1}{\pi t} * g(t)$.
- **Frequency-domain**: Find the Fourier transform $\hat{G}(f)$, then find the inverse Fourier transform
- $\hat{g}(t) = \int_{-\infty}^{\infty} \hat{G}(f)\, e^{j2\pi f t}\, df$

Some properties of the Hilbert transform

- A signal $g(t)$ and its Hilbert transform $\hat{g}(t)$ have the same energy spectral density
- $\left|\hat{G}(f)\right|^2 = |-j\, sgn(f)|G(f)||^2 = |-j\, sgn(f)|^2 \left|G(f)\right|^2$
- $= |G(f)|^2$



The consequences of this property are:

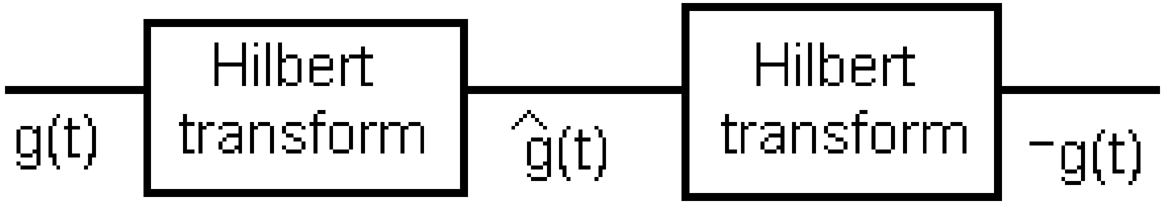
- If a signal g(t) is bandlimited to a bandwidth W Hz, then $\hat{g}(t)$ is bandlimited to the same bandwidth (note that $\left|\hat{G}(f)\right| = |G(f)|$)
- $\hat{g}(t)$ and $g(t)$ have the same total energy (or power). $E = \int_{-\infty}^{\infty} |G(f)|^2 df$
- $\hat{g}(t)$ and $g(t)$ have the same autocorrelation function (in the next lecture, we will see that the autocorrelation function and the energy spectral density form a Fourier transform pair $R_g(\tau) \leftrightarrow |G(f)|^2$)

Some properties of the Hilbert transform

- A signal g(t) and $\hat{g}(t)$ are orthogonal, i.e., $\int_{-\infty}^{\infty} \mathrm{g(t)}\, \hat{g}(t) dt = 0$
- This property can be verified using the general formula of Rayleigh energy theorem
- $\int_{-\infty}^{\infty} \mathrm{g(t)}\, \hat{g}(t) dt = \int_{-\infty}^{\infty} \mathrm{G(f)}\, \hat{G}^{*}(f) df = \int_{-\infty}^{\infty} \mathrm{G(f)}\, \{-jsgn(f)\, G(f)\}^{*} df$

$$= \int_{-\infty}^{\infty} jsgn(f)\, |G(f)|^{2} df = 0.$$

- The result above follows from the fact that $|G(f)|^2$ is an even function of f while $sgn(f)$ is an odd function of f. Their product is odd. The integration of an odd function over a symmetrical interval is zero.
- If $\hat{g}(t)$ is a Hilbert transform of $g(t)$, then the Hilbert transform of $\hat{g}(t)$ is $-g(t)$ (each Hilbert transform introduces 90 degrees phase shift).



Example on Hilbert transform

Example: Find the Hilbert transform of the impulse function $g(t) = \delta(t)$

Solution: Here, we use the convolution in the time domain

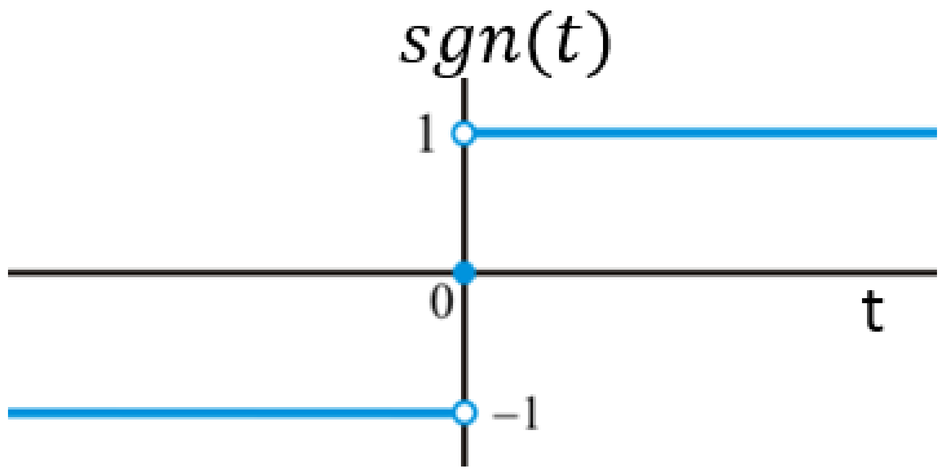
- $\hat{g}(t) \ = \frac{1}{\pi t} * \delta(t)$
- As we know, the convolution of the delta function with a continuous function is the function itself. Therefore,
- $\hat{g}(t) = \frac{1}{\pi t}.$

Example on Hilbert transform

Example: Find the Hilbert transform of $g(t) = \cos(2\pi f_0 t)$

Solution: Here, we use the frequency domain approach

- $\hat{G}(f) = -jsgn(f)G(f) = -\frac{jsgn(f)\{\delta(f-f_0)+\delta(f+f_0)\}}{2}$
- $\hat{G}(f) = -jsgn(f)G(f) = \frac{sgn(f)\{\delta(f-f_0)+\delta(f+f_0)\}}{j2} = \frac{\{\delta(f-f_0)-\delta(f+f_0)\}}{j2}$
- $\hat{g}(t) = \sin(2\pi f_0 t)$



Example on Hilbert transform

- Find the Hilbert transform of $g(t) = \frac{\sin t}{t}$
- **Solution:** Here, we will first find the Fourier transform of $g(t)$, find $\hat{G}(f)$, and then find $\hat{g}(t)$:
- $A\,rect\left(\frac{t}{\tau}\right) \leftrightarrow A\tau\, sinc\, f\tau\,;\ \tau = \frac{1}{\pi}$
- $A\ \mathrm{rect}\left(\frac{t}{1/\pi}\right) \leftrightarrow A\ \frac{1}{\pi}\frac{\sin \pi f\tau}{\pi f\tau} = \frac{1}{\pi}\frac{\sin f}{f}$
- $\pi\, rect\left(\frac{t}{1/\pi}\right) \leftrightarrow \frac{\sin f}{f}$
- So, by the duality property, we get the pair
- $\pi\, rect\left(\frac{f}{1/\pi}\right) \leftrightarrow \frac{\sin t}{t}$
- i.e. $G(f) = \pi\, rect(\frac{f}{1/\pi})$, (See figure next)

- $\hat{G}(f) = -jsgn(f)G(f) = \begin{cases} -j\pi & 0 < f < 1/2\pi \\ j\pi & -1/2\pi < f < 0 \end{cases}$
- $\hat{g}(t) \ = \int_{-\infty}^{\infty} \hat{G}(f)\, e^{j2\pi ft}\, df$
- $\quad\quad = \int_{-1/2\pi}^{0} j\pi \ \ e^{j2\pi ft}\, df \ - \int_{0}^{1/2\pi} j\pi \ \ e^{j2\pi ft}\, df$
- $= \frac{1}{2t}\left(1 - e^{-jt}\right) - \frac{1}{2t}\left(e^{jt} - 1\right)$
- $= \frac{1}{t} - \frac{1}{t}\ \frac{\left(e^{jt}+e^{-jt}\right)}{2} = \frac{1-\cos t}{t}$