Spectrum of a Single-Tone FM Signal

- The objective of this lecture is to find a meaningful definition of the bandwidth of an FM signal.
- To accomplish that, we need to find the spectrum of an FM signal with a single-tone test message signal.
- **Review of basic results from the previous lecture:**
- The expression for an angle modulated signal is: $\boldsymbol{s(t) = A_c \cos\left(2\pi f_c t + \theta(t)\right)}$
- The instantaneous frequency of s(t) is: $\boldsymbol{f_i(t) = f_c + \frac{1}{2\pi}\frac{d\theta(t)}{dt}}$
- For **phase modulation**:
  - $\theta(t) = k_p m(t)$, k_p in rad/volt.
  - $s(t)_{PM} = A_c \cos\left(2\pi f_c t + k_p m(t)\right)$.
  - $f_i(t) = f_c + \frac{k_p}{2\pi}\frac{dm(t)}{dt}$

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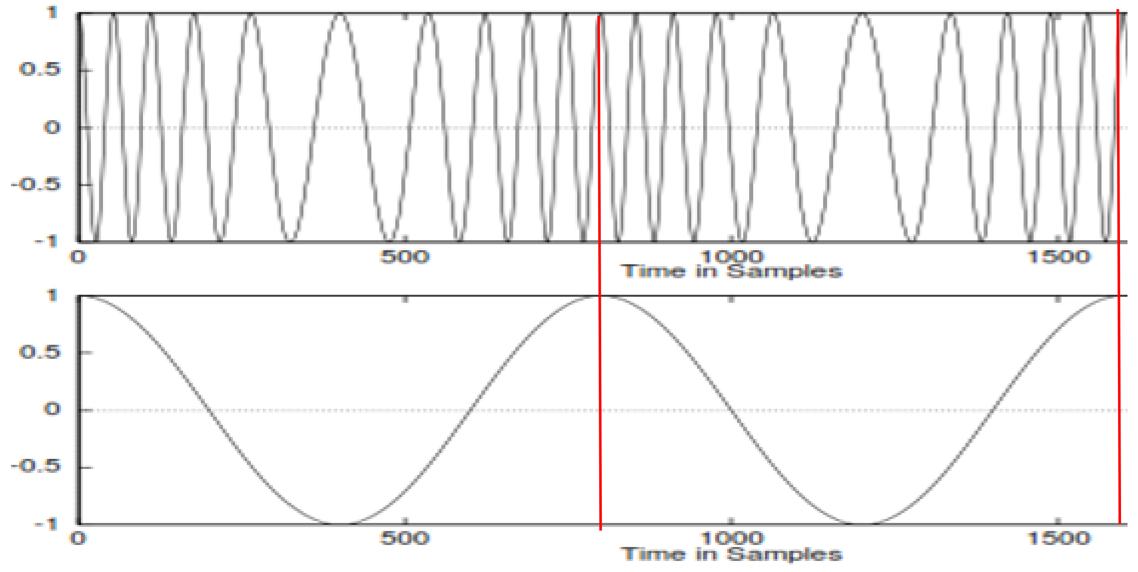
- **Review of basic results from the previous lecture:**
- The expression for an angle modulated signal is: $\boldsymbol{s(t) = A_c \cos\left(2\pi f_c t + \theta(t)\right)}$
- The instantaneous frequency of s(t) is: $\boldsymbol{f_i(t) = f_c + \frac{1}{2\pi}\frac{d\theta(t)}{dt}}$
- For **frequency modulation**:
  - $\theta(t) = 2\pi k_f \int_0^t m(\alpha)d\alpha;$ k_f in Hz/volt.
  - $s(t)_{FM} = A_c \cos\left(2\pi f_c t + 2\pi k_f \int_0^t m(\alpha)d\alpha\right).$
  - $f_i(t) = f_c + k_f m(t);$
- When $\boldsymbol{m(t) = A_m \cos 2\pi f_m t}$
  - $f_i = f_c + A_m k_f \cos 2\pi f_m t;$
  - $s(t)_{FM} = A_c \cos\left(2\pi f_c t + 2\pi k_f \int_0^t A_m \cos \omega_m \alpha \, d\alpha\right) = \boldsymbol{A_c \cos\ (2\pi f_c t + \beta \sin 2\pi f_m t)}.$
- $\boldsymbol{\beta = \frac{k_f A_m}{f_m}} = \frac{peak\ frequency\ deviation}{message\ bandwidth} = \frac{\Delta f}{f_m};$ is the **FM modulation index,**

Spectrum of a Single-Tone FM Signal

- Let $m(t) = A_m \cos 2\pi f_m t$ be the test message signal, then the FM signal is
- $\boldsymbol{s(t) = A_c \cos(2\pi f_c t + \beta \sin 2\pi f_m t)}$
- $\boldsymbol{\beta = \frac{k_f A_m}{f_m}}$; **FM modulation index**
- $s(t)$ can be rewritten as:
- $s(t) = Re\{e^{j(2\pi f_c t + \beta \sin 2\pi f_m t)}\}$
- $\quad = Re\{e^{j(2\pi f_c t)} . e^{j(\beta \sin 2\pi f_m t)}\}$



- Remember that: $e^{j\theta} = cos\theta + jsin\,\theta$ and that $cos\theta = Re\{e^{j\theta}\}$
- The sinusoidal waveform ($\beta \sin 2\pi f_m t$) is periodic with period $T_m = \frac{1}{f_m}$. The exponential function $\boldsymbol{e^{j(\beta \sin 2\pi f_m t)}}$ is also periodic with the same period $T_m = \frac{1}{f_m}$
- $e^{j\beta \sin 2\pi f_m (t+T_m)} = e^{j\beta \sin 2\pi f_m t} . e^{j\beta \sin 2\pi f_m T_m} = e^{j\beta \sin 2\pi f_m t}$; $f_m T_m = 1 \Rightarrow$ $\sin(2\pi f_m T_m) = 0$;

Spectrum of a Single-Tone FM Signal

- $s(t) = A_c \, cos(\, 2\pi f_c t + \beta \, sin \, 2\pi f_m t \,)$
- $s(t) = Re\{e^{j(2\pi f_c t)} \, \boldsymbol{e^{j(\beta \, sin \, 2\pi f_m t)}}\}$
- A periodic function g(t) can be expanded into a complex Fourier series as:
- $g(t) = \sum_{-\infty}^{\infty} C_n \, e^{jn\omega_m t}$;
- $C_n = \frac{1}{T_m} \int_0^{T_m} g(t) \, e^{-jn\omega_m t} \, dt$
- Now, let $\boldsymbol{g(t) = e^{j(\beta \, sin \, 2\pi f_m t)}}$
- $C_n = \frac{1}{T_m} \int_0^{T_m} e^{j(\beta \sin 2\pi f_m t)} e^{-jn\omega_m t} \, dt$
- It turns out that the Fourier coefficients

$$\boldsymbol{C_n = J_n(\beta)}.$$

- where $J_n(\beta)$ is the Bessel function of the first kind of order n (will describe it on next slide)
- Hence, $g(t) = \sum_{-\infty}^{\infty} J_n(\beta) \, e^{jn\omega_m t}$;
- Substituting $g(t)$ into $s(t)$, we get
- $s(t) = A_c \, Re\{e^{j(2\pi f_c t)} \, \sum_{-\infty}^{\infty} J_n(\beta) \, e^{jn\omega_m t} \, \}$

$$= A_c \;\; Re\{ \sum_{-\infty}^{\infty} J_n(\beta) e^{j2\pi(f_c + nf_m)t} \, \}$$

$$= A_c \sum_{-\infty}^{\infty} J_n(\beta) \cos(2\pi(f_c + nf_m)t)$$

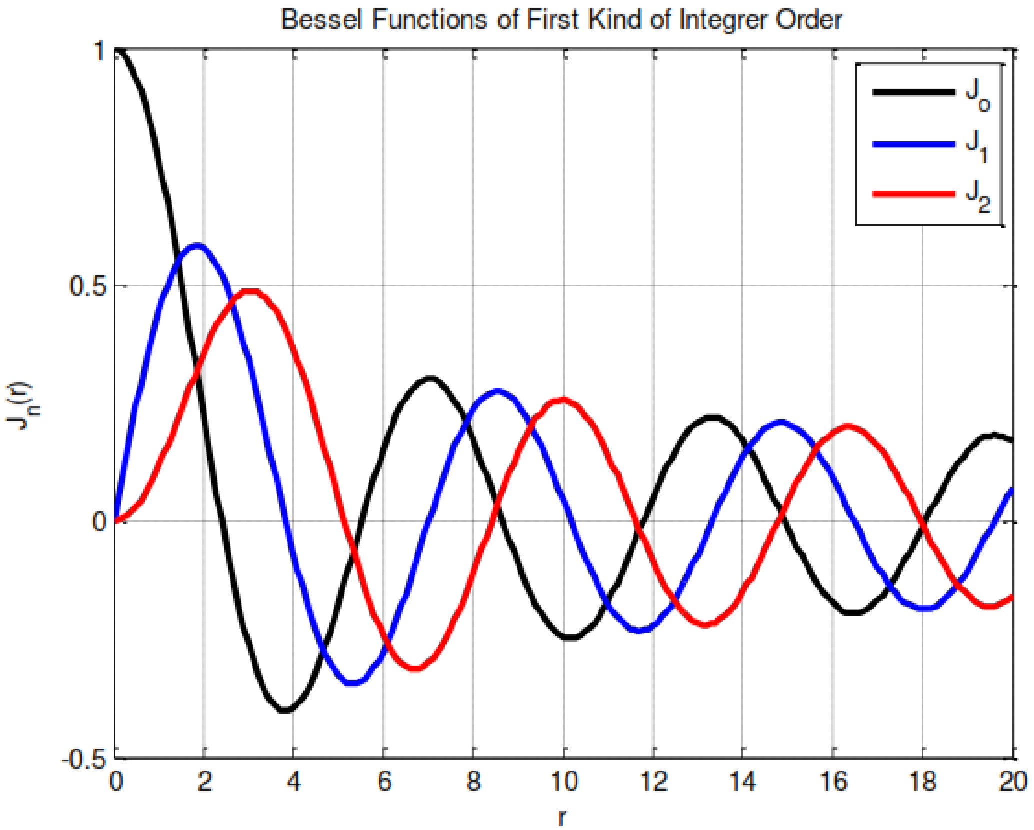
- Finally, the FM signal can be represented as
- $\boldsymbol{s(t) = A_c \, \sum_{-\infty}^{\infty} J_n(\beta) \cos(2\pi(f_c + nf_m)t)}$

Spectrum of a Single-Tone FM Signal

- **Bessel Functions**: The Bessel equation of order n is $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$
- This is a second order differential equation with variable coefficients. We can solve it using the power series method. The solution for each value of n is $J_n(x)$, the Bessel function of the first kind of order n. The figure, below, shows the first three Bessel functions.

Some Properties of Bessel Functions

- $J_n(x) = (-1)^n J_{-n}(x)$; relative to n
- $J_n(x) = (-1)^n J_n(-x)$; relative to x
- Recurrence formula $J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$.
- For small x, $J_n(x) \cong \frac{x^n}{2^n\ n!}$, $J_0(x) \cong 1, J_1(x) \cong \frac{x}{2}$.
- For large x: $J_n(x) \cong \sqrt{\frac{2}{\pi x}} \cos(x - \frac{\pi}{4} - \frac{n\pi}{2})$,
- $\sum_{n-\infty}^{\infty} (J_n(x))^2 = 1$, $for\ all\ x$.



The Fourier Series Representation of the FM Signal

- A single tone FM signal can be represented in a Fourier series as

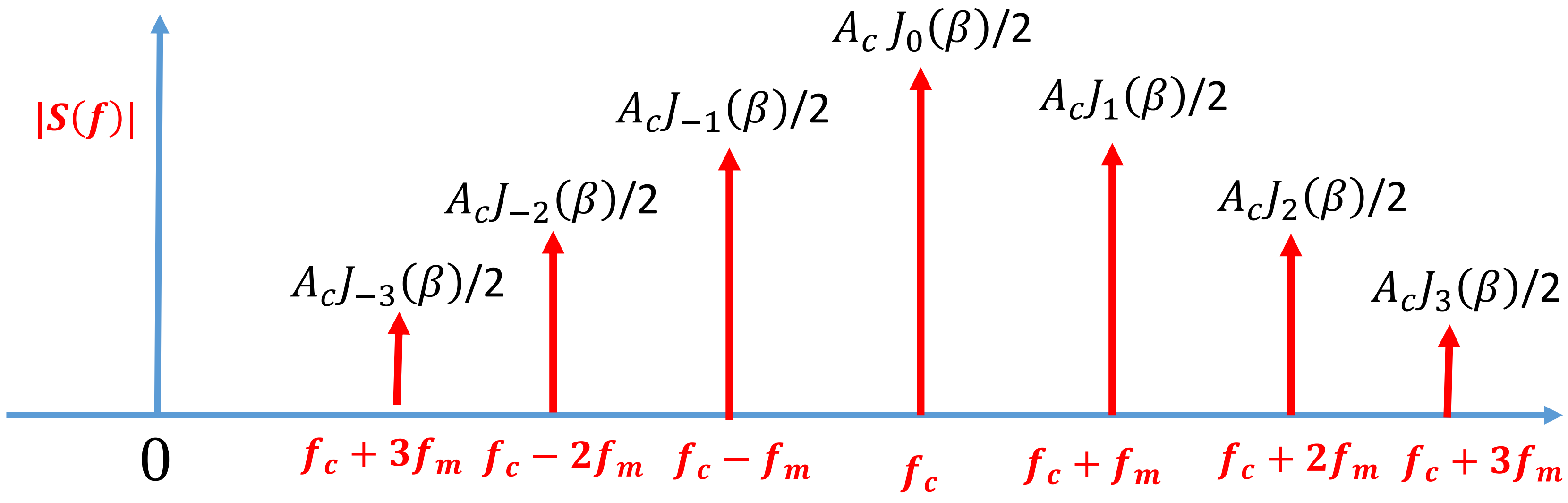
$$s(t) = A_c \sum_{-\infty}^{\infty} J_n(\beta) \cos(2\pi(f_c + nf_m)t)$$

$$S(f) = A_c/2 \sum_{-\infty}^{\infty} J_n(\beta)[\delta(f - (f_c + nf_m) + \delta(f + (f_c + nf_m)]$$

- The first few terms in this expansion are:
- $s(t) = A_c\, J_0(\beta) \cos(2\pi f_c t) + A_c J_1(\beta) \cos 2\pi(f_c + f_m)t + A_c J_{-1}(\beta) \cos 2\pi(f_c - f_m)t + A_c J_2(\beta) \cos 2\pi(f_c + 2f_m)t + A_c J_{-2}(\beta) \cos 2\pi(f_c - 2f_m)t + \cdots$
- $J_{-1}(\beta) = -J_1(\beta),\ J_{-2}(\beta) = J_2(\beta); J_{-3}(\beta) = -J_3(\beta); J_{-4}(\beta) = J_4(\beta);$

Remarks:

1. Spectral components are separated by fm.
2. The 98% power bandwidth is: $B_T = 2(\beta + 1)f_m$



The Fourier Series Representation of the FM Signal

- **Few remarks about the FM spectrum:**
- The FM signal $s(t) = A_c\, cos(\, 2\pi f_c t + \beta\, sin\, 2\pi f_m t\,)$ can be represented as
- $\boldsymbol{s(t) = A_c\, \sum_{-\infty}^{\infty} J_n(\beta) \cos(2\pi(f_c + nf_m)t)}$
- **s(t) =** $\boldsymbol{A_c J_0(\beta) \cos(2\pi(f_c)t)}$ **+**

  $\boldsymbol{A_c J_1(\beta) \cos(2\pi(f_c + f_m)t)}$ **+** $\boldsymbol{A_c J_{-1}(\beta) \cos(2\pi(f_c - f_m)t)}$

  + $\boldsymbol{A_c J_2(\beta) \cos(2\pi(f_c + 2f_m)t)}$ **+** $\boldsymbol{A_c J_{-1}(\beta) \cos(2\pi(f_c - 2f_m)t)}$

  + $\boldsymbol{A_c J_3(\beta) \cos(2\pi(f_c + 3f_m)t)}$ **+** $\boldsymbol{A_c J_{-3}(\beta) \cos(2\pi(f_c - 3f_m)t)}$ **+ …..**
- The FM signal consists of an infinite number of spectral components concentrated around f_c.
- Therefore, the theoretical bandwidth of the signal is infinity. That is to say, if we need to recover the FM signal without any distortion, all spectral components must be accommodated. This means that a channel with infinite bandwidth is needed. This is, of course, not practical since the frequency spectrum is shared by many users.
- In the following discussion we need to truncate the series so that, say 98%, of the total average power is contained within a certain bandwidth. But, first let us find the total average power using the series approach.

Power in the Spectral Components of s(t)

- A single tone FM signal $s(t) = A_c\, cos(\, 2\pi f_c t + \beta\, sin\, 2\pi f_m t\,)$ is expanded as:

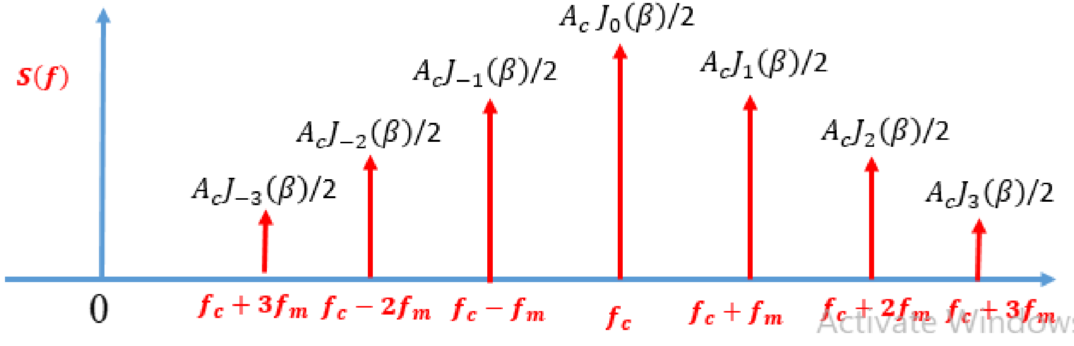
$$\boldsymbol{s(t) = A_c \sum_{-\infty}^{\infty} J_n(\beta) \cos(2\pi(f_c + nf_m)t)}; \quad \textbf{average power} = \boldsymbol{< s^2\ (t) > = \frac{A_C{}^2}{2}}.$$

- Note that s(t) consists of an infinite number of Fourier terms, and the power in s(t) will be equal the power in the respective Fourier components
- Any term in s(t) takes the form: $A_c J_n(\beta) \cos(2\pi(f_c + nf_m)t)$
- The average power in this term is: $\frac{(A_c)^2 (J_n(\beta))^2}{2}$
- Hence the total power in s(t) is
- $< s^2(t) > \; = \; \frac{A_C{}^2\, J_0{}^2(\beta)}{2} + \frac{A_C{}^2 J_1{}^2(\beta)}{2} + \frac{A_C{}^2 J_{-1}{}^2(\beta)}{2} + \frac{A_C{}^2 J_2{}^2(\beta)}{2} + \frac{A_C{}^2 J_{-2}{}^2(\beta)}{2} + \cdots$
- $= \frac{A_C{}^2}{2} \{J_0{}^2(\beta) + J_1{}^2(\beta) + J_{-1}{}^2\,(\beta) + J_2{}^2(\beta) + J_{-2}{}^2(\beta) + \cdots..\}$
- $= \frac{A_C{}^2}{2} \{J_0{}^2(\beta) + 2J_1{}^2(\beta) + 2J_2{}^2(\beta) + \cdots..\} = \frac{A_C{}^2}{2}$

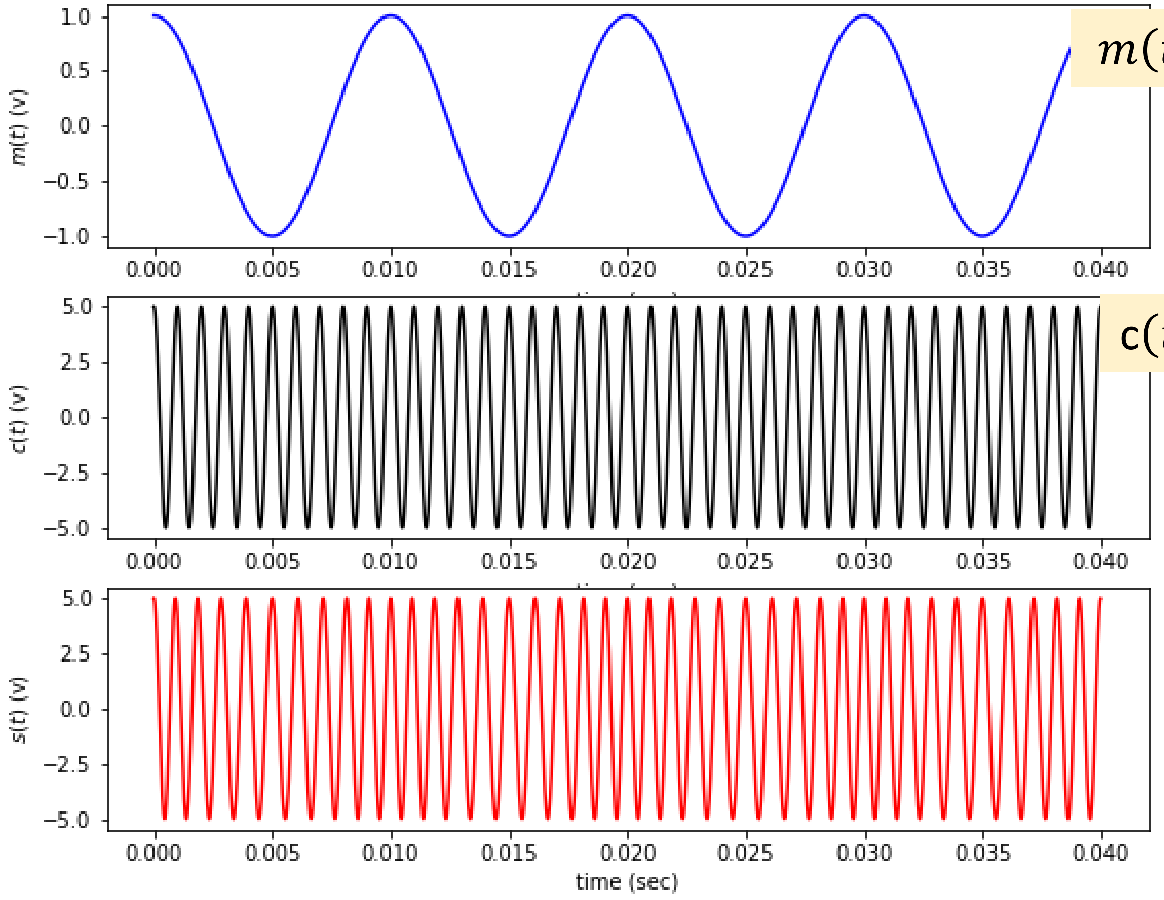
Remark: $\{\sum_{n=-\infty}^{\infty} J_n{}^2(\beta) = 1\}$, (A property of Bessel functions).

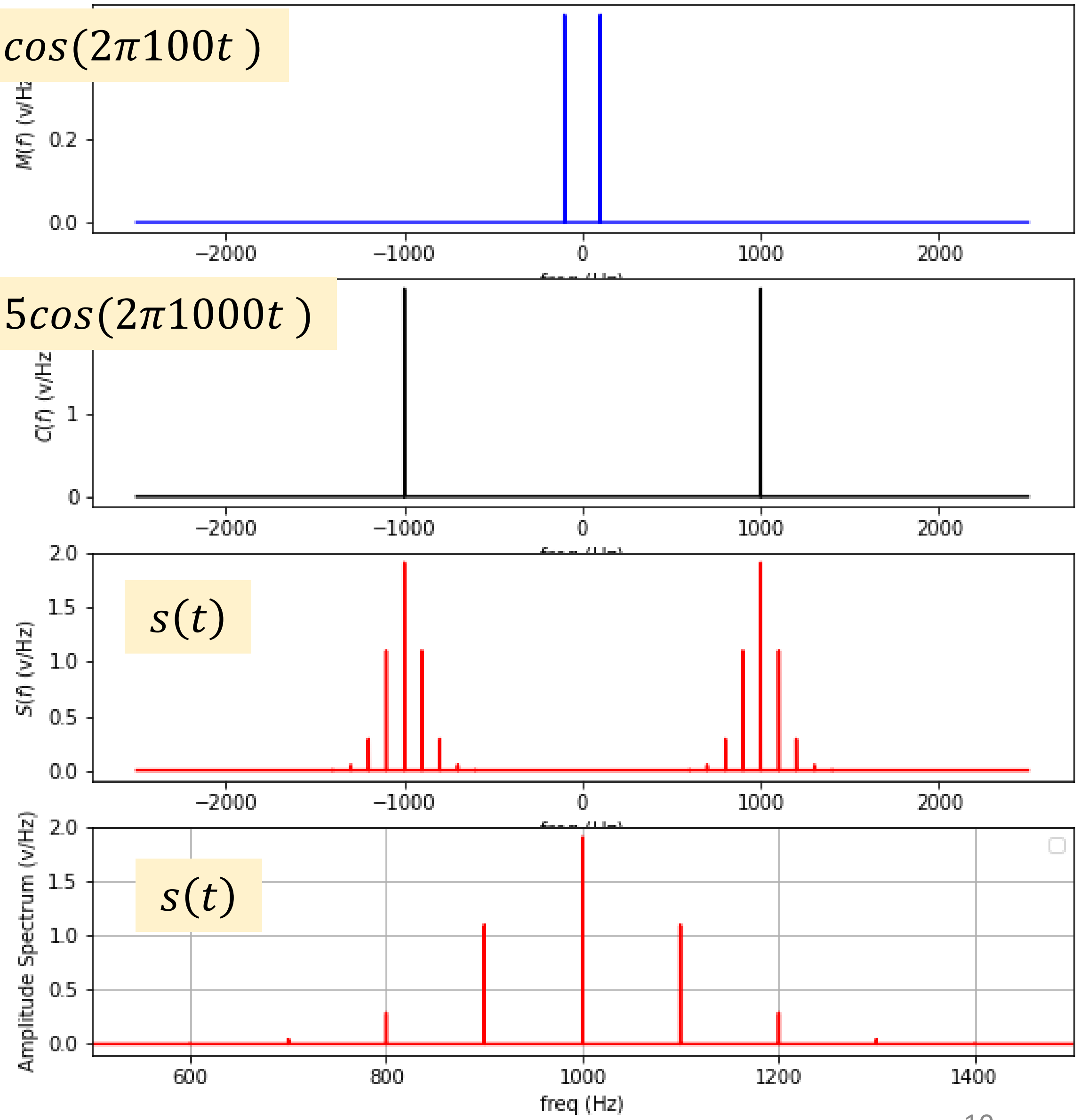
Example: 99% Power Bandwidth of an FM Signal

- Find the 99% power bandwidth of an FM signal when $\beta = 1$
- **Solution**: $s(t) = A_c \sum_{-\infty}^{\infty} J_n(\beta) \cos(2\pi(f_c + nf_m)t)$
- **Case a: $\beta = 1$ (wideband FM)**
- The first five terms corresponding to $\beta = 1$ (obtained from the table) are
- $J_0(1)$ = 0.7652, $J_1(1)$ = 0.4401, $J_2(1)$ = 0.1149, $J_3(1)$ = 0.01956, $J_4(1)$ = 0.002477
- The power in s(t) is $< S^2(t) > = \frac{A_C{}^2}{2}$
- Let us try to find the average power in the terms at (f_c), (f_c+f_m), (f_c-f_m), (f_c+2f_m), (f_c-2f_m)
- $f_c : \frac{A_C{}^2 J_0{}^2(\beta)}{2}$; $f_c + f_m$: $\frac{A_C{}^2 J_1{}^2(\beta)}{2}$; $f_c - f_m$: $\frac{A_C{}^2 J_{-1}{}^2(\beta)}{2}$; $f_c + 2f_m$: $\frac{A_C{}^2 J_2{}^2(\beta)}{2}$; $f_c - 2f_m$: $\frac{A_C{}^2 J_{-2}{}^2(\beta)}{2}$
- The average power in the five spectral components is the sum
- $P_{av} = \frac{A^2{}_c}{2}\left[J_0{}^2(1) + 2J_1{}^2(1) + 2J_2{}^2(1)\right]$; $P_{av} = \frac{A^2{}_c}{2}\left[(0.7652)^2 + 2 * (0.4401)^2 + (0.1149)^2\right] = 0.9993\ \frac{A^2{}_c}{2}$
- Hence, these terms contain 99.9 % of the total power.
- Therefore, the 99.9 % power bandwidth is
- $BW = (f_c + 2f_m) - (f_c - 2f_m) = 4f_m$



FM in the time and frequency domains: $\beta = 1$





$m(t) = cos(2\pi 100t\ )$

$\mathrm{c}(t) = 5cos(2\pi 1000t\ )$

$s(t)$

$s(t)$

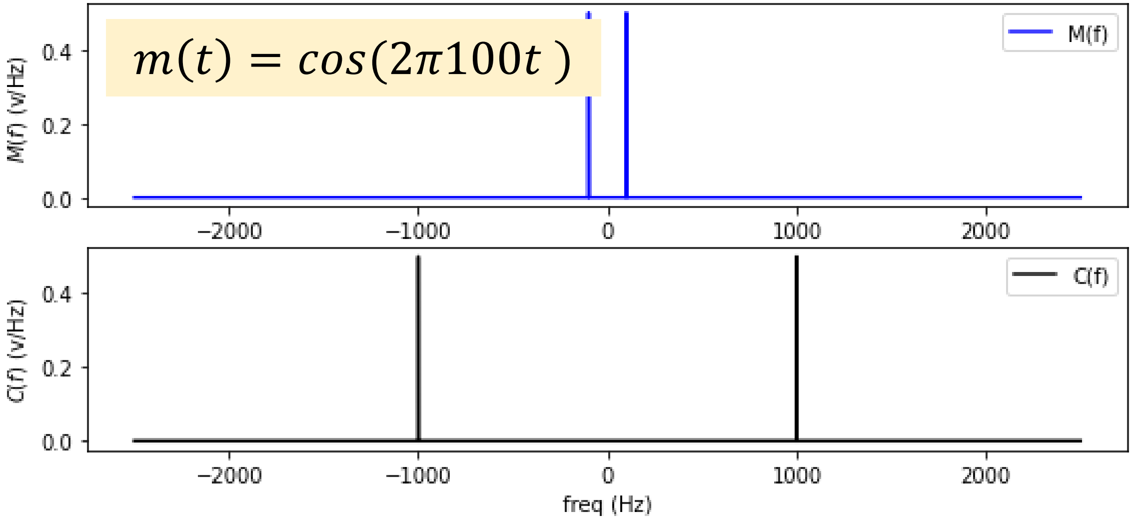
Kf = 100 Hz/V $s(t) = 5\,cos(\ 2\pi f_c t + 5\,sin\,2\pi 100t\ )$

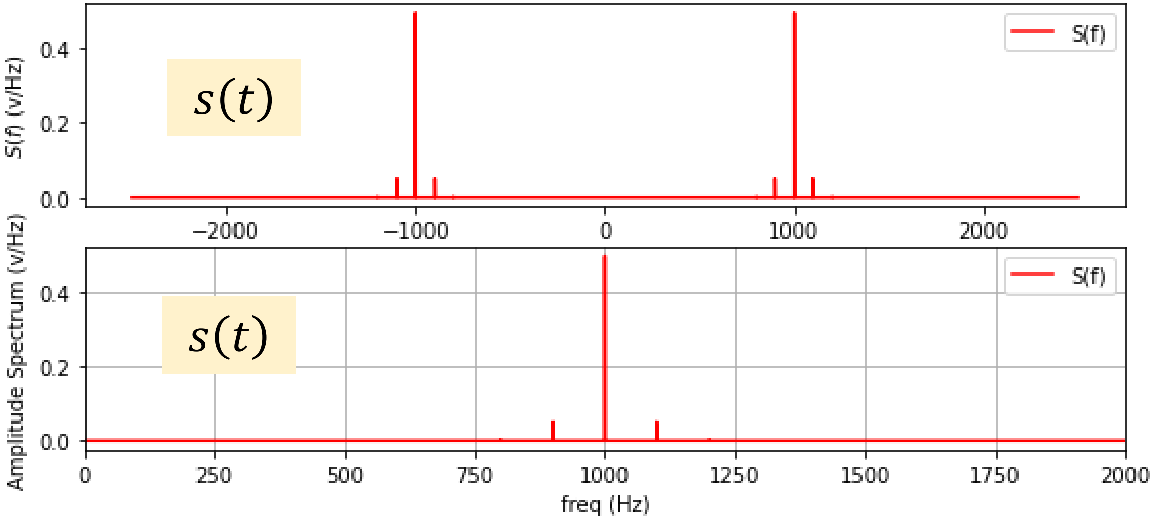
$\boldsymbol{BW = (f_c + 2f_m) - (f_c - 2f_m) = 4f_m = 400\ Hz}$

Example: 99% Power Bandwidth of an FM Signal

- Find the 99% power bandwidth of an FM signal when $\beta = 0.2$
- **Solution**: $s(t) = A_c \sum_{-\infty}^{\infty} J_n(\beta) \cos(2\pi(f_c + nf_m)t)$
- **Case b: $\boldsymbol{\beta = 0.2}$ (Narrowband FM)**
- For $\beta = 0.2,\ J_0(0.2) = 0.99,\ J_1(0.2) = 0.0995,\ J_2(0.2) = 0.00498335$
- The power in the carrier and the two sidebands at $(f_c\ , f_c + f_m, f_c - f_m)$ is
- $P = \frac{A^2{}_c}{2}\left[J_0{}^2(0.2) + 2J_1{}^2(0.2)\right]$
- $P = \frac{A^2{}_c}{2}[0.9999]$
- Therefore, 99.99% of the total power is found in the carrier and the two sidebands.
- The 99% bandwidth is: $\boldsymbol{B.W = (f_c + f_m) - (f_c - f_m) = 2f_m}$

Narrow-Band FM: $\beta = 0.2$





$c(t) = cos(2\pi 1000t\ )$

Spectrum is similar to that of normal AM

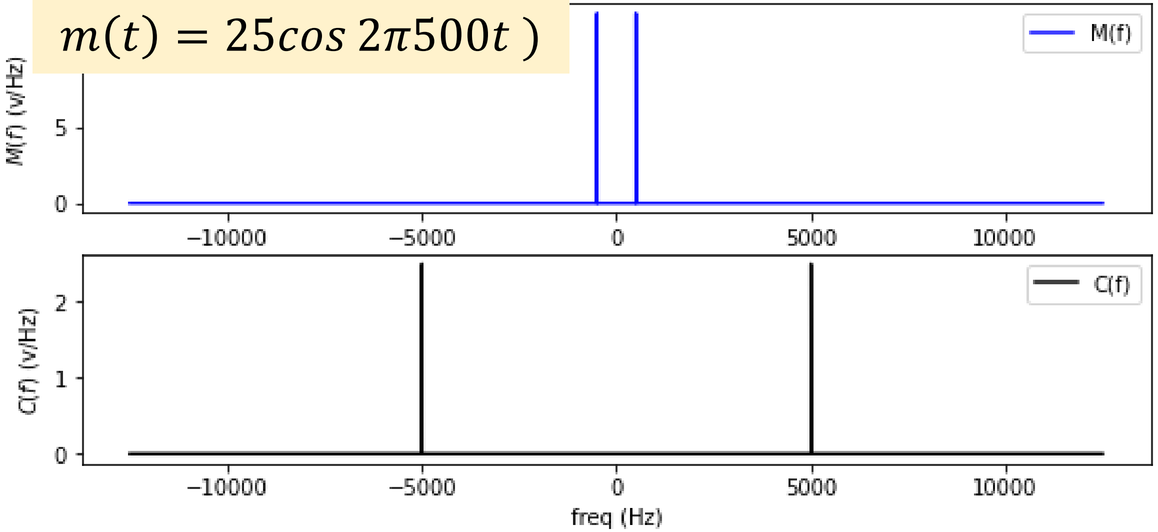
$s(t) = cos(\ 2\pi f_c t + 0.2\ sin\ 2\pi 100t\ )$

Kf = 20 Hz/V

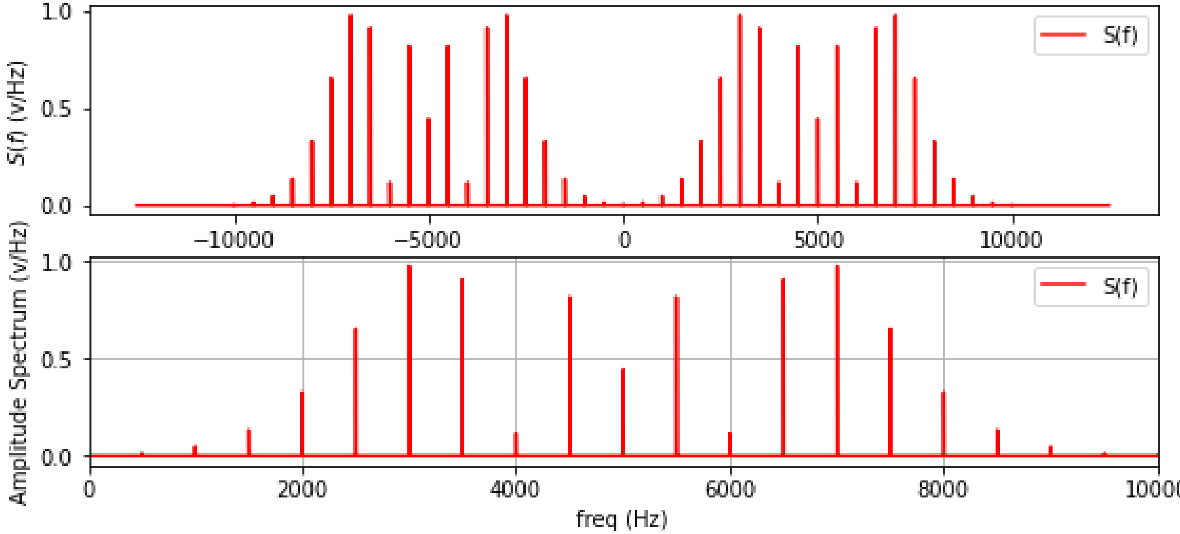
$$\boldsymbol{BW = (f_c + f_m) - (f_c - f_m) = 2f_m = 200\ Hz}$$

Wideband-Band FM : $\beta = 5$

$m(t) = 25cos\, 2\pi 500t\,)$



$\mathrm{c}(t) = 5cos\, 2\pi 5000t\,)$



$s(t) = 5cos(\,2\pi f_c t + 5\, sin\, 2\pi 500t\,)$

$\boldsymbol{BW = (f_c + 6f_m) - (f_c - 6f_m) = 12f_m = 6000\ Hz}$

Kf = 100 Hz/V

Carson’s Rule

- A 98% power B.W of an FM signal can be estimated using Carson’s rule
- $\boldsymbol{B_T = 2(\beta + 1)f_m}$
- The rule works well when the message signal is continuous (Cannot be used when the message contains discontinuities as in the case of a square function).
- **Example**: Find the bandwidth of the FM signal
- $s(t) = A_c\, cos(\, 2\pi f_c t + sin\, 2\pi f_m t\,)$
- **Solution**: $B_T = 2(\beta + 1)f_m = 2(1 + 1)f_m = 4f_m$
- **Example**: Find the bandwidth of the FM signal

  $s(t) = A_c\, cos(\, 2\pi f_c t + 5\, sin\, 2\pi f_m t\,)$
- **Solution**: $B_T = 2(\beta + 1)f_m = 2(5 + 1)f_m = 12f_m$
- **Remark**: Same result as was obtained using the spectral analysis.

Table of Bessel Functions

β	J0(β)	J1(β)	J2(β)	J3(β)	J4(β)	J5(β)
0	1	0	0	0	0	0
0.1	0.9975	0.0499	0.0012	0.0000	0.0000	0.0000
0.2	0.9900	0.0995	0.0050	0.0002	0.0000	0.0000
0.3	0.9776	0.1483	0.0112	0.0006	0.0000	0.0000
0.4	0.9604	0.1960	0.0197	0.0013	0.0001	0.0000
0.5	0.9385	0.2423	0.0306	0.0026	0.0002	0.0000
0.6	0.9120	0.2867	0.0437	0.0044	0.0003	0.0000
0.7	0.8812	0.3290	0.0588	0.0069	0.0006	0.0000
0.8	0.8463	0.3688	0.0758	0.0102	0.0010	0.0001
0.9	0.8075	0.4059	0.0946	0.0144	0.0016	0.0001
1	0.7652	0.4401	0.1149	0.0196	0.0025	0.0002