BIRZEIT UNIVERSITY

FACULTY OF ENGINEERING AND TECHNOLOGY

DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

ENEE 339

COMMUNICATION SYSTEMS

LECTURE NOTES

BY

Dr. WAEL HASHLAMOUN

SEPTEMBER, 2018

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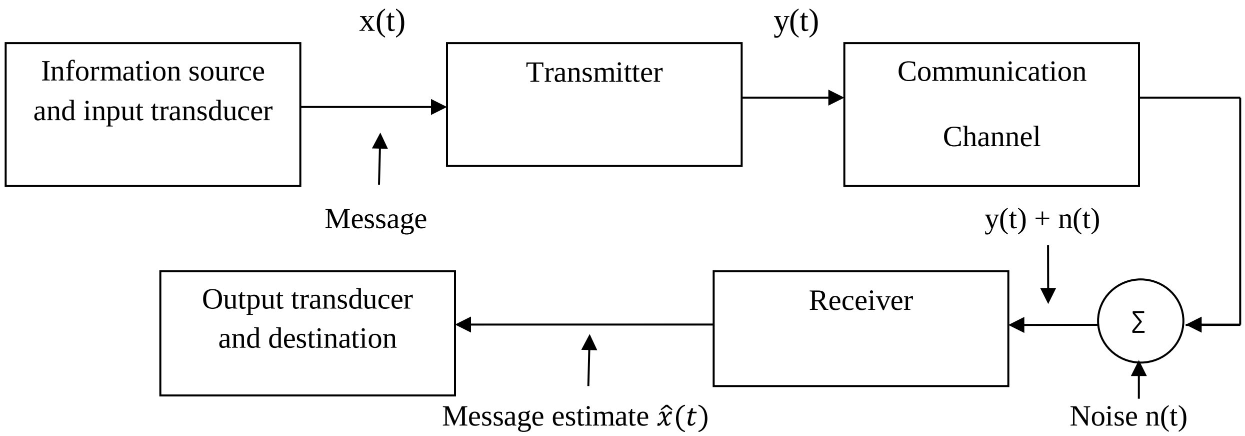
Module 1

Signal Analysis

Model of a Communication System

- *Communication* is defined as "exchange of information".
- *Telecommunication* refers to communication over a distance greater than would normally be possible without artificial aids.
- Telephony is an example of point-to-point communication and normally involves a two – way flow of information.
- Broadcast radio and television : Information is transmitted from one location but is received at many locations using different receivers (point to multi-point communication)
- Model of a communication system :



- The purpose of a communication system is to transmit information-bearing signals from a source located at one point to a user located at another end.
- The input transducer is used to convert the physical message generated by the source into a time-varying electrical signal called the *message signal*.
- The original message is recreated at the destination using an output transducer.
- The *transmitter* modifies the message signal into a form suitable for transmission over the channel. *Here modulation takes place*.
- The *channel* is the medium over which signal is transmitted, (like free space, an optical fiber, transmission lines, twisted pair of wires…). Here signal is distorted due to
    A. Nonlinearities and/or imperfections in the frequency response of the channel.
    B. Noise and interference are added to the signal during the course of transmission.
- The purpose of the *receiver* is to recreate the original signal x(t) from the degraded version x(t) + n(t) of the transmitted signal after propagating through channel .
  Here, demodulation takes place.

Milestones in Communications

- Smoke Signals: used to alert soldiers on an impending danger.
- Pigeons: Used for long distance communication.
- Pony Express: Relay messages and packages between stations.
-
- 1837: Samuel Morse developed the Morse code used in telegraph.
- 1843 – Samuel Morse builds the first long distance electric telegraph line.
- Telegraph remained in service in the US until 2006.
- 1864: Maxwell formulated the electromagnetic (EM) theory1864
- 1875: Bell invented the telephone
- 1887: Hertz demonstrated physical evidence of EM waves, meaning that EM waves can be transmitted and received.
- 1889 – Almon Strowger patents the direct dial telephone.
- 1890's-1900's: **Marconi** & Popov long-distance radio
- 1906: Radio broadcast began
- 1918: Armstrong invented superheterodyne radio receiver (and FM in 1933)
- 1925 – John Logie Baird transmits the first television signal.
- 1928: Nyquist proposed the sampling theorem.
- 1947: Microwave relay system.
- 1947 – Full-scale commercial television is first broadcast.
- 1948: Information theory formulated by Shannon.
- 1957: Era of satellite communication began.
- 1965 – First email sent (at MIT)
- 1966: **Kuen Kao** pioneered fiber-optical communications (Nobel Prize Winner).
- 1970's: Era of computer networks began.
- 1981: Analog cellular system (1-G AMPS System).
- 1988: Digital cellular system debuted in Europe.
- 1988: Digital cellular system debuted in Europe (2-G Mobile System)
- 2000: 3G Mobile network.
- 1990's: Era of internet.

Classification of Signals

Definition: A signal may be defined as a single valued function of time that conveys information.

Depending on the feature of interest, we may distinguish four different classes of signals:

1. Periodic and Non-periodic Signals

A *periodic signal* g(t) is a function of time that satisfies the condition $g(t) = g(t + T_0), \forall t$.

The smallest value of T_0 that satisfies this condition is called the period of $g(t)$.

Example: A Periodic Signal

The saw-tooth function shown below is an example of a periodic signal.

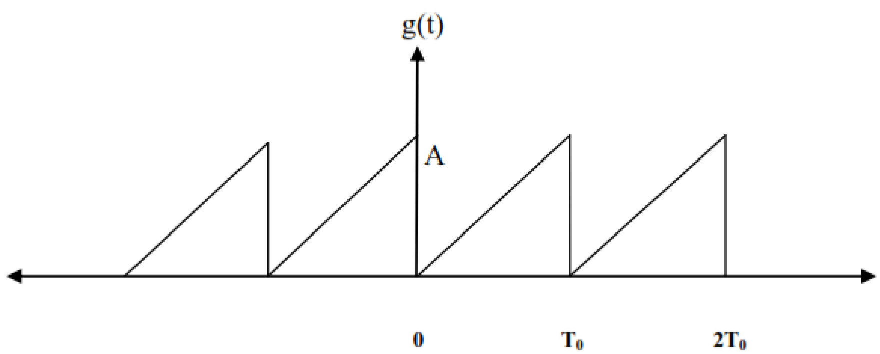


Fig. 1.1: A periodic signal with period T_0

Example: A Non-periodic Signal

The signal

$$g(t) = \begin{cases} A, & 0 \le t \le \tau \\ 0, & otherwise \end{cases}$$

is non-periodic, since there does not exist a T_0 for which the condition $g(t) = g(t + T_0)$ is satisfied.

2. Deterministic and Random Signals

A *deterministic signal* is one about which there is no uncertainty with respect to its value at any time. It is a completely specified function of time.

Example: A Deterministic Signal

$x(t) = Ae^{-at}u(t)$; A and α are constants.

A *random signal* is one about which there is some degree of uncertainty before it actually occurs. (It is a function of a random variable)

Example: A Random Signal

$x(t) = Ae^{-at}u(t)$; α is a constant and A is a random variable with the following probability density function (pdf).

$$f_A(a) = \begin{cases} 1 & 0 \le a \le 1 \\ 0 & otherwise \end{cases}$$

Example: A Random Signal

$x(t) = \cos(2\pi f_c t + \Theta)$; f_c is a constant and Θ is a random variable uniformly distributed over the interval $(0, 2\pi)$ with the following probability density function (pdf).

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi} & 0 \le \theta \le 2\pi \\ 0 & otherwise \end{cases}$$

3. Energy and Power Signals

The *instantaneous power* in a signal g(t) is defined as that power dissipated in a 1-Ω resistor, i.e.,

$$p(t) = |g(t)|^2$$

The *average power* is defined as:

$$P_{av} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |g(t)|^2 \, dt$$

The total energy of a signal $g(t)$ is

$$E = \lim_{T\to\infty} \int_{-T}^{T} |g(t)|^2 \, dt$$

A signal g(t) is classified as *energy signal* if it has a finite energy, i.e, $0 < E < \infty$.

A signal g(t) is classified as *power signal* if it has a finite power, i.e, $0 < P_{av} < \infty$.

The average power in a periodic signal $g(t)$ is

$$\boldsymbol{P_{av} = \frac{1}{T_0} \int_0^{T_0} |g(t)|^2 \, dt} \ ; \ T_0 \text{ is the period;}$$

$f_0 = 1/T_0$ is referred to as the fundamental frequency

Usually, periodic signals and random signals are power signals. Both deterministic and non-periodic signals are energy signals (we will present a counterexample shortly).

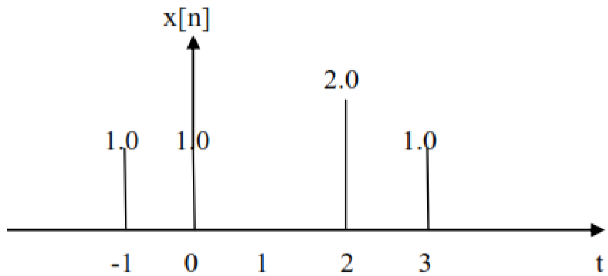
4. Analog and Digital Signals

An *analog signal* is a continuous time - continuous amplitude function of time.

Example: The sinusoidal signal $x(t) = A\cos 2\pi f_0 t$, $-\infty < t < \infty$, is an example of an analog signal.

A *discrete time- discrete amplitude* (digital) signal is defined only at discrete times. Here, the independent variable takes on only discrete values.

Example: The sequence x[n] shown below is an examples of a digital signal. The amplitudes are drawn from the finite set {1, 0, 2}.



Example: The Exponential Pulse

Find the energy in the signal $g(t) = Ae^{-\propto t}u(t)$

$E = \int_0^\infty A^2 \, e^{-2\propto t} dt = A^2 \frac{-e^{-2\propto t}}{2\propto} \Big|_0^\infty = \frac{A^2}{2\propto}$. Since E is finite, then $g(t)$ is an energy signal.

Example: The Rectangular Pulse

Find the energy in the signal:

$$g(t) = \begin{cases} A, & 0 < t < \tau \\ 0, & o.w \end{cases}$$

$E = \int_0^\tau A^2 \, dt = A^2\tau$. This is an energy signal since E is finite.

Example: The Periodic Sinusoidal Signal

Find the average power in the signal

$$g(t) = A\cos(2\pi f_0 t) \quad , \quad -\infty < t < \infty$$

Since $g(t)$ is periodic, then

$$P_{av} = \frac{1}{T_0}\int_0^{T_0} A^2 \cos^2 \omega t \, dt = \frac{A^2}{T_0}\int_0^{T_0} \left(\frac{1+\cos 2\omega t}{2}\right) = \left(\frac{A^2}{T_0}\right) . \left(\frac{T_0}{2}\right) = \frac{A^2}{2}.$$

Here, P_{av} is finite and, therefore, $g(t)$ is a power signal.

Example: The Periodic Saw-tooth Signal

Find the average power in the saw-tooth signal $g(t)$ plotted in Fig.1.

$$g(t) = \frac{A}{T_0}t, 0 \le t \le T_0$$

$$P_{av} = \frac{1}{T_0}\int_0^{T_0} \frac{A^2}{{T_0}^2} t^2 dt = \frac{1}{T_0}\frac{A^2}{{T_0}^2}\frac{t^3}{3}\Big|_0^{T_0} = \frac{A^2 {T_0}^3}{3\,{T_0}^3} = \frac{A^2}{3}.$$

Example: The Unit Step Function

Consider the signal $g(t) = Au(t)$

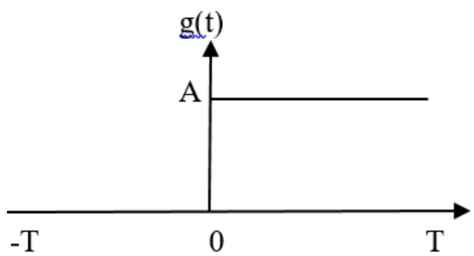


Fig. 1.2

This is a non-periodic signal. Let us first try to find its energy

$$E = \int_0^{\infty} A^2\, dt \to \infty$$

Sine E is not finite, then $g(t)$ is not an energy signal. To find the average power, we employ the definition

$$P_{av} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} |g(t)|^2\, dt,$$

where 2T is chosen to be a symmetrical interval about the origin, as in Fig. 1.2 above.

$$P_{av} = \lim_{T\to\infty}\frac{1}{2T}\int_0^{T} A^2\, dt = \lim_{T\to\infty}\frac{A^2\,T}{2T} = \frac{A^2}{2}.$$

So, even-though $g(t)$ is non-periodic, it turns out that it is a power signal.

Remark: This is an example where the general rule (periodic signals are power signals and non-periodic signals are energy signals) fails to hold.

Fourier Series

Let $g(t)$ be a periodic signal with period $T_0 = \frac{1}{f_0}$ such that it absolutely integrable over one period, i.e.,

$$\int_0^{T_0} |g(t)|\, dt < \infty.$$

The signal g(t), satisfying the above integrability condition, may be expanded in one of three possible Fourier series forms (we will not address the question of series convergence in this discussion).

The complex Form:

$$g(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

where, $$C_n = \frac{1}{T_0}\int_0^{T_0} g(t)\, e^{-jn\omega_0 t} dt.$$

Note that C_n is a complex valued quantity, which can be written as

$$C_n = |C_n| e^{j\theta n}$$

The plot of $|C_n|$ versus frequency is called the *Discrete Amplitude Spectrum*.

The plot of θ_n versus frequency is called the *Discrete Phase Spectrum*.

The term at f_0 is referred to as the fundamental frequency. The term at $2f_0$ is referred to as the second order harmonic, the term at $3f_0$ is referred to as the third order harmonic and so on.

The Trigonometric Form:

$$g(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

where,

$$a_0 = \frac{1}{T_0}\int_0^{T_0} g(t)\, dt \quad ; \quad \text{(dc or average value)}$$

$$a_n = \frac{2}{T_0}\int_0^{T_0} g(t) \cos n\omega_0 t\, dt$$

$$b_n = \frac{2}{T_0}\int_0^{T_0} g(t) \sin n\omega_0 t\, dt$$

The Polar Form

$$g(t) = c_0 + \sum_{n=1}^{\infty} 2|C_n| \cos(n\omega_0 t + \theta_n)$$

where C_n and θ_n are those terms defined in the complex form.

Remark: The above three forms are equivalent and are representations of the same waveform. If you know one representation, you can easily deduce the other.

Example: Find the trigonometric Fourier series of the periodic rectangular signal defined over one period T_0 as:

$$g(t) = \begin{cases} +A, & -T_0/4 \le t \le T_0/4 \\ 0, & otherwise \end{cases}$$

Solution:

$$a_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\,dt = \frac{1}{T_0}\int_{-T_0/4}^{T_0/4} A\,dt = A/2$$

$$b_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\sin(\frac{2\pi n}{T_0}t)\,dt = \frac{2}{T_0}\int_{-T_0/4}^{T_0/4} A\sin(\frac{2\pi n}{T_0}t)\,dt = 0$$

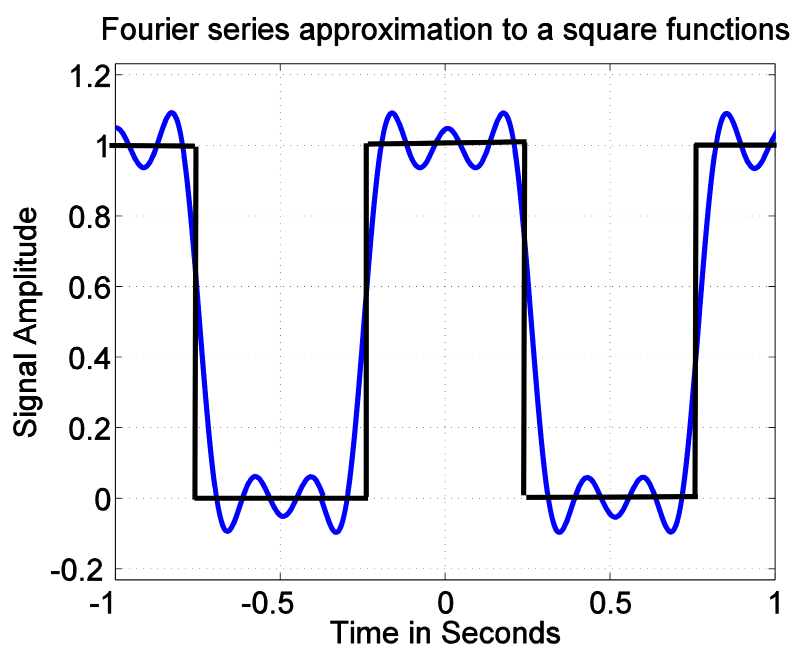
$$a_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\cos(\frac{2\pi n}{T_0}t)\,dt = \frac{2}{T_0}\int_{-T_0/4}^{T_0/4} A\cos(\frac{2\pi n}{T_0}t)\,dt$$

$$a_n = \begin{cases} \frac{2A}{n\pi}, & n = 1, 5, 9, \dots \\ \frac{-2A}{n\pi}, & n = 3, 7, 11, \dots \\ 0, & n = 2, 4, 6 \dots \end{cases}$$

The first four terms in the expansion of $g(t)$ are:

$$\tilde{g}(t) = \frac{A}{2} + \frac{2A}{\pi}\{\cos(2\pi f_0 t) - \frac{1}{3}\cos(2\pi 3 f_0 t) + \frac{1}{5}\cos(2\pi 5 f_0 t)\}$$

The function $\tilde{g}(t)$ along with $g(t)$ are plotted in the figure for $-1 \le t \le 1$ assuming $A = 1$ and $f_0 = 1$



Remark: As more terms are added to $\tilde{g}(t)$, $\tilde{g}(t)$ becomes closer to $g(t)$ and in the limit as $n \rightarrow \infty$, $\tilde{g}(t)$ becomes equal to $g(t)$ at all points except at the points of discontinuity.

Parseval's Power Theorem

The average power of a periodic signal g(t) is given by:

$$P_{av} = \frac{1}{T_0}\int_0^{T_0}|g(t)|^2\,dt = \sum_{n=-\infty}^{\infty}|C_n|^2 = |C_0|^2 + 2\sum_{n=1}^{\infty}|C_n|^2$$

$$= |a_0|^2 + \frac{1}{2}\sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2$$

Power Spectral Density

The plot of $|C_n|^2$ versus frequency is called the ***power spectral density*** (PSD). It displays the power content of each frequency (spectral) component of a signal. For a periodic signal, the PSD consists of discrete terms at multiples of the fundamental frequency.

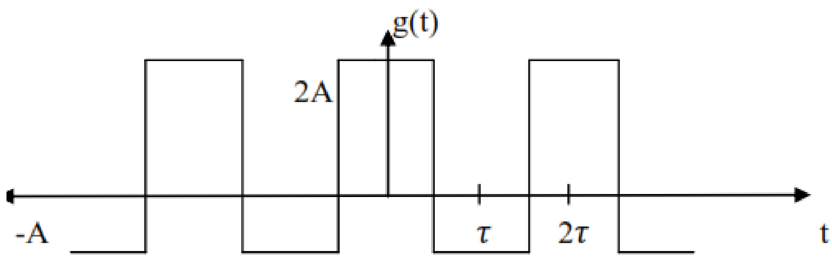
Exercise: Consider again the saw-tooth function defined over one period as $g(t) = t, 0 \leq t \leq 1$

a. Use matlab to find the dc terms and the first three harmonics(i.e., let n = 3) in the Fourier series expansion

$$\tilde{g}(t) = a_0 + \sum_{n=1}^{3}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

b. Plot $\tilde{g}(t)$ and $g(t)$ versus time for $-1 \leq t \leq 1$ on the same graph.
c. Find the fraction of the power contained in $\tilde{g}(t)$ to that in $g(t)$.
d. Sketch the power spectral density.

Example: Find the power spectral density of the periodic function $g(t)$ shown in the figure.

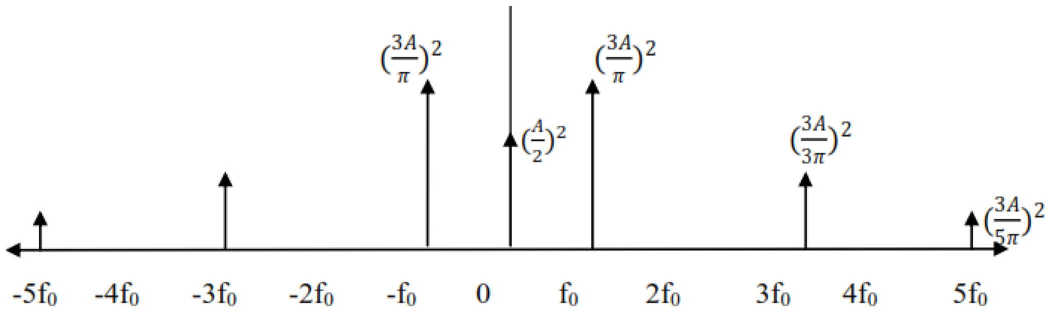


Solution: Here, we need to find the complex Fourier series expansion, where the period $T_0 = 2\tau$

$$g(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}; \qquad C_n = \frac{1}{T_0}\int_0^{T_0} g(t)\, e^{-jn\omega_0 t} dt$$

$$C_n = \begin{cases} \frac{A}{2}, & n = 0 \\ \frac{3A}{|n|\pi}, & n = \pm 1, \pm 5, \pm 9, \\ \frac{-3A}{|n|\pi}, & n = \pm 3, \pm 7, \pm 11, \ldots \\ 0, & n = \pm 2, \pm 4, \ldots \end{cases} \Longrightarrow |C_n|^2 = \begin{cases} (\frac{A}{2})^2, & n = 0 \\ (\frac{3A}{n\pi})^2, & n: odd \\ 0, & n: even \end{cases}$$

$$S_g(f) = \sum_{n=-\infty}^{\infty} |C_n|^2\, \delta(f - nf_0)$$



As can be seen, the power spectral density of this periodic signal is a discrete function in frequency.

Exercise: Verify Parseval's power theorem for this signal, i.e., show that

$$P_{av} = \frac{1}{T_0}\int_0^{T_0} |g(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |C_n|^2 = 2.5A^2$$

Fourier Transform

Let $g(t)$ be a non-periodic square integrable function of time. That is one for which

$$\int_{-\infty}^{\infty} |g(t)|^2 \, dt < \infty$$

The Fourier transform of $g(t)$ exists and is defined as:

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft} \, dt$$

The time function $g(t)$ can be recovered from $G(f)$ using the inverse Fourier Transform:

$$g(t) = \int_{-\infty}^{\infty} G(f)e^{j2\pi ft} \, df$$

Remarks:

- All energy signals for which $E = \int_{-\infty}^{\infty} |g(t)|^2 \, dt < \infty$ are Fourier transformable.
- G(f) is a complex function of frequency f, which can be expressed as:

$$G(f) = |G(f)|j^{\theta(f)}$$

where, $|G(f)|$: is the *continuous amplitude spectrum* of $g(t)$, (an even function of f).

$\theta(f)$: is the *continuous phase spectrum* of $g(t)$, (an odd function of f).

Rayleigh Energy Theorem

The energy in a signal g(t) is given by:

$$\boldsymbol{E = \int_{-\infty}^{\infty} |g(t)|^2 \, dt = \int_{-\infty}^{\infty} |G(f)|^2 df}$$

The function $|G(f)|^2$ is called the *energy spectral density*. It illustrates the range of frequencies over which the signal energy extends and the frequency bands which are significant in terms of their energy contents. For a non-period signal energy signal, the energy spectral density is a continuous function of f.

A General Form of the Rayleigh Energy Theorem

For two energy functions $g(t)$ and $v(t)$, the following result holds:

$$\int_{-\infty}^{\infty} g(t)v(t)^* \, dt = \int_{-\infty}^{\infty} G(f)V(f)^* \, df$$

Example: **Energy Spectral Density of the Exponential Signal**

$$v(t) = \begin{cases} A\,e^{-bt} & t > 0 \\ 0 & t < 0 \end{cases}$$

$$V(f) = \int_0^\infty v(t)e^{-j2\pi f t}\,dt = \int_0^\infty Ae^{-bt}\,e^{-j2\pi f t}dt$$

$$V(f) = A\int_0^\infty e^{-(b+j2\pi f)t}\,dt = A\frac{e^{-(b+j2\pi f)t}}{-(b+j2\pi f)}\Big|_0^\infty = \frac{A}{b+j2\pi f}.$$

$$V(f) = \frac{A}{b+j2\pi f}, |V(f)| = \frac{A}{(b^2+(2\pi f)^2)^{1/2}}$$

The energy spectral density is: $S_v(f) = |V(f)|^2 = \frac{A^2}{b^2+(2\pi f)^2}$

Remark: The signal $v(t)$ is called a *baseband signal* since the signal occupies the low frequency part of the spectrum. That is, the energy in the signal is found around the zero frequency. When the signal is multiplied by a high frequency carrier, the spectrum becomes centered around the carrier and the modulated signal is called a *bandpass signal*.

Exercise: For the exponential pulse, verify Rayleigh energy theorem, i.e., show that

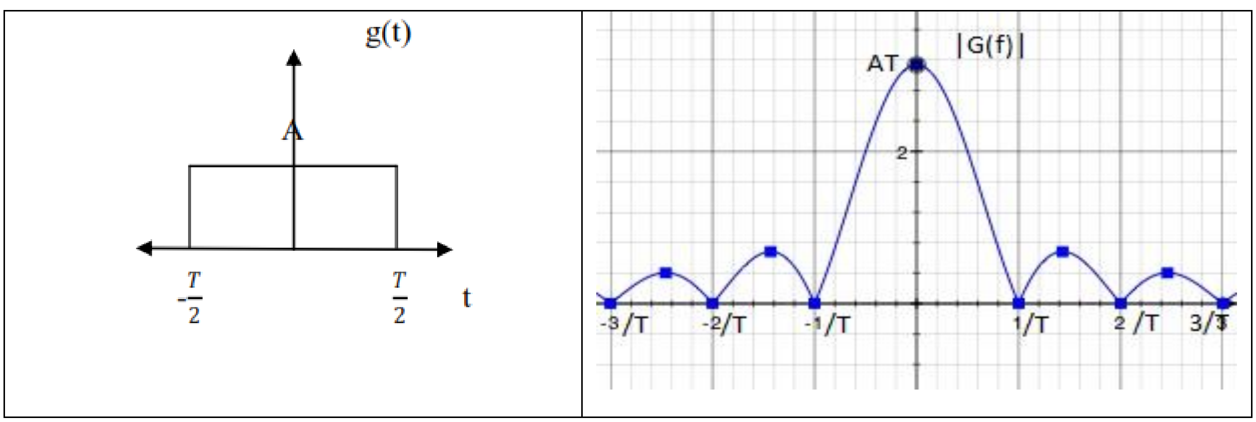
$$\int_0^\infty |v(t)|^2 dt = 2\int_0^\infty |V(f)|^2 df = \frac{A^2}{2b}.$$

Example: The Rectangular Pulse $g(t) = Arect(\frac{t}{T})$

$$G(f) = \int_{-T/2}^{T/2} Ae^{-j2\pi f t}dt = \frac{A}{\pi f}\sin\pi f T \qquad , \quad AT\frac{\sin\pi f T}{\pi f T} \triangleq AT\ sincfT$$

$$|G(f)| = AT\ |sincfT|$$

The maximum of $|G(f)|$ occurs at $f = 0$ since $\lim_{x\to 0}\frac{\sin x}{x} = 1$. Also, $G(f) = 0$ when $sin(\pi f T) = 0$, which occurs at the points that satisfy $\pi f T = n\pi, \Rightarrow fT = n, or\ f = \frac{n}{T}, n = \pm1, \pm2, \pm3, \ldots$



Remark: Time Duration and Bandwidth

Note that as the signal time duration T *increases*, the first zero crossing at $f = \frac{1}{T}$ decreases, implying that the bandwidth of the signal decreases. More on this will be said later when we discuss the time bandwidth product.

Exercise: For the rectangular pulse $g(t) = Arect(\frac{t}{T})$, verify Rayleigh energy theorem, i.e., show that

$$\int_0^{\infty} |g(t)|^2 dt = 2\int_0^{\infty} |G(f)|^2 df = A^2 T.$$

Properties of the Fourier Transform

- **Linearity (superposition)**

  Let $g_1(t) \leftrightarrow G_1(\mathrm{f})$

  and $g_2(t) \leftrightarrow G_2(\mathrm{f})$, then

  $c_1 g_1(t)+c_2 g_2(t) \leftrightarrow c_1 G_1(f)+c_2 G_2(f)$; c_1, c_2 are constants

- **Time Scaling**

$g(t) \leftrightarrow G(f)$	$g(at) \leftrightarrow \frac{1}{\lvert a \rvert} G(f/a)$

- **Duality**

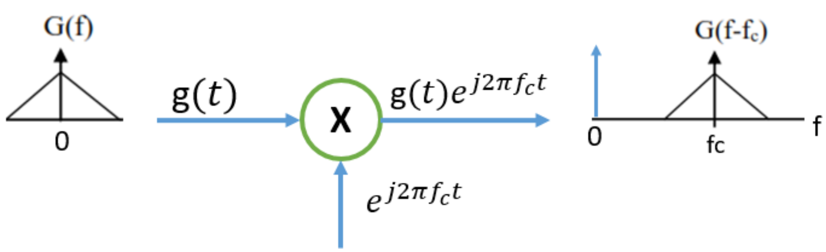
$g(t) \leftrightarrow G(f)$	$G(t) \leftrightarrow g(-f)$

- **Time Shifting**

$g(t) \leftrightarrow G(f)$	$g(t-t_0) \leftrightarrow G(f)e^{-j2\pi f t_0}$
Delay in time domain corresponds to a phase shift in frequency domain	

- **Frequency Shifting**

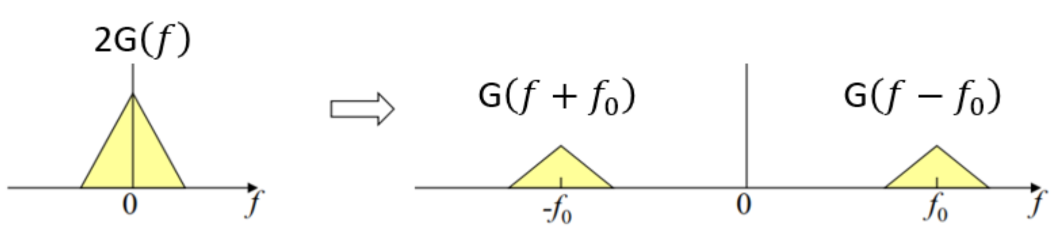
$g(t) \leftrightarrow G(f)$	$g(t)e^{j2\pi f_c t} \leftrightarrow G(f-f_c)$; f_c is constant



Frequency Shifting Property of the Fourier Transfer

- **Modulation Property**

$g(t) \leftrightarrow G(f)$	$2g(t)\cos(2\pi f_0 t) \leftrightarrow G(f-f_0) + G(f+f_0)$; f_0 is a constant



- **Area under $G(f)$**

$g(t) \leftrightarrow G(f)$	$g(t=0)= \int_{-\infty}^{\infty} G(f)df$
The value $g(t=0)$ is equal to the area under its Fourier transform function	

- **Area under $g(t)$**

$g(t) \leftrightarrow G(f)$	$G(0) = \int_{-\infty}^{\infty} g(t)dt$
The area under a function g(t) is equal to the value of its Fourier transform G(f) at f = 0, where $G(0)$ implies the presence of a dc component.	

- **Differentiation in the Time Domain**

If $g(t)$ and its derivative $g'(t)$ are Fourier transformable, then,

$$g'(t) \leftrightarrow (j2\pi f)G(f)$$

i.e., differentiation in the time domain ⟹ multiplication by j$2\pi f$ in the frequency domain.

(Differentiation in the time domain enhances high frequency components of a signal)

Also, $$\frac{d^n g(t)}{dt^n} \leftrightarrow (j2\pi f)^n G(f)$$

- **Integration in the Time Domain**

$$\int_{-\infty}^{t} g(\tau)d\tau \leftrightarrow \frac{1}{j2\pi f}G(f);\text{ assuming } G(0) = 0$$

i.e., integration in the time domain corresponds to division by (j$2\pi f$) in the frequency domain. This amounts to low pass filtering, where high frequency components are attenuated due to filtering.

When $G(0) \neq 0$, the above result becomes:

$$\int_{-\infty}^{t} g(\tau)d\tau \leftrightarrow \frac{1}{j2\pi f}G(f) + \frac{1}{2}\, G(0)\delta(f).$$

- **Conjugate Functions**

For a complex – valued time signal $g(t)$, we have:

$$g^*(t) \leftrightarrow G^*(-f) \quad ;$$

Also, $$g^*(-t) \leftrightarrow G^*(f) \quad ;$$

Therefore, $\text{Re}\{g(t)\} \leftrightarrow \frac{1}{2}\{ G(f) + G^*(-f)\}$

$\text{Im}\{g(t)\} \leftrightarrow \frac{1}{2j}\{ G(f) - G^*(-f)\}$

- **Multiplication in the Time Domain**

$$g_1(t)\, g_2(t) \leftrightarrow \int_{-\infty}^{\infty} G_1(\lambda)\, G_2(f-\lambda)d\lambda = G_1(f) * G_2(f)$$

Multiplication of two signals in the time domain is transformed into the convolution of their Fourier transforms in the frequency domain.

- **Convolution in the Time Domain**

$$g_1(t) * g_2(t) \leftrightarrow G_1(f)G_2(f)$$

Convolution of two signals in the time domain is transformed into a multiplication of their Fourier transforms in the frequency domain.

Fourier Transform of Power Signals

For a non-periodic (energy) signal g(t), the Fourier transform exists when

$$E = \int_{-\infty}^{\infty} |g(t)|^2 \, dt < \infty$$

so that $G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi f t}dt.$

For power signals, the integral $\int_{-\infty}^{\infty} g(t)e^{-j2\pi f t}dt$ **does not exist**.

However, one can still finds the Fourier transform of power signals by employing the delta function. This function is defined next.

Dirac – Delta Function (Impulse Function)

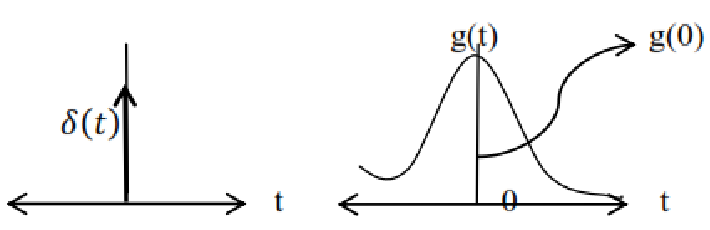
This function is defined as

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$

such that: $\int_{-\infty}^{\infty} \delta(t)dt = 1$

and $\int_{-\infty}^{\infty} g(t)\delta(t)dt = g(0)$

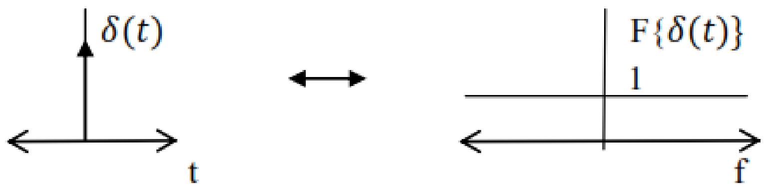
(Here, g(t) is a continuous function of time. The second property shows that the delta function samples the function g(t) at the time of its occurrence).



Some Properties of the Delta Function

1. $g(t)\delta(t-t_0) = g(t_0)\delta(t-t_0)$; (Multiplication)
2. $\int_{-\infty}^{\infty} g(t)\delta(t-t_0)dt = g(t_0)$; (Sifting or sampling property)

3. $\delta(\alpha t) = \frac{1}{|\alpha|}\delta(t)$
4. $\delta(t) * g(t) = g(t)$
5. $\delta(t) = \frac{du(t)}{dt} \quad \Rightarrow u(t) = \int_{-\infty}^{t} \delta(t)dt$
6. $\delta(t) = \delta(-t)$; an even function of its argument.
7. Fourier transform: $\Im\{\delta(t)\} = 1$



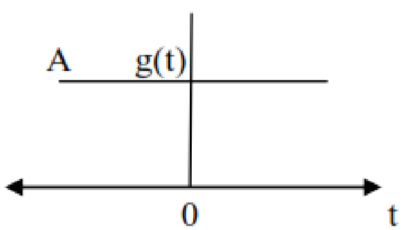
(Note how the time-bandwidth relationship holds for this pair. A narrow pulse in time extends over a large frequency spectrum)

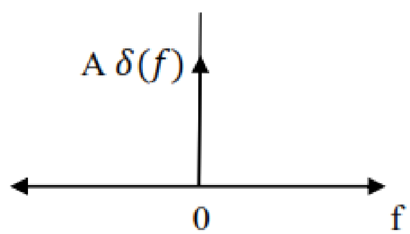
8. $\Im\{\delta(t - t_0)\} = e^{-j2\pi f t_0}$

Applications of the Delta Function

- **DC or a Constant Signal**

Since $\Im\{\delta(t)\} = 1$, then by the duality property $\Im\{1\} = \delta(f)$





(Again, note that the transform of a dc signal is an impulse at f =0)

- **Complex Exponential Function**

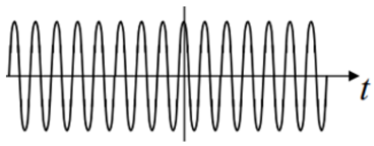
  $\Im\{Ae^{j2\pi f_c t}\} = A\delta(f - f_c)$

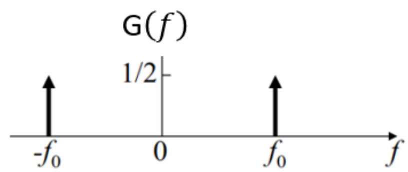
- **Sinusoidal Functions**

  $\Im\{\cos 2\pi f_0 t\} = \frac{1}{2}\{\delta(f - f_0) + \delta(f + f_0)\}$

  $\Im\{\sin 2\pi f_0 t\} = \frac{1}{2j}\{\delta(f - f_0) - \delta(f + f_0)\}$

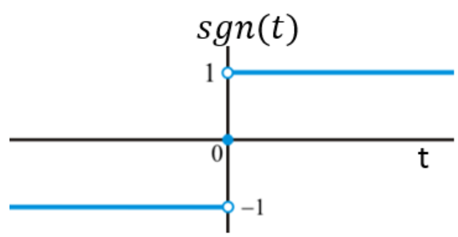
$$g(t) = \cos(2\pi f_0 t)$$





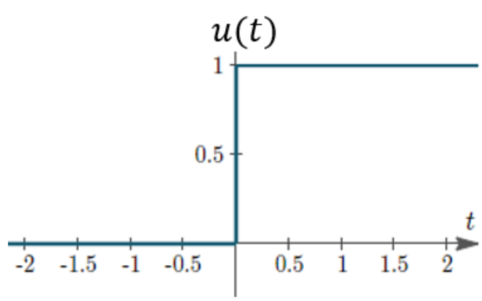
- **Signum Function**

$sgn(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$	$\Im\{sgn(t)\} = \dfrac{1}{j\pi f}$



- **Unit Step Function**

$u(t) = \begin{cases} 1 & t > 0 \\ \dfrac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases}$	$sgn(t) = 2u(t) - 1$ $u(t) = \frac{1}{2}\{sgn(t) + 1\}$ $\Im\{u(t)\} = \frac{1}{j2\pi f} + \frac{1}{2}\,\delta(f)$



- **Periodic Signals**

A periodic signal g(t) is expanded in the complex form as:

$g(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$	$\Im\{g(t) = \sum_{n=-\infty}^{\infty} C_n\,\delta(f - nf_0)$

Example: Consider the following train of impulses

$$g(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT_0)$$

The Fourier coefficients are obtained by integrating over one period of $g(t)$.

$$C_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} g(t)\, e^{-jn\omega_0 t} dt, \qquad \frac{1}{T_0} = f_0$$

Therefore, the complex Fourier series of $g(t)$ is

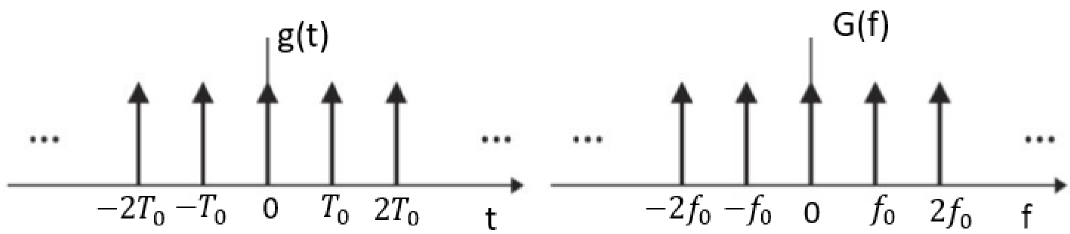
$$g(t) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}$$

Recognizing that

$$\{e^{jn\omega_0 t}\} = \delta(f - nf_0)$$

Combing the above results, we obtain

$$\Im\{g(t)\} = \sum_{\mathrm{m}=-\infty}^{\infty} \delta(t - mT_0) \ = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} \delta(f - nf_0)$$



Note that the signal is periodic in the time domain and its Fourier transform is periodic in the frequency domain.

Remark: This sequence will be found useful when the sampling theorem is considered later in the course.

Module 2
Transmission of Signals through Linear Systems

Definition: A system refers to any physical device that produces an output signal in response to an input signal.

Definition: A system is linear if the principle of superposition applies.

If	$x_1(t)$	produces output	$y_1(t)$
	$x_2(t)$	produces output	$y_2(t)$
then	$a_1x_1(t)+ a_2x_2(t)$	produces an output	$a_1y_1(t)+ a_2y_2(t)$
Also,	a zero input	should produce	a zero output.

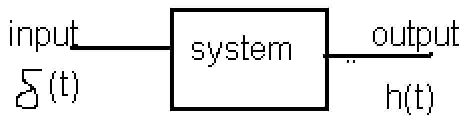
Example of linear systems include filters and communication channels.

Definition: A filter refers to a frequency selective device that is used to limit the spectrum of a signal to some band of frequencies.

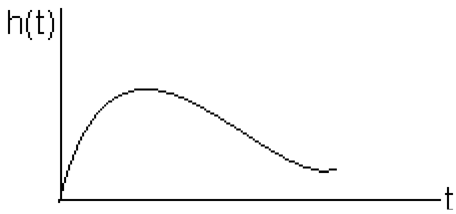
Definition: A channel refers to a transmission medium that connects the transmitter and receivers of a communication system.

Time domain and frequency domain may be used to evaluate system performance.

Time response

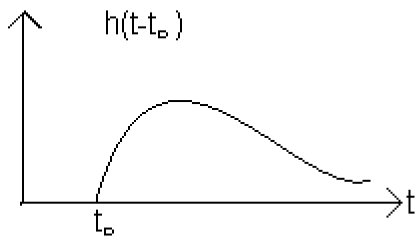


Definition: The impulse response h(t) is defined as the response of a system to an impulse $\delta(t)$ applied to the input at t=0 .



Definition: A system is time-invariant when the shape of the impulse response is the same no matter when the impulse is applied to the system.

$\delta(t) \longrightarrow h(t)$, then $\delta(t-t_d) \longrightarrow h(t-t_d)$



When the input to a linear time-invariant system in a signal x(t) , then the output is given by

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)\, d\lambda$$
$$= \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)\, d\lambda; \quad \text{convolution integral}$$

Definition: A system is said to be causal if it does not respond before the excitation is applied, i.e.,

$$h(t) = 0 \qquad t < 0$$

The causal system is physically realizable.

Definition: A system is said to be stable if the output signal is bounded for all bounded input signals.

If $|x(t)| \leq M$; M is the maximum value of the input

then $|y(t)| \leq \int_{-\infty}^{\infty} |h(\tau)||x(t-\tau)| d\tau$

$$= M \int_{-\infty}^{\infty} |h(\tau)| d\tau$$

⟹ A necessary and sufficient condition for stability (a bounded output) is

$\int_{-\infty}^{\infty} |h(t)| dt < \infty$; h(t) is absolutely integrable.

∴ zero initial conditions assumed.

Frequency response

Definition: The transfer function of a linear time invariant system is defined as the Fourier transform of the impulse response h(t)

$$H(f) = \Im\{h(t)\}$$

Since $y(t) = x(t) * h(t)$, then $Y(f) = H(f)X(f)$. The system transfer function is thus the ratio of the Fourier transform of the output to that of the input.

$$H(f) = \frac{Y(f)}{X(f)}$$

The transfer function $H(f)$ is a complex function of frequency, which can be expressed as

$$H(f) = |H(f)|e^{j\,\theta(f)}$$

where,

$|H(f)|$: Amplitude spectrum

$\theta(f)$: Phase spectrum.

System input–output energy spectral density

Let $x(t)$ be applied to a LTI system, then the Fourier transform of the output is related to the Fourier transform of the input through the relation

$$Y(f) = H(f)X(f).$$

Taking the absolute value and squaring both sides, we get

$$|Y(f)|^2 = |H(f)|^2|X(f)|^2$$

$$S_Y(f) = |H(f)|^2 S_X(f)$$

$S_Y(f)$: Output Energy Spectral Density
$S_X(f)$: Input Energy Spectral Density.

$$output\ energy\ spectral\ density\ = (|H(f|^2)\ (input\ energy\ spectral\ density)$$

The total output energy is

$$E_y = \int_{-\infty}^{+\infty} S_Y(f)df$$
$$= \int_{-\infty}^{+\infty} |\mathrm{H(f)}|^2\, S_X(f)df.$$

The total input energy is

$$E_x = \int_{-\infty}^{+\infty} S_x(f)df.$$

Example: response of a filter to a sinusoidal input

The signal $x(t) = \cos(2\pi f_0 t)\,, -\infty < t < \infty$, is applied to a filter described by the transfer function $H(f) = \frac{1}{1+jf/B}$, B is the 3-dB bandwidth. Find the filter output $y(t)$ using the frequency domain approach.

Solution: we will find the output using the frequency domain approach.

$$Y(f) = H(f)X(f)$$

$$H(f) = \frac{1}{\sqrt{1+(\frac{f}{B})^2}} e^{-j\theta}; \qquad \theta = \tan^{-1}\frac{f}{B}; \qquad \theta_0 = \tan^{-1}\frac{f_0}{B}$$

$$Y(f) = H(f)[\tfrac{1}{2}\delta(f - f_0) + \tfrac{1}{2}\delta(f + f_0)]$$

$$Y(f) = \tfrac{1}{2}H(f_0)\delta(f - f_0) + \tfrac{1}{2}H(-f_0)\delta(f + f_0)$$

$$Y(f) = \frac{1}{2}\frac{1}{\sqrt{1+(\frac{f_0}{B})^2}} e^{-j\theta_0}\delta(f - f_0) + \frac{1}{2}\frac{1}{\sqrt{1+(\frac{f_0}{B})^2}} e^{j\theta_0}\delta(f + f_0)$$

Taking the inverse Fourier transform, we get

$$y(t) = \frac{1}{\sqrt{1+(\frac{f_0}{B})^2}}\frac{1}{2}[e^{j(2\pi f_0 t - \theta_0)} + e^{-j(2\pi f_0 t - \theta_0)}]$$

$$y(t) = \frac{1}{\sqrt{1+(\frac{f_0}{B})^2}}\cos(2\pi f_0 t - \theta_0)$$

Note that in the last step we have made use of the Fourier transform pair

$$e^{j2\pi f_0 t} \leftrightarrow \delta(f - f_0)$$

Remark: Note that the amplitude of the output as well as its phase depend on the frequency of the input, f_0, and the bandwidth of the filter, B.

Assume, for instance, that $f_0 = B$, then $\theta_0 = \tan^{-1}\frac{f_0}{B} = \tan^{-1} 1 = 45°$. The output can be written as

$$y(t) = \frac{1}{\sqrt{1+1}}\cos(2\pi f_0 t - 45°)$$

$$y(t) = \frac{1}{\sqrt{2}}\cos(2\pi f_0 t - 45°)$$

Exercise: The signal $x(t) = \ cos\ w_0 t - \frac{1}{\pi} cos\ 3w_0 t$ is applied to a filter described by the transfer function $H(f) = \frac{1}{1+jf/B}$.

a. Use the result of the previous example to find the filter output $y(t)$.
b. Is the transmission through this filter distortion-less?

Exercise: Consider the periodic rectangular signal $g(t)$ defined over one period T_0 as

$$g(t) = \begin{cases} +A, & -T_0/4 \le t \le T_0/4 \\ 0, & otherwise \end{cases}$$

If $g(t)$ is applied to a filter described by the transfer function $H(f) = \frac{1}{1+jf/B}$, use the result of the previous example to find the filter output $y(t)$.

Example: input and output energy spectral densities

The signal $g(t) = Arect(\frac{t}{T})$ is applied to a filter with $(f) = \frac{1}{1+jf/B}$. Find the output energy spectral density.

Solution: system input-output energy spectral densities are related by

$$S_Y(f) = |H(f)|^2 S_X(f)$$

$$S_Y(f) = \frac{1}{1+(\frac{f}{B})^2}(AT\ |sinc\ Tf|)^2$$

Example: channel response due to two impulses

The signal $g(t) = \delta(t) - \delta(t-1)$ is applied to a channel described by the transfer function $H(f) = \frac{1}{1+jf/B}$. Use the convolution integral to find the channel output.

Solution:

The impulse response of the channel is obtained by taking the inverse Fourier transform of $H(f)$, which is

$$h(t) = 2\pi B e^{-2\pi B t} u(t)$$

Using the linearity and time invariance property, the output can be obtained as

$$y(t) = h(t) * [\delta(t) - \delta(t-1)]$$

$$y(t) = h(t)u(t) - h(t-1)u(t-1)$$

$$y(t) = 2\pi B[e^{-2\pi B t}u(t) - e^{-2\pi B(t-1)}u(t-1)]$$

Exercise: channel response due to a pulse

The signal $g(t) = u(t) - u(t-1)$ is applied to a channel described by the transfer function $H(f) = \frac{1}{1+jf/B}$. Find the channel output $y(t)$.

Signal Distortion in Transmission

As we have said before, the objective of a communication system is to deliver to the receiver almost an exact copy of what the source generates. However, communication channels are not perfect in the sense that impairments on the channel will cause the received signal to differ from the transmitted one. During the course of transmission, the signal undergoes attenuation, phase delay, interference from other transmissions, Doppler shift in the carrier frequency, and many other effects. In this introductory discussion we will explain some of the reasons that cause the received signal to be distorted.

Linear distortion

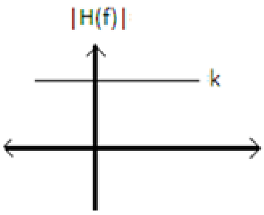
A signal transmission is said to be *distortion-less* if the output signal y(t) is an exact replica of the input signal x(t) , i.e., y(t) has the same shape as the input, except for a constant amplification (or attenuation) and a constant time delay.

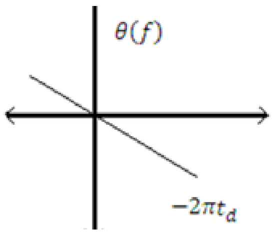
Condition for a distortion-less transmission in the time domain is:

$$\boldsymbol{y(t) = kx(t - t_d)}; \textbf{ Condition for a distortion-less transmission}$$

where k: is a constant amplitude scaling

t_d: is a constant time delay





In the frequency domain, the condition for a distortion-less transmission becomes

$$Y(f) = k\,X(f)\,e^{-j2\pi f t_d}$$

or $$H(f) = \frac{Y(f)}{X(f)} = k\,e^{-j2\pi f t_d} = ke^{-j\theta(f)}$$

That is, for a distortion-less transmission, the transfer function should satisfy two conditions:

1. **|H(f)| = k** ; The amplitude of the transfer function is constant (gain or attenuation) over the frequency range of interest.
2. $\boldsymbol{\theta(f) = -2\pi f t_d = -(2\pi t_d)f}$; The phase function is linear in frequency with a negative slope that passes through the origin (or multiples of π).

When |H(f)| is not constant for all frequencies of interest, *amplitude distortion* results.

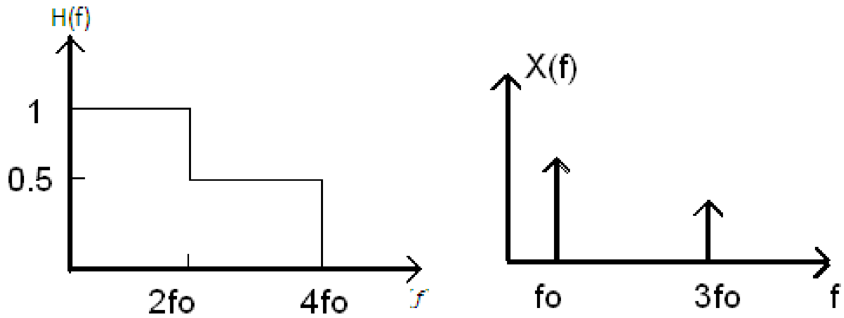
When $\theta(f) \neq -2\pi f t_d \pm 180^0$, then we have *phase distortion* (or delay distortion).

The following examples demonstrate the two types of distortion mentioned above.

Example: amplitude distortion

Consider the signal $x(t) = cos\ w_0 t\ - \frac{1}{3}\ cos\ 3w_0 t$. If this signal passes through a channel with zero time delay (i.e., $t_d = 0$) and amplitude spectrum as shown in the figure

a. Find y(t)
b. Is this a distortion-less transmission?



<u>Solution:</u>

x(t) consists of two frequency components, f_0 and $3f_0$. Upon passing through the channel, each component will be scaled by a different factor.

$$y(t) = cos\ w_0 t\ - \frac{1}{2} \cdot \frac{1}{3}\ cos\ 3w_0 t$$

Since $y(t) = \left(cos\ w_0 t\ - \frac{1}{2} \cdot \frac{1}{3}\ cos\ 3w_0 t\right) \neq k\left(cos\ w_0 t\ - \frac{1}{3}\ cos\ 3w_0 t\right)$

then this is not a distortion-less transmission.

Example: phase distortion

If x(t) in the previous example passes through a channel whose amplitude spectrum is a constant k. Each component in x(t) suffers a $-\frac{\pi}{2}$ phase shift.

a. Find y(t).
b. Is this a distortion-less transmission?

Solution:

$$x(t) = \cos w_o t - \frac{1}{3}\cos 3w_o t$$

$$y(t) = \mathrm{k}\ \cos(w_o t - \frac{\pi}{2}) - \frac{1}{3}\ k\ \cos\left(3w_o t - \frac{\pi}{2}\right)$$

$$y(t) = \text{k} \cos w_o(t - \frac{\pi}{2w_o}) - \frac{1}{3} k \cos\left(3w_o(t - \frac{\pi}{2x3w_o})\right)$$

$$y(t) = \text{k} \cos w_o(t - t_{d1}) - \frac{1}{3} k \cos(3w_o(t - t_{d2}))$$

Since $t_{d1} \neq t_{d2}$, we cannot write $y(t) = kx(t - t_d)$. Here, each component in $x(t)$ suffers from a different time delay. Hence, this transmission introduces phase (delay) distortion.

Nonlinear distortion

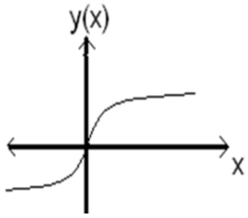
When a system contains nonlinear elements, it is not described by a transfer function H(f), but rather by a transfer characteristic of the form

$$y(t) = a_1 x(t) + a_2 x^2(t) + a_3 x^3(t) + \ldots \text{ (time domain)}$$

In the frequency domain,

$$Y(f) = a_1 X(f) + a_2 X(f)*X(f) + a_3 X(f)*X(f)*X(f) + \ldots$$

Here, the output contains new frequencies not originally present in the original signal. The nonlinearity produces undesirable frequency component for $|f| \leq W$, in which W is the signal bandwidth.



Harmonic distortion in nonlinear systems

Let the input to a nonlinear system be the single tone signal

$$x(t) = \cos(2\pi f_0 t)$$

This signal is applied to a channel with characteristic

$$y(t) = a_1 x(t) + a_2 x(t)^2 + a_3 x(t)^3$$

upon substituting x(t) and arranging terms, we get

$$y(t) = \frac{1}{2} a_2 + \left(a_1 + \frac{3}{4} a_3\right) \cos 2\pi f_0 t + \frac{1}{2} a_2 \cos 4\pi f_0 t + \frac{1}{4} a_3 cos 6\pi f_0 t$$

Note that the output contains a component proportional to x(t), which is $\left(a_1 + \frac{3}{4} a_3\right) \cos 2\pi f_0 t$, in addition to a second and a third harmonic terms (terms at twice and three times the frequency of the input). These new terms are the result of the nonlinear characteristic and are, therefore, considered as harmonic distortion. The

DC term does not constitute a distortion, for it can be removed using a blocking capacitor.

Define second harmonic distortion

$$D_2 = \frac{|amplitude\ of\ second\ harmonic\ |}{|amplitude\ of\ fundamental\ term|}$$

$$D_2 \ = \frac{|\frac{1}{2}a_2\ |}{|\left(a_1 + \frac{3}{4}a_3\right)|}\ x\ 100\%$$

In a similar way, we can define the third harmonic distortion as:

$$D_3 = \frac{|amplitude\ of\ third\ harmonic\ |}{|amplitude\ of\ fundamental\ term|}$$

Therefore,

$$D_3 \ = \frac{|\frac{1}{4}a_3\ |}{|\left(a_1 + \frac{3}{4}a_3\right)|}\ x\ 100\%$$

Remark: In the solution above, we have made use of the following two identities

$Cos^2x = \frac{1}{2}\{1 + cos2x\}$

$Cos^3x = \frac{1}{4}\{3cosx + cos3x\}$.

Filters and Filtering

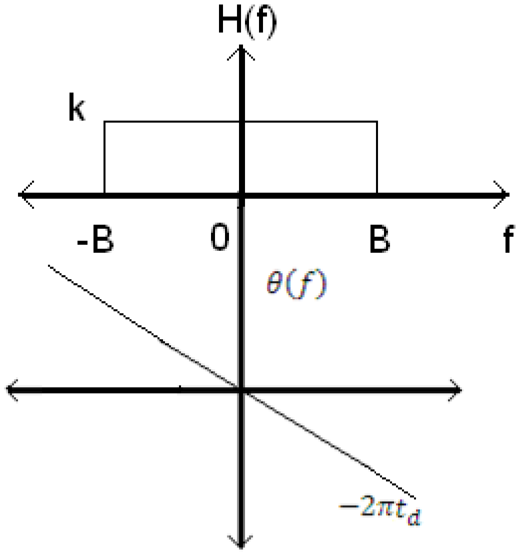
A filter is a frequency selective device. It allows certain frequencies to pass almost without attenuation while it suppresses other frequencies

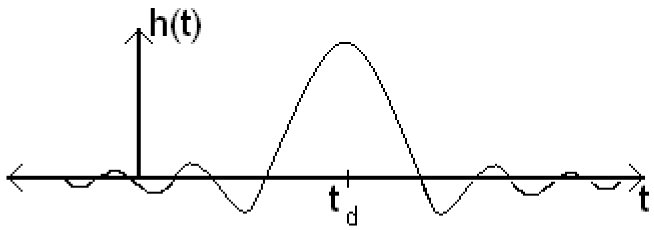
A. Ideal filter

Ideal low pass filter

$$H(f) = \begin{cases} k\, e^{-j2\pi f t_d} & |f| < B \\ 0 & o.w \end{cases}$$

$$h(t) = 2Bk\, sinc2B(t - t_d)$$





since h(t) is the response to an impulse applied at t=0, and because h(t) has nonzero values for t < 0 , the filter is *non-causal* (physically non realizable).

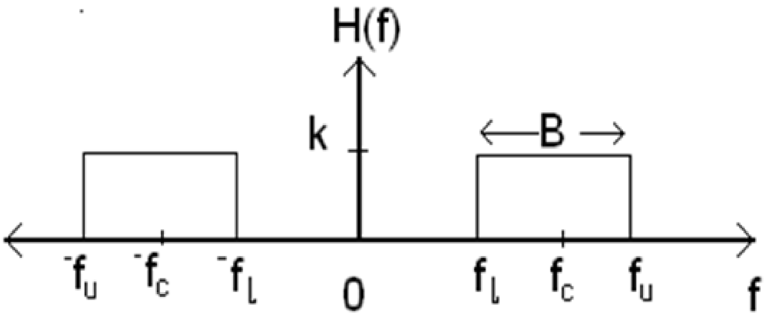
Band-pass filter

$$H(f) = \begin{cases} k\, e^{-j2\pi f t_d} & f_l < |f| < f_u \\ 0 & o.w \end{cases}$$

Filer bandwidth $B = f_u - f_l$; difference between upper and lower positive frequencies

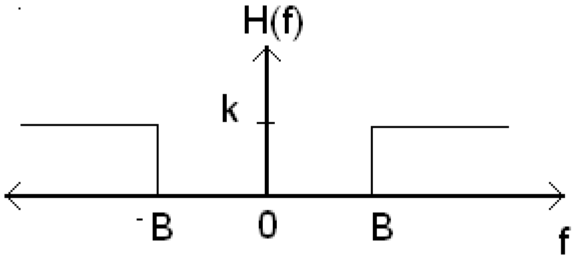
$f_c = \frac{f_u + f_l}{2}$; Center frequency of the filter

$h(t) = 2Bk\, sincB(t - t_d) cos\, w_c(t - t_d)$; impulse response.



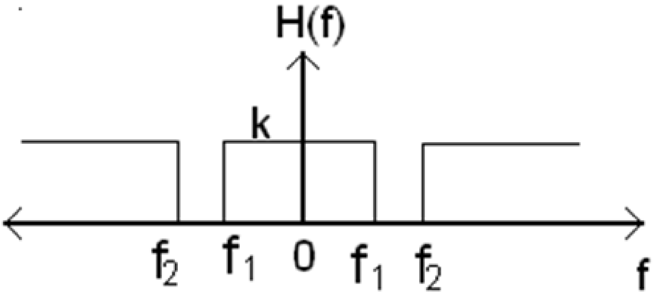
High-pass filter

$$H(f) = \begin{cases} k\, e^{-j2\pi f t_d} & |f| > B \\ 0 & o.w \end{cases}$$



Band rejection or notch filter

$$H(f) = \begin{cases} k\, e^{-j2\pi f t_d} & o.w \\ 0 & f_1 < |f| < f_2 \end{cases}$$



Real filters

Here, we only consider a Butterworth low pass filter. The transfer function of a low pass Butterworth filter is of the form.

$$H(f) = \frac{1}{P_n\left(\frac{jf}{B}\right)}$$

B is the 3-dB bandwidth of the filter and $P_n\left(\frac{jf}{B}\right)$ is a complex polynomial of order n. The family of Butterworth polynomials is defined by the property

$$\left(\left|P_n\left(\frac{jf}{B}\right)\right|\right)^2 = 1 + \left(\frac{f}{B}\right)^{2n}$$

So that

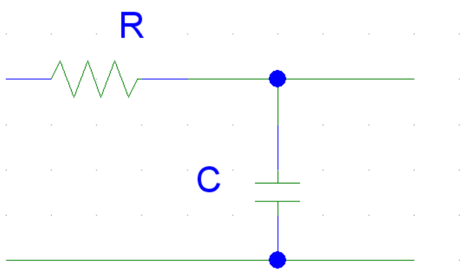
$$|H(f)| = \frac{1}{\sqrt{1+\left(\frac{f}{B}\right)^{2n}}}$$

The first few polynomials are:

$$P_1(x) = 1 + x$$
$$P_2(x) = 1 + \sqrt{2}x + x^2$$
$$P_3(x) = (1 + x)(1 + x + x^2)$$

A first order LPF

$$H(f) = \frac{\frac{1}{j2\pi f c}}{R + \frac{1}{j2\pi f c}} = \frac{1}{1 + j2\pi f RC}$$



Let $B = \frac{1}{2\pi RC}$

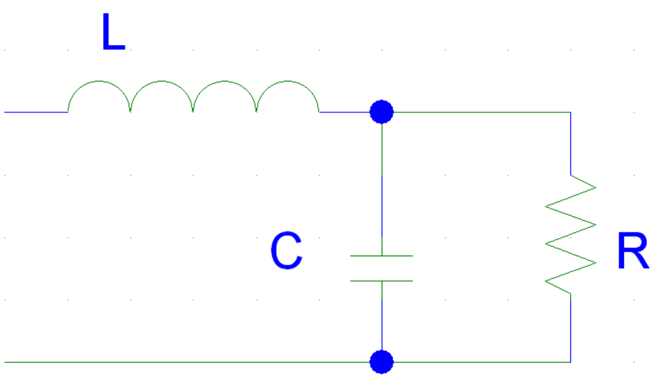
$$H(f) = \frac{1}{1 + jf/B} = \frac{1}{P_1(jf/B)} = \frac{1}{P_1(x)}$$

A second order LPF

$$H(f) = \frac{1}{1 + \frac{jwL}{R} - (2\pi\sqrt{LC} f)^2}$$

$$H(f) = \frac{1}{1 + j\sqrt{2} f/B - (f/B)^2}$$

$where\ R = \sqrt{\frac{L}{2C}},\ B = \frac{1}{2\pi\sqrt{LC}}$



$$H(f) = \frac{1}{1 + j\sqrt{2} f/B - (f/B)^2}$$

$$H(f) = \frac{1}{P_2(jf/B)}$$

Hilbert Transform

The quadrature filter is an all pass filter that shifts the phase of positive frequency by (-90°) and negative frequency by (+90°). The transfer function of such a filter is

$$H(f) = \begin{cases} -j & f > 0 \\ j & f < 0 \end{cases}$$

Using the duality property of Fourier transform, the impulse response of the filter is

$$h(t) = \frac{1}{\pi t}$$

The Hilbert transform is the output of the quadrature filter to the signal $g(t)$

$$\hat{g}(t) = \frac{1}{\pi t} * g(t) = \int_{-\infty}^{\infty} \frac{g(\lambda)}{\pi(t-\lambda)} d\lambda$$

Note that the Hilbert transform of a signal is a function of time (not frequency as in the case of the Fourier transform). The Fourier transform of $\hat{g}(t)$

$$\hat{G}(f) = -jsgn(f)G(f)$$

Hilbert transform can be found using either the time domain approach or the frequency domain approach depending on the given problem. That is

- Time-domain: Perform the convolution $\frac{1}{\pi t} * g(t)$.
- Frequency-domain: Find the Fourier transform $\hat{G}(f)$, then find the inverse Fourier transform

$$\hat{g}(t) = \int_{-\infty}^{\infty} \hat{G}(f)\, e^{j2\pi f t}\, df$$

Some properties of the Hilbert transform

1. A signal $g(t)$ and its Hilbert transform $\hat{g}(t)$ have the same energy spectral density

$$\left|\hat{G}(f)\right|^2 = |-j\, sgn(f)|G(f)||^2 = |-j\, sgn(f)|^2 |G(f)|^2 = |G(f)|^2$$

The consequences of this property are:

- If a signal g(t) is bandlimited, then $\hat{g}(t)$ is bandlimited to the same bandwidth (note that $\left|\hat{G}(f)\right| = |G(f)|$)
- $\hat{g}(t)$ and $g(t)$ have the same total energy (or power).
- $\hat{g}(t)$ and $g(t)$ have the same autocorrelation function.

2. A signal g(t) and $\hat{g}(t)$ are orthogonal, i.e.,

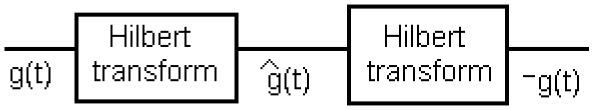
$$\int_{-\infty}^{\infty} g(t)\, \hat{g}(t) dt = 0$$

This property can be verified using the general formula of Rayleigh energy theorem

$$\int_{-\infty}^{\infty} \mathrm{g(t)}\, \hat{g}(t) dt = \int_{-\infty}^{\infty} \mathrm{G(f)}\, \hat{G}^*(f) df = \int_{-\infty}^{\infty} \mathrm{G(f)}\, \{-jsgn(f)\, G(f)\}^* df$$
$$= \int_{-\infty}^{\infty} jsgn(f)\, |G(f)|^2 df = 0.$$

The result above follows from the fact that $|G(f)|^2$ is an even function of f while $sgn(f)$ is an odd function of f. Their product is odd. The integration of an odd function over a symmetrical interval is zero.

3. If $\hat{g}(t)$ is a Hilbert transform of $g(t)$, then the Hilbert transform of $\hat{g}(t)$ is $-g(t)$ (each Hilbert transform introduces 90 degrees phase shift).



Example on Hilbert transform

Find the Hilbert transform of the impulse function $g(t) = \delta(t)$

Solution: Here, we use the convolution in the time domain

$$\hat{g}(t) = \frac{1}{\pi t} * \delta(t)$$

As we know, the convolution of the delta function with a continuous function is the function itself. Therefore,

$$\hat{g}(t) = \frac{1}{\pi t}.$$

Example on Hilbert transform

Find the Hilbert transform of $g(t) = \frac{\sin t}{t}$

Solution

Here, we will first find the Fourier transform of $g(t)$, find $\hat{G}(f)$, and then find $\hat{g}(t)$.

$$A\,rect\left(\frac{t}{\tau}\right) \xleftrightarrow{transform} A\tau\, sinc\, f\tau \quad ; \quad when\ \tau = \frac{1}{\pi}$$

$$A\,\mathrm{rect}\left(\frac{t}{1/\pi}\right) \xleftrightarrow{transform} A\,\frac{1}{\pi}\frac{\sin \pi f\tau}{\pi f\tau} = \frac{1}{\pi}\frac{\sin f}{f}$$

$$\pi\, rect\left(\frac{t}{1/\pi}\right) \xleftrightarrow{transform} \frac{\sin f}{f}$$

So, by the duality property, we get the pair

$$\pi\, rect\left(\frac{f}{1/\pi}\right) \xleftrightarrow{transform} \frac{\sin t}{t}$$

i.e. , $G(f) = \pi\, rect(\frac{f}{1/\pi})$, (See the figure below)

$$\hat{G}(f) = -jsgn(f)G(f) = \begin{cases} -j\pi & 0 < f < 1/2\pi \\ j\pi & -1/2\pi < f < 0 \end{cases}$$

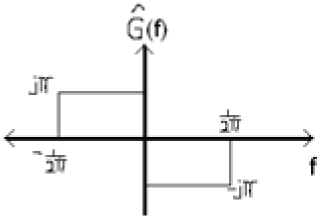
$$\hat{g}(t) = \int_{-\infty}^{\infty} \hat{G}(f)\, e^{j2\pi f t}\, df$$

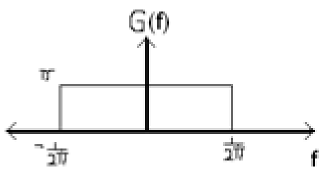
$$= \int_{-1/2\pi}^{0} j\pi \;\; e^{j2\pi f t}\, df \; - \; \int_{0}^{1/2\pi} j\pi \;\; e^{j2\pi f t}\, df$$

$$= \frac{1}{2t}\left(1 - e^{-jt}\right) - \frac{1}{2t}\left(e^{jt} - 1\right)$$

$$= \frac{1}{t} - \frac{1}{t}\,\frac{(e^{jt} + e^{-jt})}{2}$$

$$= \frac{1 - \cos t}{t}$$





Correlation and Spectral Density

Here, we consider the relationship between the autocorrelation function and the power spectral density. In this discussion, we restrict our attention to real signals. First, we consider power signals and then energy signals.

Definition: The autocorrelation function of a signal $g(t)$ is a measure of similarity between $g(t)$ and a delayed version of $g(t)$.

a. Autocorrelation function of a power signal

The autocorrelation function of a power signal $g(t)$ is defined as

$R_g(\tau) = \langle g(t)g(t-\tau)\rangle$; Where $\langle . \rangle$ denotes time average.

$$R_g(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} g(t)g(t-\tau)dt$$

Exercise: Show that for a periodic signal with period T_0, the above definition becomes

$$R_g(\tau) = \frac{1}{T_0}\int_0^{T_0} g(t)g(t-\tau)dt$$

Remark: We can take this as a definition for the autocorrelation function of a periodic signal.

Exercise: Show that if $g(t)$ is periodic with period T_0, then $R_g(\tau)$ is also periodic with the same period T_0.

Hint: Expand $g(t)$ in a complex Fourier series $g(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0\tau}$. Form the delayed signal $g(t-\tau)$, and then perform the integration over a complete period T_0. You should get the following result:

$R_g(\tau) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0\tau}$; Fourier series expansion of $R_g(\tau)$.

$D_n = |C_n|^2$ are the Fourier series coefficients of $R_g(\tau)$

C_n are the Fourier series coefficients of $g(t)$.

This formula bears two conclusions

a. $R_g(\tau)$ is periodic with period T_0.
b. The Complex Fourier coefficients D_n of $R_g(\tau)$ are related to the complex Fourier coefficients C_n of $g(t)$ by the relation $D_n = |C_n|^2$.

Properties of R(τ)

- $R_g(\tau = 0) = \frac{1}{T_0}\int_0^{T_0} g(t)^2 dt$; is the total average signal power.

- $R_g(\tau)$ is an even function of τ, i.e., $R_g(\tau) = R_g(-\tau)$.
- $R_g(\tau)$ has a maximum (positive) magnitude at τ = 0 , i.e. $|R_g(\tau)| \le R_g(0)$.
- If g(t) is periodic with period T_0, then $R_g(\tau)$ is also periodic with the same period T_0.
- The autocorrelation function of a periodic signal and its power spectral density (represented by a discrete set of impulse functions) are Fourier transform pairs

$$S_g(f) = \Im\{R_g(\tau)\}$$

$$S_g(f) = \sum_{n=-\infty}^{\infty} |C_n|^2 \delta(f - nf_0); \text{ Discrete spectrum}$$

Cross correlation function
The cross correlation function of two periodic signals $g_1(t)$ and $g_2(t)$ with the same period T_0 is defined as

$$R_{1,2}(\tau) = \frac{1}{T_0}\int_0^{T_0} g_1(t)g_2(t-\tau)dt$$

b- **Autocorrelation function of an energy signal**

When g(t) is an energy signal, $R_g(\tau)$ is defined as

$$R_g(\tau) = \int_{-\infty}^{\infty} g(t)g(t-\tau)dt$$

Properties of R(τ)

- $R_g(\tau = 0) = \int_{-\infty}^{\infty} g(t)^2 dt$; is the total signal energy.
- $R_g(\tau)$ is an even function of τ, i.e., $R_g(\tau) = R_g(-\tau)$.
- $R_g(\tau)$ has a maximum (positive) magnitude at τ = 0 , i.e. $|R_g(\tau)| \le R_g(0)$.
- The autocorrelation function of an energy signal and its energy spectral density (a continuous function of frequency) are **Fourier transform pairs**, i.e.,

$$S_g(f) = \Im\{R_g(\tau)\}$$

$$S_g(f) = \int_{-\infty}^{\infty} R_g(\tau)e^{-j2\pi f\tau}d\tau; \text{ Continuous spectrum.}$$

$$R_g(\tau) = \int_{-\infty}^{\infty} S_g(f)e^{j2\pi f\tau}df.$$

Proof:
The autocorrelation function is defined as:

$$R_g(\tau) = \int_{-\infty}^{\infty} g(\lambda)g(\lambda-\tau)d\lambda$$

In this integral we have replaced t by λ (both are dummy variables of integration). With this substitution, we can rewrite the integral as

$$R_g(\tau) = \int_{-\infty}^{\infty} g(\lambda)g(-(\tau-\lambda))d\,\lambda$$

One can realize that $R_g(\tau)$ is nothing but the convolution of $g(\tau)$ and $-g(\tau)$. That is,

$$R_g(\tau) = g(\tau) * g(-\tau)$$

Taking the Fourier transform of both sides, we get

$$F\{R_g(\tau)\} = G(f)G^*(f)$$

Therefore, $S_g(f) = \Im\{R_g(\tau)\} = |G(f)|^2$.

Cross correlation function

The cross correlation function of two energy signals $g_1(t)$ and $g_2(t)$ is defined as;

$$R_{1,2}(\tau) = \int_{-\infty}^{\infty} g_1(t)g_2(t-\tau)dt$$

Example: Autocorrelation of a periodic sinusoidal signal

Find the auto-correlation function of the sine signal $g(t) = A\cos(2\pi f_0 t + \theta)$, where A and θ are constants.

Solution

As we know, $g(t)$ is a periodic signal. Therefore, we find $R_g(\tau)$ using the definition

$$R_g(\tau) = \frac{1}{T_0}\int_0^{T_0} g(t)g(t-\tau)dt$$

$$R_g(\tau) = \frac{1}{T_0}\int_0^{T_0} A\cos(2\pi f_0 t + \theta)A\cos(2\pi f_0 t - 2\pi f_0 \tau + \theta)dt$$

$$R_g(\tau) = \frac{A^2}{2T_0}\int_0^{T_0} [\cos(4\pi f_0 t - 2\pi f_0 \tau + 2\theta) + \cos(2\pi f_0 \tau)]dt$$

$$R_g(\tau) = \frac{A^2}{2T_0}[0 + \cos(2\pi f_0 \tau)T_0]$$

$R_g(\tau) = \frac{A^2}{2}\cos(2\pi f_0 \tau)$; Periodic with period T_0.

Example: Autocorrelation of a non-periodic signal

Determine the autocorrelation function of the sinc pulse

$$g(t) = Asinc2Wt.$$

Solution:

Using the duality property of the Fourier transform, we can deduce that

$$G(f) = \frac{A}{2W} rect(\frac{f}{2W})$$

The energy spectral density of $g(t)$ is

$$S_g(f) = |G(f)|^2 = (\frac{A}{2W})^2 rect(\frac{f}{2W})$$

Taking the inverse Fourier transform, we get the autocorrelation function

$$R_g(\tau) = \frac{A^2}{2W} sinc 2W\tau$$

Exercise:

a. Find and plot the cross correlation function of the two signals

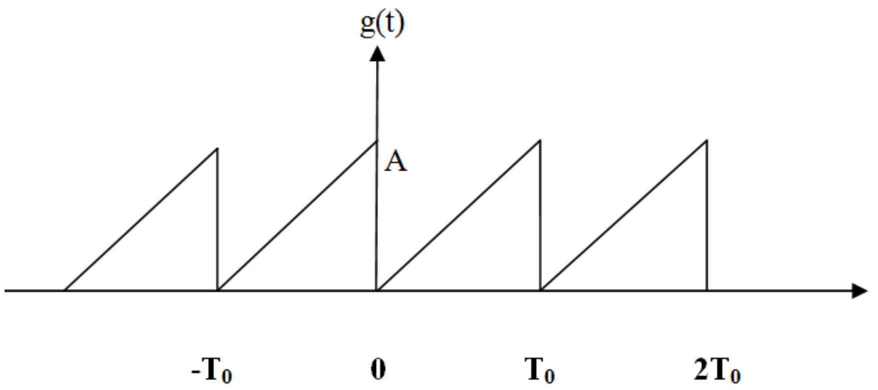
$$g_1(t) = \begin{cases} 1 & 0 \le t \le 2 \\ 0, & otherwise \end{cases}$$

$$g_2(t) = \begin{cases} 1 & 0 \le t \le 1 \\ -1 & 1 < t \le 2 \end{cases}$$

b. Are $g_1(t)$ and $g_2(t)$ orthogonal?

Exercise:

Find and plot the autocorrelation function for the periodic saw-tooth signal shown below:



Example:

Find the autocorrelation function of the rectangular pulse

$$g(t) = rect\left(\frac{t-0.5T}{T}\right),\ T = 1.$$

Solution:

As we saw earlier, this pulse is an energy signal and therefore, we can find its $R_g(\tau)$ as:

$$R_g(\tau) = \int_\tau^1 (A)(A)dt = A^2\,(1-\tau)\ ;\ 0 < \tau < 1$$

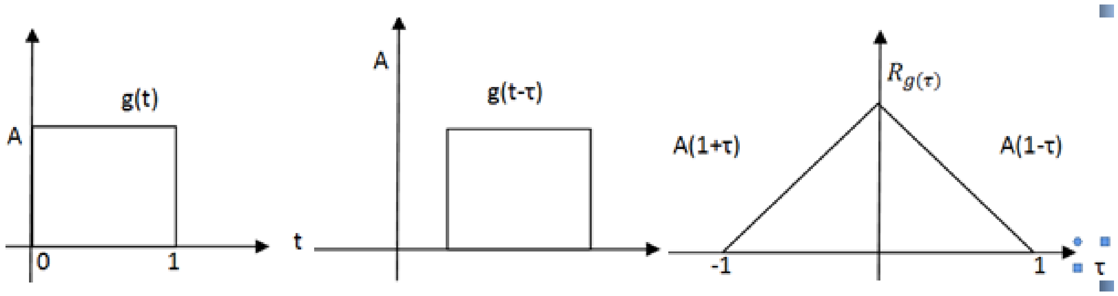
Using the even symmetry property of the autocorrelation function, we can find $R_g(\tau)$ for – ve values of τ as:

$$R_g(\tau) = = A^2\,(1+\tau)\ ;\ -1 < \tau < 0$$

This function is sketched below. Note that that the maximum value occurs at τ = 0 and that g(t) and g(t-τ) become uncorrelated for τ = 1 sec, which is the duration of the pulse.

The energy spectral density is

$$S_g(f) = \Im\{R_g(\tau)\} = A^2((sincf)^2)$$



Bandwidth of Signals and Systems

Definition: The amount of positive frequency spectrum that a signal $g(t)$ occupies is called the bandwidth of the signal.

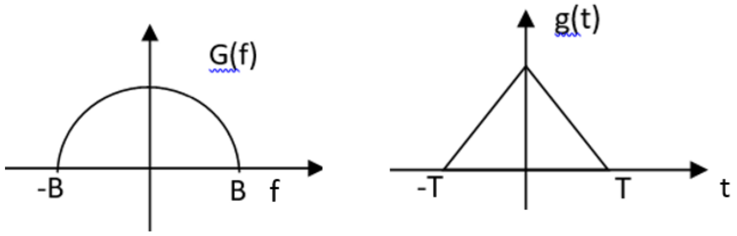
Definition: A signal g(t) is said to be (absolutely) band-limited to B Hz if

$$G(f) = 0 \qquad \text{for } |f| > B$$

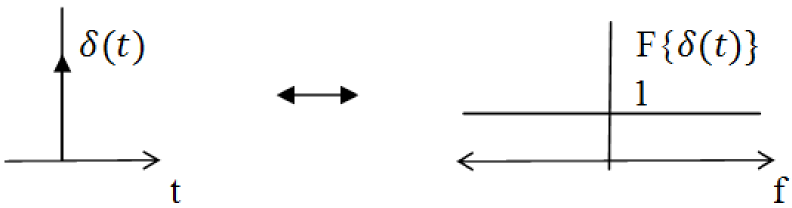
Definition: A signal g(t) is said to be (absolutely) time-limited if

$$g(t) = 0 \qquad \text{for } |t| > T$$

Theorem: An absolutely band-limited waveform cannot be absolutely time-limited and vice versa, i.e., a signal g(t) cannot be both time-limited and bandlimited.



We have earlier seen examples that support this theorem. For example, the delta function, which has an almost zero time duration, has a Fourier transform which extends uniformly over all frequencies (infinite bandwidth). Also, a constant value in the time domain (a dc) has a Fourier transform, which is an impulse in the frequency domain. This is repeated here for convenience.



In general, there is an inverse relationship between the signal bandwidth and the time duration. The bandwidth and the time duration are related through a relation, called the *time bandwidth product*, of the form

$$(\textbf{Bandwidth})(\textbf{Time Duration}) \geq \textbf{Constant}$$

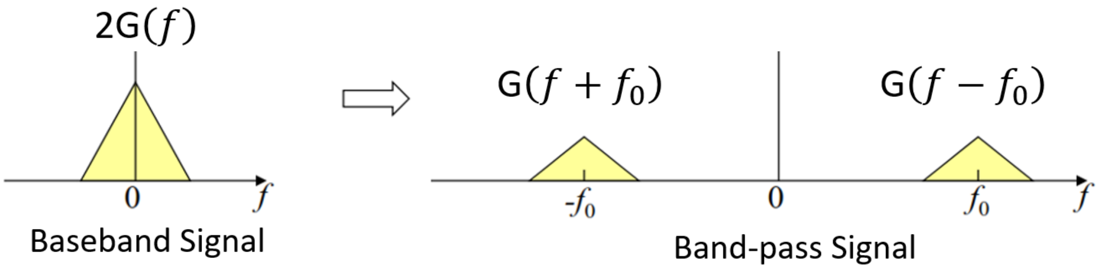
The value of the constant depends on the way we define the bandwidth and the time duration. Two possible values of the constant, that we will encounter in this chapter, are $\frac{1}{2}$ (for the equivalent rectangular bandwidth) and $\frac{1}{4\pi}$ (for the root mean square bandwidth).

Remarks:

1. The bandwidth of a signal provides a measure of the extent of significant frequency content of the signal.
2. The bandwidth of a signal is taken to be the width of a positive frequency band.
3. For baseband signals or networks, where the spectrum extends from –B to B, the bandwidth is taken to be B Hz.
4. For bandpass signals or systems where the spectrum extends between (f_1, f_2) and ($-f_1$, $-f_2$), the B.W = $f_2 - f_1$.

Definition of a baseband signal: A baseband signal is one for which most of the energy is contained within a band centered around the zero frequency and negligible elsewhere. Another term synonymous with baseband is ***low-pass***. In the communication systems, the message to be transmitted is a baseband signal.

Definition of a band-pass signal: A band-pass signal is one for which the energy is concentrated around some high frequency carrier f_0 and negligible elsewhere. This type of signal will arise in this course when the baseband message signal $m(t)$ modulates a high frequency carrier $c(t)$ to produce the modulated signal $s(t)$.

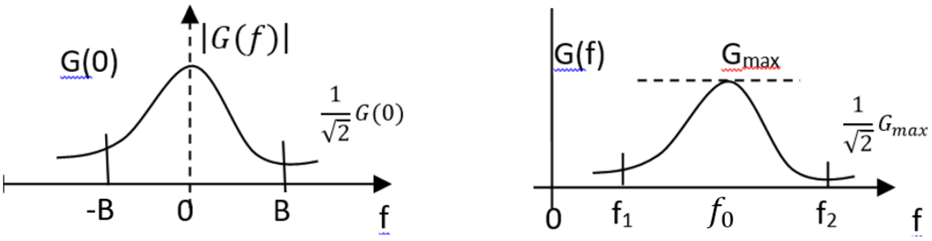


Some definitions of bandwidth

1- Absolute bandwidth
Here, the Fourier transform of a signal is non-zero only within a certain frequency band. If G(f) = 0 for |f| > B, then g(t) is absolutely band-limited to BHz. When G(f) ≠ 0 for $f_1 < |f| < f_2$, then the absolute bandwidth is $f_2 - f_1$.

2- 3-dB (half power points) bandwidth
The range of frequencies from 0 to some frequency B at which |G(f)| drops to $\frac{1}{\sqrt{2}}$ of its maximum value (for a low pass signal). As for a band pass signal, the B.W = $f_2 - f_1$.

3- The 95 % (energy or power) bandwidth.
Here, the B.W is defined as the band of frequencies where the area under the energy spectral density (or power spectral density) is at least 95% (or 99%) of the total area.



Total Signal Energy $E = \int_{-\infty}^{\infty} |G(f)|^2\, df = 2\int_0^{\infty} |G(f)|^2 df$

The 95% energy bandwidth B should satisfy the relationship

$$\int_{-B}^{B} |G(f)|^2 df = 0.95 \int_{-\infty}^{\infty} |G(f)|^2\, df = 0.95\, E$$

4- Equivalent rectangular bandwidth.
It is the width of a fictitious rectangular spectrum such that the power in that rectangular band is equal to the energy associated with the actual spectrum over positive frequency. Let B_{eq} be the equivalent rectangular bandwidth. To find B_{eq} we set

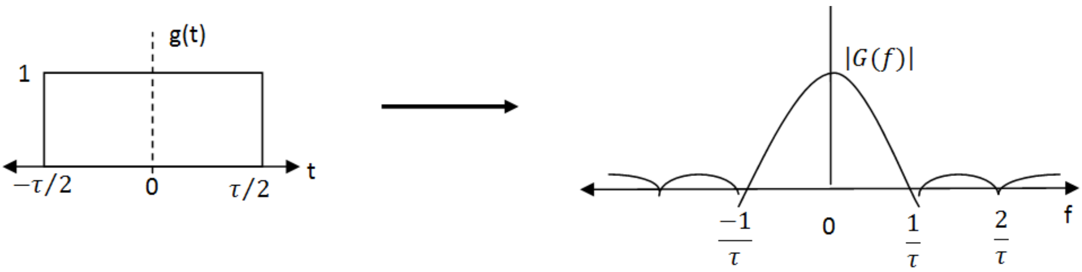
Area under fictitious rectangle = Total Signal Energy E

$$|G(0)|^2 * 2B_{eq} = \int_{-\infty}^{\infty} |G(f)|^2 df = E$$

$$|G(0)|^2 * 2B_{eq} = 2\int_0^{\infty} |G(f)|^2 df$$

$$B_{eq} = \frac{1}{|G(0)|^2} \int_0^{\infty} |G(f)|^2 df$$

5- <u>Null – to – null bandwidth</u>
For baseband signals, the null bandwidth is taken to be the band from zero to the first null in the envelope of the magnitude spectrum.



For example, consider the rectangular pulse g(t), for which the Fourier transform is G(f). Note that

$$rect\left(\frac{t}{\tau}\right) \rightarrow \tau\ \mathrm{sinc} f\tau = \tau \frac{\sin \pi f\tau}{\pi f\tau}.$$

The zero crossings occur when sin(πfτ) = 0

$\pi f\tau = n\pi \;\rightarrow\; f = \frac{n}{\tau}$; n = 1,2, … The smallest value of $n = 1$, gives

$Null\ Bandwidth = \frac{1}{\tau}$.

For a band pass signal, B.W= $f_2 - f_1$

6- Bounded spectrum bandwidth

Range of frequencies (0, B) such that outside the band, the power spectral density must be down by say 50 dB below the maximum value.

$$-50\ \text{dB} = 10 \log \frac{|G(B)|^2}{|G(0)|^2}.$$

7- RMS bandwidth

$$B_{rms} = \sqrt{\left(\frac{\int_{-\infty}^{\infty} f^2 |G(f)|^2 \mathrm{df}}{\int_{-\infty}^{\infty} |G(f)|^2 \mathrm{df}}\right)}$$

The corresponding rms duration of g(t) is

$$T_{rms} = \sqrt{\left(\frac{\int_{-\infty}^{\infty} t^2 |g(t)|^2 \mathrm{dt}}{\int_{-\infty}^{\infty} |g(t)|^2 \mathrm{dt}}\right)}$$

(here g(t) is assumed to be centered around the origin).

Remark: The time bandwidth product is $(T_{rms})(B_{rms}) \geq \frac{1}{4\pi}$ (the proof is beyond the scope of this presentation).

Time – bandwidth product

To illustrate the time – bandwidth product, consider the equivalent rectangular bandwidth defined earlier as

$$B_{eq} = \frac{\int_{-\infty}^{\infty} |G(f)|^2 \mathrm{df}}{2|G(0)|^2}$$

Analogous to this definition, we define an equivalent rectangular time duration as

$$T_{eq} = \frac{\left(\int_{-\infty}^{\infty} |g(t)| dt\right)^2}{\int_{-\infty}^{\infty} |g(t)|^2 \mathrm{dt}}$$

The time bandwidth product is

$$B_{eq}T_{eq} = \left(\frac{\int_{-\infty}^{\infty} |G(f)|^2 \mathrm{df}}{2|G(0)|^2}\right)\left(\frac{\left(\int_{-\infty}^{\infty} |g(t)| dt\right)^2}{\int_{-\infty}^{\infty} |g(t)|^2 \mathrm{dt}}\right)$$

Note that $\int_{-\infty}^{\infty} |g(t)|^2\, \mathrm{dt} = \int_{-\infty}^{\infty} |G(f)|^2\, \mathrm{df}$; Rayleigh energy theorem.

Note also that $G(0) = \int_{-\infty}^{\infty} g(t)dt.$

Using these two relations, we get

$$B_{eq}T_{eq} = \frac{1}{2}\frac{\left(\int_{-\infty}^{\infty}|g(t)|dt\right)^2}{|\int_{-\infty}^{\infty} g(t)\mathrm{dt}|^2}$$

Case 1: When g(t) is positive for all time t, then |g(t)| = g(t) and $B_{eq}T_{eq}$ becomes

$$\boldsymbol{B_{eq}T_{eq} = \frac{1}{2}}$$

Case 2 : For a general g(t) that can take on positive as well as negative values, $B_{eq}T_{eq}$ satisfies the inequality

$$\boldsymbol{B_{eq}T_{eq} \geq \frac{1}{2}}$$

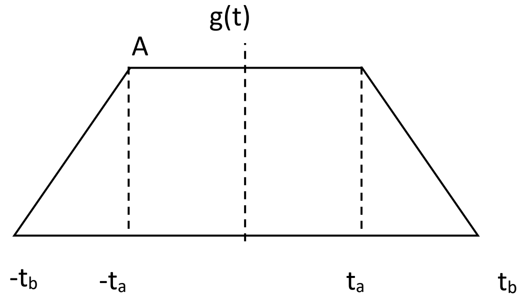
Note : For B_{rms} and T_{rms} , the time – bandwidth product satisfies the inequality

$$\boldsymbol{B_{rms}T_{rms} \geq \frac{1}{4\pi}}$$

Example: bandwidth of a trapezoidal signal

Find the equivalent rectangular bandwidth, B_{eq}, for the trapezoidal pulse shown.

Solution



$$T_{eq} = \frac{\left(\int_{-\infty}^{\infty}|g(t)|dt\right)^2}{\int_{-\infty}^{\infty}|g(t)|^2\mathrm{dt}}$$

$$\int_{-\infty}^{\infty} |g(t)|dt = A\,(t_a + t_b)$$

$$\int_{-\infty}^{\infty}|g(t)|^2 dt = \frac{2A^2}{3}\,(2t_a + t_b)$$

$$T_{eq} = \frac{3}{2}\frac{(t_a+t_b)^2}{(2t_a+t_b)}$$

$$B_{eq} = \frac{0.5}{T_{eq}} = \frac{2t_a+t_b}{3(t_a+t_b)^2}.$$

Remark: Note that using this method we were able to determine the signal bandwidth without the need to go through the Fourier transform.

Exercise: Use the above method to find the equivalent rectangular bandwidth for the triangular signal $g(t) = tri(\frac{t}{T})$.

Example: bandwidth of a periodic signal

Find the 93% power bandwidth for the periodic square function define over one period as

$$g(t) = \begin{cases} 2A, & \frac{-T_0}{4} \le t \le \frac{T_0}{4} \\ -A, & o.w \end{cases}$$

Solution: The average power, computed using the time average, is

$$P_{av} = \frac{1}{T_0}\int_0^{T_0} |g(t)|^2 \; dt$$

$$= \frac{1}{T_0}\left[4A^2\frac{T_0}{2} + A^2\frac{T_0}{2}\right] = \frac{5A^2T_0}{2T_0} = \frac{5A^2}{2} = 2.5A^2$$

Also, by using the Parseval's theorem, the average power can be computed as:

$$P_{av} = |C_0|^2 + 2\sum_{n=1}^{\infty}|C_n|^2$$

We recall that the Fourier coefficients for this signal were found in Chapter 1. Using these values, we get

$$P_{av} = \left(\frac{A}{2}\right)^2 + 2\sum_{n=1}^{\infty}\frac{(3A)^2}{(n\pi)^2}$$

$$P_{av} = \frac{A^2}{4} + 2A^2\sum_{n=1}^{\infty}\frac{(3)^2}{(n\pi)^2}$$

Let us take n = 1, then the power in the DC and the fundamental frequency is

$$P_1 = A^2\left\{0.25 + 2\left(\frac{9}{\pi^2}\right)\right\} = 2.073A^2 \Rightarrow \frac{P_1}{P_{av}} = \frac{2.073A^2}{2.5A^2} = 82.95\%$$

The fraction of power in these two terms relative to the total average power is only 82.95%. The 93% power limit is not yet reached. So, let us add one more term.

When n = 3, the power in the DC, the fundamental term, and the third harmonic is

$$P_3 = A^2\left\{0.25 + 2\left(\frac{3^2}{\pi^2} + \frac{3^2}{3^2\pi^2}\right)\right\} = 2.276A^2 \Rightarrow \frac{P_3}{P_{av}} = \frac{2.276A^2}{2.5A^2} = 91.05\%.$$

The fraction of power in these three terms relative to the total average power is now 91.05%. Still, the 93% power limit is not reached yet. So, let us add one more term.

For n = 5, the power in the DC, the fundamental term, the third harmonic, and the fifth harmonic is

$$P_5 = A^2\left\{0.25 + 2\left(\left(\frac{3}{\pi}\right)^2 + \left(\frac{3}{3\pi}\right)^2 + \left(\frac{3}{5\pi}\right)^2\right)\right\} = 2.349A^2$$

$$\Rightarrow \frac{P_5}{P_{av}} = \frac{2.349A^2}{2.5A^2} = 93.97\%.$$

With n=5, the 93% power limit has been reached. Therefore, the 93% power bandwidth is $\boldsymbol{B_{93\%} = 5f_0}$.

Example: bandwidth of an energy signal

Find the 95% energy bandwidth for the exponential pulse $g(t) = Ae^{-\alpha t}\ u(t)$.

Solution: The Fourier transform of g(t) is

$$G(f) = \frac{A}{\alpha + j2\pi f}$$

The total energy in g(t) (calculated in the time domain) is

$$E_g = \int_0^\infty |g(t)|^2\, dt = \int_0^\infty A^2 e^{-2\alpha t}\, dt = \frac{A^2}{2\alpha}$$

Let B be the 95% energy bandwidth, then the energy contained within B is

$$E_B = \int_{-B}^{B} |G(f)|^2\, df = \int_{-B}^{B} \frac{A^2}{(\alpha^2 + (2\pi f)^2)}\, df$$

$$E_B = \frac{2A^2}{2\pi\alpha} tan^{-1}\frac{2\pi B}{\alpha}$$

B should be chosen such that it satisfies the condition

$$E_B = 0.95 E_g$$

$$\frac{2A^2}{2\pi\alpha} tan^{-1}\frac{2\pi B}{\alpha} = 0.95\left(\frac{A^2}{2\alpha}\right)$$

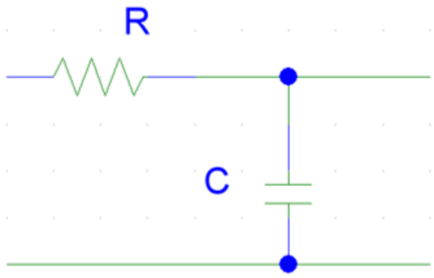
The table below shows the ratio E_B/E_g for various values of B.

B	$(E_B/E_g) \times 100\%$
$\alpha/4$	63.9%
$\alpha/2$	80.38%
α	89.95%
$\mathbf{2\alpha}$	**94.94%**

Thus, the 95% energy bandwidth is $B_{95\%} = 2\alpha$

Example: 3-dB bandwidth of a first order RC low pass filter

Consider the RC circuit shown in the figure



The transfer function of the circuit is

$$H(f) = \frac{\frac{1}{j2\pi fC}}{R + \frac{1}{j2\pi fC}} = \frac{1}{1 + j2\pi fRC}$$

The magnitude of $H(f)$ is

$$|H(f)| = \frac{1}{\sqrt{1 + (2\pi fRC)^2}}$$

The 3-dB bandwidth is some frequency $f = B$ at which $|H(f)|$ drops to $1/\sqrt{2}$ of its maximum value. Note that the maximum value of $|H(f)|$ is 1 and occurs at $f = 0$. Therefore, B should satisfy

$$|H(B)| = \frac{1}{\sqrt{1 + (2\pi BRC)^2}} = \frac{1}{\sqrt{2}}$$

From this relationship, we notice that B occurs when

$$2\pi BRC = 1$$

Therefore,

$$\boldsymbol{B = \frac{1}{2\pi RC}}$$

Pulse Response and Rise-time

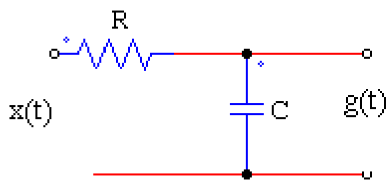
A rectangular pulse contains significant high frequency components. When that pulse is passed through a LPF, the high frequency components will be attenuated resulting in signal distortion.

We need to investigate the relationship that should exist between the pulse bandwidth and the channel bandwidth. This subject is of particular importance, especially, when we study the transmission of data over band-limited channels. In the simplest form, a binary digit 1 may be represented by a pulse , $0 \le t \le T_b$, while binary digit 0 may be represented by the negative pulse $-A,\ \ 0 \le t \le T_b$. Therefore, in order to retrieve the transmitted data, the channel bandwidth must be wide enough to accommodate the transmitted data.

To convey this idea in a simple form, we first consider the response of a first order low pass filter to a unit step function and then to a pulse.

Step response of a first order LPF (channel)

Let $x(t) = u(t)$ be applied to a first order RC circuit. This first order filter is a fair representation of a low-pass communication channel



The system differential equation (D.E.) is

$$x(t) = Ri(t) + g(t) = RC\frac{dg(t)}{dt} + g(t)$$

where g(t) is the channel output. Now let $x(t) = u(t)$. The system D.E. becomes $x(t)$

$$RC\frac{dg(t)}{dt} + g(t) = u(t)$$

The solution to this first order system is

$$g(t) = (1 - e^{-t/RC})u(t)$$

The 3- dB bandwidth of the channel (was derived in a previous example in this chapter) is

$$B_{ch} = \frac{1}{2\pi RC}$$

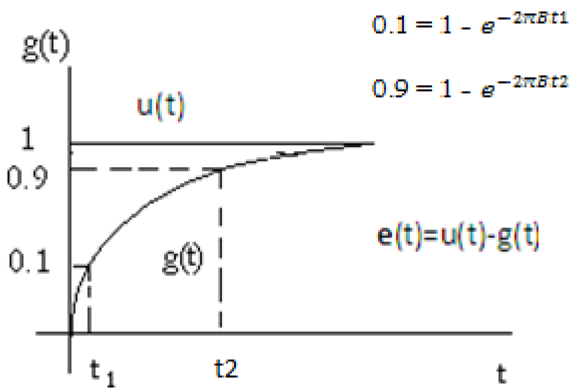
The output $g(t)$, expressed in terms of B_{ch} becomes

$$g(t) = (1 - e^{-2\pi B_{ch}t})u(t)$$

Define the difference between the input and the output as

$$e(t) = u(t) - g(t) = e^{-2\pi B_{ch}t}$$

Note that $e(t)$ decreases as B_{ch} increases. This means that as the channel bandwidth increases, the output becomes closer and closer to the input. In the ideal case, when the channel bandwidth becomes infinity, the output becomes a step function. In essence, to reproduce a step function (or a rectangular pulse), a channel with infinite bandwidth is needed.



The rise-time

The rise-time is a measure of the speed of rise of the output of a system due to step function applied at its input. One common measure is the 10-90 % rise-time, defined as the time it takes for the output to rise between 10% to 90% of the final steady state value when a unit step function is applied to the system input. The 10% - 90% rise-time for the first order RC circuit considered above is

$$\boldsymbol{T_r = t_2 - t_1 = \frac{0.35}{B_{ch}}}$$

From this result, we conclude that **increasing the bandwidth of the channel will decrease the rise-time**, implying a faster response.

Exercise: For the system above, verify that the rise-time is given as $T_r = \frac{0.35}{B_{ch}}$.

Pulse Response

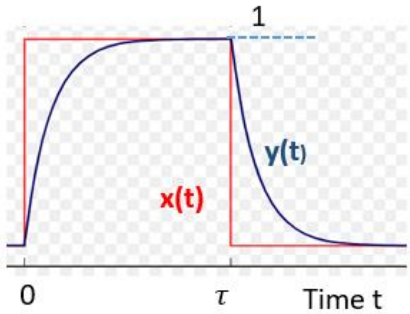
It is the response of the circuit to a pulse of duration τ. For the same RC circuit, considered above, let us apply the pulse

$$x(t) = u(t) - u(t-\tau)$$

Using the linearity and time invariance properties, the output due to the pulse can be obtained from the step response as

$$y(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-2\pi B_{ch} t} & 0 < t < \tau \\ (1 - e^{-2\pi B_{ch}\tau})e^{-2\pi B_{ch}(t-\tau)} & t > \tau \end{cases}$$

This response is sketched in the figure below.



From the equation above, we observe that the output $y(t)$ approximates the input $x(t)$ provided that

$$B_{ch}\tau \geq 1 \text{ or}$$

$$\boldsymbol{B_{ch} \geq \frac{1}{\tau}}$$

Relationship to data transmission

In digital communication systems, data are transmitted at a rate of R_b bits/sec. The time allocated for each bit is $\tau = \frac{1}{R_b}$. To enable the receiver to recognize the transmitted bit within its allocated slot and to prevent cross talk between neighboring time slots, we require that

$$\boldsymbol{B_{ch} \geq \frac{1}{\tau} = R_b}$$

Result: the channel bandwidth in binary digital communication systems should be larger than the rate of the data sent over the channel.

Band-pass Signals and Systems

(Details are not required for ENEE 339)

A signal g(t) is called a *band pass signal* if its Fourier transform G(f) is non-negligible only in a band of frequencies of total extent 2W centered about f_c .

A signal is called *narrowband* if 2W is small compared with f_c.

A band pass signal g(t) can be represented in the canonical form

$$g(t) = g_I(t) \cos \omega_c t - g_Q(t) \sin\omega_c t.$$

$g_I(t)$ is a low pass signal of B.W = W Hz called the *in phase component* of $g(t)$.

$g_Q(t)$ is a low pass signal of B.W = W Hz called the *quadrature component* of $g(t)$

g(t) appears as a modulated signal in which $g_I(t)$ and $g_Q(t)$ are the low pass signals and f_c is the carrier frequency. Recall the modulation property of the Fourier transform

$$x(t) \cos \omega_c t \rightarrow \frac{1}{2}(X(f- f_c)+ X(f+ f_c))$$

$$x(t) \sin \omega_c t \rightarrow \frac{1}{j2}(X(f- fc)- X(f+ fc))$$

Define the *complex envelope* of a signal g(t) as:

$$\tilde{g}(t) = g_I(t) + j\, g_Q(t)$$

$\tilde{g}(t)$ is a low pass signal of B.W =W. The signals g(t) and $\tilde{g}(t)$ are related by :

$$g(t) = \text{Re}\{\tilde{g}(t)\, e^{j\omega ct} \}$$

How to get $g_I(t)$ and $g_Q(t)$ from g(t)

If we multiply g(t) by cos $\omega_c t$, we get

$$g(t) \cos\omega_c t = g_I(t) \cos^2 \omega_c t - g_Q(t) \sin \omega_c t \cos \omega_c t$$

$$= \frac{1}{2} g_I(t) + \frac{1}{2} g_I(t) \cos 2\omega_c t - \frac{1}{2} g_Q(t) \sin 2\omega_c t .$$

The first term is the desired low pass signal. The second and third terms are high frequency components centered about 2 f_c.

$$g_I(t) = \text{lowpass}\{2g(t) \cos\omega_c t\}$$

Or, in the frequency domain

$$G_I(f) = \begin{cases} G(f - fc) + G(f + fc) & -w \leq f \leq w \\ 0 & otherwise \end{cases}$$

Now if we multiply g(t) by $\sin\omega_c t$, we get

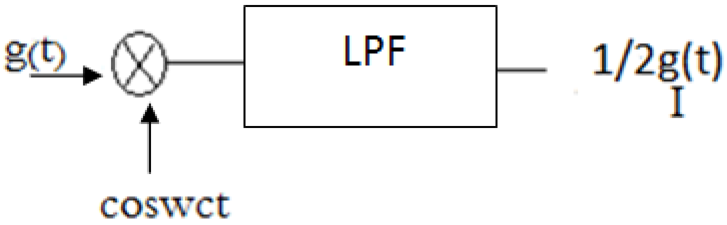
$$g(t)\sin\omega_c t = g_I(t)\sin\omega_c t\cos\omega_C t - g_Q(t)\sin^2\omega_c t$$

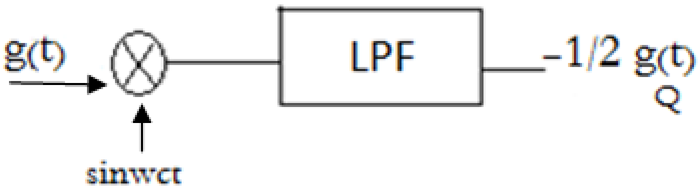
$$= -\frac{1}{2}g_Q(t) + \frac{1}{2}g_I(t)\sin 2\omega_c t + \frac{1}{2}g_Q(t)\cos 2\omega_c t$$

Again, the first term is a low pass signal, while the second and third are high frequency terms centered about 2 f_c.

$g_Q(t)$ = - low pass{$2g(t)\sin\omega_c t$}In the frequency domain, this is equivalent to

$$G_Q(f) = \begin{cases} j[G(f-fc) - G(f+fc) & -w \le f \le w \\ 0 & otherwise \end{cases}$$





<u>Band pass systems</u>

The analysis of band pass systems can be simplified by using the complex envelope concept. Here, results and techniques from low pass systems can be easily applied to band pass systems.

The problem to be addressed is:

The input x(t) is a band pass signal

$$x(t)=x_I(t)\cos\omega_c t - x_Q(t)\sin\omega_c t$$

x(t) is applied to a band pass filter with impulse response

$$h(t) = h_I(t)\cos\omega_c t - h_Q(t)\sin\omega_c t$$

The objective is to find the filter output y(t). The output is, of course, the convolution of x(t) and h(t)

$$y(t) = x(t)*h(t)),$$

which can also be expressed as

$$y(t) = y_I(t)\cos\omega_c t - y_Q(t)\sin\omega_c t$$

Due to the band-pass nature of the problem, carrying out the direct convolution will be a tedious task due to the presence of the sin and cos functions in all terms. The complex envelope concept simplifies the problem to a very great extent. The procedure is summarized as follows:

a. Form the complex envelope for both the input and the channel:

$$\tilde{x}(t) = x_I(t) + jx_Q(t)$$

$$\tilde{h}(t) = h_I(t) + jh_Q(t)$$

b. Carry out the convolution between $\tilde{x}(t)$ and $\tilde{h}(t)$. Note that both signals are low-pass signals and so is $\tilde{y}(t)$.

$$2\,\tilde{y}(t) = \tilde{h}(t) * \tilde{x}(t)$$

$$\tilde{y}(t) = y_I(t) + jy_Q(t)$$

c. The band-pass filter output is obtained from the low pass signal $\tilde{y}(t)$ through the relation

$$y(t) = \mathrm{Re}\{\tilde{y}(t)\, e^{jw_c t}\}$$

or through the relation

$$y(t) = y_I(t)\cos\omega_c t - y_Q(t)\sin\omega_c t$$

Example :

The rectangular radio frequency (RF) pulse

$$x(t) = \begin{cases} A\cos 2\pi f_c t & 0 \le t \le T \\ 0 & otherwise \end{cases}$$

is applied to a linear filter with impulse response (We will see later that this is a filter matched to x(t), called the *matched filter*).

$$h(t) = x(T - t)$$

Assume that T= nT_C; n is an integer, $T_C = \frac{1}{fc}$. Determine the response of the filter and sketch it.

<u>Solution:</u> We follow the three steps outlined above.

$h(t) = A\cos 2\pi f_c(T - t)$

$= A\cos 2\pi f_c T\cos 2\pi f_c t + A\sin 2\pi f_c T\sin 2\pi f_c\, t$

$= A\cos 2\pi\left(\frac{nT_c}{T_c}\right)\cos 2\pi f_c t + A\sin 2\pi\left(\frac{nT_c}{T_c}\right)\sin 2\pi f_c t$

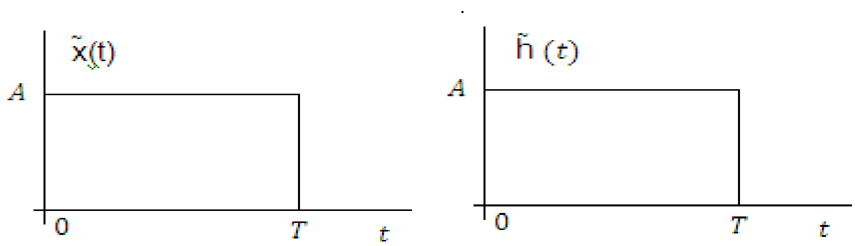
$\cos 2n\pi \equiv 1$ $\qquad$ $\sin 2n\pi \equiv 0$

Therefore, $h(t) = \begin{cases} A\cos 2\pi f_c t & 0 \le t \le T \\ 0 & otherwise \end{cases}$

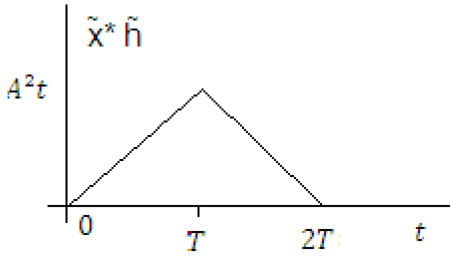
The complex envelopes of x(t) and h(t) are (step a)

$$\tilde{x}(t) = \begin{cases} A & 0 \le t \le T \\ 0 & o.w \end{cases}$$

$$\tilde{h}(t) = \begin{cases} A & 0 \le t \le T \\ 0 & o.w \end{cases}$$



$\tilde{y}(t) = \tilde{x}(t) * \tilde{h}(t)$ is the triangular signal shown in the Figure (step b).

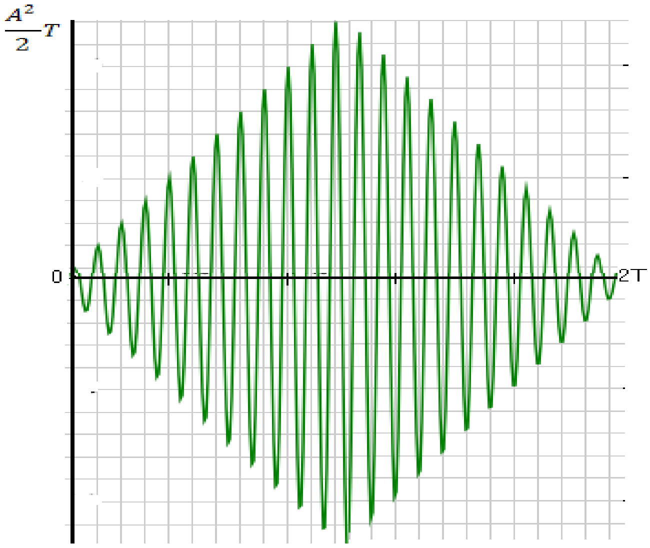


$$2\tilde{y}(t)=\begin{cases} A^2t & 0 \le t \le T \\ A^2(2T-t) & T \le t \le 2T \end{cases}$$

The bandpass signal is obtained as (step c)

$$y(t) = \begin{cases} \frac{A^2}{2} t\, cos w_c t & 0 \le t \le T \\ \frac{A^2}{2}(2T-t)\, cos w_c t & T \le t \le 2T \end{cases}$$

$y(t)$ is sketched as in the figure below.



Exercise

The band-pass signal $x(t) = e^{-\frac{t}{\tau}}\cos(2\pi f_c t)\, u(t)$ is applied to a band-pass filter with impulse response $h(t)$ given as:

$$h(t) = \begin{cases} A\, cos 2\pi f_c t & 0 \le t \le T \\ 0 & otherwise \end{cases}$$

Find and sketch the filter output.

Module 3
Normal Amplitude Modulation

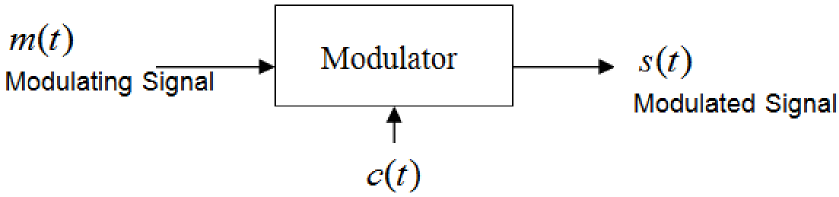
Modulation: is the process by which some characteristic of a carrier $c(t)$ is varied in accordance with a message signal $m(t)$.

Amplitude modulation is defined as the process in which the amplitude of the carrier $c(t)$ is varied linearly with $m(t)$. Four types of amplitude modulation will be considered in this chapter. These are normal amplitude modulation, double sideband suppressed carrier modulation, single sideband modulation, and vestigial sideband modulation.

A common form of the *carrier*, in the case of continuous wave modulation, is a sinusoidal signal of the form

$$c(t) = A_C \cos(2\pi f_C t + \varphi)$$

The baseband (message) signal $m(t)$ is referred to as the *modulating signal* and the result of the modulation process is referred to as the *modulated signal* $s(t)$. The following block diagram illustrates the modulation process.



We should point out that modulation is performed at the transmitter and demodulation, which is the process of extracting $m(t)$ from $s(t)$, is performed at the receiver.

ADD a section explaining why we need modulation

Normal Amplitude Modulation

A *normal AM* signal is defined as

$$s(t) = A_c(1 + k_a m(t)) \cos(2\pi f_c t)$$

where, k_a is the sensitivity of the AM modulator (units in 1/volt). $s(t)$ can be also be written in the form:

$$s(t) = A(t) \cos 2\pi f_C t$$

where, $A(t) = A_C + A_C k_a m(t)$. In this representation, we observe that $A(t)$ is related to $m(t)$ in a linear relationship of the form $y = a + bx$.

The *envelope* of $s(t)$ is defined as

$$|A(t)| = A_C |1 + k_a m(t)|$$

Notice that the envelope of $s(t)$ has the same shape as $m(t)$ provided that:

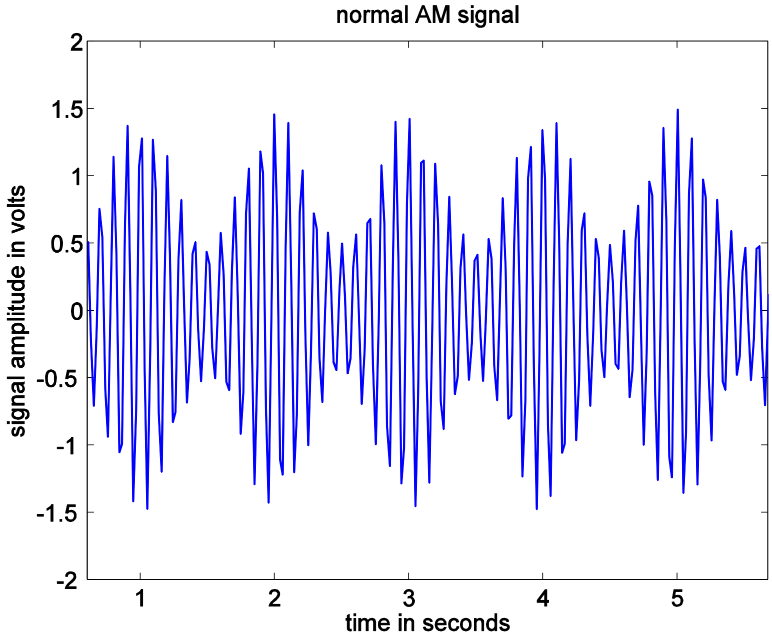
1. $|1+k_a m(t)| \geq 0$ or, equivalently, $|k_a m(t)| \leq 1$. Over-modulation occurs when $|k_a m(t)| > 1$, resulting in envelope distortion.
2. $f_C >> w$, where w is bandwidth of $m(t)$. f_C has to be at least $10\,w$. This ensures the formation of an envelope, whose shape resembles the message signal.

Matlab Demonstration

The figure below shows the normal AM signal $s(t) = (1 + 0.5\cos 2\pi t)\cos(2\pi(10t)$

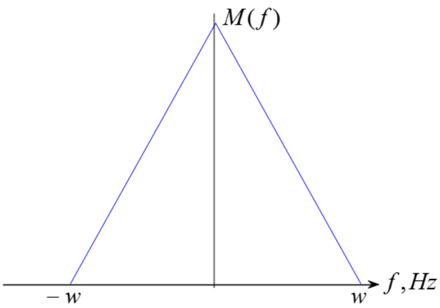
a. Make similar plots for the cases (μ = 0.5, 1, and 1.5)

b. Show the effect of f_C on the envelope. (Take f_C = 4 Hz, and f_C = 25Hz)



Spectrum of the Normal AM Signal

Let the Fourier transform of $m(t)$ be as shown (The B.W of $m(t)$ = w Hz).



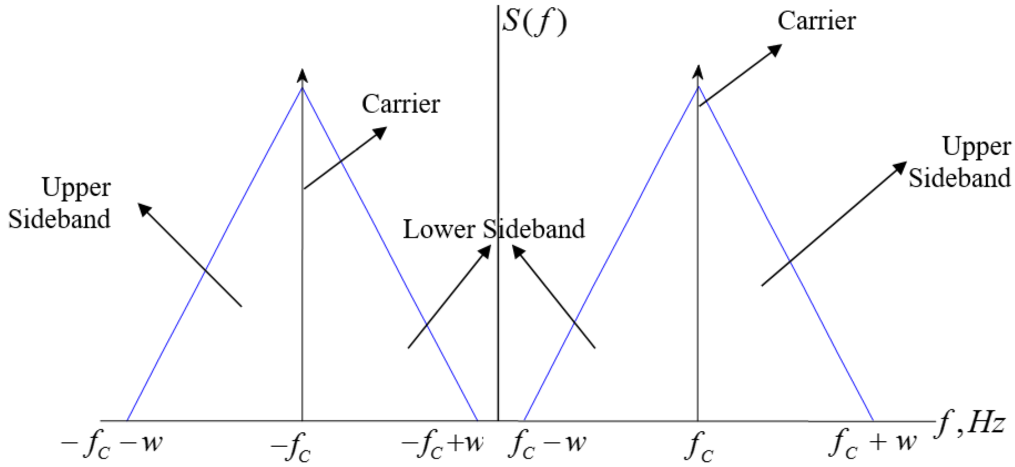
$s(t) = A_C(1 + k_a m(t))\cos 2\pi f_C t$ (dc + message)*carrier

$s(t) = A_C \cos 2\pi f_C t + A_C k_a m(t)\cos 2\pi f_C t$ (carrier + message*carrier)

Taking the Fourier transform, we get

$$S(f) = \frac{A_C}{2}\delta(f - f_C) + \frac{A_C}{2}\delta(f + f_C) + \frac{A_C k_a}{2}M(f - f_C) + \frac{A_C k_a}{2}M(f + f_C)$$

The spectrum of $s(t)$ is shown below



Remarks

a. The baseband spectrum $M(f)$, of the message has been shifted to the bandpass region centered around the carrier frequency f_C.
b. The spectrum $S(f)$ consists of two sidebands (upper sideband and lower sideband) and a carrier.
c. The transmission bandwidth of $s(t)$ is:

$$B.W = (f_C + w) - (f_C - w) = 2w$$

Which is twice the message bandwidth.

Power Efficiency

The *power efficiency* of a normal AM signal is defined as:

$$\eta = \frac{power\ in\ the\ sidebands}{power\ in\ the\ sidebands + power\ in\ the\ carrier}$$

Now, we find the power efficiency of the AM signal for the single tone modulating signal $m(t) = A_m \cos(2\pi f_m t)$. Let $\mu = A_m k_a$, then $s(t)$ can be expressed as

$$s(t) = A_C(1 + \mu\cos 2\pi f_m t)\cos 2\pi f_C t$$
$$s(t) = A_C \cos 2\pi f_C t + A_C \mu \cos 2\pi f_C t \cos 2\pi f_m t$$
$$s(t) = A_C \cos 2\pi f_C t + \frac{A_C \mu}{2}\cos 2\pi(f_C + f_m)t + \frac{A_C \mu}{2}\cos 2\pi(f_C - f_m)t$$

$$\text{Power in carrier } = \frac{A_C{}^2}{2}$$

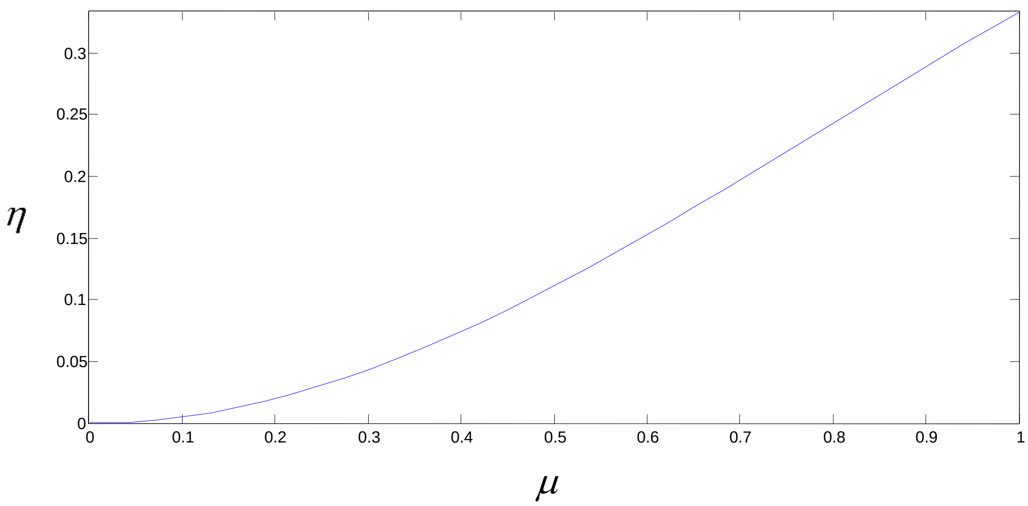
$$\text{Power in sidebands } = \frac{1}{2}\left(\frac{A_C\mu}{2}\right)^2 + \frac{1}{2}\left(\frac{A_C\mu}{2}\right)^2$$

$$= \frac{1}{8}A_C{}^2\mu^2 + \frac{1}{8}A_C{}^2\mu^2 = \frac{1}{4}A_C{}^2\mu^2$$

Therefore,

$$\eta = \frac{\frac{1}{4}A_C{}^2\mu^2}{\frac{A_C{}^2}{2} + \frac{1}{4}A_C{}^2\mu^2} = \frac{\mu^2}{2+\mu^2} \qquad ; \quad 1 \geq \mu \geq 0$$

The following figure shows the relationship between η and μ



The maximum efficiency occurs when μ = 1, i.e. for a 100% modulation index. The corresponding maximum efficiency is only η = 1/3. As a result, 2/3 of the transmitted power is wasted in the carrier.

Remark: Normal AM is not an efficient modulation scheme in terms of the utilization of the transmitted power.

Exercise

a. Show that for the general AM signal $s(t) = A_C \; 1 + k_a m(t) \; \cos(2\pi f_c t)$, the power efficiency is given by $\eta = \dfrac{\frac{1}{2}A_C{}^2\left\langle k_a{}^2 m(t)^2\right\rangle}{\frac{A_C{}^2}{2} + \frac{1}{2}A_C{}^2\left\langle k_a{}^2 m(t)^2\right\rangle} = \dfrac{\left\langle k_a{}^2 m(t)^2\right\rangle}{1+\left\langle k_a{}^2 m(t)^2\right\rangle}$, where $\left\langle k_a{}^2 m(t)^2\right\rangle$ is the average power in $k_a m(t)$

b. Apply the above formula for the single tone modulated signal $s(t) = A_C(1+\mu\cos 2\pi f_m t)\cos 2\pi f_C t$

AM Modulation Index

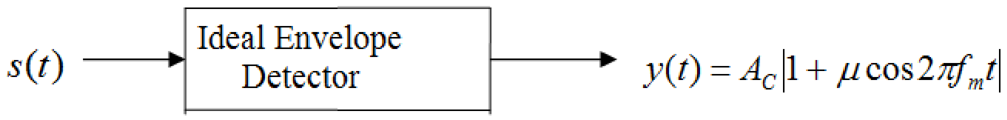
Consider the AM signal

$$s(t) = A_C\left(1 + k_a m(t)\right)\cos 2\pi f_C t = A(t)\cos 2\pi f_C t$$

The envelope of $s(t)$ is defined as:

$$|A(t)| = A_C|1 + k_a m(t)|$$

The following block diagram illustrate the envelope detection process for a sinusoidal message signal.



To avoid distortion, the following condition must hold

$$|1 + k_a m(t)| \geq 0 \qquad \text{or} \qquad |k_a m(t)| \leq 1$$

The modulation index of an AM signal is defined as:

$$Modulation\ \ Index\ (M.I) = \frac{|A(t)|_{\max} - |A(t)|_{\min}}{|A(t)|_{\max} + |A(t)|_{\min}}$$

Example (single tone modulation)

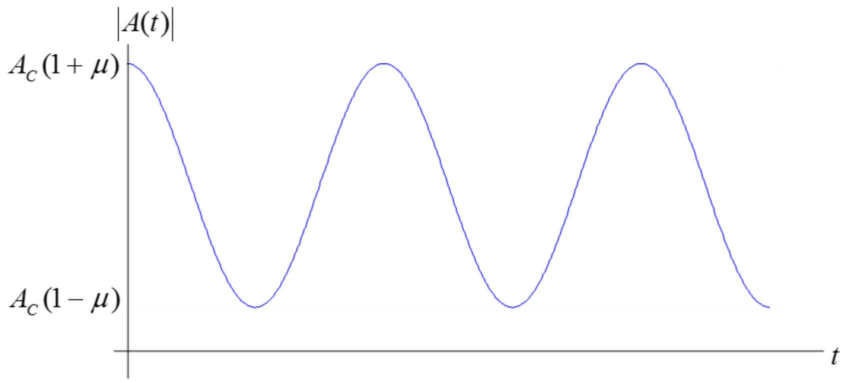
Let $m(t) = A_m \cos 2\pi f_m t$

$$then,\ s(t) = A_C(1 + k_a A_m \cos 2\pi f_m t)\cos 2\pi f_C t$$

$$= A_C(1 + \mu\cos 2\pi f_m t)\cos 2\pi f_C t \qquad where,\ \mu = k_a A_m$$

To avoid distortion $k_a A_m = \mu < 1$

The envelope $|A(t)| = A_C|(1 + \mu\cos 2\pi f_m t)|$ is plotted below



$$|A(t)|_{\max} = A_C(1+\mu), \qquad |A(t)|_{\min} = A_C(1-\mu)$$

$$M.I = \frac{A_C(1+\mu) - A_C(1-\mu)}{A_C(1+\mu) + A_C(1-\mu)} = \frac{2A_C\mu}{2A_C} = \mu$$

Therefore, the modulation index is μ.

Over-modulation

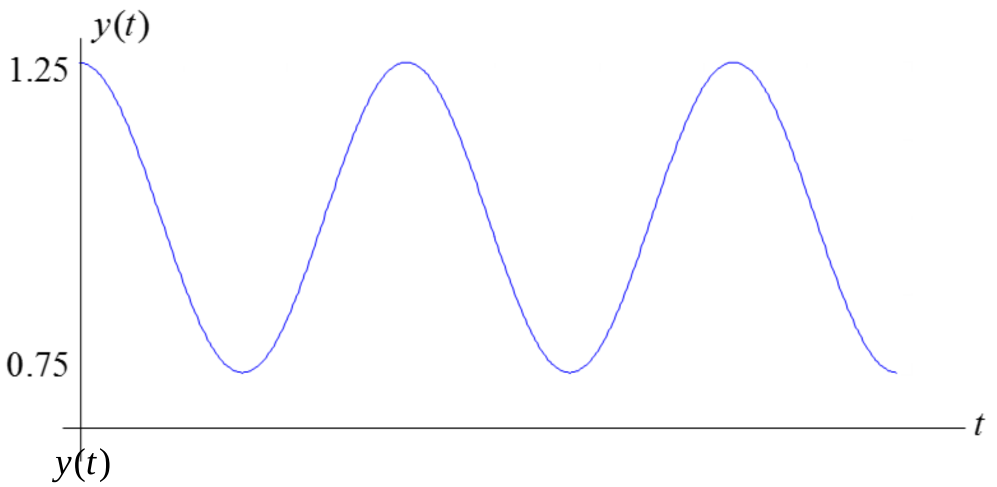
When the modulation index $\mu > 1$, an ideal envelope detector cannot be used to extract $m(t)$ from s(t) and *distortion* takes place.

Example: Let $s(t) = A_C(1 + \mu\cos 2\pi f_m t)\cos 2\pi f_C t$ be applied to an ideal envelope detector, sketch the demodulated signal for $\mu = 0.25, 1.0, \textit{and } 1.25$.

As was mentioned before, the output of the envelope detector is $y(t) = A_C\left|1 + \mu\cos 2\pi f_m t\right|$

Case1 : (μ = 0.25)

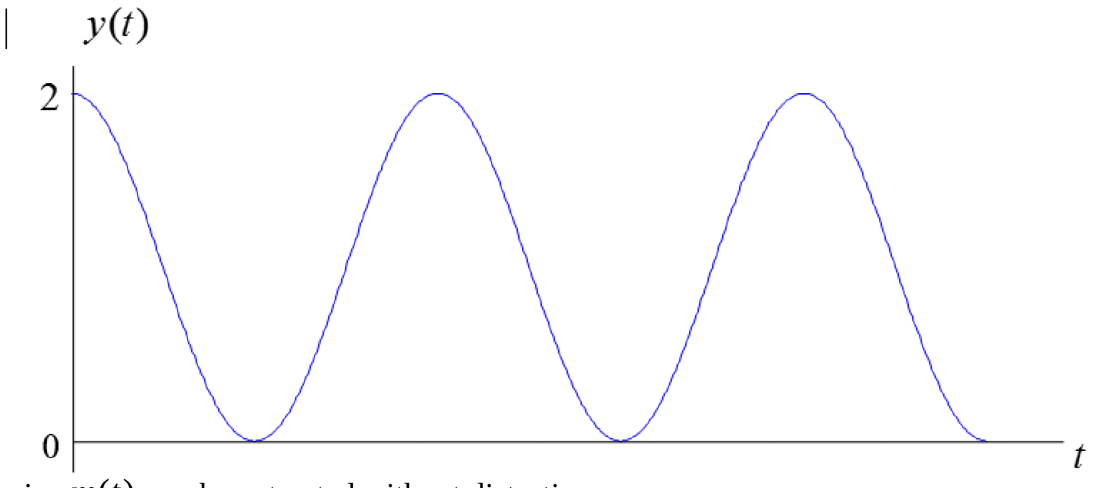
$$y(t) = A_C\left|1 + 0.25\cos 2\pi f_m t\right|$$



Here, $m(t)$ can be extracted without distortion.

Case2: (μ = 1.0)

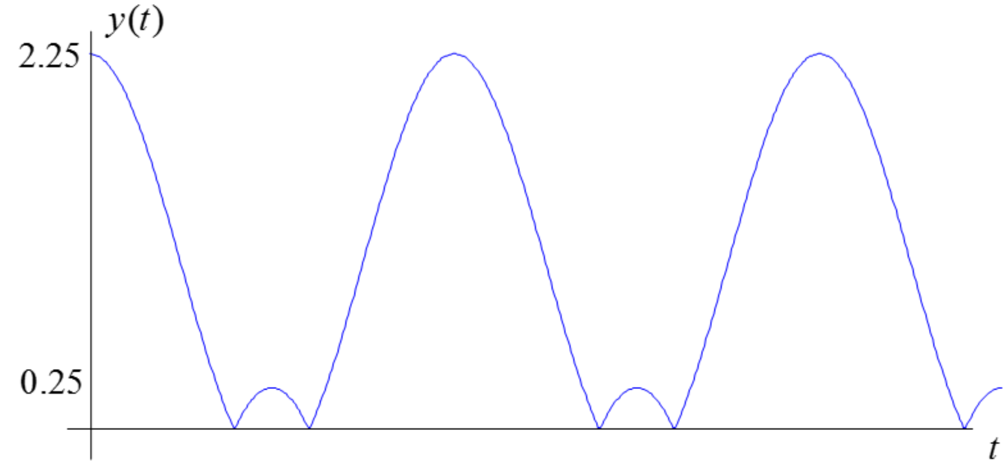
$$y(t) = A_C\left|1 + \cos 2\pi f_m t\right|$$



Here again, $m(t)$ can be extracted without distortion.

Case3: (μ = 1.25)

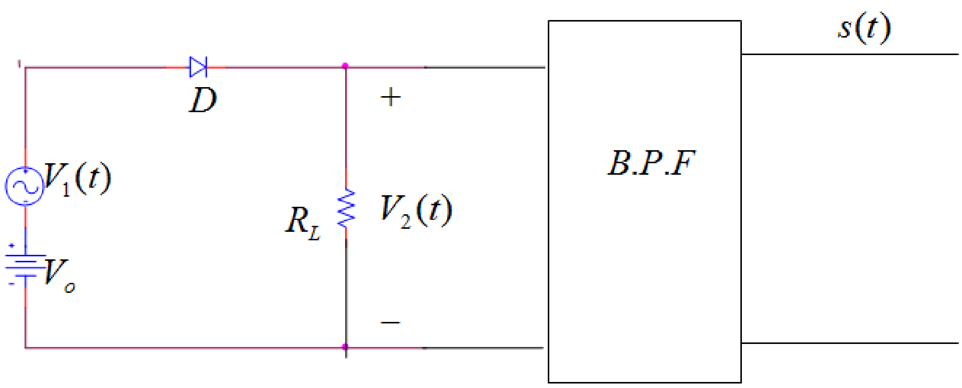
$$y(t) = A_C \left|1 + 1.25 \cos 2\pi f_m t\right|$$



Here, $m(t)$ cannot be recovered without distortion.

Generation of Normal AM

Square Law Modulator (will not be covered for ENCS students)
Consider the following circuit



For small variations of $V_1(t)$ around a suitable operating point, $V_2(t)$ can be expressed as:

$V_2 = \alpha_1 V_1 + \alpha_2 V_1^2$; Where α_1 and α_2 are constants.

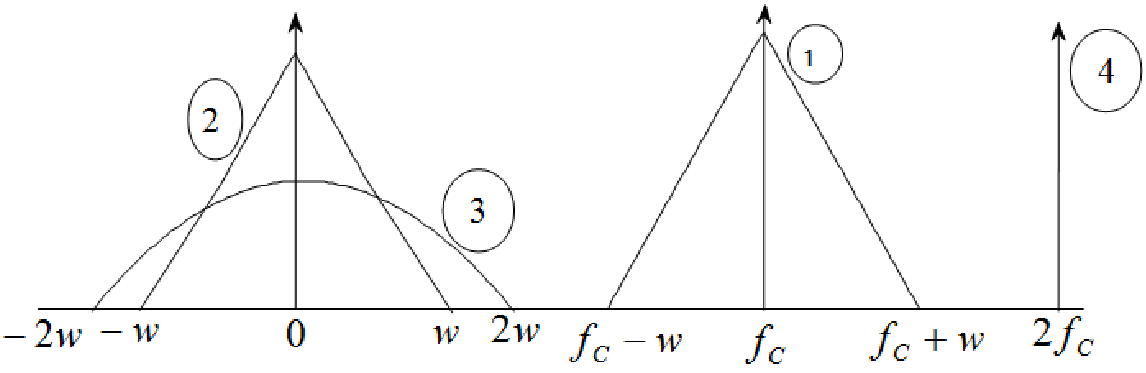
Let $V_1(t) = m(t) + A_C' \cos 2\pi f_C t$
Substituting $V_1(t)$ into the nonlinear characteristics and arranging terms, we get

$$V_2(t) = \alpha_1 A_C' \left[1 + \frac{2\alpha_2}{\alpha_1} m(t)\right] \cos 2\pi f_C t + \alpha_1 m(t) + \alpha_2 m(t)^2 + \alpha_2 A_C'^2 \cos^2(2\pi f_C t)$$

$$V_2(t) = (1) + (2) + (3) + (4)$$

The first term is the desired AM signal obtained by passing $V_2(t)$ through a bandpass filter.

$$s(t) = \alpha_1 A_C' \left[1 + \frac{2\alpha_2}{\alpha_1} m(t)\right] \cos 2\pi f_C t$$



Note: the numbers shown in above figure represent the number of term in $V_2(f)$.

(1) = The desired normal AM signal

(2) = $M(f)$

(3) = $M(f)*M(f)$

(4) = The cosine square term amounts to a term at $2f_c$ and a DC term.

Limitations of this technique:

a. Variations of $V_1(t)$ should be small to justify the second order approximation of the nonlinear characteristic.

b. The bandwidth of the filter should be such that $f_C - w > 2w \Rightarrow f_C \geq 3w$

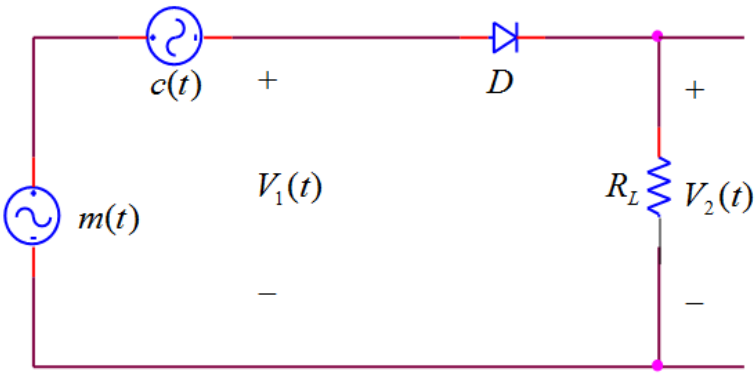
When $f_C >> w$, a bandpass filter with reasonable edge could be used.

When f_C is of the order $3w$, a filter with sharp edges should be used.

Generation of Normal AM

The Switching Modulator (will be covered)

Assume that the carrier $c(t)$ is large in amplitude so that the diode –shown in the figure below- acts like an ideal switch.



When $m(t)$ is small compared to $|c(t)|$,

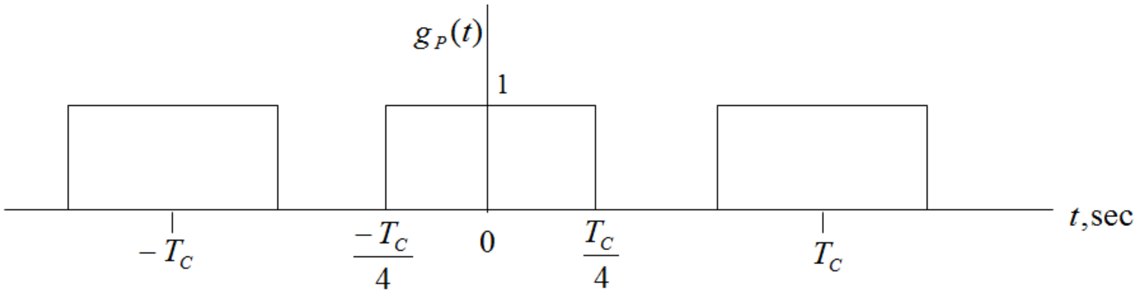
$$V_2(t)=\begin{cases} m(t)+A_C\cos\omega_C t & ; \quad c(t)>0 \\ 0 & ; \quad c(t)<0 \end{cases}$$

Here, the diode opens and closes at a rate equals to the carrier frequency f_C. This switching mechanism can be modeled as:

$$V_2(t)=[A_C\cos\omega_C t+m(t)]g_P(t)$$

where $g_P(t)$ is the periodic square function, expanded in a Fourier series as

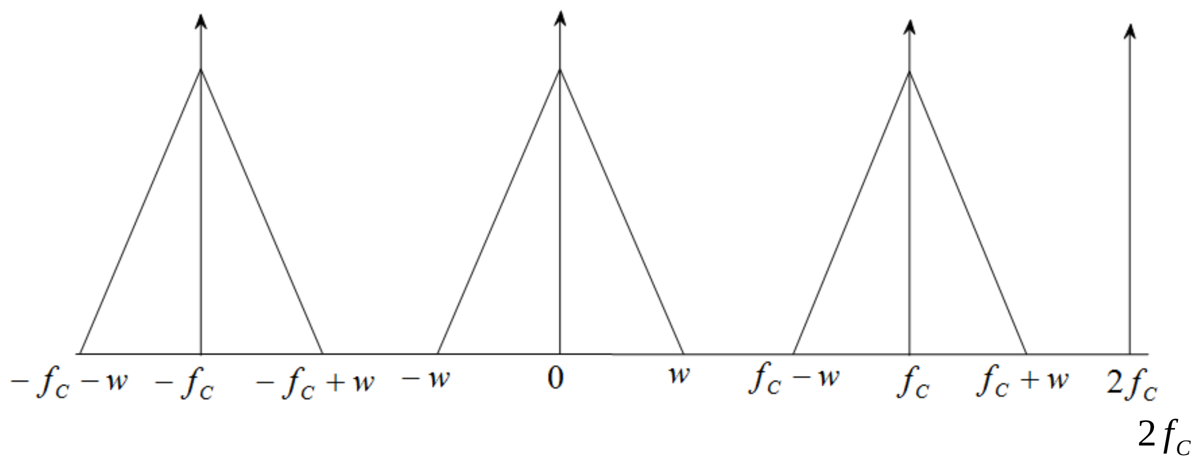
$$g_P(t)=\frac{1}{2}+\frac{2}{\pi}\left(\cos\omega_C t-\frac{1}{3}\cos 3\omega_C t+\frac{1}{5}\cos 5\omega_C t+\ldots\right)$$



$$V_2(t)=[A_C\cos\omega_C t+m(t)]\left(\frac{1}{2}\right)+\left(\frac{2}{\pi}\cos\omega_C t\right)(A_C\cos\omega_C t+m(t))-\left(\frac{2}{3\pi}\cos 3\omega_C t\right)$$
$$(m(t)+A_C\cos\omega_C t)+\ldots$$

$\Longrightarrow$

$$V_2(t)=\frac{m(t)}{2}+\frac{A_C}{2}\cos\omega_C t+\frac{2}{\pi}m(t)\cos\omega_C t+\frac{A_C}{\pi}+\frac{A_C}{\pi}\cos 2\omega_C t+\frac{2}{3\pi}m(t)\cos 3\omega_C t$$
$$+\frac{2}{3\pi}A_C\cos 2\omega_C t+\ldots$$



A bandpass filter with a bandwidth $2w$, centered at f_C, passes the second term (a carrier) and the third term (a carrier multiplied by the message). The filtered signal is

$$s(t) = \frac{A_C}{2}\cos\omega_C t + \frac{2}{\pi} m(t)\cos\omega_C t$$

$$s(t) = \frac{A_C}{2}\left(1 + \frac{4}{\pi A_C} m(t)\right)\cos\omega_C t \quad ; \qquad \text{Desired AM signal.}$$

Modulation Index $= M.I = \frac{4}{\pi A_C}|m(t)|_{\max}$

Demodulation of AM Signal: The Ideal Envelope Detector

The ideal envelope detector responds to the envelope of the signal, but is insensitive to phase variation. If

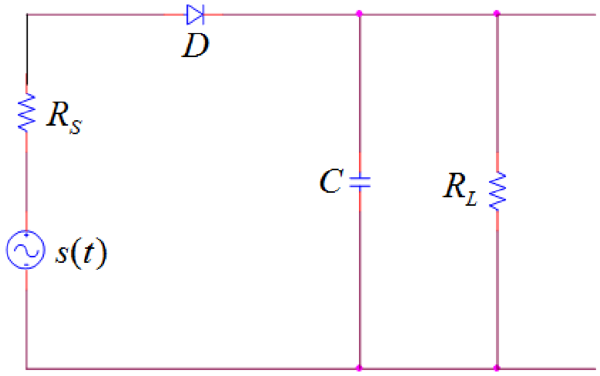
$$s(t) = A_C\left(1 + k_a m(t)\right)\cos 2\pi f_C t$$

then, the output of the ideal envelope detector is

$$y(t) = A_C|1 + k_a m(t)|$$

A Simple Practical Envelope Detector

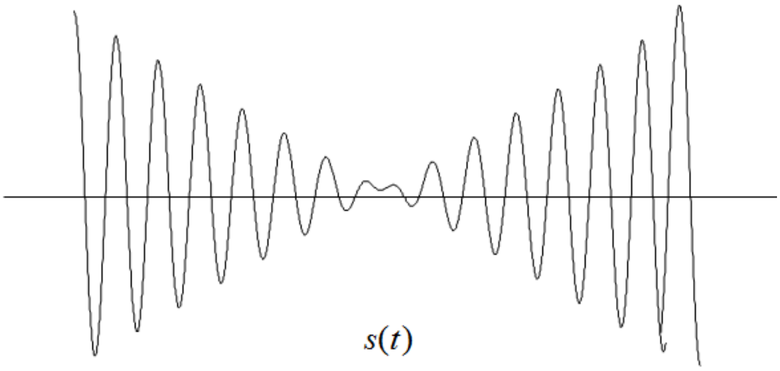
It consists of a diode followed by an RC circuit that forms a low pass filter.

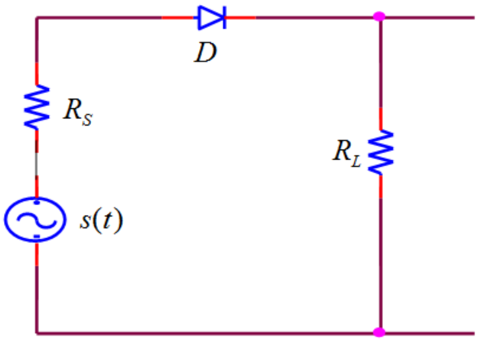


During the positive half cycle of the input, the diode is forward biased and C charges rapidly to the peak value of the input. When $s(t)$ falls below the maximum value, the diode becomes reverse biased and C discharges slowly through R_L. To follow the envelope of $s(t)$, the circuit time constant should be chosen such that :

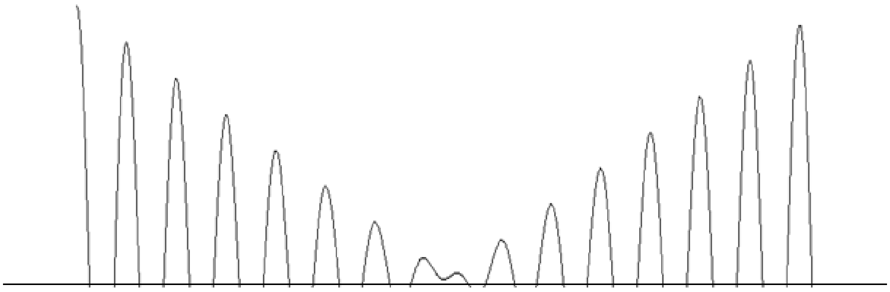
$$\frac{1}{f_C} << R_L C << \frac{1}{w}$$

Where w is the message B.W and f_C is the carrier frequency.





Half – Wave Rectifier

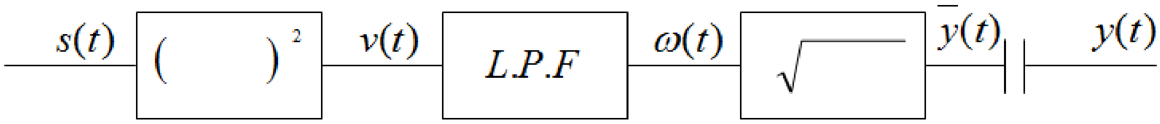


Output of half wave rectifier (without C)

When a capacitor C is added to a half wave rectifier circuit, the output follows the envelope of $s(t)$. The circuit output (with C connected) follows a curve that connects the tips of the positive half cycles, which is the envelope of the AM signal.

Example: Demodulation of AM Signal

Let $s(t) = (1 + k_a m(t))\cos \omega_C t$ be applied to the scheme shown below, find $y(t)$.



$$v(t) = s(t)^2 = \left(1 + k_a m(t)\right)^2 cos^2(2\pi f_c t)$$

$$v(t) = 0.5\left(1 + k_a m(t)\right)^2 + 0.5\left(1 + k_a m(t)\right)^2 cos(4\pi f_c t)$$

The filter suppresses the second term and passes only the first term. Hence,

$$\omega(t) = 0.5\left(1 + k_a m(t)\right)^2$$

$$\bar{y}(t) = \sqrt{\omega(t)} = \frac{1}{\sqrt{2}}\left(1 + k_a m(t)\right)$$

The capacitor blocks the dc term (first term on the RHS) and the output becomes

$$y(t) = \frac{1}{\sqrt{2}} k_a m(t)$$

<u>Concluding remarks about AM</u>:

i. Modulation is accomplished using a nonlinear device.
ii. Demodulation is accomplished using a simple envelope detector.
iii. AM is wasteful of power; most power resides in the carrier (not in the sidebands).
iv. The transmission B.W = twice message B.W

Module 3

Double Sideband, Single Sideband, and Vestigial Sideband Modulation

In this chapter, we will study in detail three other types of amplitude modulation techniques, namely, Double Sideband Suppressed Carrier Modulation, Single Sideband Suppressed Carrier Modulation, and Vestigial Sideband Modulation. For each type, we will study the generation, demodulation, spectrum, transmission bandwidth, required power, and other relevant aspects. First, we start with Double Sideband Suppressed Carrier Modulation.

Double Sideband Suppressed Carrier Modulation (DSB-SC)

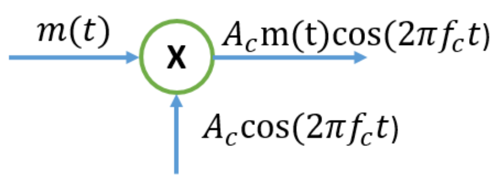
A DSB-SC signal is an amplitude-modulated signal that has the form

$\boldsymbol{s(t) = A_c m(t) \cos(2\pi f_c t)}$, where

$c(t) = A_c \cos(2\pi f_c t)$: is the carrier signal

$m(t)$: is the baseband message signal

$f_c >> W$, W is the bandwidth of the baseband message signal $m(t)$

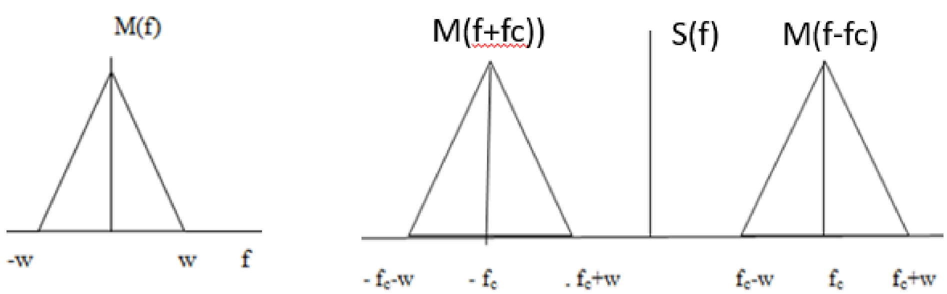


Generation of a DSB-SC Signal

The spectrum of $s(t)$ is

$$S(f) = \Im\{A_c m(t) \cos(2\pi f_c t)\}$$

$$= \frac{A_c}{2}[M(f - f_c) + M(f + f_c)]$$



Remarks*:*

1. No impulses are present in the spectrum at $\pm f_c$, i.e., no carrier is transmitted.

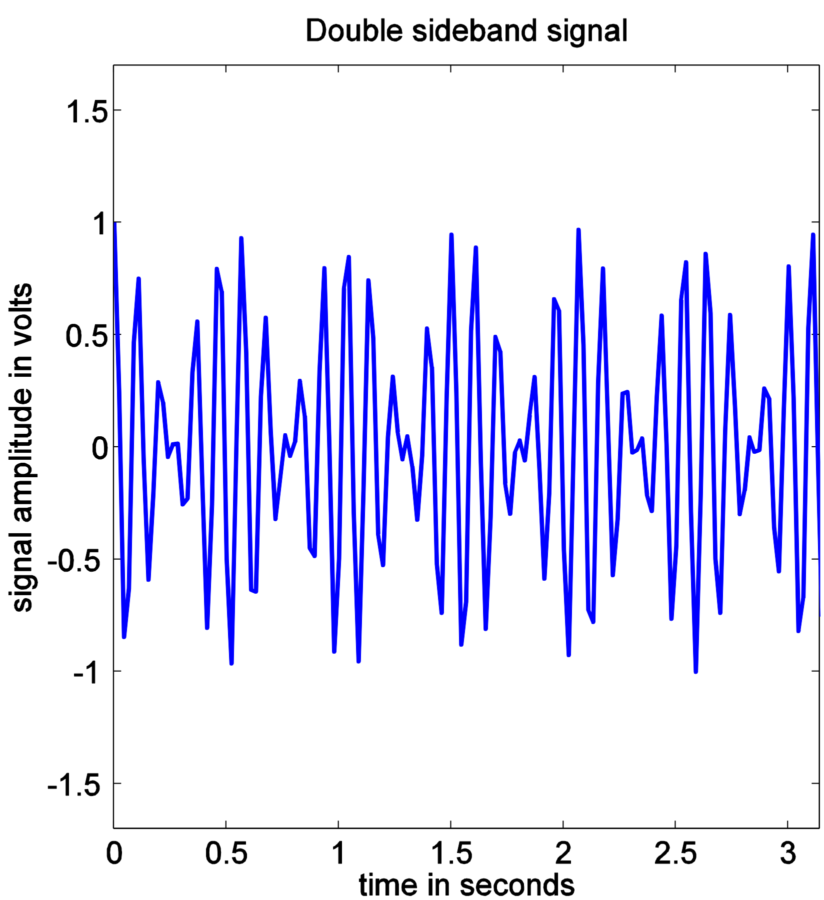
2. The transmission B.W of s(t) = 2W, twice the message bandwidth (same as that of normal AM).

3. Power efficiency = $\frac{power\ in\ the\ side\ bands}{total\ transmitted\ power}$ = 100%. This is a power efficient modulation scheme.

4. Coherent detector is required to extract m(t) from s(t), as we shall demonstrate shortly.

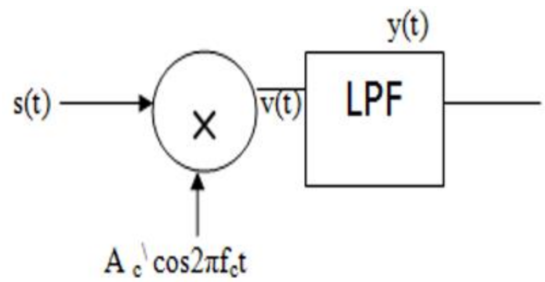
5. Envelope detection cannot be used for this type of modulation.

Computer Simulation: The next figure shows a DSB-SC signal when $m(t) = \cos(2\pi t)$ and $c(t) = \cos(2\pi(10)t)$.



Demodulation of a DSB-SC Signal

A DSB-SC signal is demodulated using what is known as ***coherent demodulation***. This means that the modulated signal s(t) is multiplied by a locally generated signal at the receiver which has the same frequency and phase as that of the carrier c(t) at the transmitting side.



a. **Perfect Coherent Demodulation**

Let $c(t) = A_c \cos(2\pi f_c t)$, $c'(t) = A_c' \cos(2\pi f_c t)$

Mixing the received signal with the version of the carrier at the receiving side, we get

$$v(t) = s(t)\ A_c' \cos 2\pi f_c t = A_c A_c' m(t) \cos^2 2\pi f_c t$$

$$= \frac{A_c A_c'}{2} m(t)\ [1 + \cos 2\ (2\pi f_c t)]$$

$$= \frac{A_c A_c'}{2} m(t) + \frac{A_c A_c'}{2} m(t) \cos 2(2\pi f_c t)$$

The first term on the RHS is proportional to $m(t)$, while the second term is a DSB signal modulated on a carrier with frequency $2f_c$. The high frequency component can be eliminated using a LPF with B.W = W. The output is

$$y(t) = \frac{A_c A_c'}{2} m(t)$$

Therefore, $m(t)$ has been recovered from $s(t)$ without distortion, i.e., the whole modulation-demodulation process is distortion-less.

b. Effect of Carrier Non-Coherence on Demodulated Signal

Here we consider two cases.

Case 1: A constant phase difference between c(t) and c\(t)

Let $c(t) = A_c \cos 2\pi f_c t$, $c'(t) = A_c' \cos(2\pi f_c t + \varnothing)$

We use the demodulator considered above

$v(t) = A_c m(t)\cos 2\pi f_c t \,.\, A_c^{\backslash} \cos(2\pi f_c t + \emptyset)$

$= \frac{A_c A_c^{\backslash}}{2} m(t)[\cos(4\pi f_c t + \emptyset) + \cos\emptyset]$

$= \frac{A_c A_c^{\backslash}}{2} m(t)\cos(4\pi f_c t + \emptyset) + \frac{A_c A_c^{\backslash}}{2} m(t)\cos\emptyset$

The low pass filter suppresses the first high frequency term and admits only the second low frequency term. The output is

$$y(t) = \frac{A_c A_c'}{2} m(t)\cos\emptyset$$

For $0 < \emptyset < \frac{\pi}{2}$, $0 < \cos\emptyset < 1$, y(t) suffers from an attenuation due to Ø.

However, for $\emptyset = \frac{\pi}{2}$, $\cos\emptyset = 0$ and y(t) = 0, i.e., receiver loses the signal.

The disappearance of a message component at the demodulator output is called *quadrature null effect.* This highlights the importance of maintaining synchronism between the transmitting and receiving carrier signals $c^{\backslash}(t)$ and c(t).

Case 2: Constant Frequency Difference between c(t) and $c^{\backslash}(t)$

Let $c(t) = A_c\cos 2\pi f_c t$, $c^{\backslash}(t) = A_c^{\backslash}\cos 2\pi(f_c + \Delta f)t$

In an analysis similar to that carried out in case 1, we get

$v(t) = A_c m(t)\cos 2\pi f_c t \,.\, A_c^{\backslash}\cos 2\pi(f_c + \Delta f)t$

$= \frac{A_c A_c^{\backslash}}{2} m(t)[\cos(4\pi f_c t + 2\pi\Delta f t) + \cos 2\pi\Delta f t]$

After low-pass filtering,

$$y(t) = \frac{A_c A_c'}{2} m(t)\cos(2\pi\Delta f t)$$

As you can see, $y(t) \neq km(t)$, but rather $m(t)$ is multiplied by a time function. Hence, the system is not distortion-less. In addition, y(t) appears as a double side band modulated signal with a carrier with magnitude Δf.

Example: Let $m(t) = \cos 2\pi(1000)t$ and let $\Delta f = 100$ Hz

From the analysis in case 2 above,

$y(t) = \frac{A_c A_c^{\backslash}}{2}\cos 2\pi(1000)t\cos 2\pi(100)t$

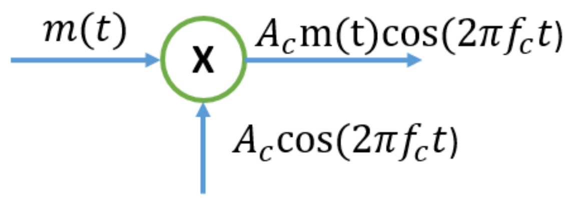
$$= \frac{Ac\ A\ c\backslash}{4}[\cos 2\pi(1100)t + \cos 2\pi(900)t]$$

The original message was a signal with a single frequency of 1000 Hz while the output consists of a signal with two frequencies at f_1 =1100 Hz and f_2 = 900 Hz ($\Rightarrow Distortion$)

Exercise: Use Matlab to plot both m(t) and y(t) and see the distortion caused by the lack of synchronization between the transmitting and receiving oscillators in the previous example.

Generation of DSB-SC

a. **Product Modulator**: It multiplies the message signal m(t) with the carrier c(t). This technique is usually applicable when low power levels are possible and over a limited carrier frequency range.

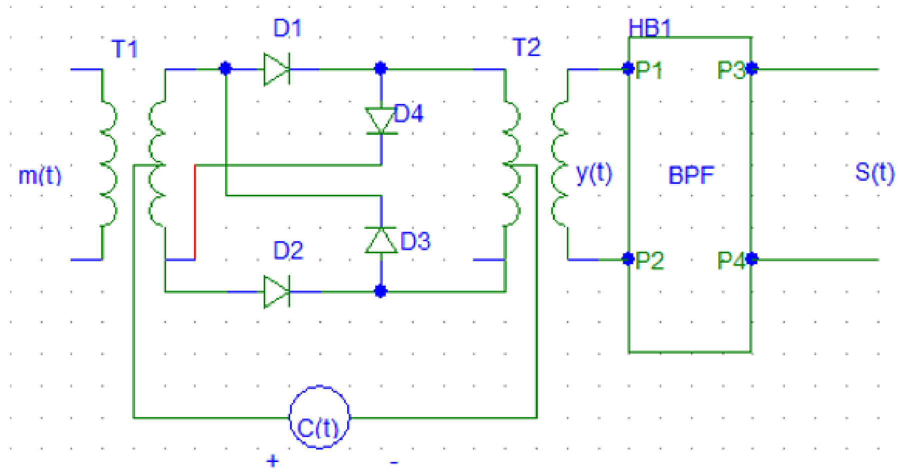


Generation of a DSB-SC Signal

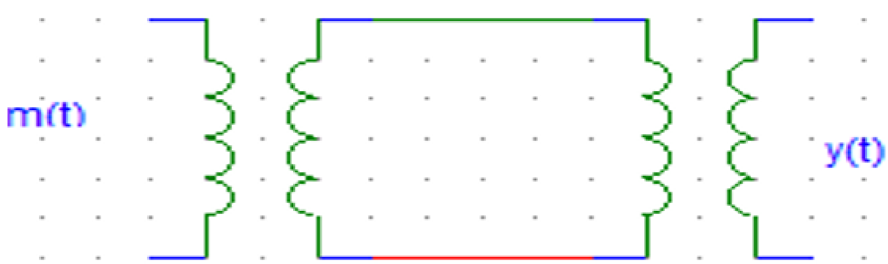
b . **Ring Modulator**

Consider the scheme shown in the figure.

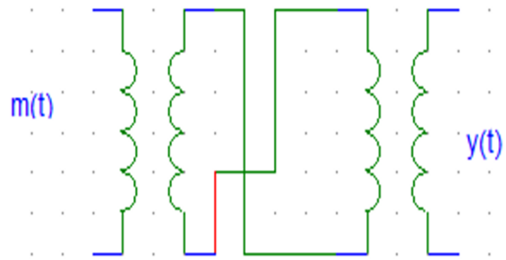
Let c(t) >> m(t). Here the carrier c(t) controls the behavior of the diodes .



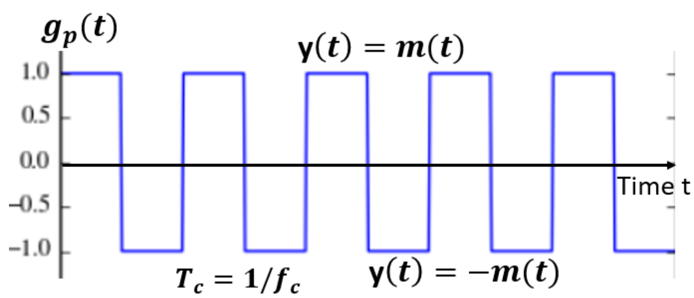
During the positive half cycle of c(t), c(t) > 0, and D1 and D2 are ON while D3 and D4 are OFF. Here, **y(t) = m(t)** and the circuit appears like this



During the negative half cycle of c(t), c(t) < 0 and D3 and D4 are ON while and D1 and D2 are OFF. Here, **y(t) = -m(t)** , and the circuit appears like this



So m(t) is multiplied by +1 during the +ve half cycle of c(t) and m(t) is multiplied by -1 during the -ve half cycle of c(t). Mathematically, y(t) behaves as if multiplied by the switching function $g_p(t)$ where $g_p(t)$ is the square periodic function with period $T_c = \frac{1}{fc}$, where f_c the period of c(t). By expanding $g_p(t)$ in a Fourier series, we get



$$y(t) = m(t)[\frac{4}{\pi}\cos 2\pi f_c t - \frac{4}{3\pi}\cos 3(2\pi f_c t) + \frac{4}{5\pi}\cos 5(2\pi f_c t)]$$

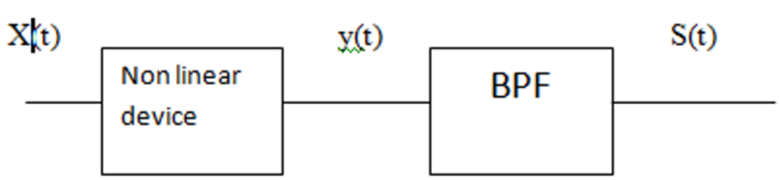
$$= m(t)\frac{4}{\pi}\cos 2\pi f_c t - m(t)\frac{4}{3\pi}\cos 3(2\pi f_c t) + m(t)\frac{4}{5\pi}\cos 5(2\pi f_c t)$$

When y(t) passes through the BPF with center frequency f_c, and bandwidth = 2W, the only component that appears at the output is the desired DSB-SC signal, which is

$$s(t) = \frac{4}{\pi}m(t)\cos 2\pi f_c t$$

C. Nonlinear Characteristic

Consider the scheme shown in the figure



Let the non-linear characteristic be of the form

$$y(t) = a_0x(t) + a_1x^3(t)$$

Let $x(t) = A\cos 2\pi f_c t + m(t)$, ($m(t)$ is the message signals)

$$Y = a_0(A\cos 2\pi f_c t + m(t)) + a_1 (A\cos 2\pi f_c t + m(t))^3$$

$$= a_0 A\cos 2\pi f_c t + a_0 m(t)+ a_1A^3\cos^3 2\pi f_c t + a_1 m(t)^3 + 3 a_1A^2 m(t)\cos^2 2\pi f_c t$$

$$+ 3A a_1\cos 2\pi f_c t$$

After some algebraic manipulations, a DSB-SC term appear in x(t) along with other undesirable terms. The band pass filter will admit the desired signal, which is

$$s(t) = \frac{3(A)^2 a1}{2} m(t) \cos(4\pi f_c t).$$

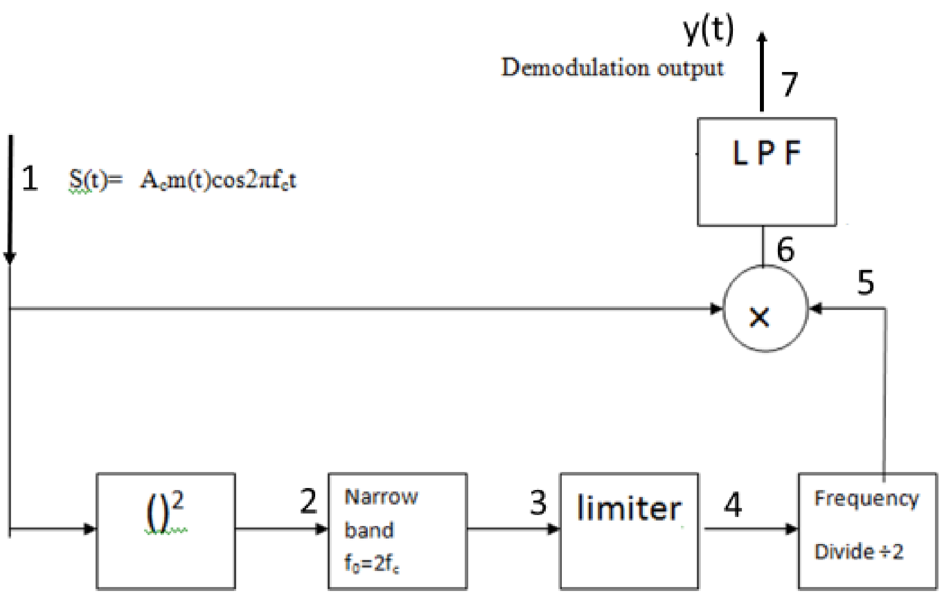
Note that the carrier frequency=$2f_c$ in this case.

Carriers Recovery for Coherent Demodulation

We consider briefly two circuits, which are used to extract the carrier f_c from the incoming DSB-SC signal. We recall that demodulation of DSB-SC signal requires the availability of a signal with the same frequency and phase as that of the carrier c(t) at the transmitter side.

a. **Squaring Loop**

The basic elements of squaring loop are shown in the figure below. The incoming signal is the DSB-SC signal $\boldsymbol{s(t) = A_c m(t)\cos(2\pi f_c t)}$. In the figure, we mark seven points that demonstrate the operation of the loop. In summary the signals at the points are:



1- $s(t) = A_c m(t)\cos(2\pi f_c t)$; The incoming DSB-SC

2- $s(t)^2 = (A_c m(t)\cos(2\pi f_c t))^2 = \frac{(A_c m(t))^2}{2}(1 + \cos(4\pi f_c t))$

$$s(t)^2 = \frac{(A_c m(t))^2}{2} + \frac{(A_c m(t))^2}{2}\cos(4\pi f_c t)$$

$$lowpass\ term + bandpass\ term\ around\ 2f_c$$

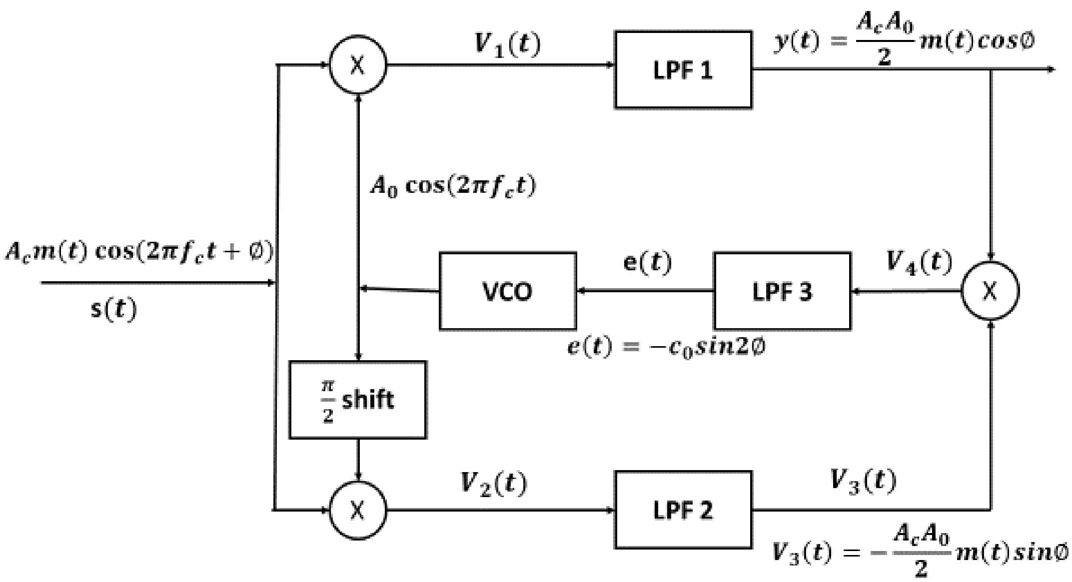
3- $\mathrm{K}\cos(4\pi f_c t)$ (The BPF suppresses the first term and admits the second term. When the bandwidth of the filter is narrow, its output is a signal with frequency $2f_c$)

4- $\mathrm{K}\cos(4\pi f_c t)$ (The limiter removes any variation in the amplitude but keeps the frequency unchanged).

5- $A_c{}'\cos(2\pi f_c t)$; The frequency divider produces a signal with frequency f_c, as required.
6- $s(t)A_c{}'\cos(2\pi f_c t) = A_c A_c{}' m(t)\cos(2\pi f_c t)\cos(2\pi f_c t)$
$= \frac{A_c A_c{}'}{2} m(t) + \frac{A_c A_c{}'}{2} m(t)\cos(4\pi f_c t)$; Product Demodulator
7- $\boldsymbol{y(t) = \frac{A_c A_c{}'}{2} m(t)}$**;** Demodulator output. The LPF suppresses the second term.

Even though the receiver did not have a copy of the carrier, yet it was able to perform coherent demodulation by generating its own version of the carrier via this loop.

Costas Loop:



The VCO: is an oscillator that produces a signal whose frequency is proportional to the voltage $e(t)$. In this analysis, it is assumed that the instantaneous frequency of the VCO is related to $e(t)$ by $\boldsymbol{f_r(t) = f_c - k_a e(t)}$.

When $e_c(t) = 0$, the frequency of the VCO $f_r(t) = f_c$. This is called the free running frequency. In the initialization stage, the frequency of the VCO is set to f_c, which should be the same as that used by the transmitter. During transmission, the frequency (or phase) of the received signal may change. The objective of the loop is to keep track of this change. Let the frequency of the received $s(t)$ be $f_s = f_c + \Delta f$

Initial Frequency Difference is: $f_s - f_r = (f_c + \Delta f) - f_c = \Delta f$.

Initial phase shift $\Delta\emptyset = 2\pi\Delta f$

Assume that the received DSB signal takes the form,

$$s(t) = A_c m(t) \cos(2\pi f_c t + 2\pi\Delta f t) = A_c m(t) \cos(2\pi f_c t + \emptyset)$$

At the initial stage, the VCO produces the signal $c'(t) = A_0 \cos(2\pi f_c t)$

$$V_1 = A_c A_0 m(t) \cos(2\pi f_c t) \cos(2\pi f_c t + \emptyset)$$

$$y(t) = Low\ pass\ \{V_1\} = Low\ pass\ \{A_c A_0 m(t) \cos(2\pi f_c t) \cos(2\pi f_c t + \emptyset)\}$$

$y(t) = \frac{A_c A_0}{2} m(t) cos\emptyset$

$V_2 = A_c A_0 m(t) \sin(2\pi f_c t) \cos(2\pi f_c t + \emptyset)$

$V_3(t) = Low\ pass\ \{V_2\} = Low\ pass\ \{A_c A_0 m(t) \sin(2\pi f_c t) \cos(2\pi f_c t + \emptyset)\}$

$V_3(t) = -\frac{A_c A_0}{2} m(t) sin\emptyset$

$V_4(t) = (\frac{A_c A_0}{2} m(t) cos\emptyset)(-\frac{A_c A_0}{2} m(t) sin\emptyset)$

When the bandwidth of LPF 3 is very narrow, the output can be approximated as

$e(t) = -c_0 sin2\emptyset$, c_0 is a constant.

The received frequency of $s(t)$ is $f_s(t) = f_c + \Delta f$. The frequency produced by the VCO is $f_r(t) = f_c - k_a e(t)$. Note that if $\emptyset = 2\pi\Delta f > 0$, then $e(t) < 0$. Meaning that $f_r(t) = f_c + k_a c_0 sin2\emptyset$. That is $f_r(t)$ increases trying to be the same as that of $s(t)$.

When $\emptyset$ is small, then $\sin(2\emptyset) \cong 2\emptyset$. Substituting into $f_r(t)$, we get

$f_r(t) = f_c + k_a c_0 (2\emptyset)$; New frequency of the VCO

Also, $f_s(t) = f_c + \emptyset/\boldsymbol{2\pi}$; Received frequency of $s(t)$

One can chose the loop constants so that $2k_a c_0 = 1/2\pi$. In this case,

$f_r(t) = f_s(t)$; Both frequencies become equal.

New Frequency Difference is: $f_s - f_r = 0$ (practically, it is not zero but a small difference)

New phase shift $\Delta\emptyset = 0$ (practically, it is not zero but a small difference)

Here, any change that takes in the frequency of the incoming signal s(t) is immediately felt by the VCO, which changes its frequency accordingly keeping the difference between the two frequencies very small. The loop output $y(t)$ will always be proportional to $m(t)$.

Single Sideband Modulation

In this type of modulation, only one of the two sidebands of DSB-SC is retained while the other sideband is suppressed. This mean that B.W of the SSB signal is one half that of DSB-SC. The saving in the bandwidth comes at the expense of increasing modulation complexity.

The time-domain representation of a SSB signal is

$$s(t) = A_C m(t) cos\omega_c t \pm A_C \hat{m}(t) sin\omega_c t$$

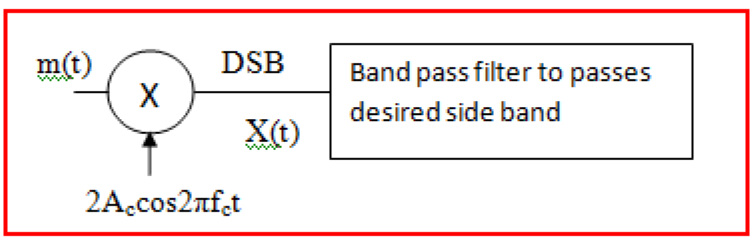
$\hat{m}(t)$: Hilbert transform of $m(t)$ obtained by passing $m(t)$ through a 90-degree phase shifter.

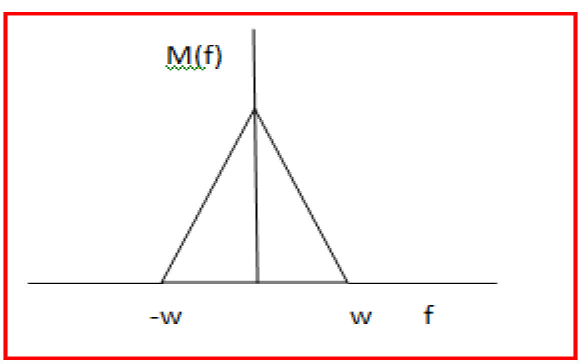
- sign: upper sideband is retained.

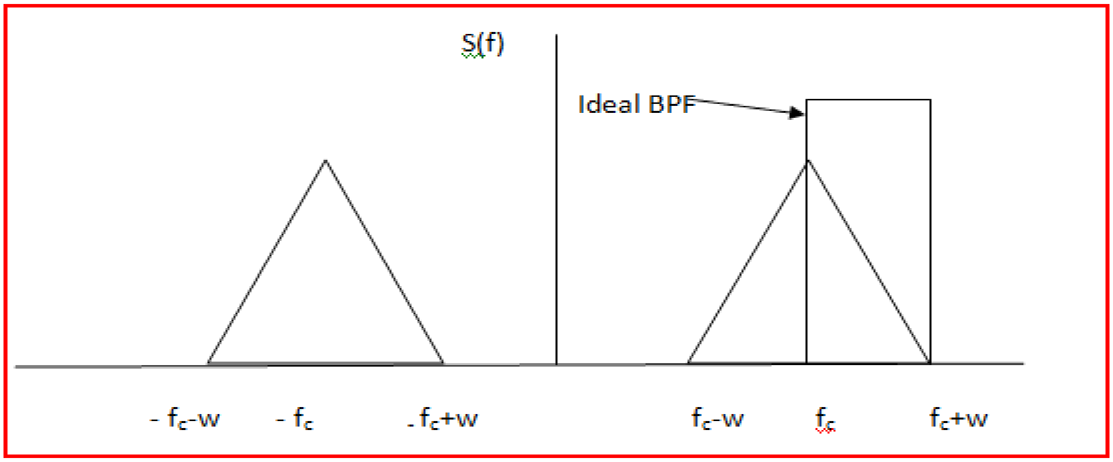
+ sign: lower sideband is retained.

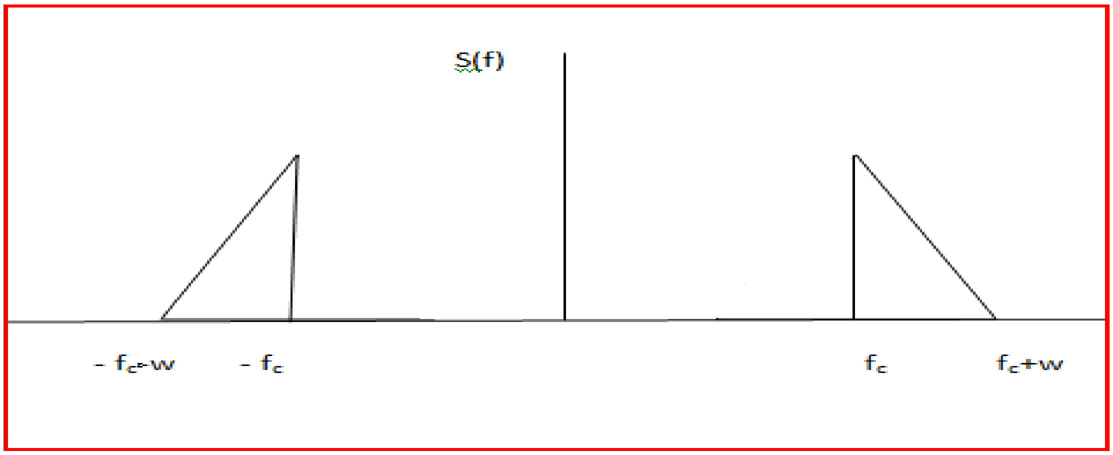
Generation of SSB: Filtering Method

A DSB-SC signal $x(t) = 2A_C m(t) cos\omega_c t$ is generated first. A band pass filter with appropriate B.W and center frequency is used to pass the desired side band only and suppress the other sideband.



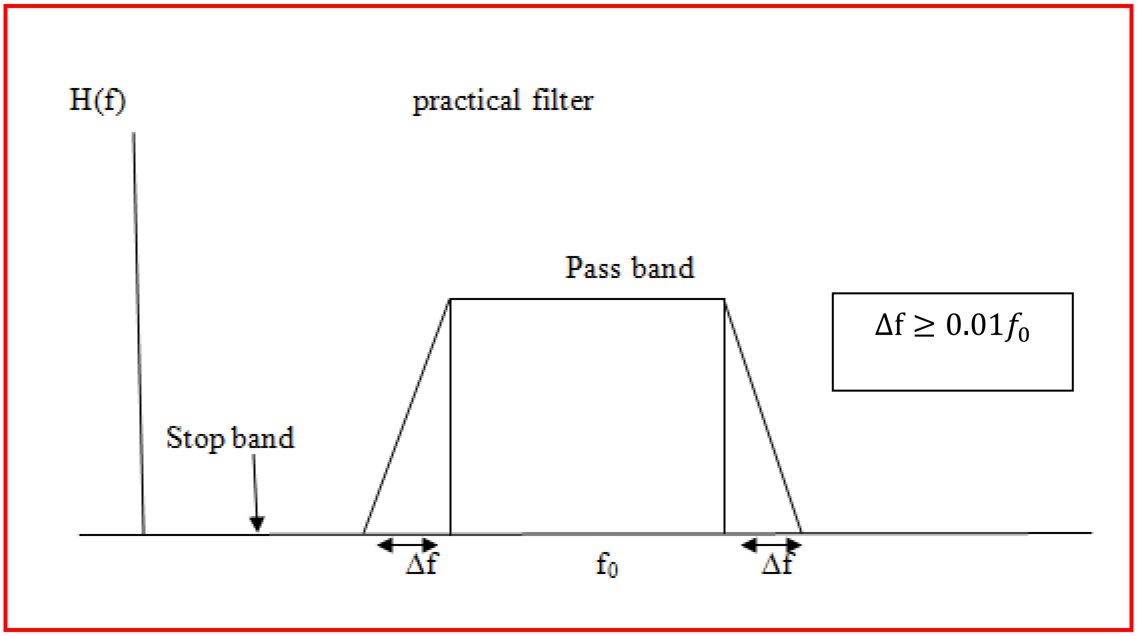






The band pass filter must satisfy two conditions:

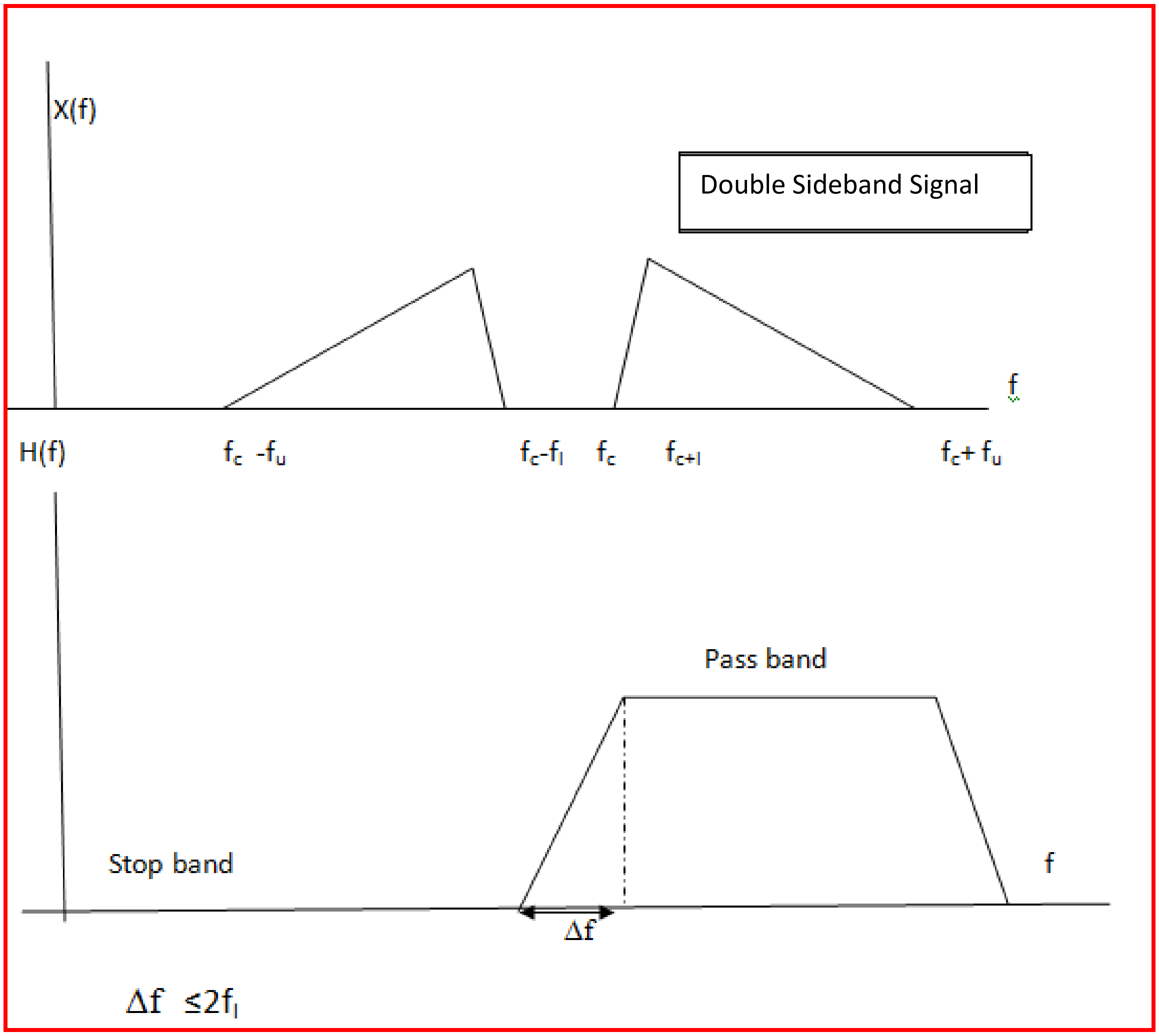
a. The pass band of the filter must occupy the same frequency range as the desired sideband.
b. The width of the transition band of the filter separating the pass band and the stop band must be at least 1% of the center frequency of the filter. i.e., $0.01f_0 \leq \Delta f$. This is sort of a rule of thumb for realizable filters on the relationship between the transition band and the center frequency.



Two remarks should be considered when generating a SSB signal.

1. Ideal filter do not exist in practice meaning that a complete elimination of the undesired side band is not possible. The consequence of this is that either part of the undesired side band is passed or the desired one will be highly attenuated. SSB modulation is suitable for signals with low frequency components that are not rich in terms of their power content.

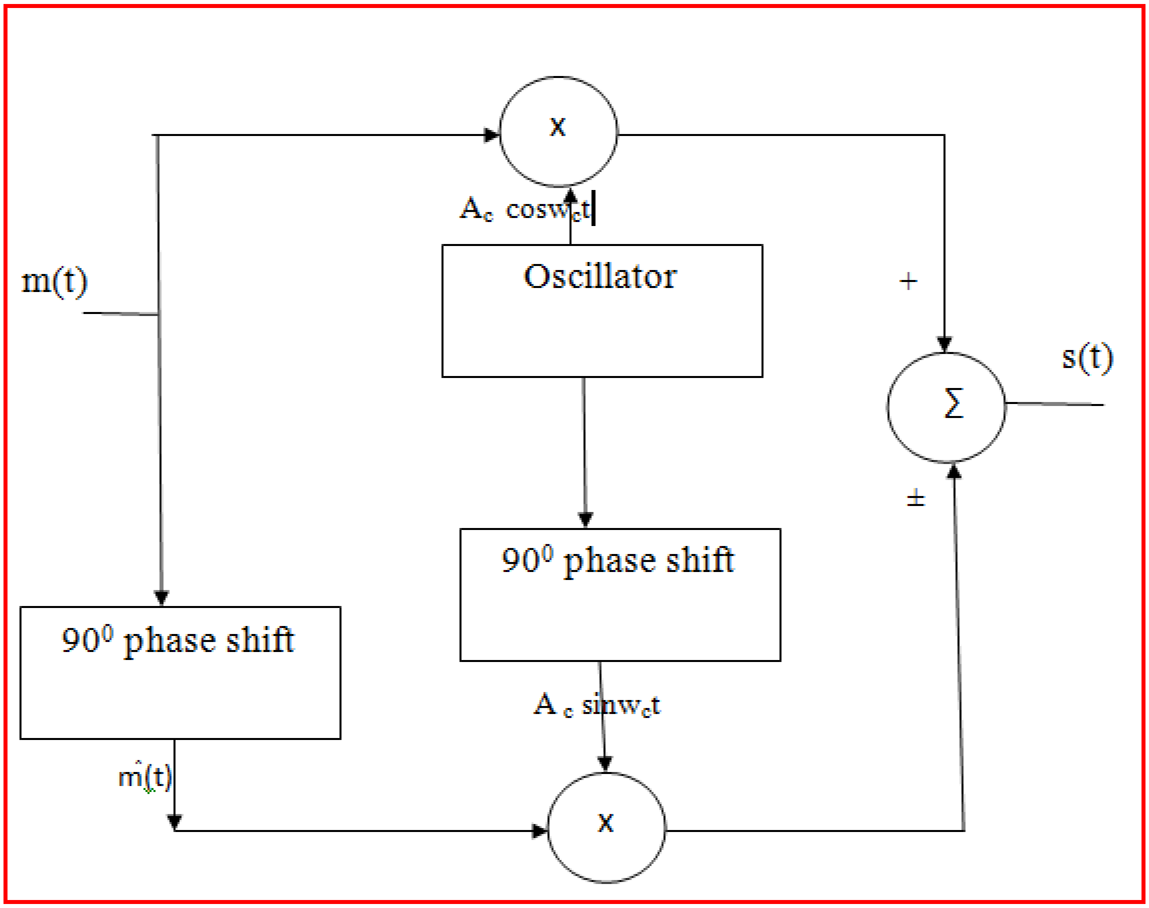
2. The width of the transition band of the filter should be at most twice the lowest frequency components of the message signal so that a reasonable separation of the two side band is possible. If the message significant frequency components extends between (f_l, f_u), then $2f_l \geq \Delta f$.



Generation of SSB Signal: Phase Shift Method

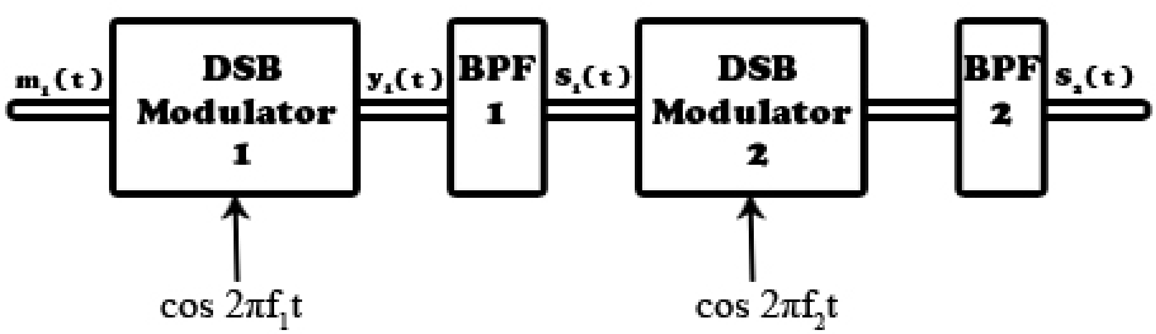
The method is based on the time –domain representation of the SSB signal

$$s(t) = A_C m(t) cos\omega_c t \pm A_C \hat{m}(t) sin\omega_c t$$



Two- Stage Generation of SSB Signal

When the conditions on the filter cannot be met in a single-stage SSB system, a two-stage scheme is used instead where less stringent conditions on the filters can be imposed. The block diagram illustrates this procedure.



$m_1(t)$ is the base band signal with a gap in its spectrum extending over $(0, f_l)$.

$y_1(t)$: is a DSB-SC signal on a carrier frequency f_1.

BPF_1 selects the upper side band of $y_1(t)$. The parameters of the filter are f_{01} (center frequency) and the transition band length Δf_1.

We must maintain that

$$\Delta f_1 \geq 0.01\, f_{01} \qquad \text{and} \qquad \Delta f_1 \leq 2\, f_I$$

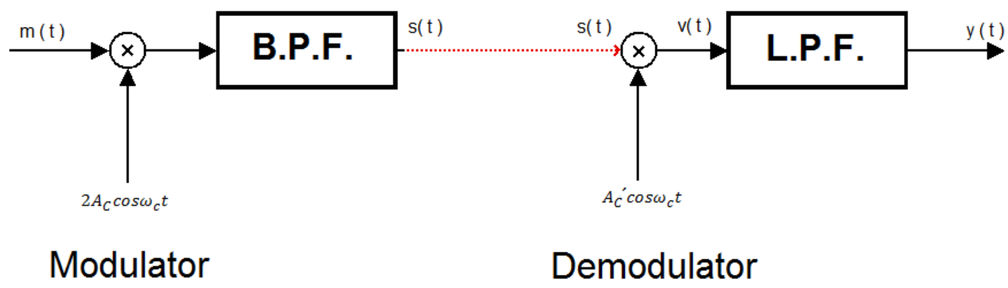
$s_1(t)$ is a single side band signal. The frequency gap of this signal extends over $(0, f_{1+}\, f_L)$. The second modulator views this signal as the baseband signal to be modulated on a carrier with frequency f_2.

The second modulator generates a DSB signal. The second BPF with center frequency f_{02} and transition band Δf_2 selects the upper side band. Again, we maintain that

$$\Delta f_2 \geq 0.01\, f_{02} \qquad \text{and} \qquad \Delta f_2 \leq 2\,(f_{1+} f_I)$$

Demodulation of SSB: Time-Domain Analysis

A SSB signal can be demodulated using coherent demodulation (oscillator at the receiver should have the same frequency and phase as those of transmitter carrier) as shown in the figure:



Let the received signal be the upper single sideband

$$s(t) = A_C m(t) cos\omega_c t - A_C \hat{m}(t) sin\omega_c t$$

At the receiver, $s(t)$ is mixed with the carrier signal. The result is

$$v(t) = s(t) A_C' cos\omega_c t$$

$$= A_C' \left[A_C m(t) cos\omega_c t - A_C \hat{m}(t) sin\omega_c t\right] cos\omega_c t$$

$$= A_C A_C' m(t) cos2\omega_c t - A_C A_C' \hat{m}(t)\, sin\omega_c t\, cos\omega_c t$$

$$= \frac{A_C A_C'}{2} m(t) + \frac{A_C A_C'}{2} m(t) cos2\omega_c t - \frac{A_C A_C'}{2} \hat{m}(t)\, sin2\omega_c t$$

The low pass filter admits only the first terms. The output is:

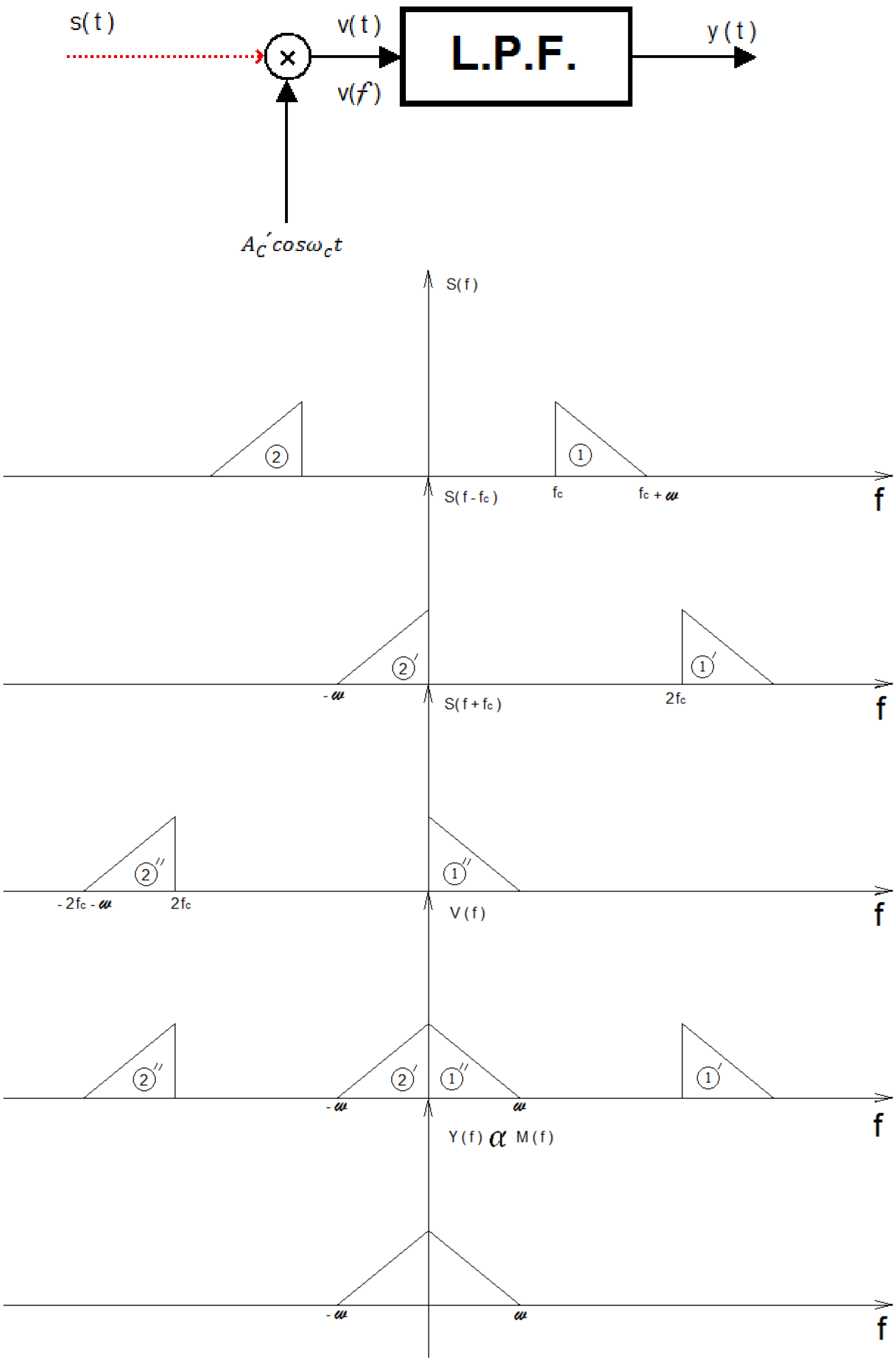
$$\boldsymbol{y(t) = \frac{A_C A_C'}{2} m(t)}$$

The following steps demonstrate the demodulation process viewed in the frequency domain .

$$V(f) = \frac{A_C'}{2} S(f - f_C) + \frac{A_C'}{2} S(f + f_C)$$

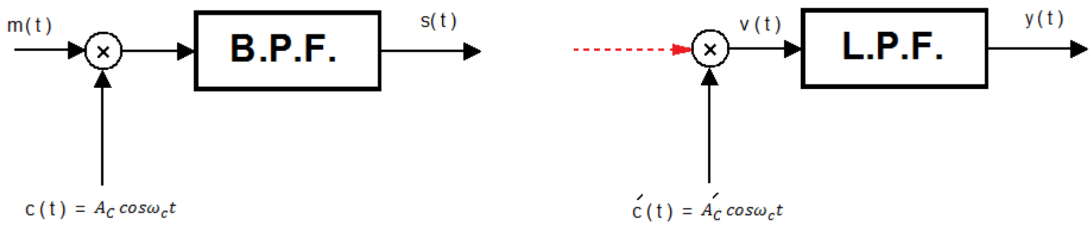
$$Y(f) = Low\ pass \left\{\frac{A_C'}{2} S(f - f_C) + \frac{A_C'}{2} S(f + f_C)\right\}$$

Why One Side Band is Enough: A Frequency-Domain Perspective



Demodulation of SSB: Coherent Demodulation

a. Perfect Coherence



when $c(t) = A_c \cos\omega_c t$, $\acute{c}(t) = \acute{A}_c \cos\omega_c t$,

we have perfect coherence and

$$\boldsymbol{y(t) = \frac{A_c\acute{A}_c}{2} m(t)}$$

as was derived earlier.

b. Constant Phase Difference

The local oscillator takes the form

$$\acute{c}(t) = \acute{A}_c \cos(\omega_c t + \emptyset);$$

$$v(t) = [A_c m(t)\cos\omega_c t - A_c\hat{m}(t)\sin\omega_c t]\acute{A}_c \cos(\omega_c t + \emptyset)$$

$$= A_c\acute{A}_c m(t)\cos\omega_c t\cos(\omega_c t + \emptyset) - A_c\acute{A}_c\hat{m}(t)\sin\omega_c t\cos(\omega_c t + \emptyset)$$

$$= \frac{A_c\acute{A}_c}{2} m(t)\cos(2\omega_c t + \emptyset) + \frac{A_c\acute{A}_c}{2} m(t)\cos(\emptyset)$$

$$-\frac{A_c\acute{A}_c}{2}\hat{m}(t)\cos(2\omega_c t + \emptyset) - \frac{A_c\acute{A}_c}{2}\hat{m}(t)\sin(\emptyset)$$

➔ $$\boldsymbol{y(t) = \frac{A_c\acute{A}_c}{2} m(t)\cos(\emptyset) - \frac{A_c\acute{A}_c}{2}\hat{m}(t)\sin(\emptyset)}$$

Note that there is a distortion due to the appearance of the Hilbert transform of the message signal at the output.

c. $\acute{c}(t) = \acute{A}_c \cos\ 2\pi(f_c + \Delta f)t;$ Constant frequency shift

$$v(t) = [A_c m(t)\cos\omega_c t - A_c\hat{m}(t)\sin\omega_c t]\acute{A}_c \cos\ 2\pi(f_c + \Delta f)t$$

$$= \frac{A_c \acute{A}_c}{2} m(t)[\cos(2\omega_c + \Delta\omega)t + \cos 2\pi\Delta f t]$$

$$- \frac{A_c \acute{A}_c}{2} \widehat{m}(t)[\sin(2\omega_c + \Delta\omega)t - \sin 2\pi\Delta f t]$$

➔ $$\boldsymbol{y(t) = \frac{A_c \acute{A}_c}{2} m(t) \cos 2\pi\Delta f t + \frac{A_c \acute{A}_c}{2} \widehat{m}(t) \sin 2\pi\Delta f t}$$

Once again we have distortion and m(t) appears as if single sideband modulated on a carrier frequency = Δ f.

Example :

Let m(t) = $cos2\pi(1000)t$, $\Delta f =$ 100Hz and let s (t) be an upper sideband signal. Then,

$$y(t) = \frac{A_c \acute{A}_c}{2} cos2\pi(1000)t\ cos2\pi(100)t + \frac{A_c \acute{A}_c}{2} sin2\pi(1000)t\ \ sin2\pi(100)t$$

$$\cos(a - b) = \cos a \cos b + \sin a \sin b$$

$$y(t) = \cos 2\pi(900)t \neq \cos 2\pi(1000)t$$

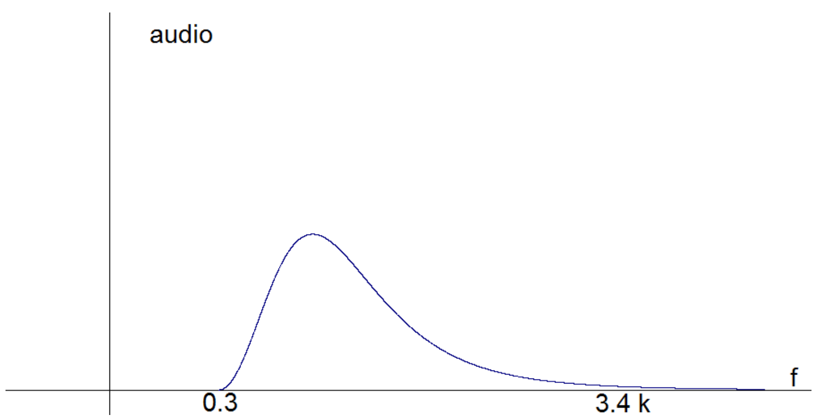
→ Distortion

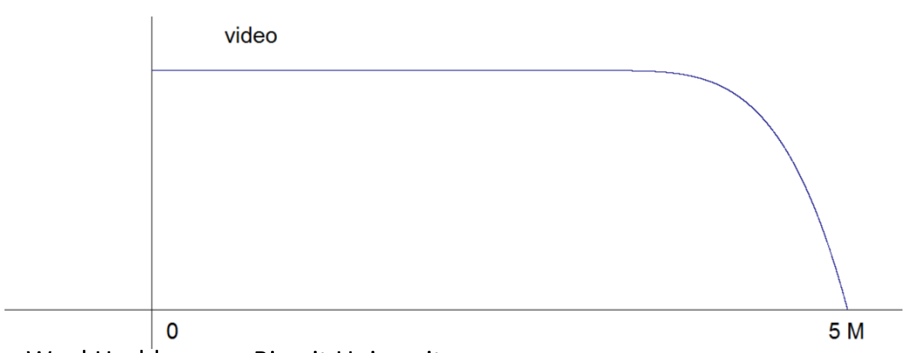
So, a message component with f = 1000Hz appears as a 900Hz component at the demodulator output.

Again, distortion occurs because of failing to synchronize the transmitter and receiver carrier frequencies.

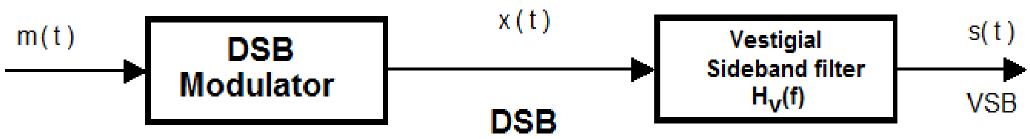
Vestigial Sideband (VSB) Modulation

- This type of modulation finds applications in the transmission of video signal.
- Unlike the audio signal, the video signal is rich in low frequency components around the zero frequency.
- The B.W of a video signal is about 5MHz.
- If a video signal is to be transmitted using DSB, it requires a 10 MHz B.W; too large.
- If a video signal is to be transmitted using SSB (B.W = 5MHz) distortion will results due to the inability to suppress one of the sidebands completely using practical filters.
- A compromise between DSB and SSB was proposed called vestigial sideband modulation.
- Here, a DSB-SC signal is first generated The DSB is applied to a band pass filter (called a *vestigial filter*) that has an asymmetrical frequency response about (-+fc).
- The filter allows one of the sidebands to pass almost without attenuation , while a trace or a vestige of the second sideband is allowed to pass (most of the second sideband is attenuated)
- A typical spectral density of an audio and a video signal is shown below.





Generation of a VSB: Filtering Method

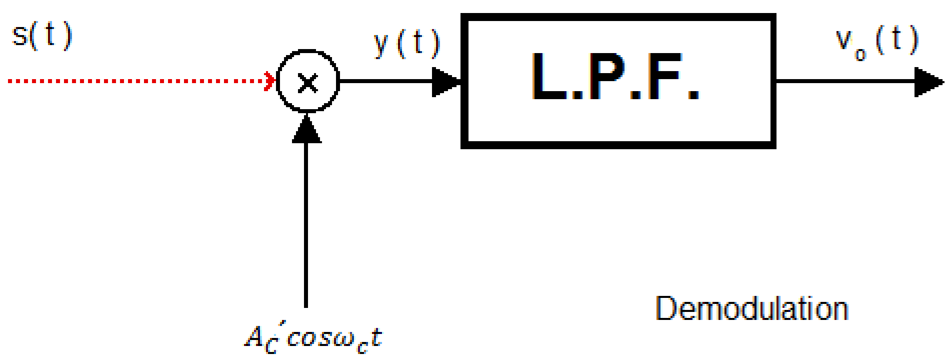


Let $H_v(f)$ be the transfer function of the vestigial filter. We need to find a condition on the characteristic of the filter such that the demodulated signal at the receiver is proportional to the message signal. Now we proceed to find such a condition.

$x(t) = A_c m(t) \cos \omega_c t$; A DSB-SC signal

$S(f) = X(f) H_v(f)$; The Fourier transform of the filter output.

$= \frac{A_c}{2}\{M(f - f_c) + M(f + f_c)\} H_v(f)$; VSB signal



The objective is to specify a condition on $H_v(f)$ such that $V_0(t)$ is an exact replica of m(t) .

$$y(t) = \acute{A}_c s(t) \cos \omega_c t$$

$$Y(f) = \frac{\acute{A}_c}{2}\{S(f - f_c) + S(f + f_c)\}$$

$$= \frac{A_c \acute{A}_c}{4}\{M(f - 2f_c) + M(f)\} H_v(f - f_c)$$

$$+ \frac{A_c \acute{A}_c}{4}\{M(f + 2f_c) + M(f)\} H_v(f + f_c)$$

The LPF will eliminate the high frequency component and retains only the low frequency terms.

$$Vo(f) = \frac{A_c A'_c}{4}\{H_v(f - f_c) + H_v(f + f_c)\}M(f)$$

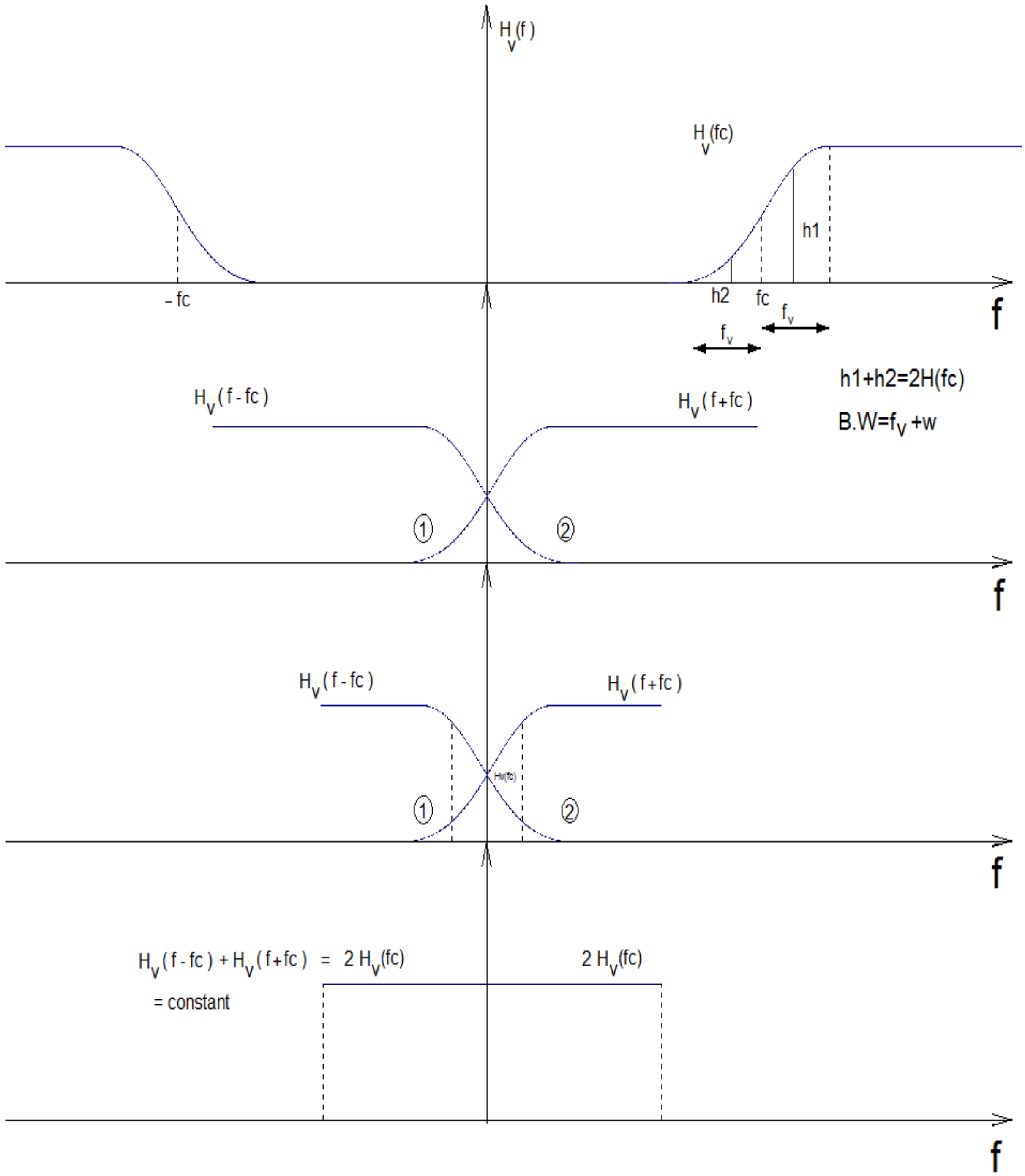
In order for $V_0(f)$ to be proportional to M(f) , we require that

$$H_v(f - f_c) + H_v(f + f_c) = constant = 2H_v(f_c)$$

When this condition is imposed on the filter, the output becomes

$$V_o(f) = \frac{A_c A'_c}{2} H_v(f_c)M(f)$$

$$v_o(t) = \frac{A_c A'_c}{2} H_v(f_c)m(t)$$



Two remarks :

- B.W = $W + f_v$; f_v is the size of the vestige.
- VSB can be demodulated using coherent demodulation.

Generation of VSB: Phase Discrimination Method

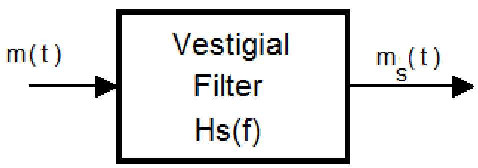
The time-domain representation of a VSB signal is

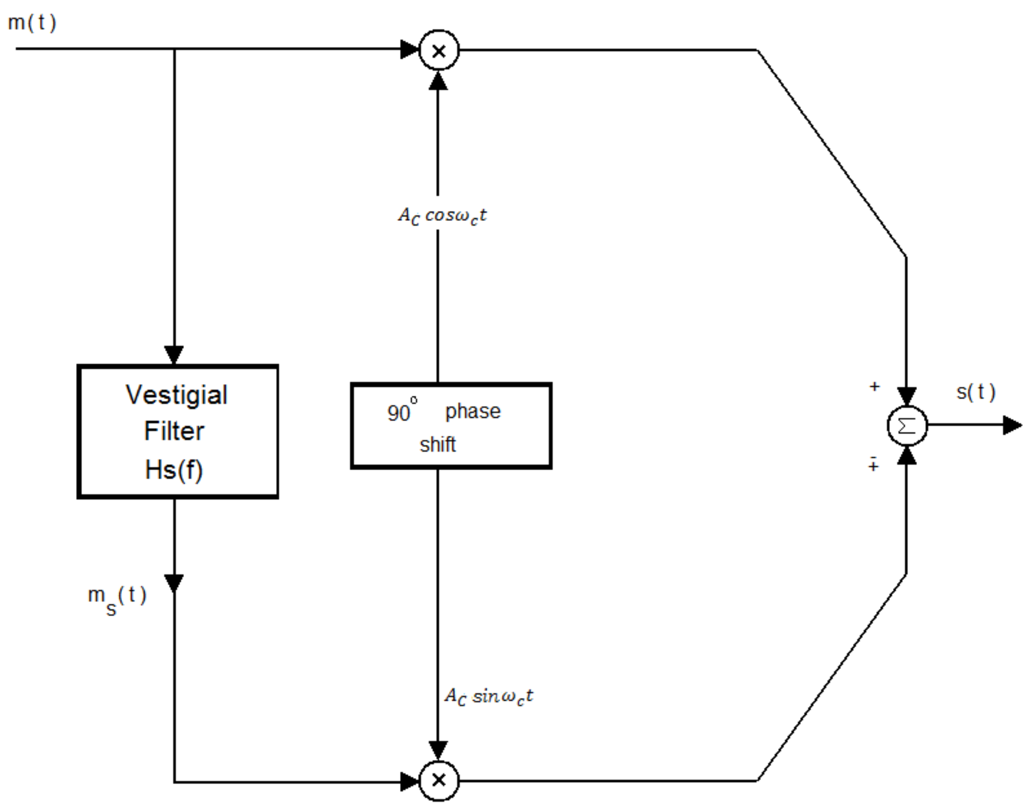
$$s(t) = A_c m(t) \cos \omega_c t \mp A_c m_s(t) \sin \omega_c t$$

Where $m_s(t)$ is the response of a vestigial filter (in the base band spectrum) to the message m(t). Using the time-domain representation ,

the following scheme may be used

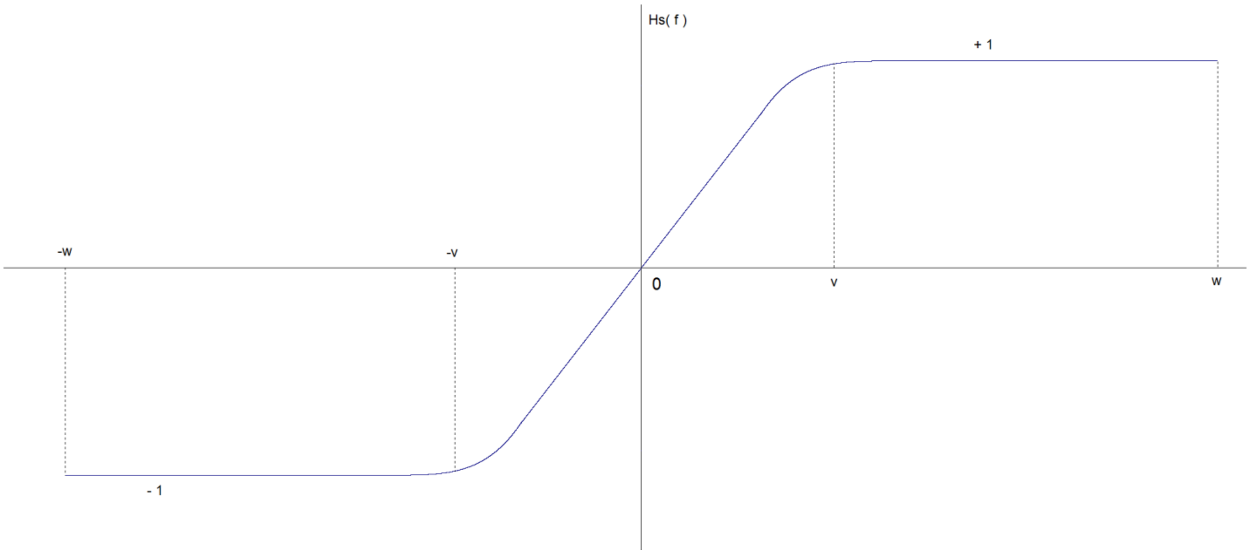
to generate a VSB signal





The – sign means that most of the upper sideband is admitted

+ sign means that most of the lower sideband is admitted

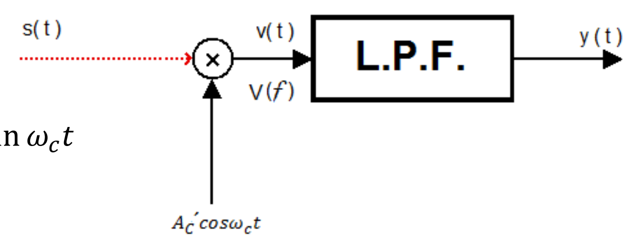


The transfer function HS(f) of the low pass filter is related to the band pass characteristic by:

$$H_s(f) = Low\ pass\ \{H_v(f + f_c) - H_v(f - f_c)\}$$

Coherent Detection of VSB: Time Domain Analysis

Let the received VSB signal be given as:



$$s(t) = A_c m(t) \cos \omega_c t - A_c m_s(t) \sin \omega_c t$$

This signal is mixed with a version of the

transmitted carrier of the same phase and frequency.

$$v(t) = s(t) \acute{A}_c \cos 2\pi f_c t$$

$$= A_c \acute{A}_c [m(t) \cos \omega_c t - m_s(t) \sin \omega_c t] \cos \omega_c t$$

$$= A_c \acute{A}_c m(t) \cos^2 \omega_c t - A_c \acute{A}_c m_s(t) \sin \omega_c t \cos \omega_c t$$

$$= \frac{A_c \acute{A}_c}{2} m(t) + \frac{A_c \acute{A}_c}{2} m(t) \cos 2\omega_c t - \frac{A_c \acute{A}_c}{2} m_s(t) \sin 2\omega_c t$$

The low pass filter admits only the low pass component, which is nothing but a scaled version of the message signal.

$$y(t) = \frac{A_c \acute{A}_c}{2} m(t)$$

Envelope Detection of VSB + Carrier

This type of modulation takes the form:

$$s(t) = carrier + VSB$$

$$s(t) = A_c \cos\omega_c t + A_c \beta m(t) \cos\omega_c t \mp A_c \beta m_s(t) \sin\omega_c t$$

β is a scaling factor chosen to minimize envelope distortion. The addition of the carrier is meant to simplify the demodulation of the video signal in practical TV systems and avoids the complexity of coherent demodulation. It is also less expensive since a simple envelope detector, of the type described in demodulating a normal AM signal, can be used.

$$s(t) = \left(A_c + A_c \beta m(t)\right) \cos\omega_c t \mp A_c \beta m_s(t) \sin\omega_c t$$

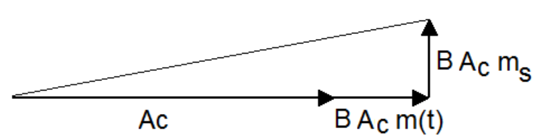
$$s(t) = \sqrt{\left(A_c + A_c \beta m(t)\right)^2 + {A_c}^2 \beta^2 {m_s}^2(t)}\ \cos(\omega_c t + \emptyset)$$

If s(t) is applied to an envelope detector (which is insensitive to phase variations), the output is

$$y(t) = \sqrt{\left(A_c\left(1 + \beta m(t)\right)\right)^2 + {A_c}^2 \beta^2 {m_s}^2(t)}$$

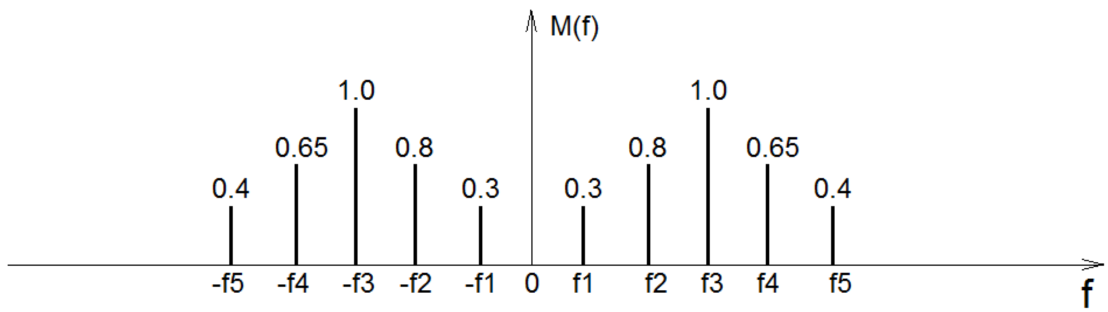
If $A_c \gg \beta m(t)$, then

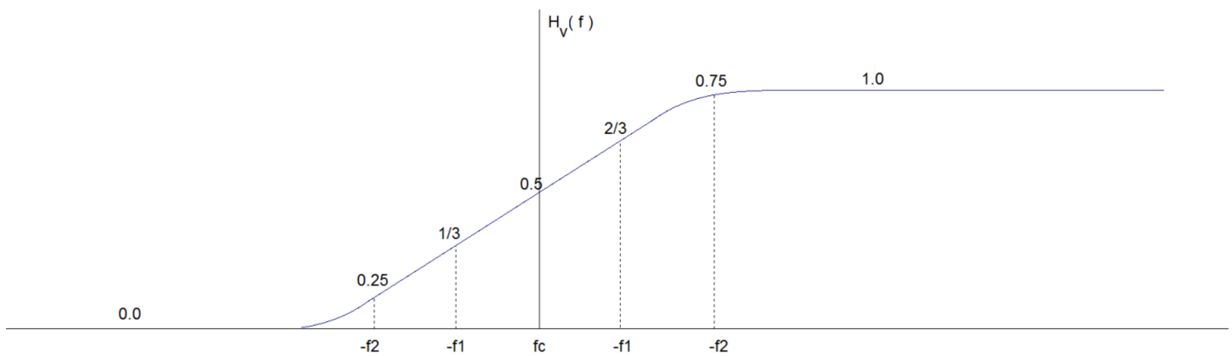
$$y(t) \cong A_c(1 + \beta m(t))$$

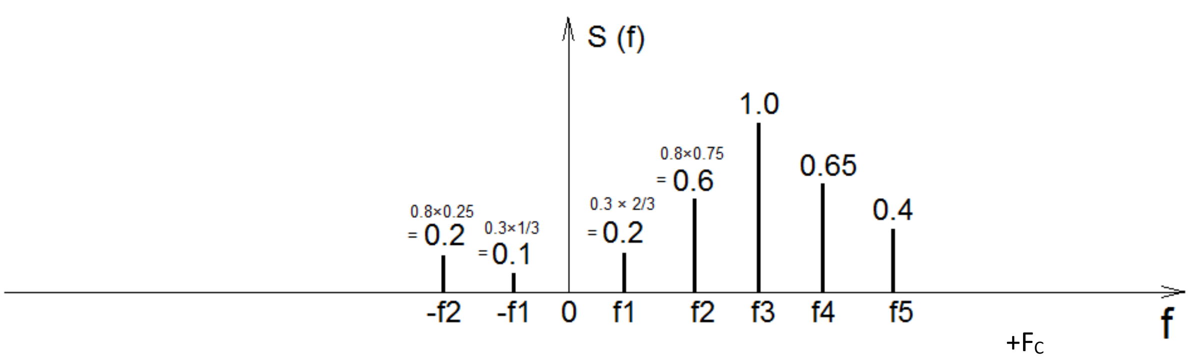


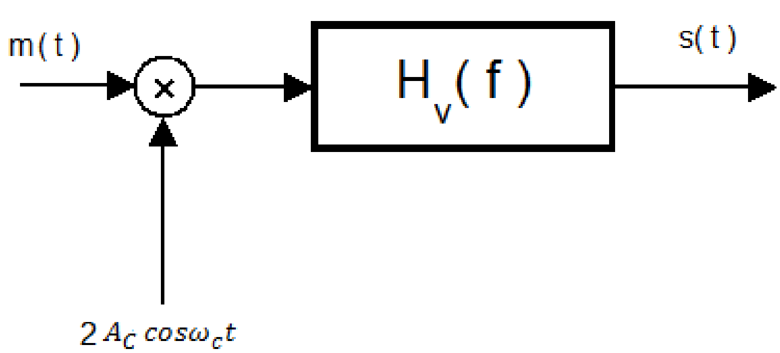
Hence , m(t) can be demodulated, almost without distortion, using simple envelope detection techniques if the above condition is satisfied .

Example: A VSB is generated from the DSB-SC signal $2m(t)\cos\omega_c t$. M(f) and $H_v(f)$ are shown below . Find the spectrum of the transmitted signal s(t).









Baseband Signal

The input signal consists of five frequency components. It is represented as:

$$m(t) = 0.6\cos 2\pi f_1 t + 1.6\cos 2\pi f_2 t$$
$$+2\cos 2\pi f_3 t + 1.3\cos 2\pi f_4 t + 0.8\cos 2\pi f_5 t$$

Transmitted Signal

The spectrum of the transmitted signal is:

$$S(f) = H_v(f)M(f - f_c) + H_v(f)M(f + f_c)$$

If we perform the multiplication in the frequency domain and take the inverse Fourier transform, we get the time domain representation of the transmitted signal.

$$s(t) = 0.4\cos 2\pi(f_c - f_2)t + 0.2\cos 2\pi(f_c + f_1)t$$
$$+0.4\cos 2\pi(f_c + f_1)t + 1.2\cos 2\pi(f_c + f_2)t + 2\cos 2\pi(f_c + f_3)t$$
$$+ 1.3\ \cos 2\pi(f_c + f_4)t + 0.8\ \cos 2\pi(f_c + f_5)t$$

Module 5

Frequency and Phase Modulation

To generate an angle modulated signal, the amplitude of the modulated carrier is held constant, while either the phase or the time derivative of the phase is varied linearly with the message signal m(t).

The expression for an angle modulated signal is

$$s(t) = A_c \cos\big(2\pi f_c t + \theta(t)\big), \qquad f_c \text{ is the carrier frequency in Hz.}$$

The instantaneous frequency of s(t) is :

$$f_i(t) = \frac{1}{2\pi}\frac{d}{dt}\big(2\pi f_c t + \theta(t)\big) = f_c + \frac{1}{2\pi}\frac{d\theta(t)}{dt}$$

For **phase modulation**, the phase is directly proportional to the modulating signal

$$\theta(t) = k_p m(t), \qquad k_p \text{ is the phase sensitivity measured in rad/volt.}$$

The peak phase deviation is

$$\Delta\theta_{max} = k_p \times \max\big(m(t)\big).$$

For **frequency modulation**, the frequency deviation of the carrier is proportional to the modulating signal:

$$\frac{1}{2\pi}\frac{d\theta(t)}{dt} = k_f m(t) \implies \boldsymbol{f_i(t) = f_c + k_f m(t).}$$

The frequency deviation from the un-modulated carrier is

$$f_i(t) - f_c = \frac{1}{2\pi}\frac{d\theta}{dt}$$

The peak frequency deviation is

$$\Delta f = \max\{\frac{1}{2\pi}\frac{d\theta}{dt}\}.$$

The time domain representation of a phase modulated signal is

$$s(t) = A_c \cos\big(2\pi f_c t + k_p m(t)\big).$$

The time domain representation of a frequency modulated signal is

$$s(t) = A_c \cos\Big(2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} m(\alpha)d\alpha\Big).$$

where $\theta(t) = 2\pi k_f \int_{-\infty}^{t} m(\alpha)d\alpha$

The average power in $s(t)$, for frequency modulation (FM) or phase modulation (PM) is:

$$p_{av} = \frac{(A_c)^2}{2} = constant.$$

Example: Binary Frequency Shift Keying

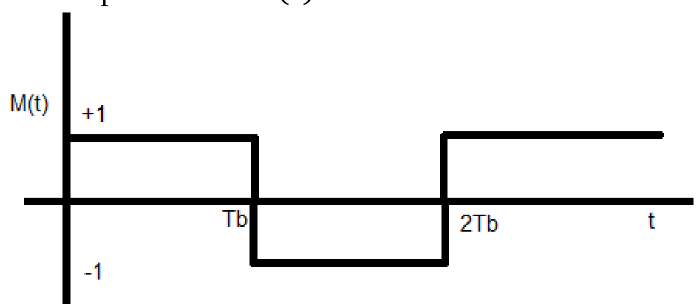
The periodic square signal m(t), shown below, frequency modulates the carrier

$$c(t) = A_c \cos(2\pi 100 t)$$

to produce the FM signal

$$s(t) = A_c \cos\left((2\pi 100 t) + 2\pi k_f \int m(\alpha)d\alpha\right) \text{ where } k_f = 10Hz/V.$$

a. Find and plot the instantaneous frequency $f_i(t)$.
b. Find the time domain expression for $s(t)$.



Solution:

a) The instantaneous frequency is

$f_i = f_c + k_f \times m(t)$

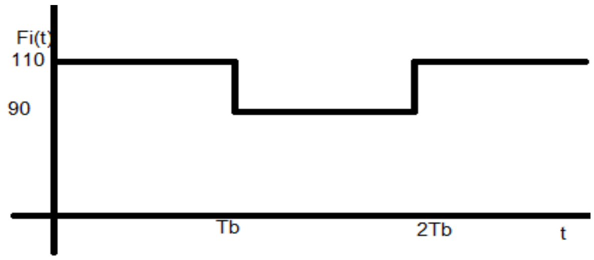
$f_i =$ 100+10 =110 Hz when $m(t) = +1$

$f_i =$ 100 -10 = 90Hz when $m(t) = -1$

For $0 < t \le T_b$, $f_i = 110$ Hz

For $T_b \le t \le 2T_b$, $f_i = 90$ Hz

The instantaneous frequency hops between the two values 110 Hz and 90 Hz as shown below



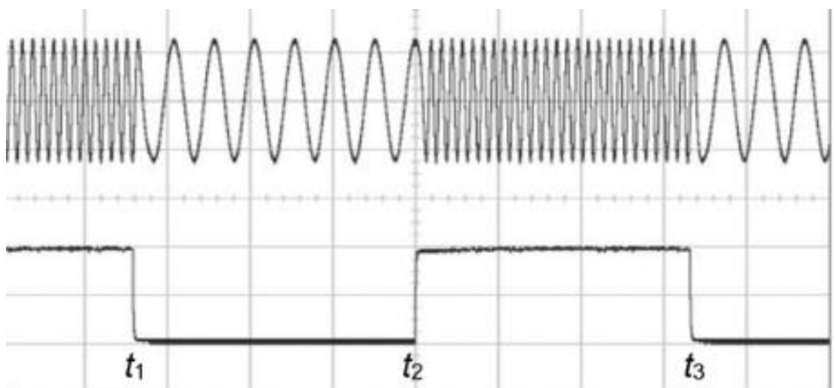
In digital transmission, we will see that a binary (1) may be represent by a signal of frequency f_1 for $0 \le t \le T_b$ and a binary (0) by a signal of frequency f_2 for $0 \le t \le T_b$.

b) Depending on the input binary digit, $s(t)$ may take any one of the following expressions

$$s(t) = A\cos(2\pi(110)t), \quad when\, m(t) = +1$$

$$s(t) = A\cos(2\pi(90)t), \quad when\, m(t) = -1$$

Exercise: Plot the transmitted signal $s(t)$ for $0 \le t \le 4T_b$ assuming $T_b = 10T_c$. You should obtain a figure similar to this figure



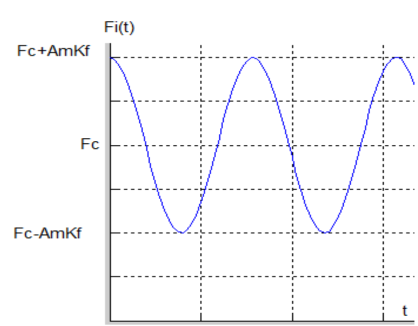
Single Tone Frequency Modulation:

Assume that the message $m(t) = A_m \cos \omega_m t$.

The instantaneous frequency is:

$$f_i = f_c + k_f m(t) = f_c + A_m k_f \cos 2\pi f_m t.$$

This frequency is plotted in the figure.



The peak frequency deviation (from the un-modulated carrier) is :

$$\Delta f = k_f A_m.$$

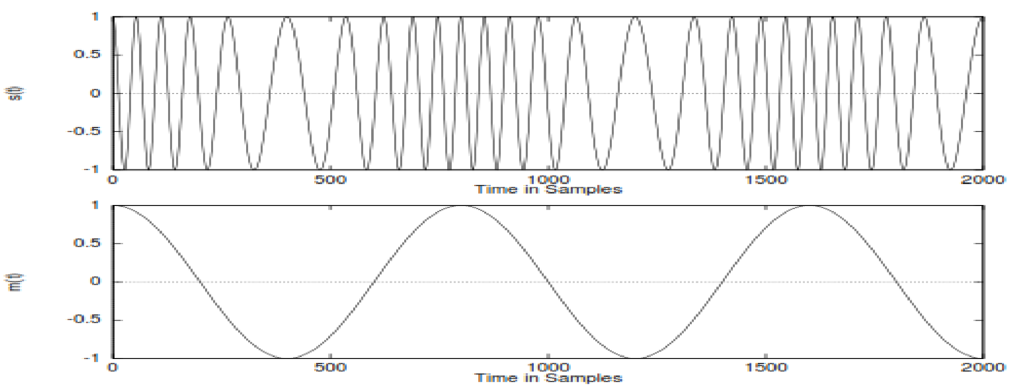
The FM signal is:

$$s(t) = A_c \cos\ (\omega_c t + \beta \sin 2\pi f_m t).$$

Where β is the FM modulation index, defined as

$$\beta = \frac{k_f A_m}{f_m} = \frac{peak\ frequency\ deviation}{message\ bandwidth} = \frac{\Delta f}{f_m}$$

In the figure below, we show a sinusoidal message signal $m(t)$ and the resulting FM signal $s(t)$.



Spectrum of a Single-Tone FM Signal

The objective is to find a meaningful definition of the bandwidth of an FM signal. To accomplish that we need to find the spectrum (the Fourier expansion) of $s(t)$.

Let $m(t) = A_m \cos 2\pi f_m t$ be the message signal, then the FM signal is

$$s(t) = A_c \cos(\ 2\pi f_c t + \beta \sin 2\pi f_m t\ )$$

$s(t)$ can be rewritten as:

$$s(t) = Re\{e^{j(2\pi f_c t + \beta \sin 2\pi f_m t}\}$$

$$= Re\{e^{j(2\pi f_c t)} \times e^{j(\beta \sin 2\pi f_m t)}\}$$

Remember that: $e^{j\theta} = cos\theta + jsin\ \theta$ and that $cos\theta = Re\{e^{j\theta}\}$

The sinusoidal waveform ($\beta \sin 2\pi f_m t$) is periodic with period $T_m = \frac{1}{f_m}$. The exponential function $e^{j(\beta \sin 2\pi f_m t)}$ is also periodic with the same period $T_m = \frac{1}{f_m}$ (but is not sinusoidal)

As we know, a periodic function g(t) can be expanded into a complex Fourier series as:

$$g(t) = \sum_{-\infty}^{\infty} C_n \, e^{jn\omega_m t} \quad .$$

where,

$$C_n = \frac{1}{T_m} \int_0^{T_m} g(t) \, e^{-j\omega_m nt} \, dt$$

Now, let $g(t) = e^{j(\beta \sin 2\pi f_m t)}$

then, $C_n = \frac{1}{T_m} \int_0^{T_m} e^{j(\beta \sin 2\pi f_m t)} \, (e^{-j\omega_m nt}) \, dt$

It turns out that the Fourier coefficients $\boldsymbol{C_n = J_n(\beta)}$.

where $J_n(\beta)$ is the Bessel function of the first kind of order n.

Hence,

$$g(t) = \sum_{-\infty}^{\infty} J_n(\beta) \, e^{jn\omega_m t} \, ;$$

Substituting $g(t)$ into $s(t)$, we get

$$s(t) = A_c \, Re\{e^{j(2\pi f_c t)} \times \sum_{-\infty}^{\infty} J_n(\beta) \, e^{jn\omega_m t} \}$$

$$= A_c \, Re\{ \sum_{-\infty}^{\infty} J_n(\beta) \times e^{j2\pi(f_c + nf_m)t} \}$$

$$= A_c \sum_{-\infty}^{\infty} J_n(\beta) \times \cos(2\pi(f_c + nf_m)t)$$

Finally, the FM signal can be represented as

$$\boldsymbol{s(t) = A_c \sum_{-\infty}^{\infty} J_n(\beta) \cos(2\pi(f_c + nf_m)t)}$$

Bessel Functions

The Bessel equation of order n is

$$x^2 \frac{dy^2}{dx_2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

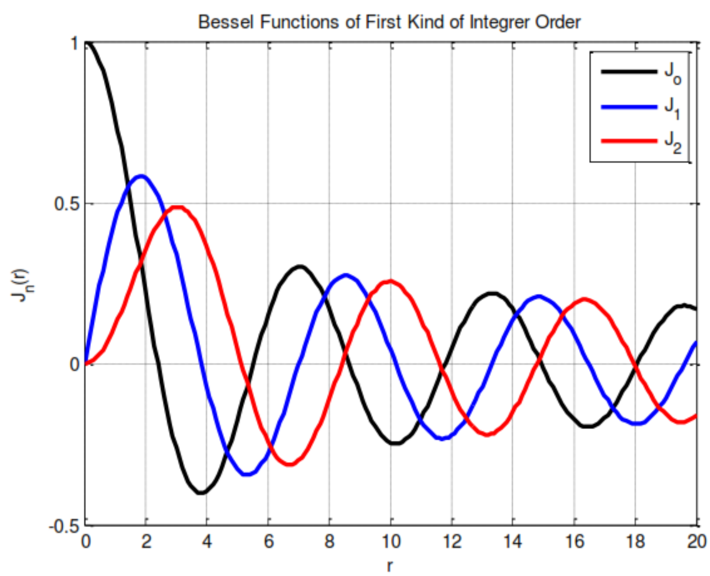
This is a second order differential equation with variable coefficients. We can solve it using the power series method. In this method, we let

$$y = \sum_{n=0}^{\infty} C_n\ x^n,\ \ \frac{dy}{dx} = \sum_{n=1}^{\infty} n\, C_n\ x^{n-1},\ \ \frac{dy^2}{dx_2} = \sum_{n=2}^{\infty} n(n-1)C_n\ x^{n-2}\,.$$

Substituting $y, \frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ into the differential equation and equating terms of equal power results in

$$y(x) = J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \times (\frac{1}{2}x)^{n+2m}}{m!\,(n+m)!}$$

The solution for each value of n (see the D.E where n appears) is $J_n(x)$, the Bessel function of the first kind of order n. The figure, below, shows the first three Bessel functions.



Some Properties of $J_n(x)$:

1- $J_n(x) = (-1)^n J_{-n}(x)$.

2- $J_n(x) = (-1)^n J_n(-x)$.

3- Recurrence formula

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)\,.$$

4- For small values of x, $J_n(x) \cong \dfrac{x^n}{2^n\ n!}$

Therefore, $J_0(x) \cong 1$

$J_1(x) \cong \frac{x}{2}$.

$J_n(x) \cong 0 \;\; for\; n > 1$

5- For large value of x:

$J_n(x) \cong \sqrt{\frac{2}{\pi x}}\cos(x - \frac{\pi}{4} - \frac{n\pi}{2})$, $J_n(x)$ behaves like a sine function with progressively decreasing amplitude.

6- For real x and fixed, $J_n(x) \longrightarrow 0 \; as\; n \longrightarrow \infty$.

7- $\sum_{-\infty}^{\infty}(J_n(x))^2 = 1$, $for\; all\; x$.

Table of Bessel Functions

β	$J_0(\beta)$	$J_1(\beta)$	$J_2(\beta)$	$J_3(\beta)$	$J_4(\beta)$	$J_5(\beta)$	$J_6(\beta)$	$J_7(\beta)$	$J_8(\beta)$	$J_9(\beta)$	$J_{10}(\beta)$
0	1	0	0	0	0	0	0	0	0	0	0
0.1	0.9975	0.0499	0.0012	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.2	0.9900	0.0995	0.0050	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.3	0.9776	0.1483	0.0112	0.0006	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.4	0.9604	0.1960	0.0197	0.0013	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.5	0.9385	0.2423	0.0306	0.0026	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.6	0.9120	0.2867	0.0437	0.0044	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.7	0.8812	0.3290	0.0588	0.0069	0.0006	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.8	0.8463	0.3688	0.0758	0.0102	0.0010	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
0.9	0.8075	0.4059	0.0946	0.0144	0.0016	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.7652	0.4401	0.1149	0.0196	0.0025	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000
1.1	0.7196	0.4709	0.1366	0.0257	0.0036	0.0004	0.0000	0.0000	0.0000	0.0000	0.0000
1.2	0.6711	0.4983	0.1593	0.0329	0.0050	0.0006	0.0001	0.0000	0.0000	0.0000	0.0000
1.3	0.6201	0.5220	0.1830	0.0411	0.0068	0.0009	0.0001	0.0000	0.0000	0.0000	0.0000
1.4	0.5669	0.5419	0.2074	0.0505	0.0091	0.0013	0.0002	0.0000	0.0000	0.0000	0.0000
1.5	0.5118	0.5579	0.2321	0.0610	0.0118	0.0018	0.0002	0.0000	0.0000	0.0000	0.0000
1.6	0.4554	0.5699	0.2570	0.0725	0.0150	0.0025	0.0003	0.0000	0.0000	0.0000	0.0000
1.7	0.3980	0.5778	0.2817	0.0851	0.0188	0.0033	0.0005	0.0001	0.0000	0.0000	0.0000
1.8	0.3400	0.5815	0.3061	0.0988	0.0232	0.0043	0.0007	0.0001	0.0000	0.0000	0.0000
1.9	0.2818	0.5812	0.3299	0.1134	0.0283	0.0055	0.0009	0.0001	0.0000	0.0000	0.0000
2	0.2239	0.5767	0.3528	0.1289	0.0340	0.0070	0.0012	0.0002	0.0000	0.0000	0.0000
2.1	0.1666	0.5683	0.3746	0.1453	0.0405	0.0088	0.0016	0.0002	0.0000	0.0000	0.0000
2.2	0.1104	0.5560	0.3951	0.1623	0.0476	0.0109	0.0021	0.0003	0.0000	0.0000	0.0000
2.3	0.0555	0.5399	0.4139	0.1800	0.0556	0.0134	0.0027	0.0004	0.0001	0.0000	0.0000
2.4	0.0025	0.5202	0.4310	0.1981	0.0643	0.0162	0.0034	0.0006	0.0001	0.0000	0.0000
2.5	-0.0484	0.4971	0.4461	0.2166	0.0738	0.0195	0.0042	0.0008	0.0001	0.0000	0.0000
2.6	-0.0968	0.4708	0.4590	0.2353	0.0840	0.0232	0.0052	0.0010	0.0002	0.0000	0.0000
2.7	-0.1424	0.4416	0.4696	0.2540	0.0950	0.0274	0.0065	0.0013	0.0002	0.0000	0.0000
2.8	-0.1850	0.4097	0.4777	0.2727	0.1067	0.0321	0.0079	0.0016	0.0003	0.0000	0.0000
2.9	-0.2243	0.3754	0.4832	0.2911	0.1190	0.0373	0.0095	0.0020	0.0004	0.0001	0.0000
3	-0.2601	0.3391	0.4861	0.3091	0.1320	0.0430	0.0114	0.0025	0.0005	0.0001	0.0000
3.1	-0.2921	0.3009	0.4862	0.3264	0.1456	0.0493	0.0136	0.0031	0.0006	0.0001	0.0000
3.2	-0.3202	0.2613	0.4835	0.3431	0.1597	0.0562	0.0160	0.0038	0.0008	0.0001	0.0000
3.3	-0.3443	0.2207	0.4780	0.3588	0.1743	0.0637	0.0188	0.0047	0.0010	0.0002	0.0000
3.4	-0.3643	0.1792	0.4697	0.3734	0.1892	0.0718	0.0219	0.0056	0.0012	0.0002	0.0000
3.5	-0.3801	0.1374	0.4586	0.3868	0.2044	0.0804	0.0254	0.0067	0.0015	0.0003	0.0001
3.6	-0.3918	0.0955	0.4448	0.3988	0.2198	0.0897	0.0293	0.0080	0.0019	0.0004	0.0001
3.7	-0.3992	0.0538	0.4283	0.4092	0.2353	0.0995	0.0336	0.0095	0.0023	0.0005	0.0001
3.8	-0.4026	0.0128	0.4093	0.4180	0.2507	0.1098	0.0383	0.0112	0.0028	0.0006	0.0001
3.9	-0.4018	-0.0272	0.3879	0.4250	0.2661	0.1207	0.0435	0.0130	0.0034	0.0008	0.0002
4	-0.3971	-0.0660	0.3641	0.4302	0.2811	0.1321	0.0491	0.0152	0.0040	0.0009	0.0002
4.1	-0.3887	-0.1033	0.3383	0.4333	0.2958	0.1439	0.0552	0.0176	0.0048	0.0011	0.0002
4.2	-0.3766	-0.1386	0.3105	0.4344	0.3100	0.1561	0.0617	0.0202	0.0057	0.0014	0.0003
4.3	-0.3610	-0.1719	0.2811	0.4333	0.3236	0.1687	0.0688	0.0232	0.0067	0.0017	0.0004
4.4	-0.3423	-0.2028	0.2501	0.4301	0.3365	0.1816	0.0763	0.0264	0.0078	0.0020	0.0005
4.5	-0.3205	-0.2311	0.2178	0.4247	0.3484	0.1947	0.0843	0.0300	0.0091	0.0024	0.0006
4.6	-0.2961	-0.2566	0.1846	0.4171	0.3594	0.2080	0.0927	0.0340	0.0106	0.0029	0.0007
4.7	-0.2693	-0.2791	0.1506	0.4072	0.3693	0.2214	0.1017	0.0382	0.0122	0.0034	0.0008
4.8	-0.2404	-0.2985	0.1161	0.3952	0.3780	0.2347	0.1111	0.0429	0.0141	0.0040	0.0010
4.9	-0.2097	-0.3147	0.0813	0.3811	0.3853	0.2480	0.1209	0.0479	0.0161	0.0047	0.0012
5	-0.1776	-0.3276	0.0466	0.3648	0.3912	0.2611	0.1310	0.0534	0.0184	0.0055	0.0015

The Fourier Series Representation of the FM Signal

We saw earlier that a single tone FM signal can be represented in a Fourier series as

$$\boldsymbol{s(t) = A_c \sum_{-\infty}^{\infty} J_n(\beta)\cos(2\pi(f_c + nf_m)t)}$$

The first few terms in this expansion are:

$$s(t) = A_c\,\{J_0(\beta)\cos(2\pi f_c t) + J_1(\beta)\cos 2\pi(f_c + f_m)t + J_{-1}(\beta)\cos 2\pi(f_c - f_m)t \ + \ J_2(\beta)\cos 2\pi(f_c + 2f_m)t \ + J_{-2}(\beta)\cos 2\pi(f_c - 2f_m)t + \cdots..$$

The FM signal consists of an infinite number of spectral components concentrated around f_c. Therefore, the theoretical bandwidth of the signal is infinity. That is to say, if we need to recover the FM signal without any distortion, all spectral components must be accommodated. This means that a channel with infinite bandwidth is needed. This is, of course, not practical since the frequency spectrum is shared by many users.

In the following discussion we need to truncate the series so that say 99% of the total average power is contained within a certain bandwidth. But, first let us find the total average power using the series approach.

Power in the Spectral Components of s(t)

Note that s(t) consists of an infinite number of Fourier terms, and the power in s(t) will be equal the power in the respective Fourier components .

Any term in s (t) takes the form: $A_c J_n(\beta)\cos(2\pi(f_c + nf_m)t)$

The average power in this term is: $\frac{(A_c)^2 (J_n(\beta))^2}{2}$

Hence the total power in s(t) is

$$< S^2(t) > \ = \ \frac{{A_C}^2\,{J_0}^2(\beta)}{2} + \frac{{A_C}^2{J_1}^2(\beta)}{2} + \frac{{A_C}^2{J_{-1}}^2(\beta)}{2} + \frac{{A_C}^2{J_2}^2(\beta)}{2} + \frac{{A_C}^2{J_{-2}}^2(\beta)}{2} + \cdots$$

$$= \frac{{A_C}^2}{2}\,\{{J_0}^2(\beta) + {J_1}^2(\beta) + {J_{-1}}^2(\beta) + {J_2}^2(\beta) + {J_{-2}}^2(\beta) + \cdots..\}$$

$= \frac{{A_C}^2}{2}\,\{\sum_{n=-\infty}^{\infty} {J_n}^2(\beta)\}$, where $\sum_{n=-\infty}^{\infty} {J_n}^2(\beta) \ = 1$, (A property of Bessel Functions).

The average power becomes

$$< S^2(t) > = \frac{A_C{}^2}{2}.$$

Spectrum of an FM Signal

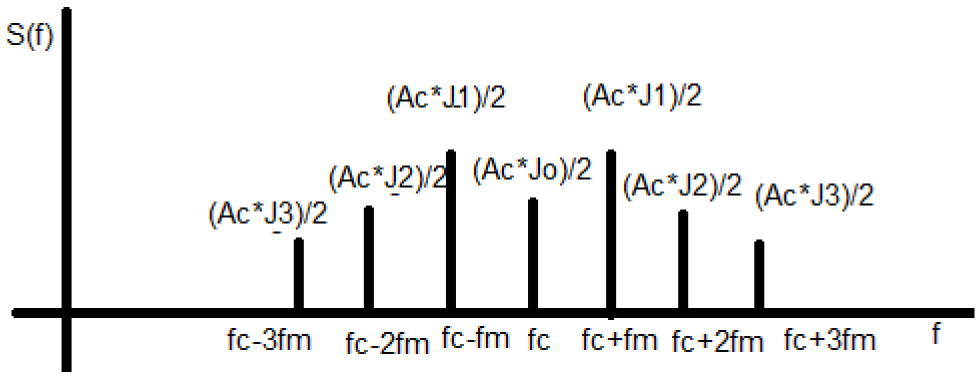


Figure: Fourier transform of $s(t) = A_c \cos\ (\omega_c t + \beta \sin 2\pi f_m t)$ (only + ve frequencies are shown)

Note that in the figure above, as f_m *decreases*, the spectral lines become closely clustered about f_c.

Example: 99% Power Bandwidth of an FM Signal

Find the 99% power bandwidth of an FM signal when $\beta = 1$ and when $\beta = 0.2$

Solution:

$$s(t) = A_c \sum_{-\infty}^{\infty} J_n(\beta) \cos(2\pi(f_c + nf_m)t)$$

Case a: $\boldsymbol{\beta = 1}$ (wideband FM)

The first five terms corresponding to $\beta = 1$ (obtained from the table) are
$J_0(1) = 0.7652$, $J_1(1) = 0.4401$, $J_2(1) = 0.1149$, $J_3(1) = 0.01956$, $J_4(1) = 0.002477$

The power in s(t) is $< S^2(t) > = \frac{A_C{}^2}{2}$

Let us try to find the average power in the terms at (f_c), $(f_c + f_m)$, $(f_c - f_m)$, $(f_c + 2f_m)$, $(f_c - 2f_m)$

1. f_c : $\frac{A_C{}^2 J_0{}^2(\beta)}{2}$

2. $f_c + f_m$: $\frac{A_C{}^2 J_1{}^2(\beta)}{2}$
3. $f_c - f_m$: $\frac{A_C{}^2 J_{-1}{}^2(\beta)}{2}$
4. $f_c + 2f_m$: $\frac{A_C{}^2 J_2{}^2(\beta)}{2}$
5. $f_c - 2f_m$: $\frac{A_C{}^2 J_{-2}{}^2(\beta)}{2}$

The average power in the five spectral components is the sum

$$P_{av} = \frac{A^2{}_c}{2}\left[J_0{}^2(1) + 2J_1{}^2(1) + 2J_2{}^2(1)\right]$$

$$P_{av} = \frac{A^2{}_c}{2}\ [(0.7652)^2 + 2 * (0.4401)^2 + (0.1149)^2] = \ 0.9993\ \frac{A^2{}_c}{2}$$

Hence, these terms contain 99.9 % of the total power. Therefore, the 99.9 % power bandwidth is

$$BW = (f_c + 2f_m) - (f_c - 2f_m) = 4f_m$$

Case b: $\beta = 0.2$ (Narrowband FM)

For $\beta = 0.2,\ J_0(0.2) = 0.99,\ J_1(0.2) = 0.0995,\ J_2(0.2) = 0.00498335$

The power in the carrier and the two sidebands at (f_c , $f_c + f_m$, $f_c - f_m$) is

$$P = \frac{A^2{}_c}{2}\left[J_0{}^2(0.2) + 2J_1{}^2(0.2)\right]$$

$$P = \frac{A^2{}_c}{2}[0.9999]$$

Therefore, 99.99% of the total power is found in the carrier and the two sidebands. The 99% bandwidth is

$$B.W = (f_c + f_m) - (f_c - f_m) = 2f_m$$

Remark:

We observe that the spectrum of an FM signal when $\beta \ll 1$ (called narrow band FM) is "similar" to the spectrum of a normal AM signal, in the sense that it consists of a carrier and two sidebands. The transmission bandwidth of both signals is $2f_m$.

Carson's Rule

A 98% power B.W of an FM signal can be estimated using Carson's rule

$$\boldsymbol{B_T = 2(\beta + 1)f_m}$$

Generation of an FM Signal

The Narrowband FM Signal

Consider an angle-modulated signal

$$s(t) = A_c \cos(2\pi f_c t + \theta(t))$$

When $s(t)$ is an FM signal,

$$\theta(t) = 2\pi k_f \int m(t)dt$$

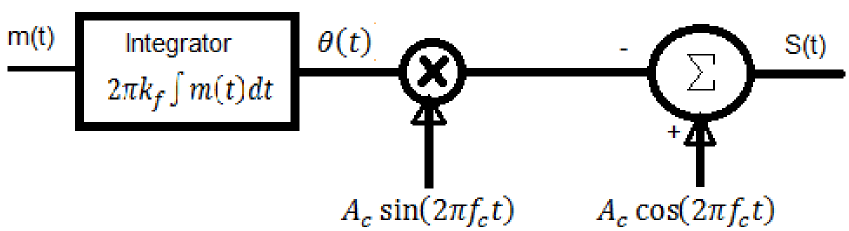
$s(t)$ can be expanded as

$$s(t) = A_c \cos(2\pi f_c t) \cos\big(\theta(t)\big) - A_c \sin(2\pi f_c t) \sin\big(\theta(t)\big)$$

When $|\theta(t)| \ll 1$, $cos\ \theta \cong 1$, $\sin(\theta) \cong \theta$ and $s(t)$, termed narrowband, can be approximated as

$$s(t) \cong A_c \cos(2\pi f_c t) - A_c\theta \sin(2\pi f_c t)$$

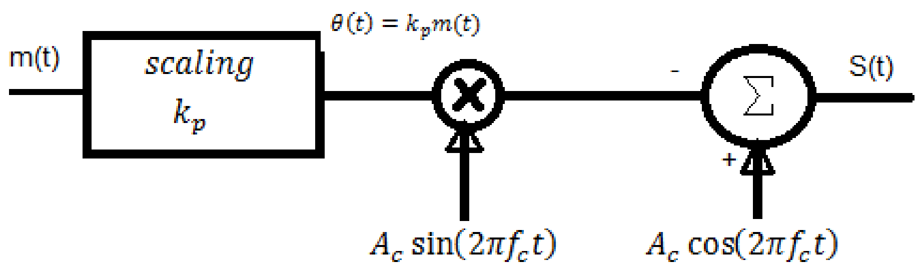
Using this expression, one can generate a narrowband FM or PM signals. This is illustrated in the block diagram below:



When $m(t) = A_m \cos(2\pi f_m t)$, $\theta(t) = \beta \sin(2\pi f_m t)$. The modulated signal takes the form

$$s(t) = A_c \cos(2\pi f_c t) - A_c\beta \sin(2\pi f_m t) \sin(2\pi f_c t)\ ;\ NBFM$$

To generate a narrow band PM signal, we can use the scheme:



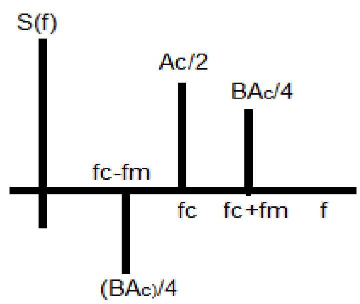
Spectrum of a Single-Tone NBFM

For an FM signal, $\theta(t) = \beta \sin(2\pi f_m t)$

$$s(t) = A_c \cos(2\pi f_c t) - A_c \beta \sin(2\pi f_m t) \sin(2\pi f_c t)$$

$$s(t) = A_c \cos(2\pi f_c t) - \frac{A_c \beta}{2} [\cos(2\pi(f_c - f_m)t) - \cos(2\pi(f_c + f_m)t)]$$

The spectrum of $s(t)$ is shown below



The spectrum consists of a component at the carrier frequency f_c and two components at $(f_c + f_m)$ and $f_c - f_m)$. Note the negative sign at the lower sideband.

The bandwidth of this NBFM signal is $2f_m$.

Frequency Multiplier

It is a device for which the frequency of the output signal is an integer multiple of the frequency of the input signal. It is primarily a nonlinear characteristic followed by a band pass filter. Now we illustrate the operation of this device.

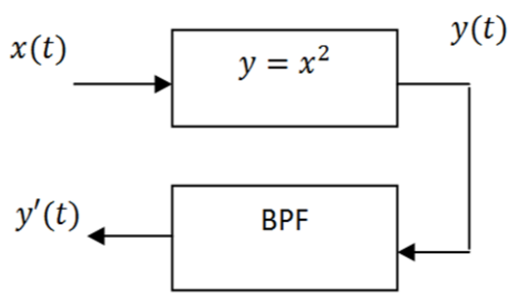
The Square Law Device

Let the input be an FM signal of the form:

$$x(t) = A_c \cos(2\pi f'_c + \beta' \sin 2\pi f_m t) = A_c \cos(\phi)$$

The output of the square law characteristic is:

$$y(t) = x(t)^2 = A_c^2 cos^2(\phi) = \frac{A_c^2}{2}[1 + \cos(2\phi)] = \frac{A_c^2}{2} + \frac{A_c^2}{2}\cos(2\phi)$$
$$= \frac{A_c^2}{2} + \frac{A_c^2}{2} cos\left[2\pi\left(2f'_c\right) + 2\beta' \sin(2\pi f_m t)\right]$$



If $y(t)$ is passed through a BPF of center frequency $2f_c$, then the DC term will be suppressed and the filter output is

$$y'(t) = \frac{A_c^2}{2} cos\left[2\pi\left(2f'_c\right) + 2\beta' \sin(2\pi f_m t)\right]$$

$$y'(t) = \frac{A_c^2}{2} cos[2\pi(f_c) + \beta \sin(2\pi f_m t)]$$

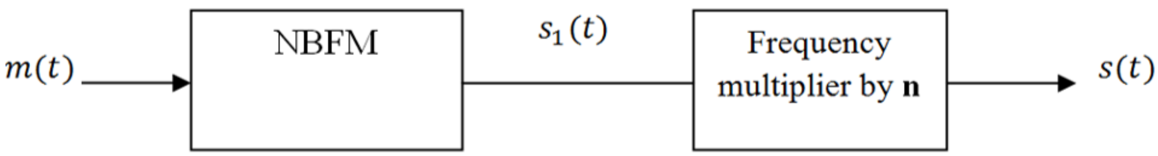
As can be seen from this result, the output is a signal with twice the frequency of the input signal and a modulation index twice that of the input.

$$f_c = 2f'_c; \beta = 2\beta'$$

To get frequency multiplication higher than two, a cascade of units, similar to what was described above, can be formed with the number of stages that achieve the desired frequency.

Indirect Method for Generating a Wideband FM

A wideband FM can be generated indirectly using the block diagram below (Armstrong Method). First, a narrowband FM is generated. Then, the wideband FM is obtained by using frequency multiplication. Next, we analyze the operation of this modulator.



Let $m(t) = A_m \cos 2\pi f_m t$ be the baseband signal, then

$$s_1(t) = A_c \cos(2\pi f_c' t + \beta' \sin 2\pi f_m t) \;\; ; \;\; \beta' = \frac{k_f A_m}{f_m}$$

is a narrowband FM with $\beta' \ll 1$. The frequency of $s_1(t)$ is

$$f'_i = f_c' + k_f A_m \cos 2\pi f_m t$$

Multiplying f_i by n (through frequency multiplication), we get the frequency of $s(t)$ as

$$f_i = n f_c' + n k_f A_m cos 2\pi f_m t$$

This result is

$$s(t) = A_c \cos[2\pi(n f_c')t + n\beta' \sin 2\pi f_m t]$$

$$= A_c \cos[2\pi f_c t + \beta \sin 2\pi f_m t]$$

Where $\beta = n\beta'$ is the desired modulation index of WBFM

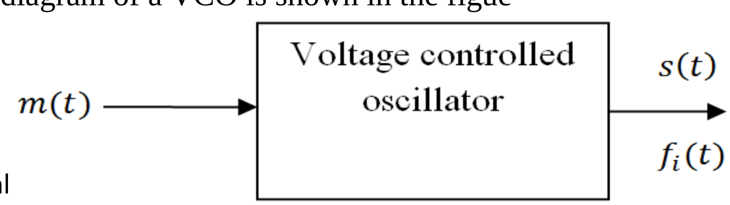
$f_c = n f_c'$ is the desired carrier frequency of WBFM

Direct Method for Generating an FM Signal

In a direct FM system, the instantaneous frequency of the carrier is varied in accordance with a message signal by means of a voltage-controlled oscillator (VCO). The voltage – frequency characteristic of a VCO is given by

$$f_i = f_c + k_f \, m(t)$$

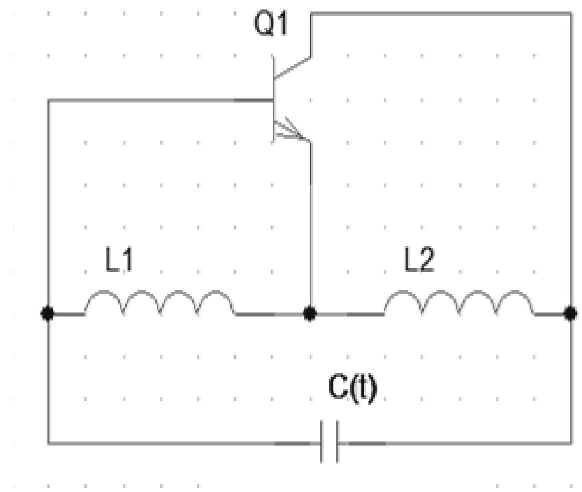
A schematic diagram of a VCO is shown in the figue



A realization of the CVO may be obtained by considering an oscillator (like the Hartley oscillator) shown below in which a varactor ((voltage variable capacitor) is used. The capacitance of the varactor varies in response to variations in the message signal. The variation is linear when the variation in the message is too small.

The frequency of the oscillator is

$$f_i(t) = \frac{1}{2\pi\sqrt{(L_1 + L_2)C(t)}}$$



Hartley Oscillator

Let $C(t) = C_0 - k\, m(t)$ (A diode operating in the reverse bias region can act like a variable capacitor); k is a constant,

When m(t) = 0, $C(t) = C_0$, and the unmodulated frequency of oscillation is

$$f_c = \frac{1}{\sqrt{(L_1+L_2)C_0}}$$

When m(t) has a finite value, the frequency of oscillation is

$$f_i(t) = \frac{1}{2\pi\sqrt{(L_1+L_2)(C_0 - k\, m(t))}}$$

$$= f_c\left(1 - \frac{k\, m(t)}{C_0}\right)^{-1/2}$$

When $\frac{k\, m(t)}{C_0} \ll 1$, we can make the approximation (using the formula $[(1+x)^n \cong 1 + nx]$ when x is small)

$$f_i(t) = f_c\left(1 + \frac{k\, m(t)}{2C_0}\right) = f_c + k_f\, m(t)$$

Here it is clear that the instantaneous frequency varies linearly with the message signal.

Demodulation of the FM Signal

An FM signal may be demodulated by means of what is called a *discriminator*.

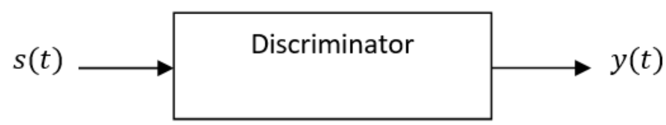
Let $s(t) = A_c \cos\big(\omega_c t + \theta(t)\big)$ be an angle-modulated signal. The output of an ideal discriminator is defined as

$$y(t) = \frac{1}{2\pi} k_D \frac{d\theta}{dt}$$

When $\theta = 2\pi k_f \int_{-\infty}^{t} m(\alpha) d\alpha$, then $\frac{d\theta}{dt} = 2\pi k_f\, m(t)$ and y(t) becomes

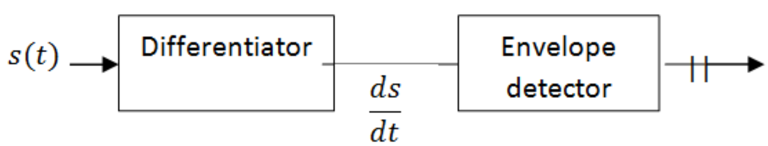
$$y(t) = k_D k_f\, m(t)$$

Hence, a discriminator can demodulate an FM signal. In this section, we will consider three techniques through which an FM (or PM) signal can be demodulated.



First Method: Differentiator Followed by an Envelope Detector

One practical realization of a discriminator is a differentiator followed by an envelope detector, as illustrated in the figure. The operation of this discriminator can be explained as follows



Let $s(t) = A_c \cos\big(\omega_c t + \theta(t)\big)$

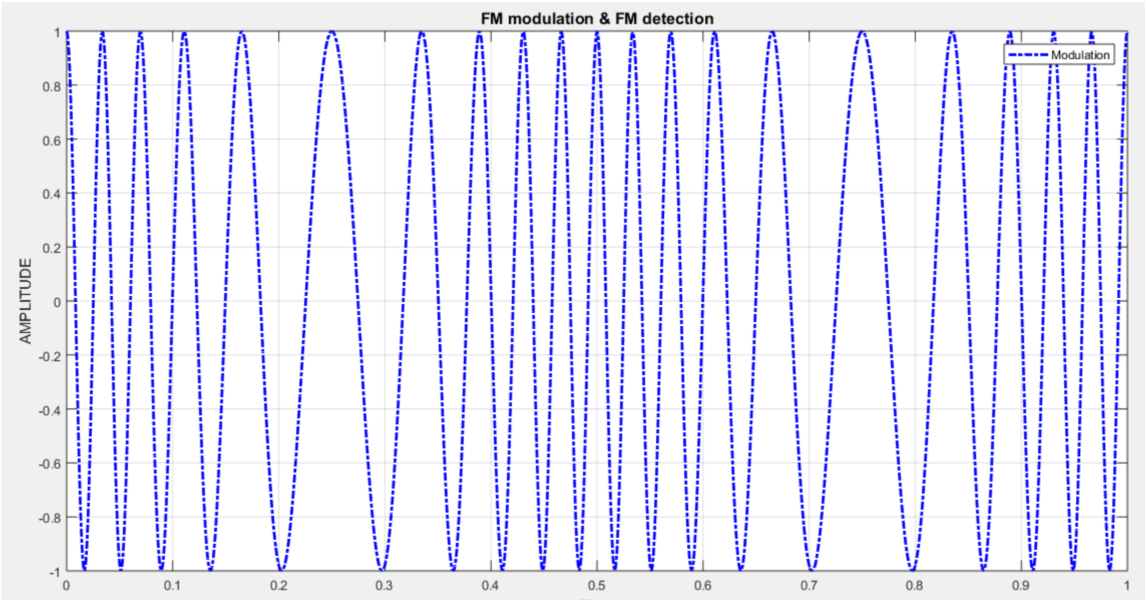
$$\frac{ds(t)}{dt} = -A_c\left(\omega_c + \frac{d\theta}{dt}\right)\sin\big(\omega_c t + \theta(t)\big)$$

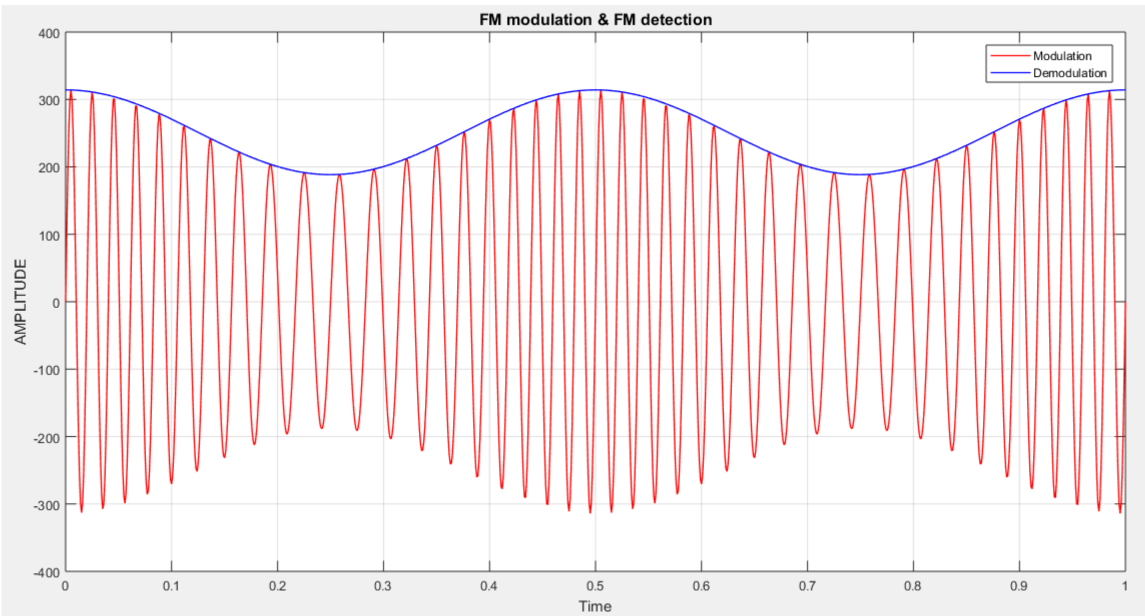
The output of the envelope detector is $A_c \left|\left(\omega_c + \frac{d\theta}{dt}\right)\right|$

The capacitor blocks the DC term and so output is:

$$V_0 = A_c \frac{d\theta}{dt} = 2\pi k_f\, A_c m(t)$$

A typical FM signal and its derivative are shown in the figure below.





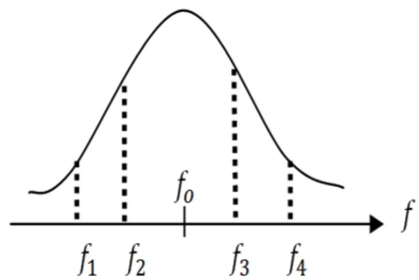
We already know what an envelope detector is (recall the material on the demodulation of the normal AM signal). Now we explain how differentiation is accomplished.

Realization of the Differentiator

From the properties of Fourier transform, we know that if

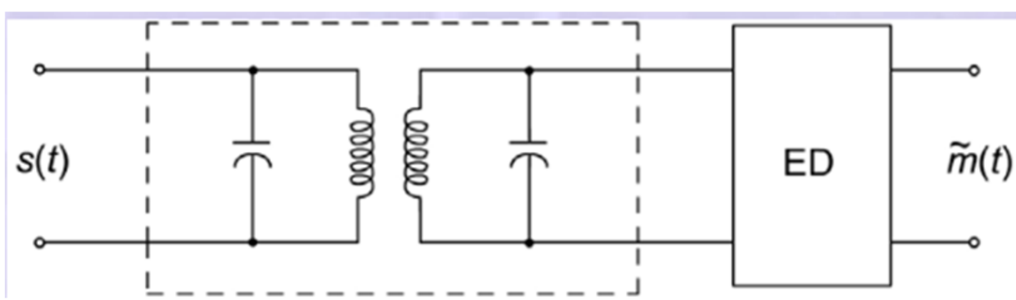
$$\Im\{g(t)\} = G(f), \text{ then}$$

$$\Im\left\{\frac{dg(t)}{dt}\right\} = j2\pi f G(f)$$



This means that multiplication by $j2\pi f$ in the frequency domain amounts to differentiating the signal in the time-domain. Hence, we need a circuit whose frequency response is linear in f to perform time differentiation. A circuit that performs this task is a tuned circuit, provided that the signal frequency variation falls within the linear part of the characteristic, i.e., either between (f_1, f_2) or (f_3, f_4).

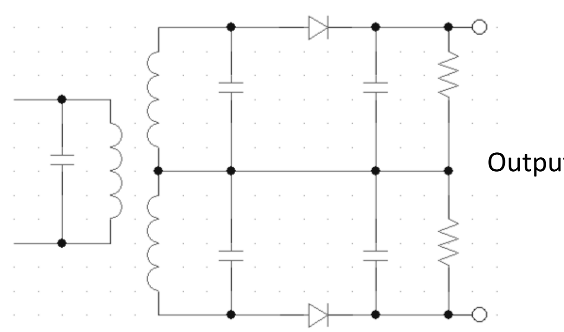
The circuit below is a realization of an FM demodulator. The primary and secondary tuned circuits perform the task of differentiation, while the envelope detector extracts the envelope, which is supposed to be proportional to the message signal.



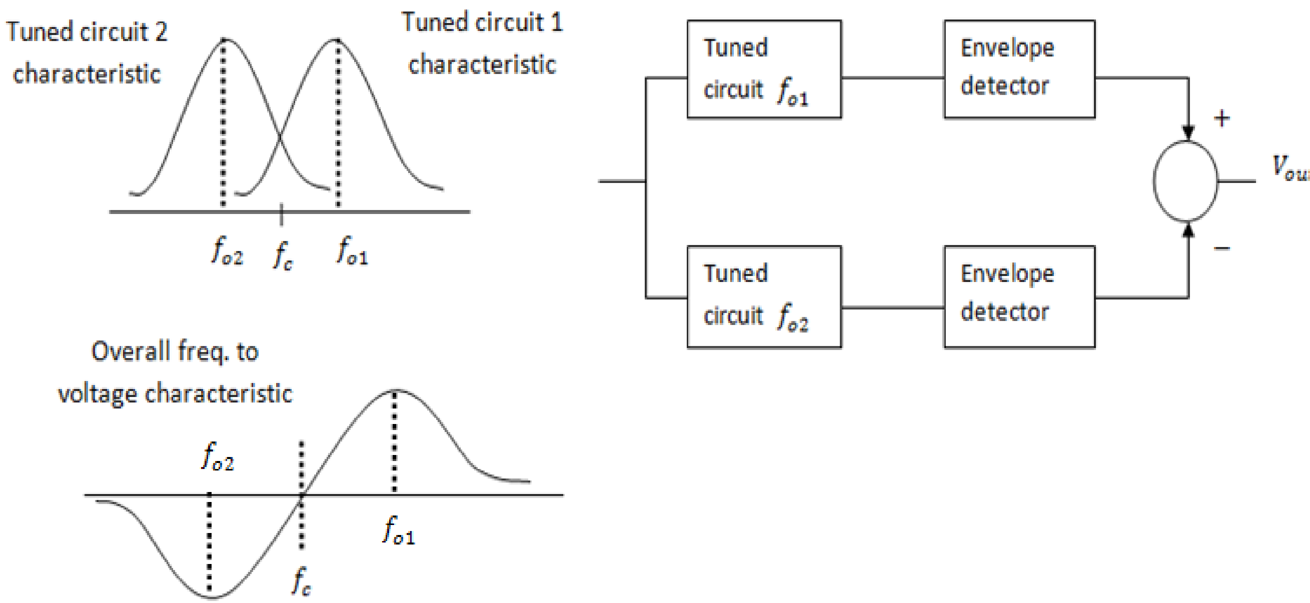
Primary circuit tuned to f_c Secondary circuit tuned to $f_0 > f_c$

Balanced Slope Detector

To extend the dynamic range of the differentiating circuit, two tuned circuits with center frequencies f_{o1} and f_{o2} are used as shown in the figure.



- This circuit has a wider range of linear frequency response.
- No DC blocking is necessary.

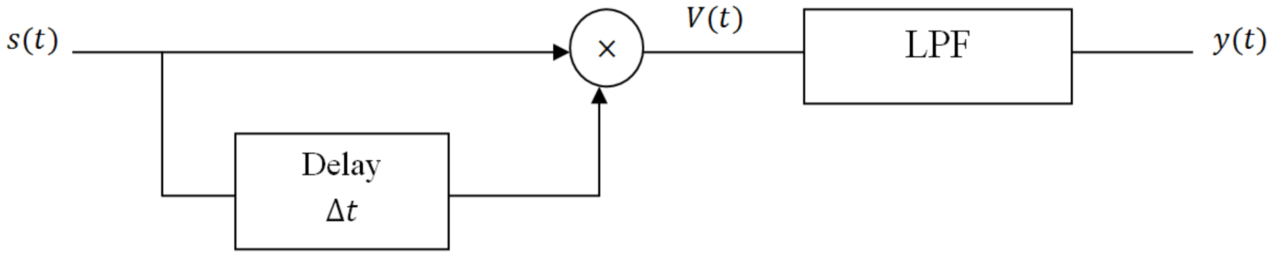

Tuned circuit 2 characteristic
Tuned circuit 1 characteristic
f_{o2}
f_c
f_{o1}
Overall freq. to voltage characteristic
f_{o2}
f_c
f_{o1}
Tuned circuit f_{o1}
Envelope detector
Tuned circuit f_{o2}
Envelope detector
+
−
V_{out}

Method 2: Phase-Locked Loop

ADD A NEW SECTION on PLL

Method 3: Phase Shift Discriminator

The Quadrature Detector: This demodulator converts frequency variations into phase variation and detects the phase changes. The block diagram of the demodulator is shown below



Let $s(t) = A_c \cos\big(2\pi f_c t + \varphi(t)\big)$; $\varphi(t) = 2\pi k_f \int_0^t m(\alpha)\, d\alpha$

$$s(t-\Delta t) = A_c \cos[2\pi f_c (t-\Delta t) + \varphi(t-\Delta t)]$$

$$= A_c \cos[2\pi f_c t - 2\pi f_c \Delta t + \varphi(t-\Delta t)]$$

The delay Δt is chosen such that $2\pi f_c \Delta t = \pi/2$

Hence,

$$s(t-\Delta t) = A_c \cos\left[2\pi f_c t - \frac{\pi}{2} + \varphi(t-\Delta t)\right]$$

$$= A_c \sin[2\pi f_c t + \varphi(t-\Delta t)]$$

$$V(t) = s(t)s(t-\Delta t) = A_c^2 \sin[2\pi f_c t + \varphi(t-\Delta t)]\, cos[2\pi f_c t + \varphi(t)]$$

$$= \frac{A_c^2}{2} \sin[2\pi(2f_c)t + \varphi(t) + \varphi(t-\Delta t)] + \frac{A_c^2}{2} \sin[\varphi(t) - \varphi(t-\Delta t)]$$

The high frequency component is suppressed by the LPF. What remains is the second term

$$\frac{A_c^2}{2} \sin[\varphi(t) - \varphi(t-\Delta t)] \cong \frac{A_c^2}{2} [\varphi(t) - \varphi(t-\Delta t)]$$

where Δt is small to justify the approximation $\sin(x) \cong x$. Hence,

$$y(t) = \frac{A_c^2}{2} [\varphi(t) - \varphi(t-\Delta t)]$$

$$y(t) = \frac{A_c^2}{2} \ \Delta t \frac{\varphi(t) - \varphi(t-\Delta t)}{\Delta t}$$

The second term is the derivative $\frac{d\varphi(t)}{dt}$. The output then becomes

$$y(t) = \frac{A_c^2}{2}\Delta t \frac{d\varphi}{dt}$$

But $\varphi(t) = 2\pi k_f \int_0^t m(\alpha)\, d\alpha$ and $\frac{d}{dt}\varphi(t) = 2\pi k_f m(t)$

$$y(t) = \frac{A_c^2}{2}\Delta t 2\pi k_f m(t)$$

$$y(t) = K\, m(t)$$

Therefore, $m(t)$ has been demodulated.

Transfer Function of the Delay

From the Fourier transform properties

$$g(t) \to G(f)$$

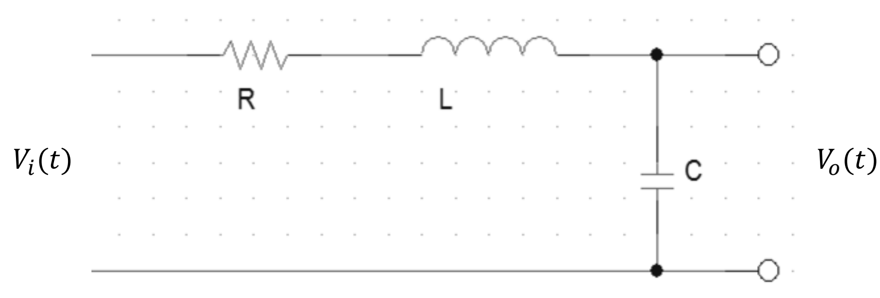
$$g(t - \Delta t) \to G(f)e^{-j2\pi f\Delta t}$$

The transfer function of the time delay is

$$H(f) = e^{-j2\pi f\Delta t}$$

Therefore, a circuit whose phase characteristic is linear in f can provide time delay of the type that we need.

A circuit with linear phase characteristic is the network shown



If $f_o = \frac{1}{2\pi\sqrt{LC}}$, $f_b = \frac{R}{2\pi L}$

then it can be shown that arg $(H(f))$ for this circuit is

$$\arg\big(H(f)\big) = -\frac{\pi}{2} - \frac{2Q}{f_o}(f - f_c)\ ,\ Q = \frac{f_o}{f_b}$$

$$\Theta(\mathrm{f}) = a - bf$$

Remarks:

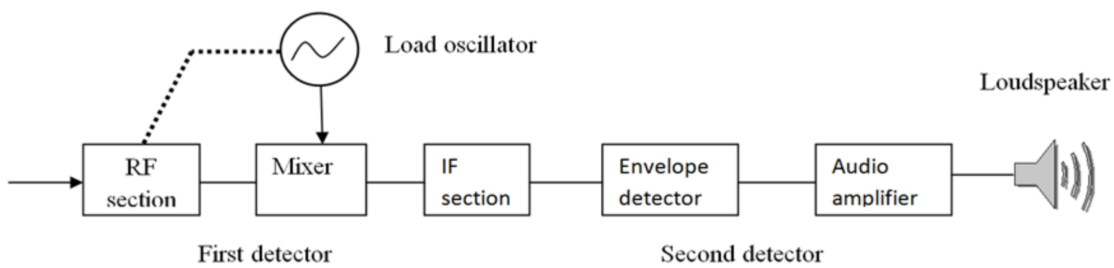
1. To perform time differentiation, we searched for a circuit whose amplitude spectrum varies linearly with frequency
2. To perform time delay, we searched for a circuit with a linear phase spectrum.

The Super Heterodyne Receiver

Practically, all radio and TV receivers are made of the super heterodyne type. The receiver performs the following functions :

- Carrier frequency tuning: The purpose of which is to select the desired signal.
- Filtering: the desired signal is to be separated from other modulated signals.
- Amplification: to compensate for the loss of signal power incurred in the course of transmission.

The description of the receiver is summarized as follows:



- The incoming signal is picked up by the antenna and amplified in the RF section that is tuned to the carrier frequency of the incoming signal.
- The incoming RF section is down converted to a fixed intermediate frequency (IF). $f_{IF} = f_{lO} - f_{RF}$
- The IF section provides most of the amplification and selectivity in the receiver. The IF bandwidth corresponds to that required for the particular type of modulation.
- The IF output is applied to a demodulator, the purpose of which is to recover the baseband signal.
- The final operation in the receiver is the power amplification of the recovered signal.
- The basic difference between AM and FM super heterodyne lies in the use of an FM demodulator such as a discriminator (differentiator followed envelope detector)

Quadrature Carrier Multiplexing (QAM)

Modulation

This scheme enables two DSB-SC modulated signals to occupy the same transmission B.W and yet allows for the separation of the message signals at the receiver.

$m_1(t)$ and $m_2(t)$ are low pass signals each with a B.W = W Hz .

The composite signal is:

$$s(t) = A_c m_1(t) cos2\pi f_c t + A_c m_2(t) sin2\pi f_c t$$

$$s(t) = s_1(t) + s_2(t)$$

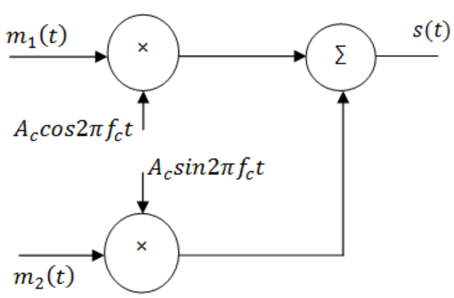
where $s_1(t)$ and $s_2(t)$ are both DSB-SC signals.

B.W of $s_1(t)$ = 2W

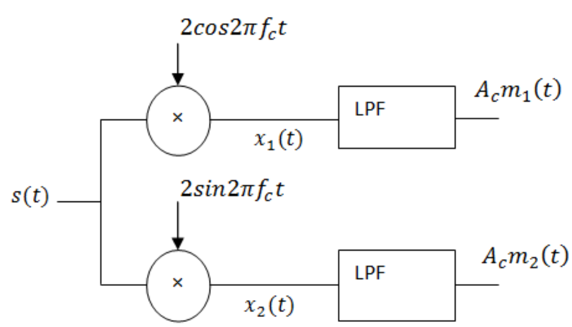
B.W of $s_2(t)$ = 2W

B.W of $s(t)$ = 2W

This method provides bandwidth conservation. That is, two DSB-SC signals are transmitted within the bandwidth of one DSB-SC signal. Therefore, this multiplexing technique provides bandwidth reduction by one half.



Modulation



Demodulation

Demodulation

Given $s(t)$, the objective is to recover $m_1(t)$ and $m_2(t)$ from $s(t)$. Consider first the in-phase channel

$$x_1(t) = 2cos2\pi f_c t \; s(t)$$
$$= 2\cos 2\pi f_c t \; (A_c m_1(t) cos2\pi f_c t + A_c m_2(t) sin2\pi f_c t)$$

$$= 2A_c m_1(t)cos^2 2\pi f_c t + 2A_c m_2(t) sin\omega_c t \; cos\omega_c t$$
$$= 2A_c m_1(t)\left(\frac{1+cos2\omega_c t}{2}\right) + A_c m_2(t) sin2\omega_c t$$

$$= A_c m_1(t) + A_c m_1(t)cos2\omega_c t + A_c m_2(t) sin2\omega_c t$$

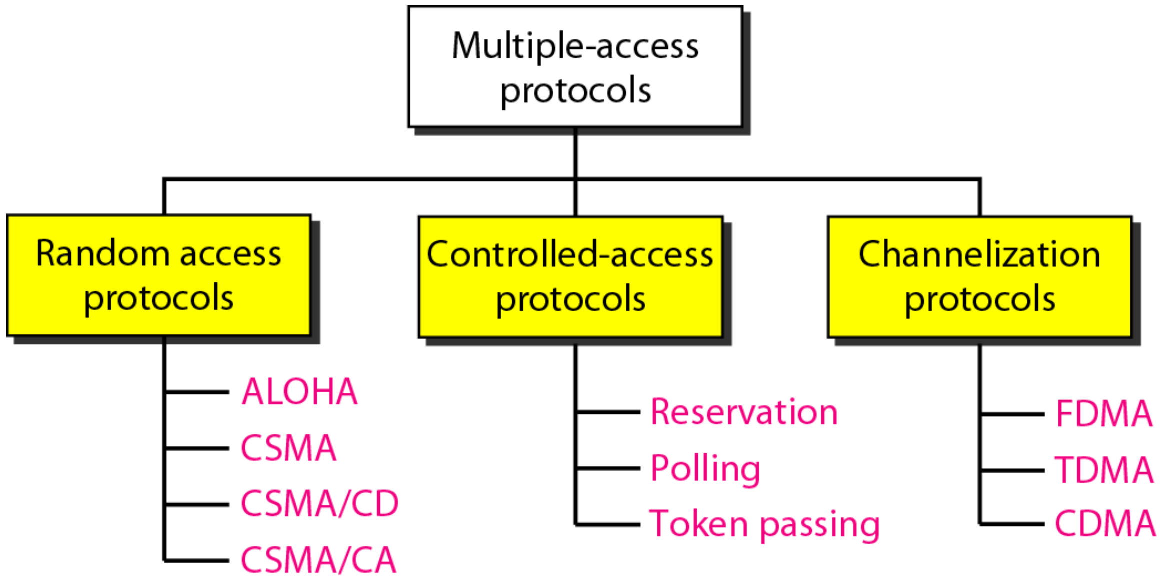
After low pass filtering, the output of the in-phase channel is

$$y_1(t) = A_c m_1(t).$$

Likewise, it can be shown that

$$y_2(t) = A_c m_2(t) .$$

Note: Synchronization is a problem. That is to recover the message signals it is important that the two carrier signals (the sine and the cosine functions) at the receiver should have the same phase and frequency as the signals at the transmitting side. A phase error or a frequency error will result is an interference type of distortion. That is, A component of $m_2(t)$ will appear in the in-phase channel in addition to the desired signal $m_1(t)$ and a component of $m_1(t)$ will appear at the quadrature output.



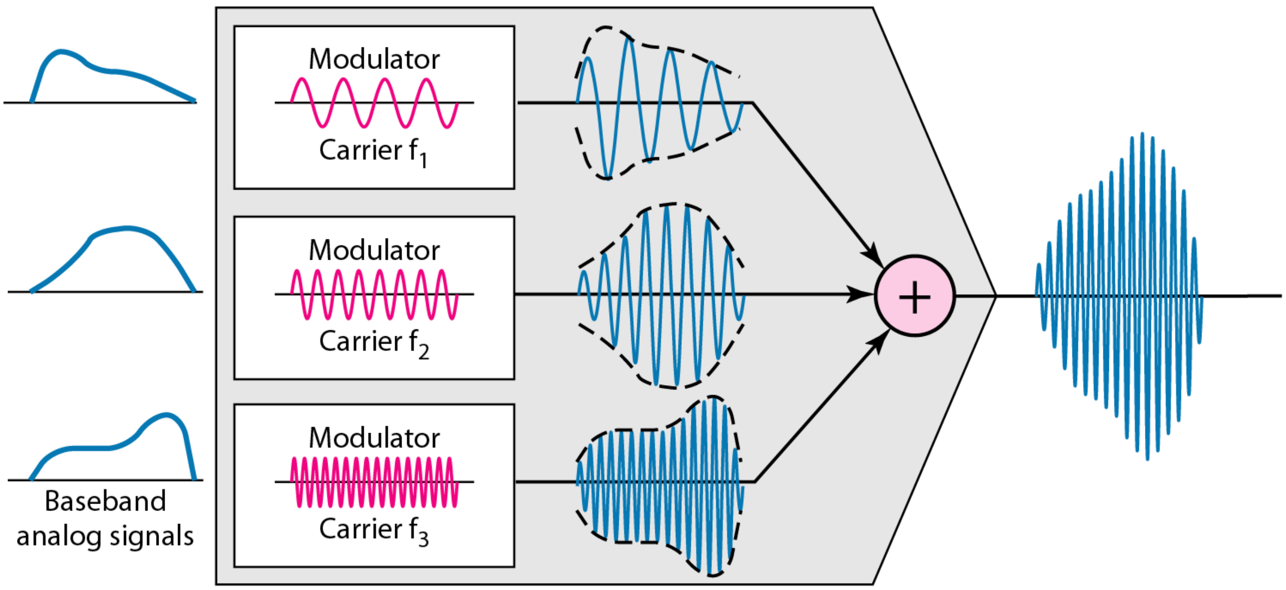
- A **multiple access channel** is one where a set of users at one end want to communicate with another set of users at the other end.
- **Channel Allocation**: The coordination of the usage of a single channel among multiple source – destination pairs.
- The algorithm which implements the channel allocation are called **medium access control** (MAC) or multiple access protocols.
- MAC protocols can be classified into
  - Conflict free protocols
  - Random access protocols

- A **Conflict free protocols**: Collisions are completely avoided by allocating the channel access to sources in a predetermined manner. Examples are TDMA, FDMA and CDMA. This is equivalent to circuit switching and is inefficient for bursty type of loads.
- **Random access protocols**: These are classified as contention systems where the stations compete to access the channel. The contention could be completely random or controlled.
- Collisions can occur between transmitted packets of different users trying to access the channel.
- A collided packet has to be transmitted until it is received properly at the destination.

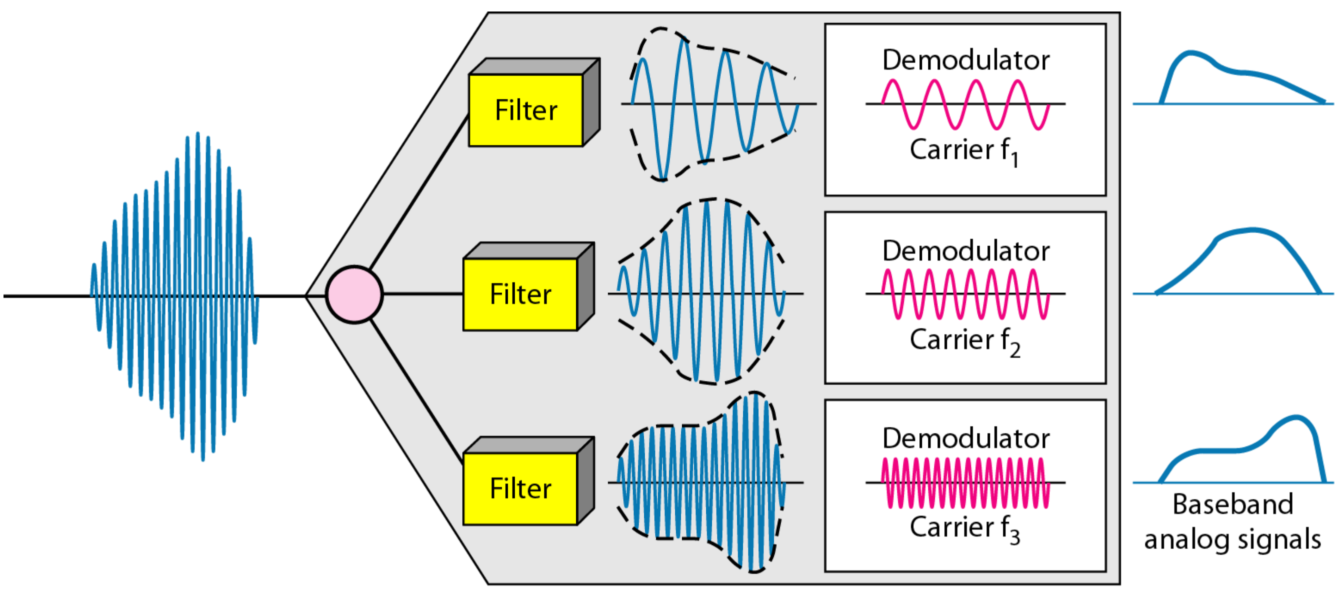
Frequency Division Multiplexing

A number of independent signals can be combined into a composite signal suitable for transmission over a common channel. The signals must be kept apart so that they do not interfere with each other and thus they can be separated at the receiving end.

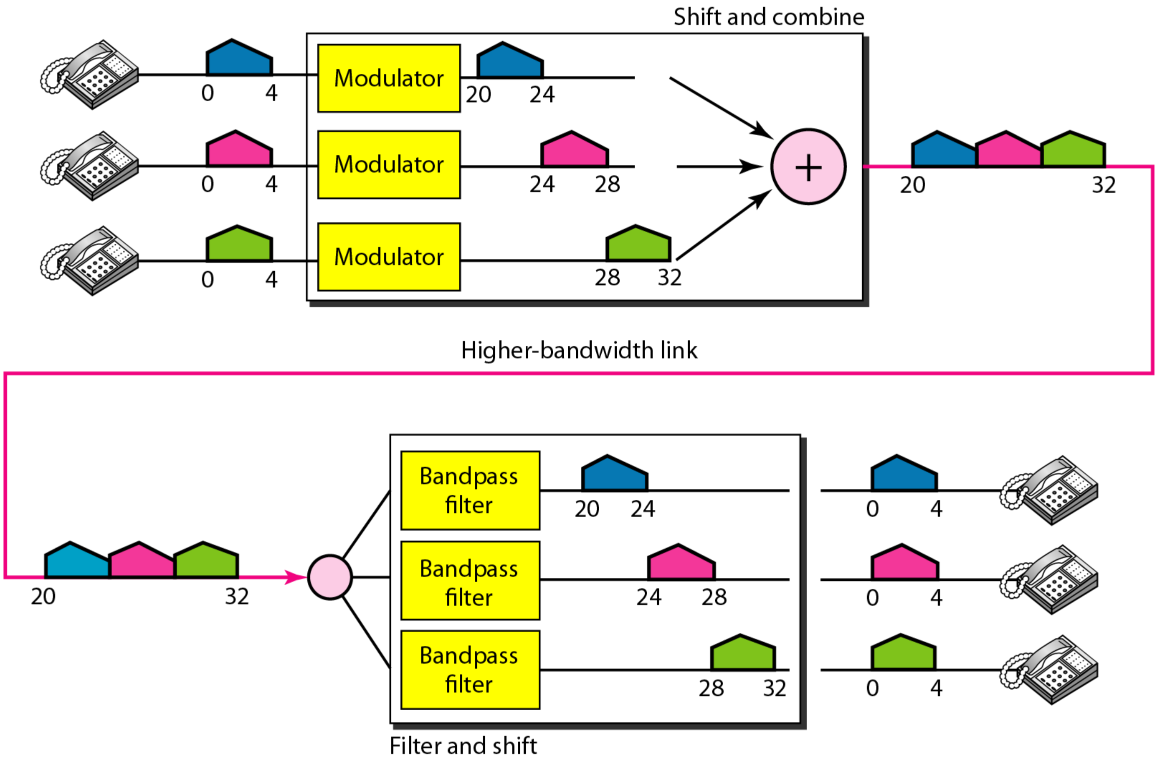
Transmitter:



Receiver:



Example: Single Sideband Frequency Division Multiplexed Signals



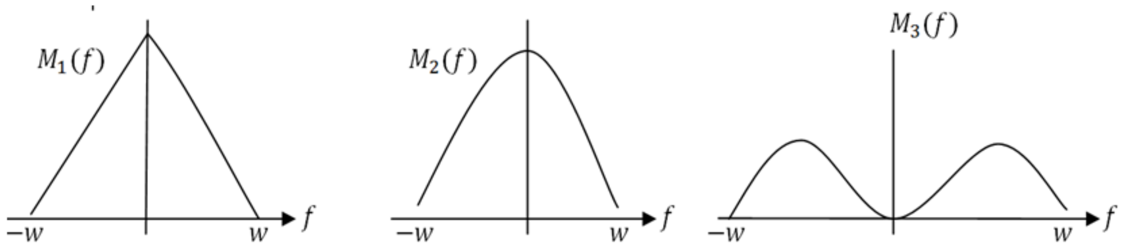
Example: Double Sideband Frequency Division Multiplexed Signals

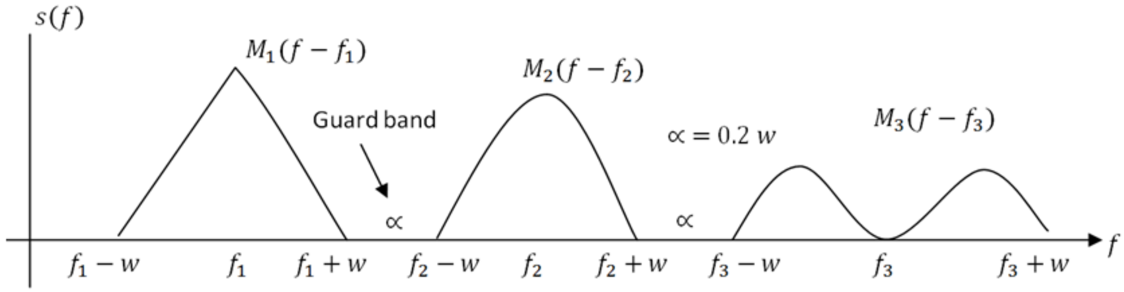
Let m_1, m_2 *and* m_3 be three baseband message signals each with a B.W = w.

The composite modulated signal $s(t)$ is

$$s(t) = A_{c_1} m_1(t) cos2\pi f_1 t + A_{c_2} m_2(t) cos2\pi f_2 t + A_{c_3} m_3(t) cos2\pi f_3 t$$
$$= s_1(t) + s_2(t) + s_3(t)$$

s_1, s_2 *and* s_3 are DSB-SC signals with carrier frequencies $f_1,\ f_2$ *and* f_3 , respectively. If the spectrum of $m_1(t),\ m_2(t)$ *and* $m_3(t)$ are as shown, the spectrum of $s(t)$ can be found as shown below.





To prevent interference we demand that

$$f_2 - w \geq f_1 + w \quad or \quad f_2 - f_1 \geq 2w$$

$$f_3 - w \geq f_2 + w \quad or \quad f_3 - f_2 \geq 2w$$

The structure of the receiver is as follows: