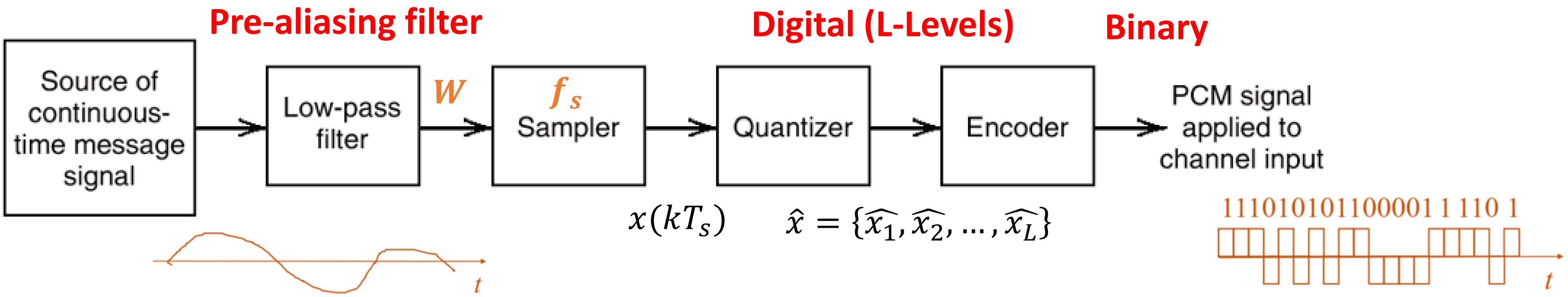
Sampling Theorem
Lecture Outline

- In this this, and the next few lecture, we will address the subject of **pulse code modulation**, where an analog source can be converted into a digital waveform via **sampling**, **quantization**, and **binary encoding**.
- This lecture focuses on Ideal Sampling and the Sampling Theorem.
- The phenomenon of aliasing is explained in detail.
- In the next lecture, we will present two other sampling techniques, which are natural and flat-topped sampling.

Pulse Code Modulation

- Sources are of two types; **analog and digital**. For an analog source, the input transducer is used to convert the physical message generated by the source into a time-varying electrical signal called the *message signal* (like the human voice). This is a continuous time continuous amplitude signal.
- An analog source can be converted into digital via sampling, quantization, and encoding. This process is called **pulse code modulation**



➢ **Sampler:** If W is the highest frequency component in a signal, then the sampling rate required to reconstruct the message from its samples should follow the Nyquist Rate where $\boldsymbol{f_s > 2W}$.

➢ The output of the sampler is a continuous amplitude discrete time signal.

➢ **Quantizer:** Converts the continuous amplitude samples $x(kT_s)$ into **discrete** level samples $\hat{x}(kT_s)$ taken from a finite set of L possible values $\hat{x} = \{\widehat{x_1}, \widehat{x_2}, \dots, \widehat{x_L}\}$.

➢ **Binary Encoder:** Each quantized level is represented by $\boldsymbol{r = log_2 L}$ binary digits

Sampling Techniques

- **Sampling:** is the process by which a continuous time continuous amplitude signal is converted into a discrete time continuous amplitude signal.

There are three types of sampling:

- Ideal sampling: To be presented in this lecture
- Natural sampling: To be discussed in the next lecture
- Flat-topped sampling (sample and hold): To be discussed in the next lecture.

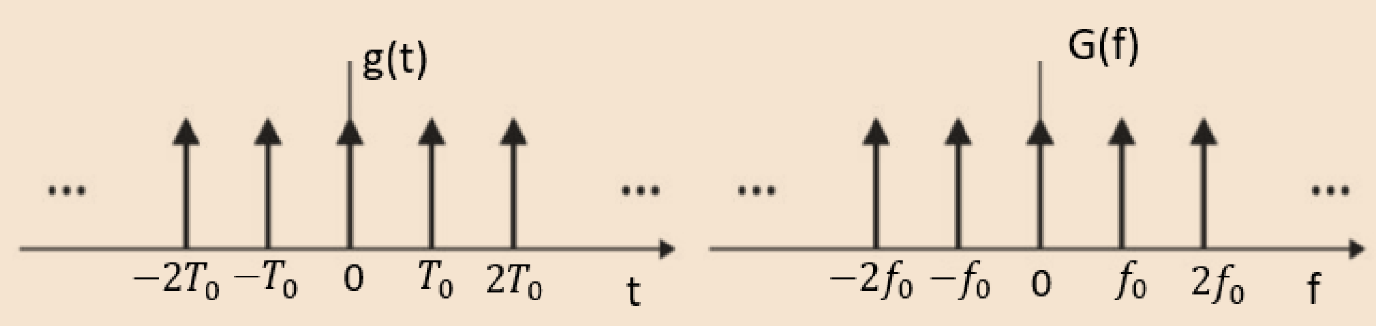
Ideal Sampling: The Periodic Train of Impulses

- **Periodic Signals:** A periodic signal g(t) is expanded in the complex Fourier series form as:
- $g(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \Rightarrow \Im\{g(t) = \sum_{n=-\infty}^{\infty} C_n\, \delta(f - nf_0)$

Example: Consider the following train of impulses $\boldsymbol{g(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT_0)}$

Solution: The Fourier coefficients are obtained by integrating over one period of $g(t)$.

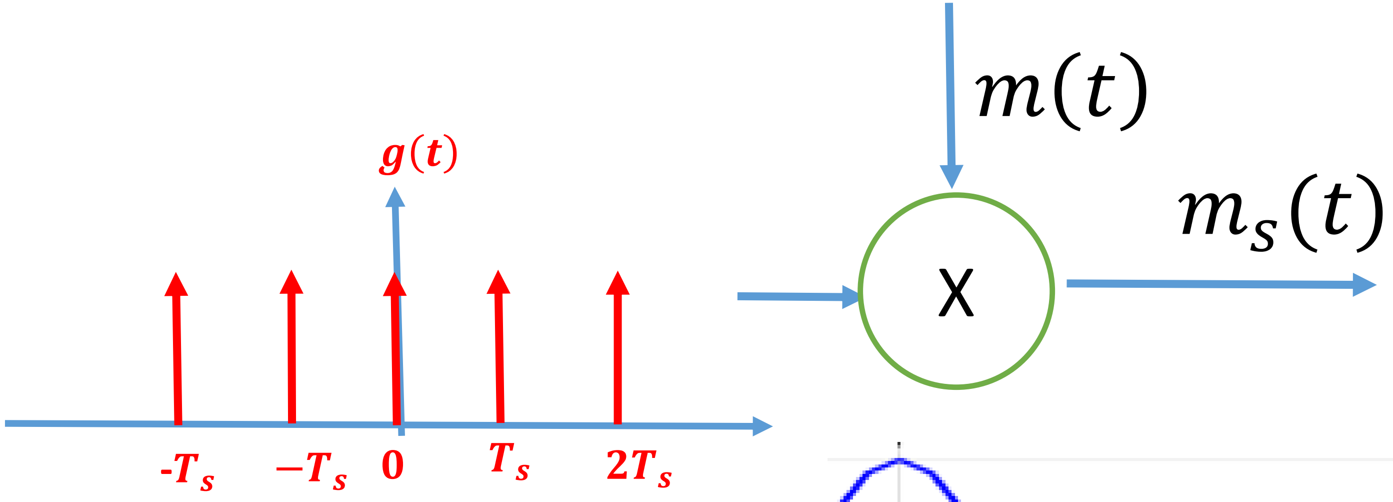
- $C_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\, e^{-jn\omega_0 t} dt = \frac{1}{T_0} = f_0$; Note that the sifting property has been used.
- Therefore, the complex Fourier series of $g(t)$ is
- $g(t) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}; \Rightarrow \Im\{g(t)\} = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} \Im\{e^{jn\omega_0 t}\} = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} \delta(f - nf_0)$
- $\boldsymbol{\Im \sum_{m=-\infty}^{\infty} \delta(t - mT_0) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} \delta(f - nf_0)}$.



Remark 1: Note that the signal is periodic in the time domain and its Fourier transform is periodic in the frequency domain.
Remark 2: This sequence will be found useful when the sampling theorem is considered later in this lecture.

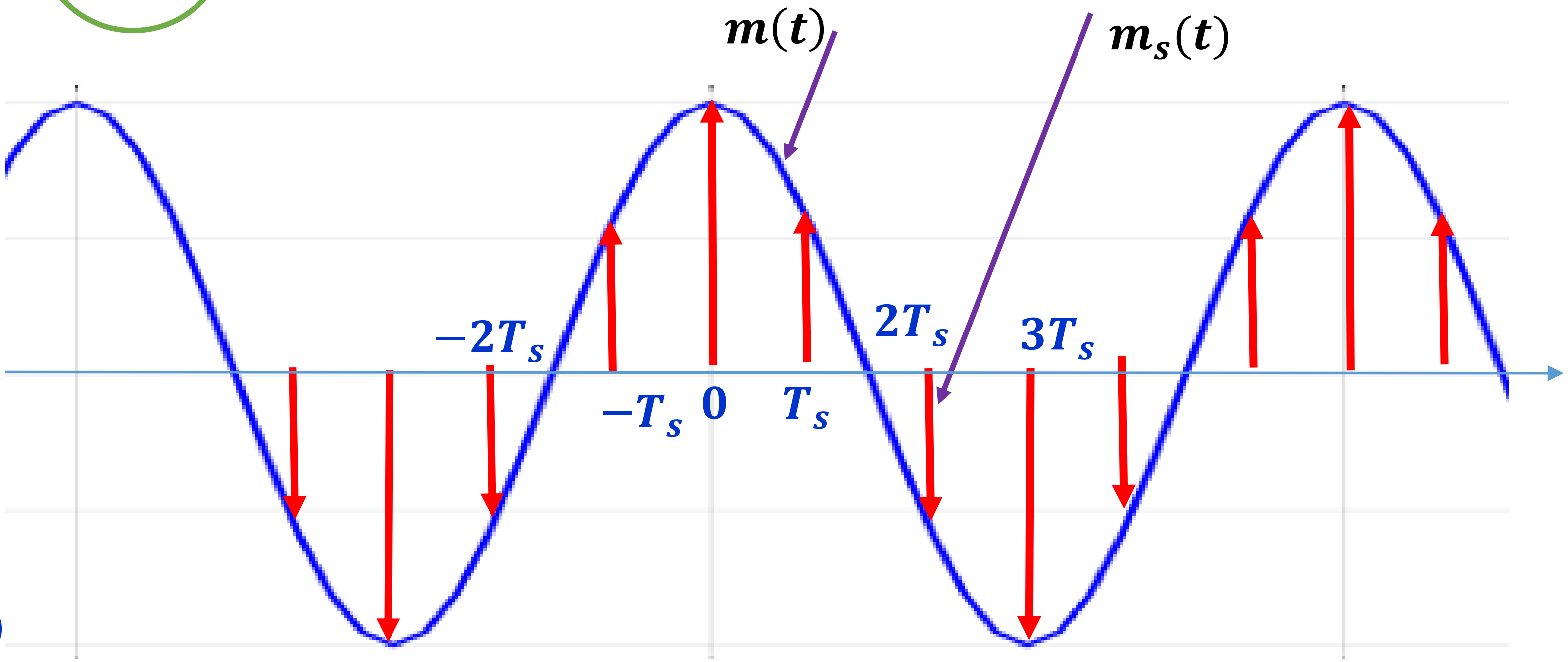
Ideal Sampling

- **Ideal Sampling**: The message m(t), with Fourier transform M(f), which is band-limited to W Hz, is multiplied by a periodic sequence of ideal impulses with period T_s to produce the sampled signal $m_s(t)$.



$$m_s(t) = \sum_{k=-\infty}^{\infty} m(kT_s)\delta(t - kT_s)$$

$$\sum_{m=-\infty}^{\infty} \delta(t - mT_s)$$



$$\delta(t)m(t) = m(0)\delta(t)$$

$$\delta(t - T_s)m(t) = m(T_s)\delta(t - T_s)$$

$$\delta(t - kT_s)m(t) = m(kT_s)\delta(t - kT_s)$$

Ideal Sampling

- $m_s(t) = m(t)g(t) = m(t)\sum_{m=-\infty}^{\infty}\delta(t - mT_s) = \sum_{\mathrm{k}=-\infty}^{\infty} m(kT_s)\delta(t - kT_s)$
- Recall the Fourier transform pair: $\boldsymbol{G(f) = \mathfrak{J}(g(t)) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty}\delta(f - nf_0)}$
- Hence, $m_s(t) = m(t)\frac{1}{T_0}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}$; Product of two functions in the time domain.
- The Fourier transform of $m_s(t)$ is the convolution of M(f) and G(f)

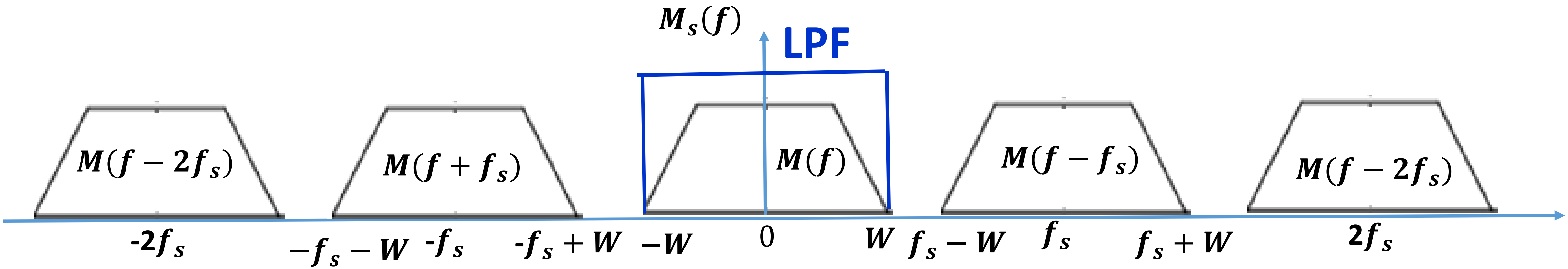
- $M_s(f) = M(f) * G(f) = M(f) * \frac{1}{T_s}\sum_{n=-\infty}^{\infty}\delta(f - nf_s)$

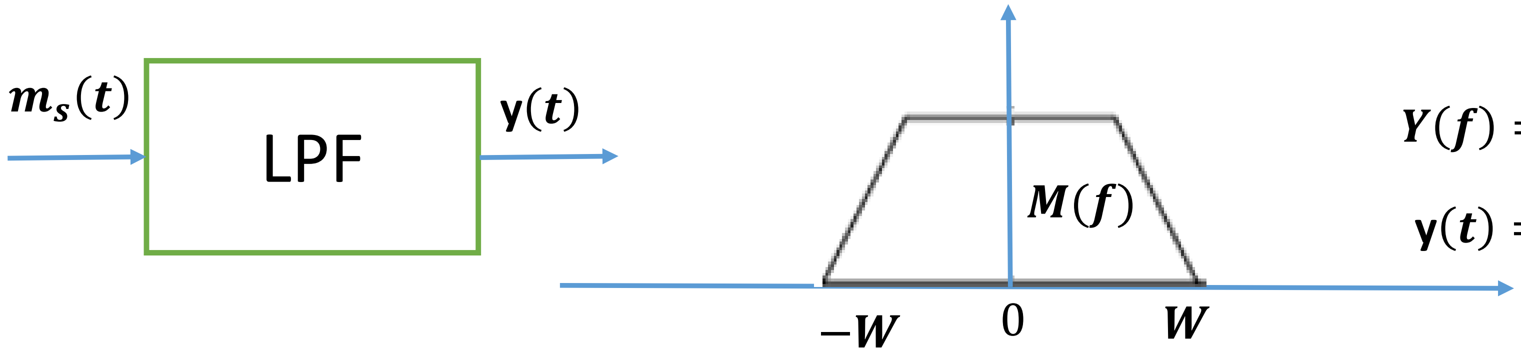
$$\boldsymbol{M_s(f) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} M(f - nf_s)}$$

Ideal Sampling: $f_s > 2W$

$$M_s(f) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} M(f - nf_s)$$

- Let M(f) be as given in the figure. When $f_s > 2W$, $M_s(f)$ will look like





$$Y(f) = \frac{1}{T_s} M(f)$$

$$y(t) = \frac{1}{T_s} m(t)$$

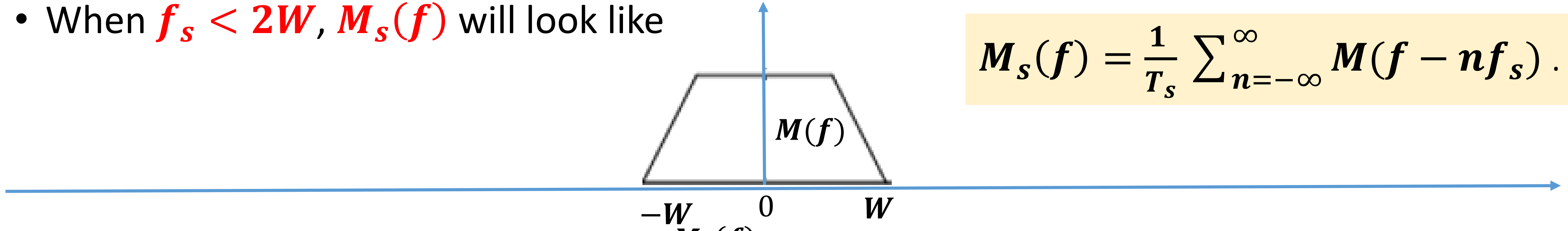
The Sampling Theorem

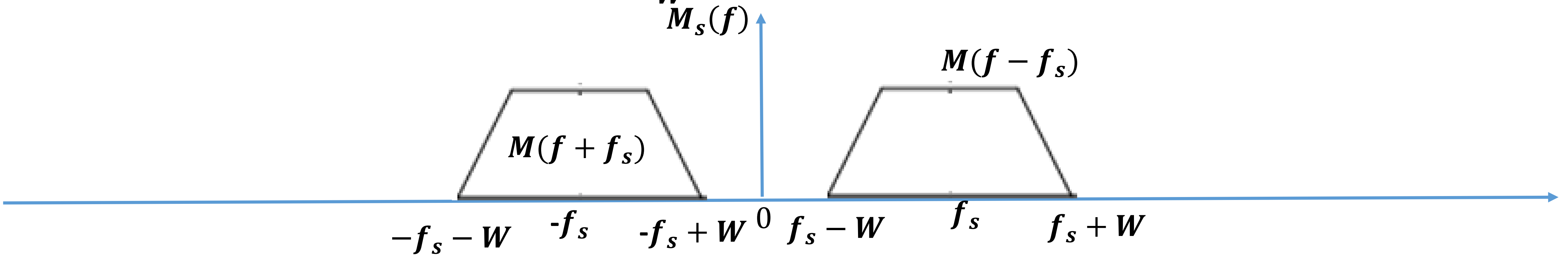
- A bandlimited signal with no frequency components above W Hz can be recovered uniquely from its samples taken every $\boldsymbol{T_s}$ seconds provided that $\boldsymbol{f_s \geq 2W}$, where $\boldsymbol{f_s = 1/T_s}$ is the sampling rate is samples/sec.
- The message m(t) can be recovered from $\boldsymbol{m_s(t)}$ using an ideal LPF with bandwidth W.
- The Sampling frequency $\boldsymbol{f_s = 2W}$, is called the Nyquist rate. It represents the minimum rate at which a signal must be sampled in order to reconstruct it from its samples without distortion.
- When the sampling rate is less than the Nyquist rate, a distortion type of noise called **Aliasing** results.

Sampling Theorem and Aliasing

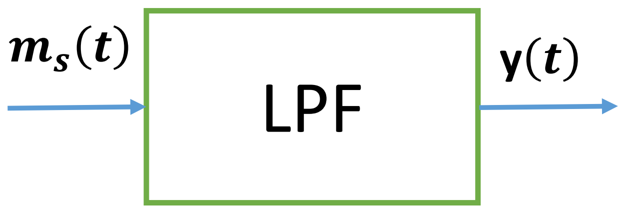
- Let M(f) be the Fourier transform of the message m(t).
- When $f_s < 2W$, $M_s(f)$ will look like

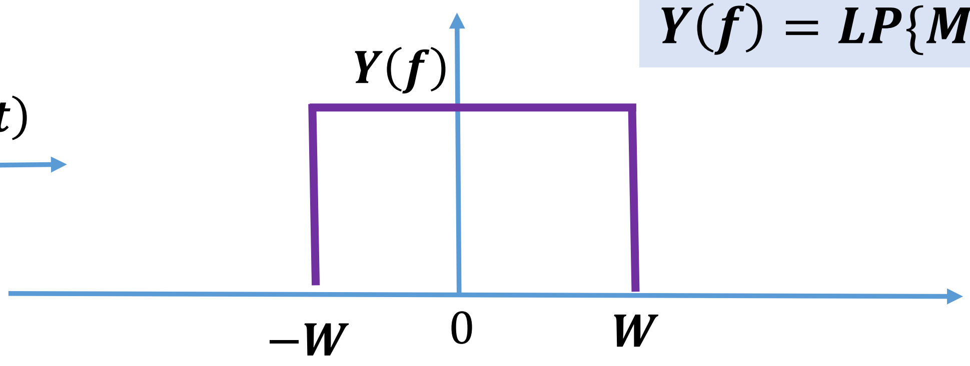
$$M_s(f) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} M(f - nf_s).$$





$$Y(f) = LP\{M(f) + M(f - f_s) + M(f + f_s)\}$$

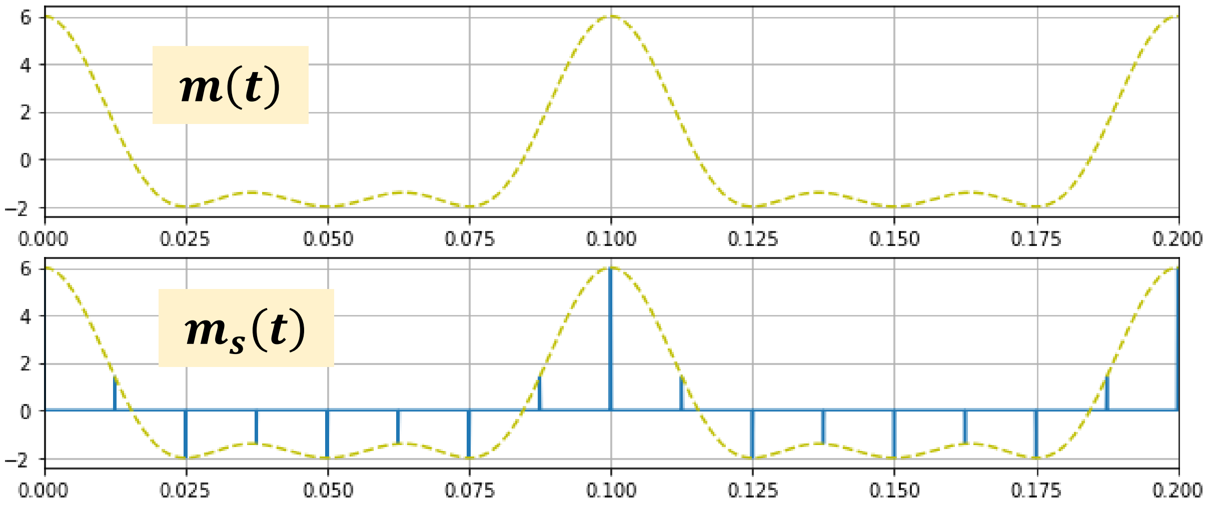




$$Y(f) \neq \frac{1}{T_s} M(f)$$

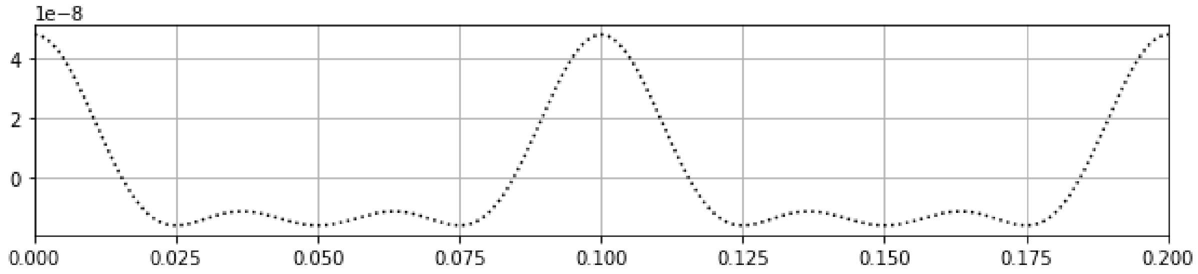
$$y(t) \neq \frac{1}{T_s} m(t)$$

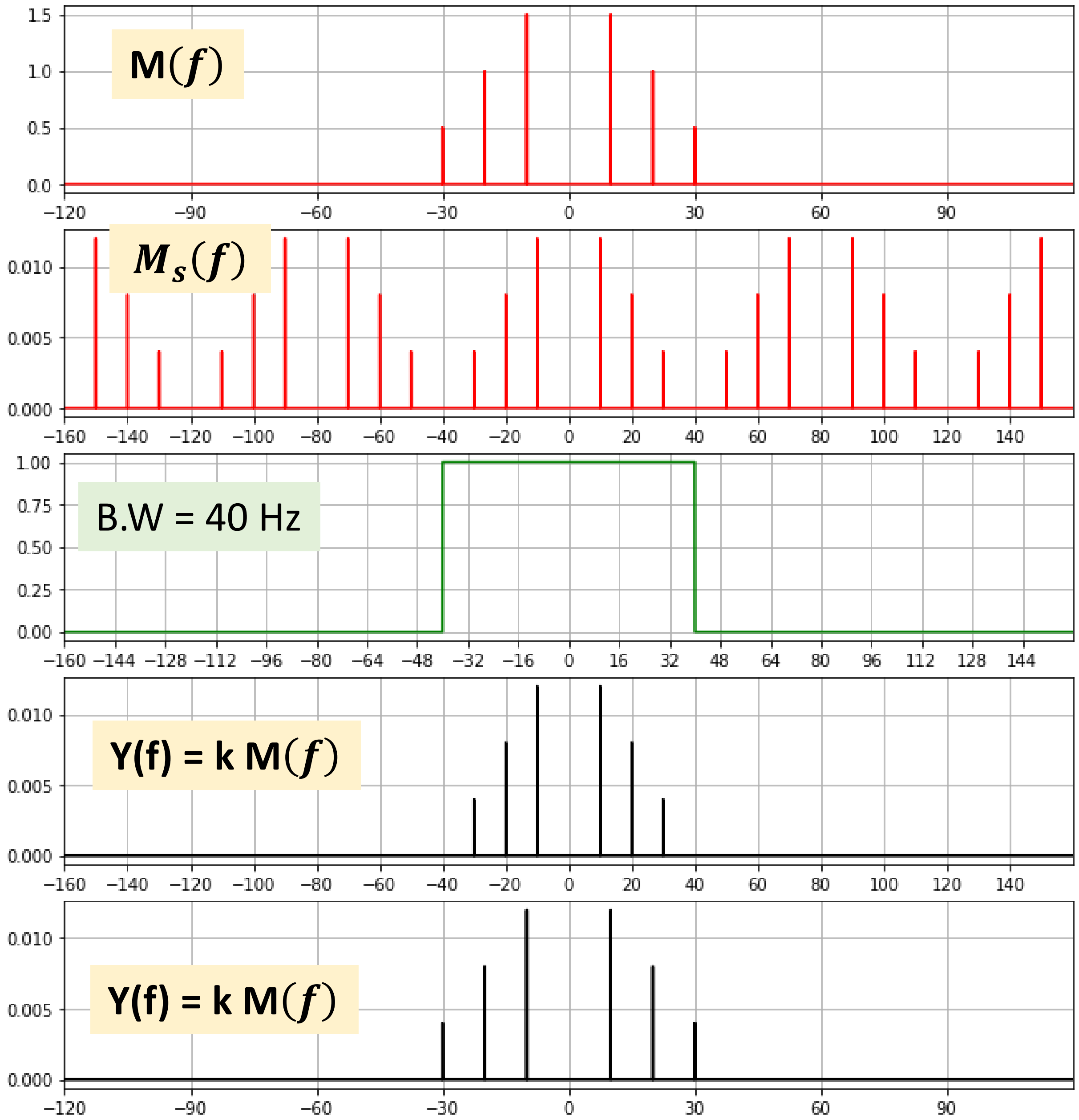
Sampling Theorem: Example $f_s > 2W$



$m(t) = 3\cos 2\pi(10)t + 2\cos 2\pi(20)t + \cos 2\pi(30)t$

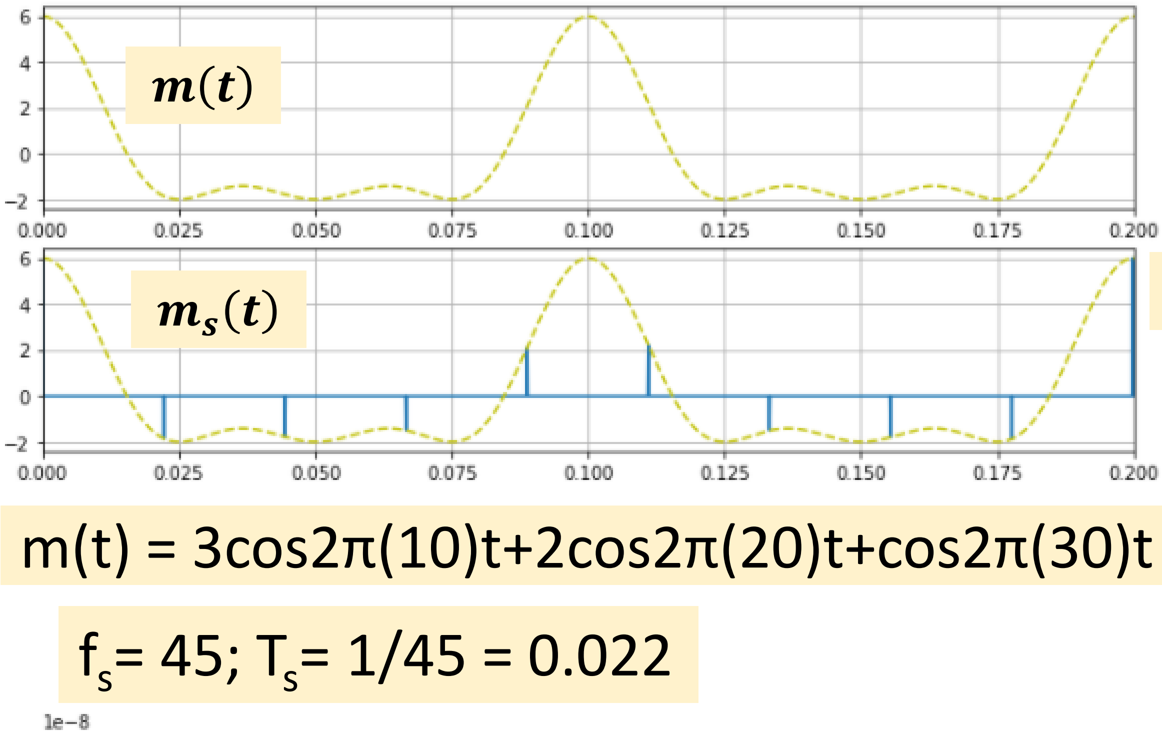
$f_s = 80$; $T_s = 0.0125$

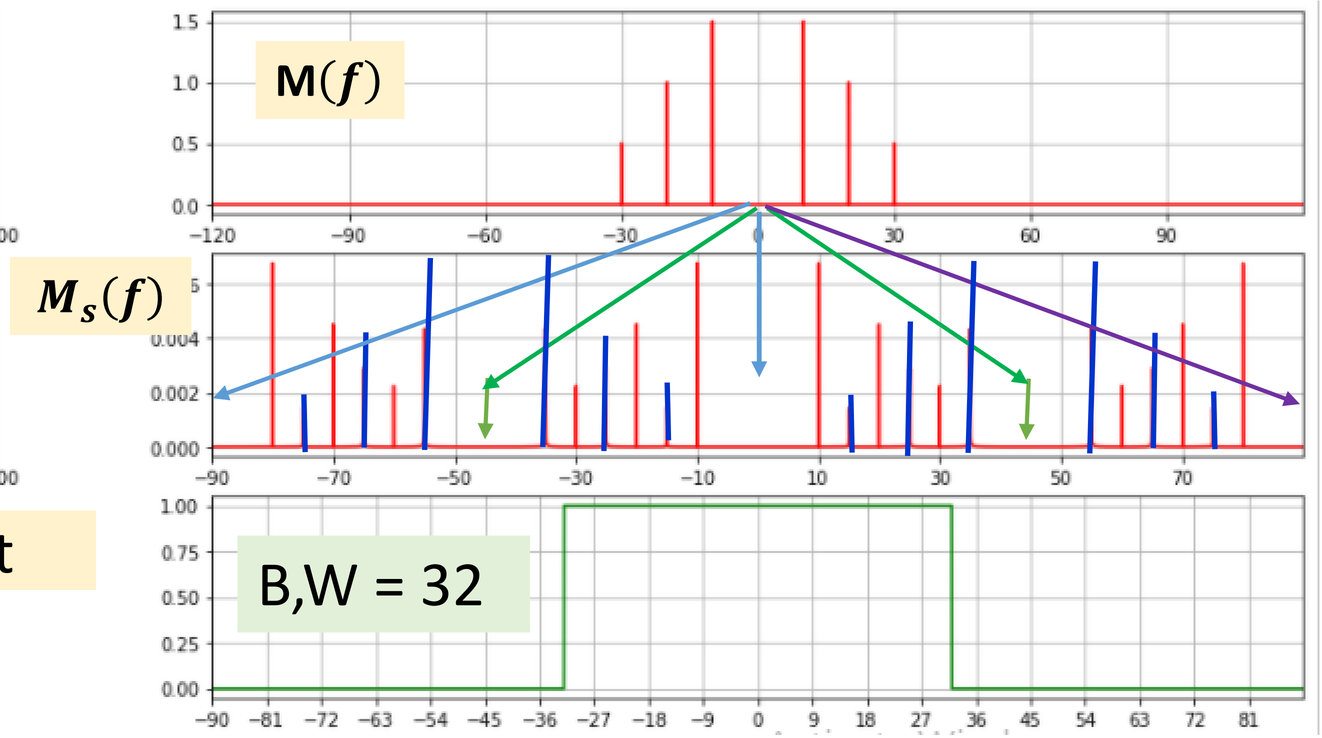




- **Since the sampling rate is greater than the Nyquist rate, the original signal is recovered without distortion.**
- **Output contains the message frequencies: 10, 20, 30**

Sampling Theorem and Aliasing : $f_s < 2W$





m(t) = 3cos2π(10)t+2cos2π(20)t+cos2π(30)t

f_s= 45; T_s= 1/45 = 0.022

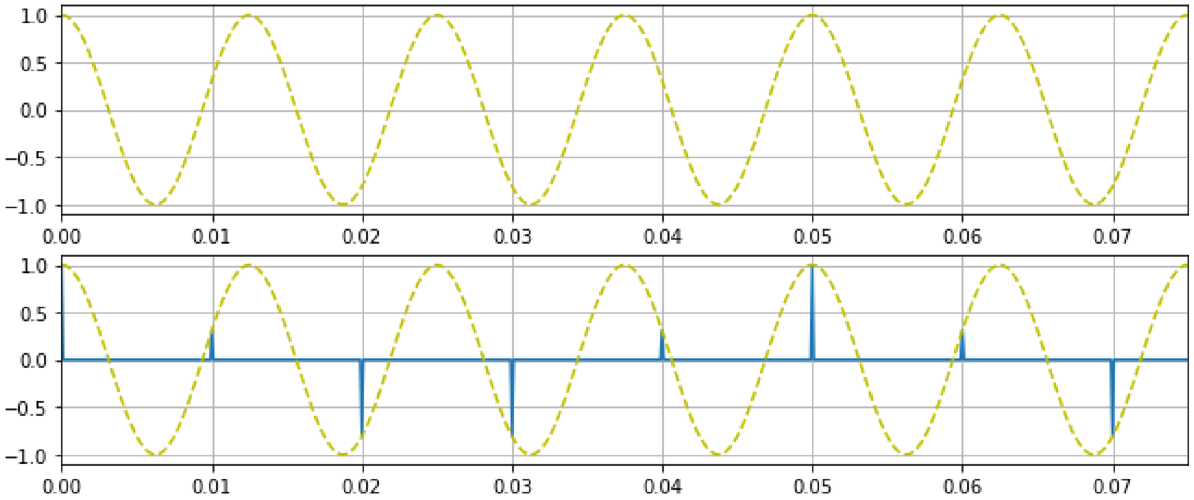
$$M_s(f) = \frac{1}{T_s}\{M(f) + M(f - f_s) + M(f + f_s) + \cdots\}$$

Output contains the message frequencies: 10, 20, 30 Hz. In addition to aliasing frequencies within message bandwidth (45 – 20) = 25 and (45 – 30) = 15

y(t) = k{3cos2π(10)t+2cos2π(20)t+cos2π(30)t

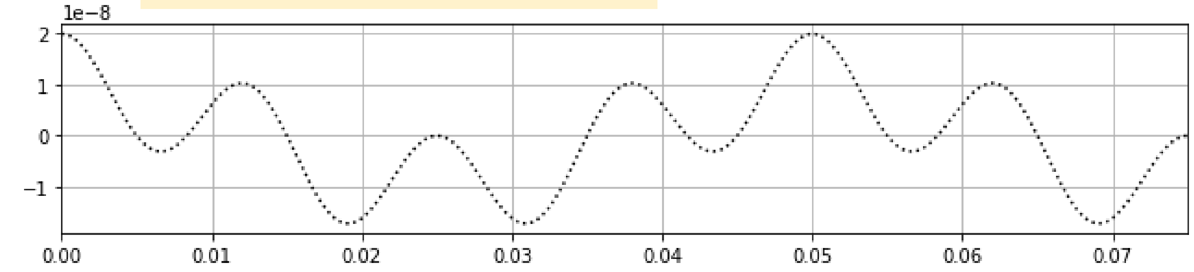
+2cos2π(25)t + cos2π(15)t

Sampling Theorem and Aliasing : Example $f_s < 2W$

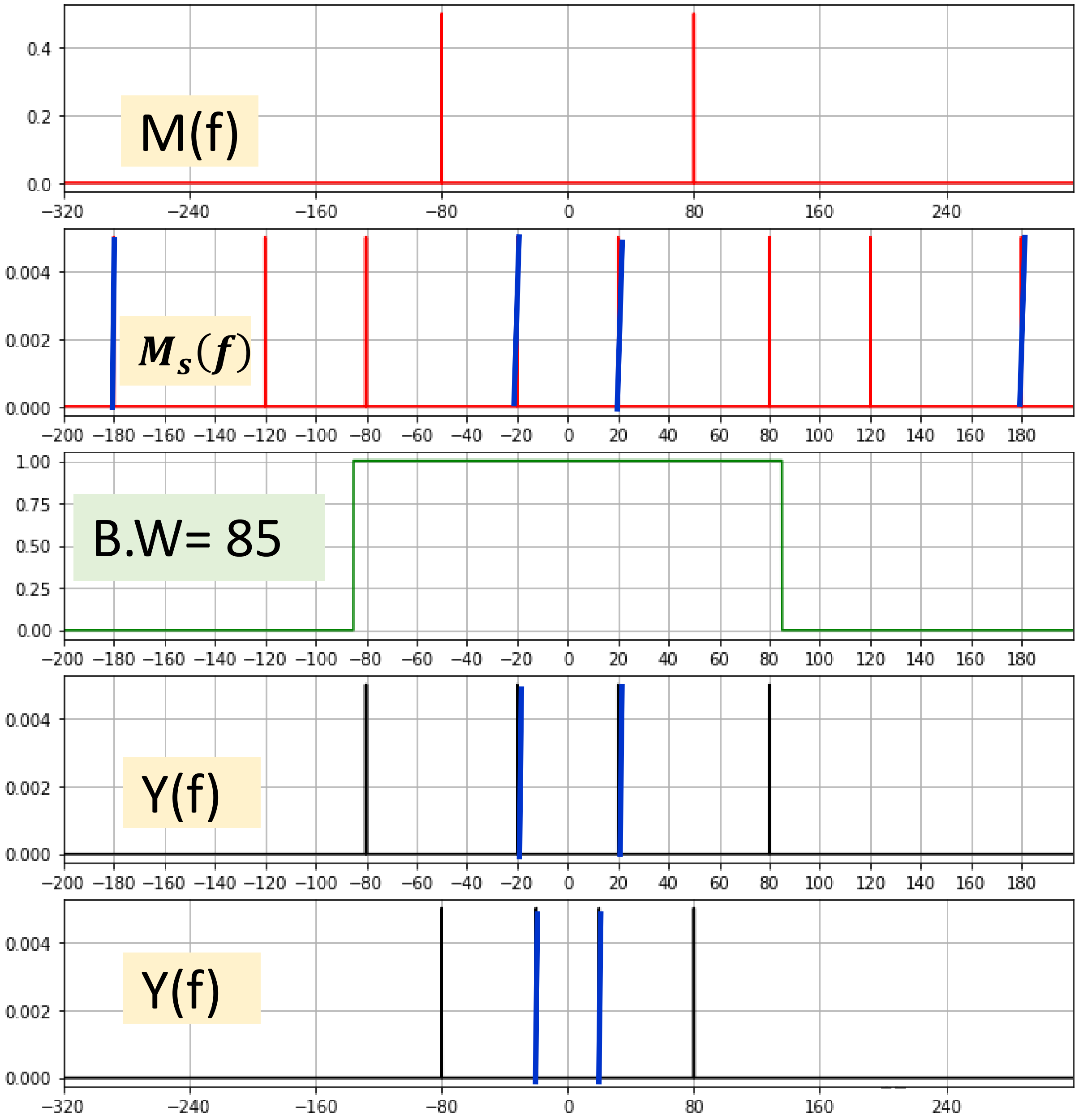


$m(t) = \cos 2\pi(80)t$

$f_s = 100;\ T_s = 0.01 =$



$y(t) = k\cos 2\pi(80)t + k\cos 2\pi(20)t$



Natural Sampling
Lecture Outline

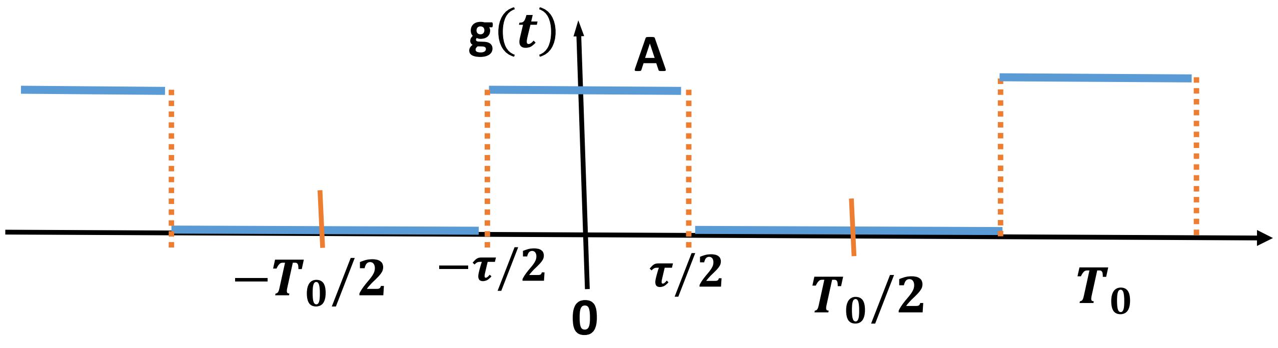
- **Sampling:** is the process by which a continuous time continuous amplitude signal is converted into a discrete time continuous amplitude signal.
- **There are three types of sampling:** Ideal sampling, natural sampling, and flat-topped sampling.
- Ideal sampling, the sampling theorem, and the phenomenon of aliasing were presented in the previous lecture.
- This lecture focuses on natural sampling and the sampling theorem.
- In the next lecture, we consider
  - Flat-topped sampling (sample and hold).
  - Time division multiplexing (TDM)

The Periodic Train of Rectangular Pulses: Fourier Series

- **Example**: Find the trigonometric Fourier series of the periodic rectangular signal defined over one period T_0 as:

$$g(t) = \begin{cases} +A, & -\tau/2 \le t \le \tau/2 \\ 0, & otherwise \end{cases}$$

- **Solution**: The FS is given as $g(t) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$
- $a_0 = \frac{1}{T_0}\int_{-\tau/2}^{\tau/2} g(t)\,dt \Rightarrow \boldsymbol{a_0 = A\tau/T_0}$; $\boldsymbol{\tau/T_0}$ is called the duty cycle of the pulse train.
- $b_n = \frac{2}{T_0}\int_{-\tau/2}^{\tau/2} g(t)\sin(\frac{2\pi n}{T_0}t)\,dt$ = 0; g(t) is an even function of t
- $a_n = \frac{2}{T_0}\int_{-\tau/2}^{\tau/2} g(t)\cos(\frac{2\pi n}{T_0}t)\,dt = \frac{4}{T_0}\int_{0}^{\tau/2} A\cos(\frac{2\pi n}{T_0}t)\,dt \Rightarrow \boldsymbol{a_n = \frac{2A}{n\pi}\sin(\frac{n\pi\tau}{T_0})}$
- $a_n = 0$ when $\mathrm{n} = \frac{T_0}{\tau}, \frac{2T_0}{\tau}, \frac{3T_0}{\tau}, \ldots$
- This is demonstrated on the next slide

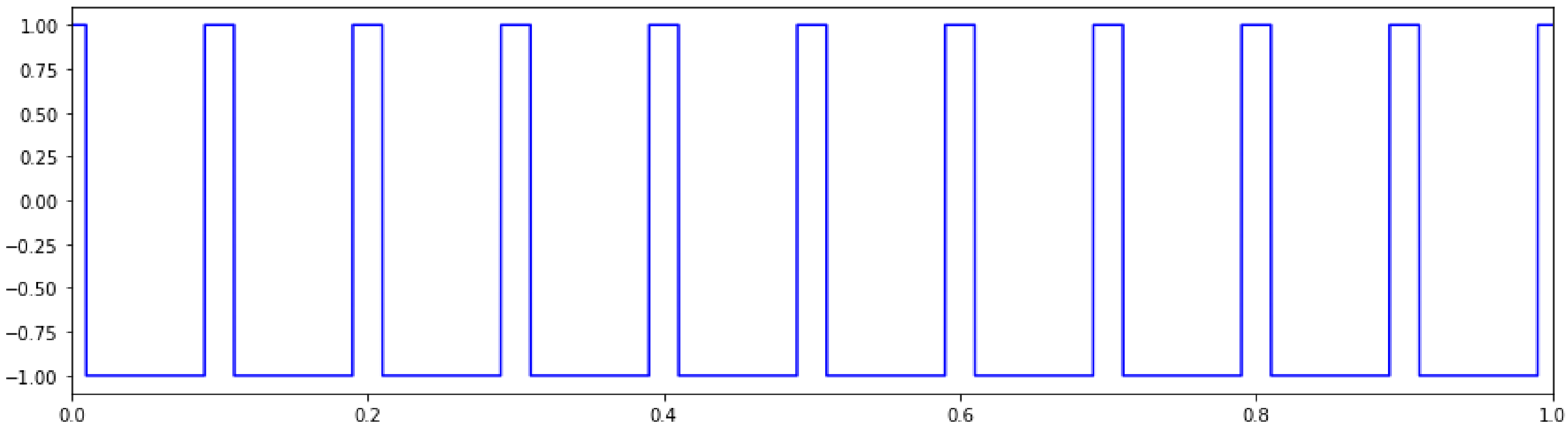


The Periodic Train of Rectangular Pulses: Time and Frequency

The Fourier series of the pulse train is: $g(t) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos 2\pi n f_0 t)$

The Fourier transform is : $\mathbf{G(f)} = a_0\delta(f) + \sum_{n=1}^{\infty} \frac{a_n}{2}[\delta(f - nf_0) + \delta(f + nf_0)]$

$g(t)$

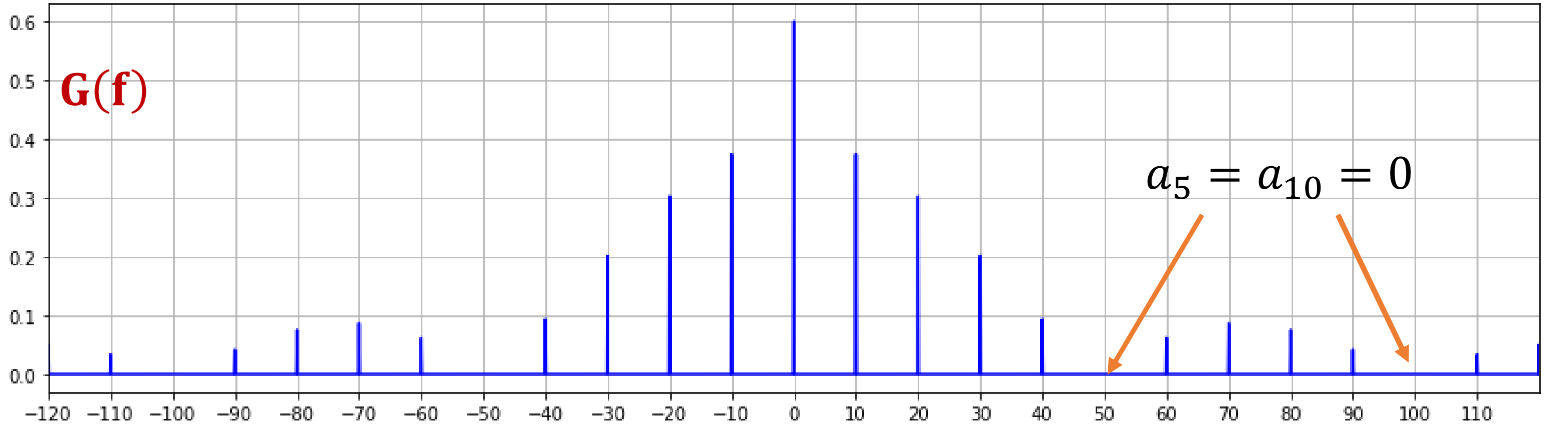


Example:

- $f_0 = 10$ Hz; $T_0 = 0.1$
- Duty cycle $\frac{\tau}{T_0} = 0.2$
- $a_n = 0$ when:
  - $n = \frac{T_0}{\tau}, \frac{2T_0}{\tau}, \frac{3T_0}{\tau}, \ldots$
  - $n = 5, 10, 15, \ldots$
- Spectral lines at $5f_0, 10f_0, \ldots$ vanish

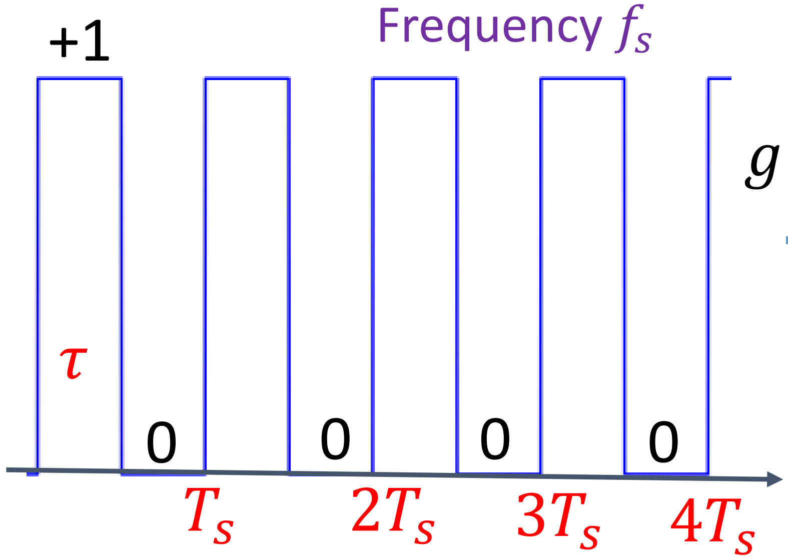
$a_0 = A\tau/T_0$

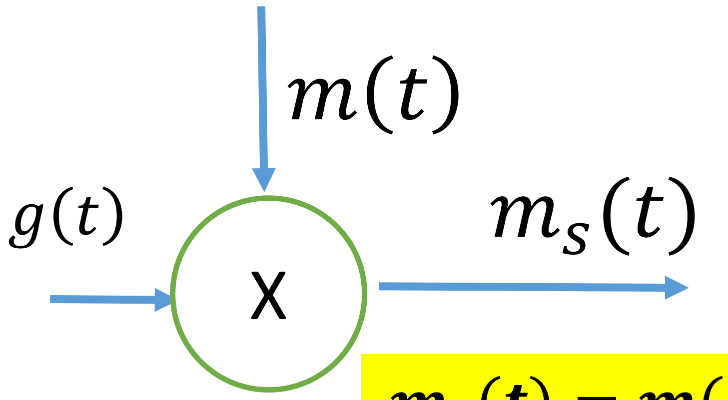
$a_n = \frac{2A}{n\pi}\sin(\frac{n\pi\tau}{T_0})$

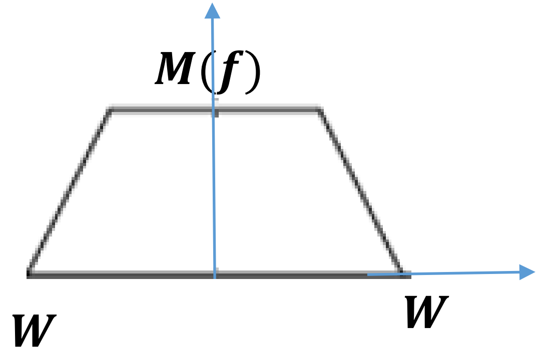


Natural Sampling

- **Natural Sampling**: The message m(t), with Fourier transform M(f), which is band-limited to W Hz, is multiplied by a periodic sequence of pulses with period T_s to produce the sampled signal $m_s(t)$.



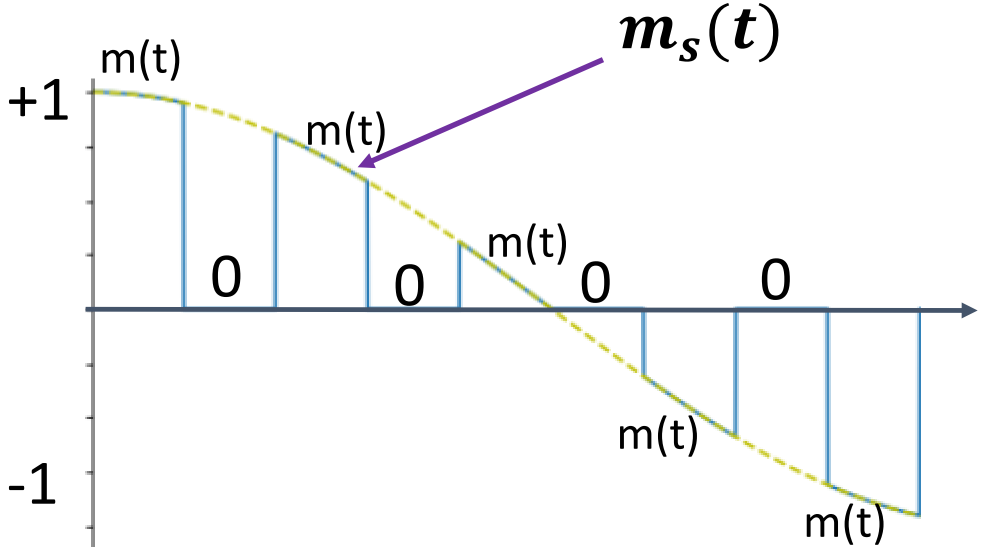




$$\boldsymbol{m_s(t) = m(t)g(t)}$$

$m_s(t)$ = m(t) when g(t) = 1

$m_s(t)$ = 0 when g(t) = 0

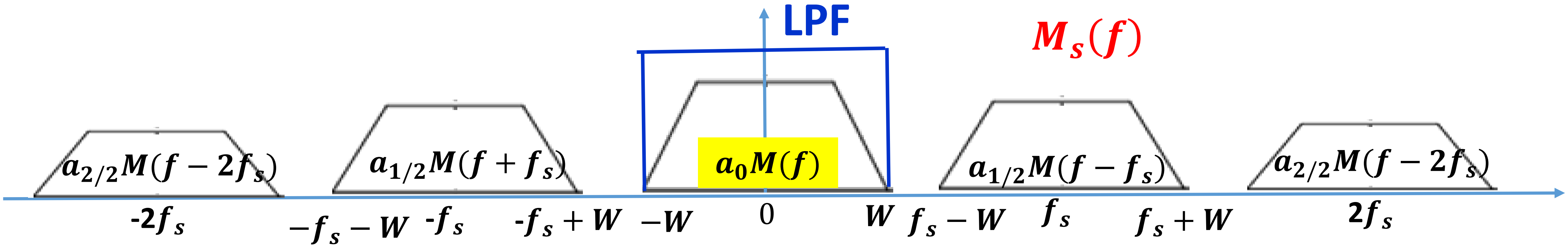


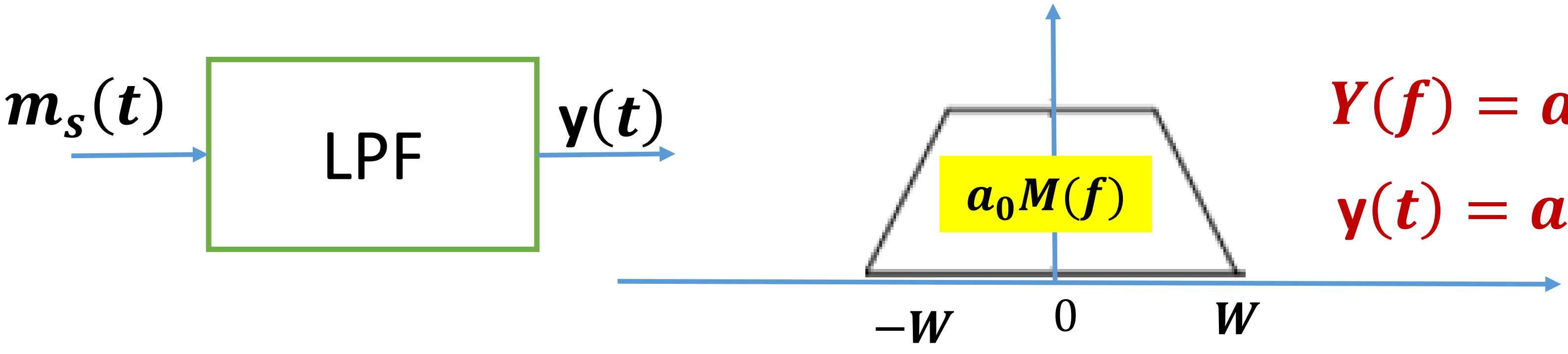
Next, we find M_s(f) in terms of M(f) and $f_s = 1/T_s$

Natural Sampling: $f_s > 2W$

- $m_s(t) = m(t)g(t) = m(t)\{a_0 + \sum_{n=1}^{\infty} a_n \cos 2\pi n f_s t\}$
- $m_s(t) = m(t)g(t) = a_0 m(t) + \sum_{n=1}^{\infty} a_n\, m(t) \cos 2\pi n f_s t$
- $\boldsymbol{M_s(f) = a_0 M(f) + \sum_{n=1}^{\infty} \frac{a_n}{2}[M(f - nf_s) + M(f + nf_s)]}$
- When $f_s > 2W$, $M_s(f)$ will look like

$f_s - W \geq W \Rightarrow f_s \geq 2W$





$Y(f) = a_0 M(f)$

$y(t) = a_0 m(t)$

Message recovered without distortion

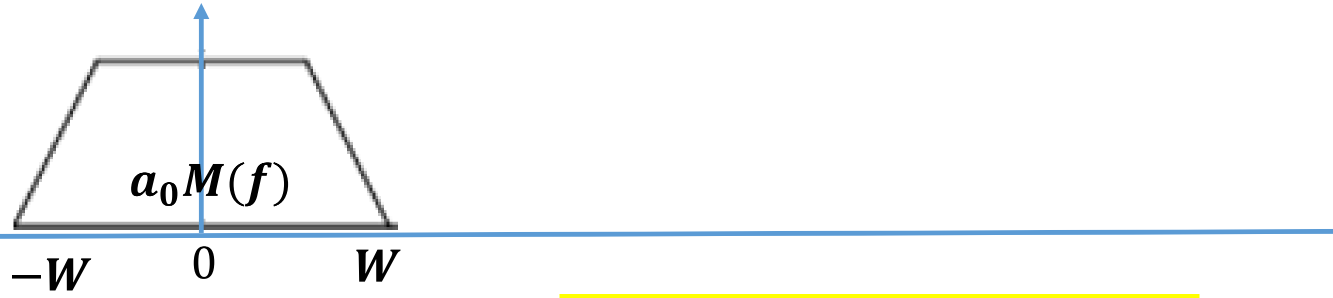
The Sampling Theorem

- A bandlimited signal with no frequency components above W Hz can be recovered uniquely from its samples taken every T_s seconds provided that $f_s \geq 2W$, where $f_s = 1/T_s$ is the sampling rate is samples/sec.
- The message m(t) can be recovered from $m_s(t)$ using an ideal LPF with bandwidth W.
- The Sampling frequency fs = 2W, is called the Nyquist rate. It represents the minimum rate at which a signal must be sampled in order to reconstruct it from its samples without distortion.
- When the sampling rate is less than the Nyquist rate, a distortion type of noise called **Aliasing** results.

Natural Sampling: $f_s < 2W$

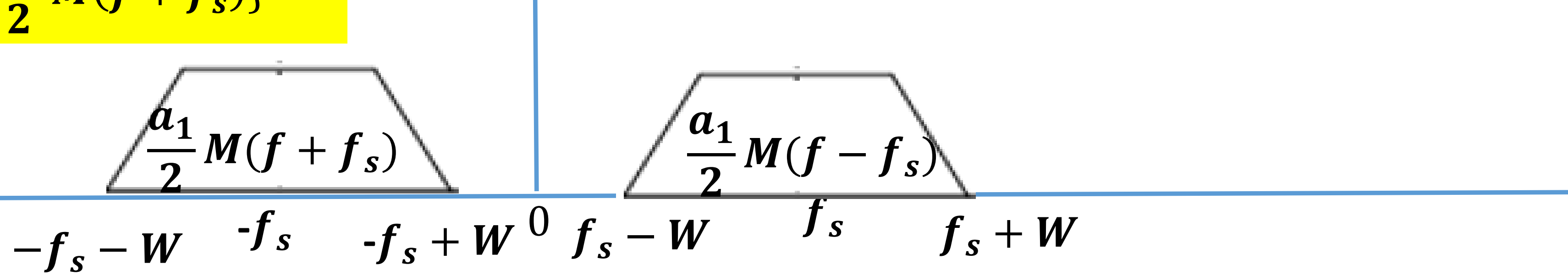
- Let m(t) be the baseband signal with bandwidth W Hz. Assume that $f_s < 2W$
- Need to find $M_s(f)$ and the filtered signal.

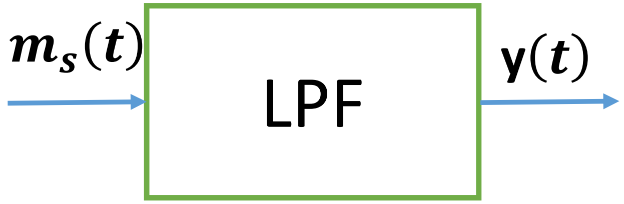
$$M_s(f) = a_0 M(f) + \sum_{n=-\infty}^{\infty} \frac{a_n}{2}\left[M(f - nf_s) + M(f + nf_s)\right]$$



$$Y(f) = LP\left\{a_0 M(f) + \frac{a_1}{2} M(f - f_s) + \frac{a_1}{2} M(f + f_s)\right\}$$

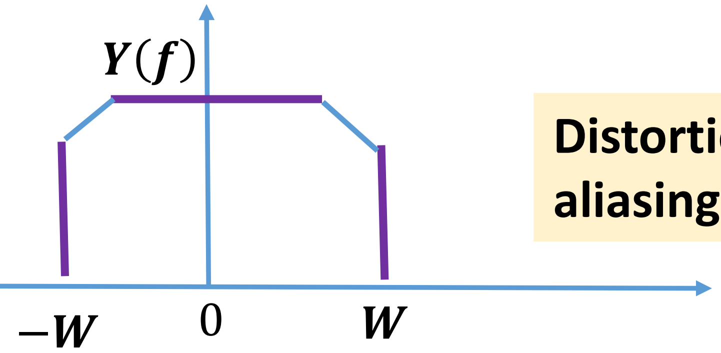
$$f_s - W \leq W \Rightarrow f_s \leq 2W$$





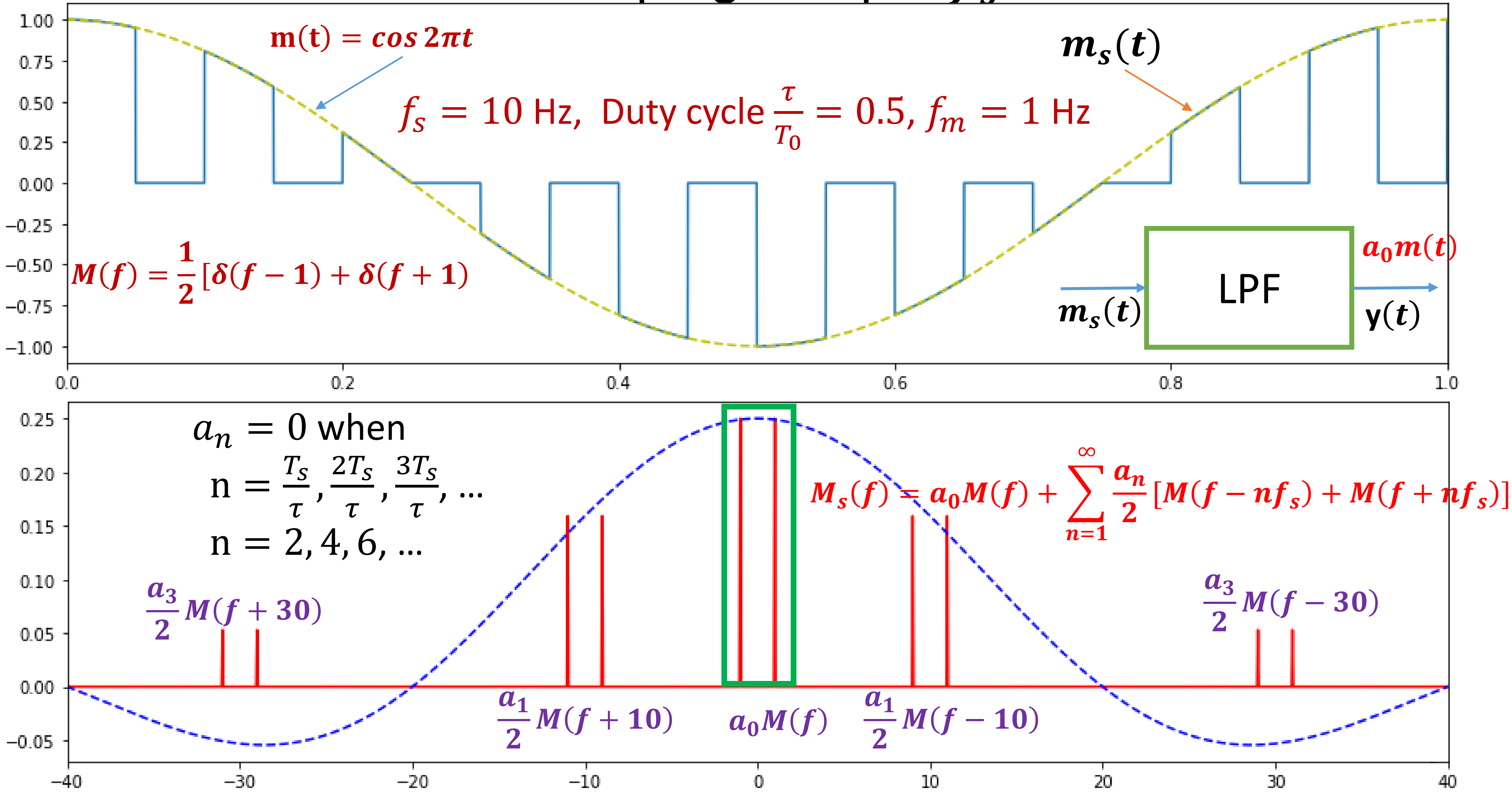
$$Y(f) \neq a_0 M(f)$$

$$y(t) \neq a_0 m(t)$$



Distortion due to aliasing is observed

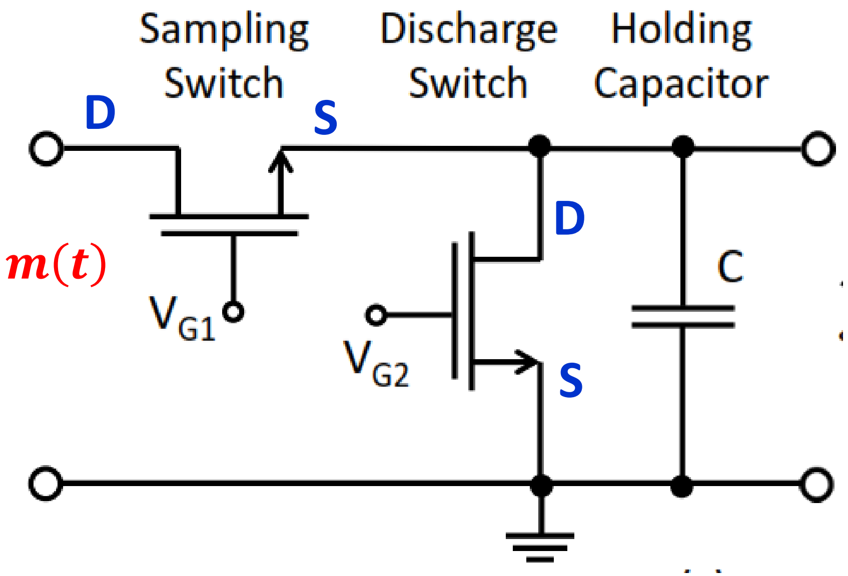
Natural Sampling Example: $f_s > 2W$



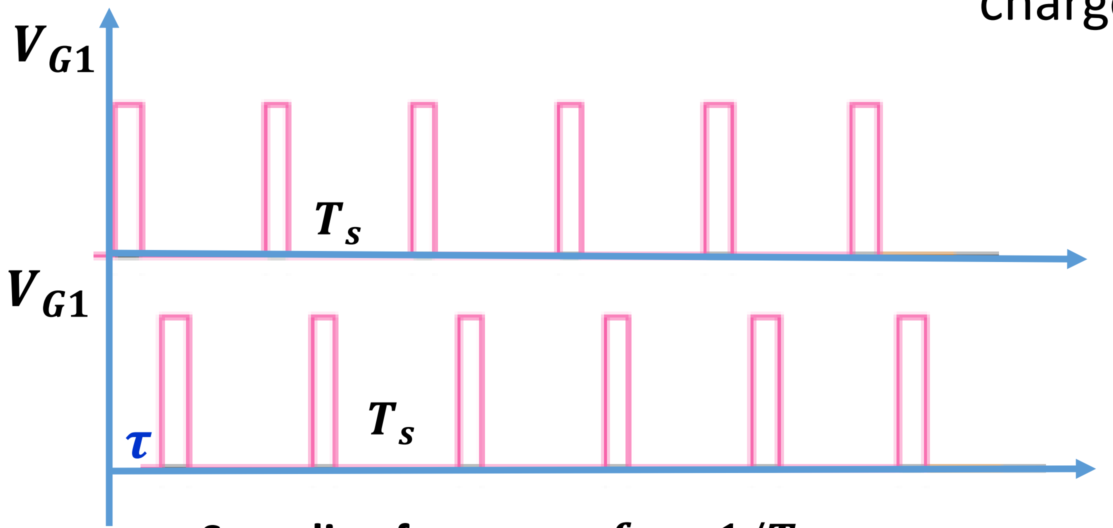
Flat-Topped Sampling
Lecture Outline

- This lecture continues the coverage of the techniques via which a continuous message signal can be sampled, as part of a PCM system
- **Sampling:** is the process by which a continuous time continuous amplitude signal is converted into a discrete time continuous amplitude signal.
- **There are three types of sampling:** Ideal sampling, natural sampling, and flat-topped sampling.
- Ideal sampling, natural sampling, the sampling theorem, and the phenomenon of aliasing were presented in the previous two lectures.
- In this lecture, we address the following topics
  - Flat-topped sampling (sample and hold).
  - Time division multiplexing (TDM)

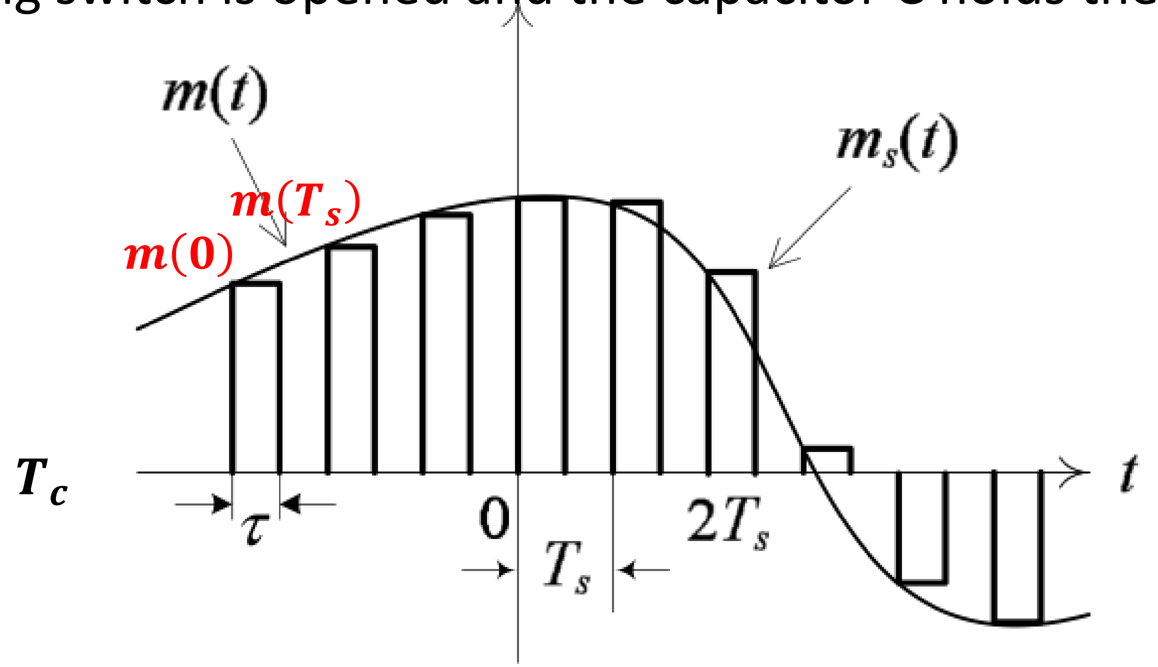
Flat-topped sampling (zero order hold sampling)



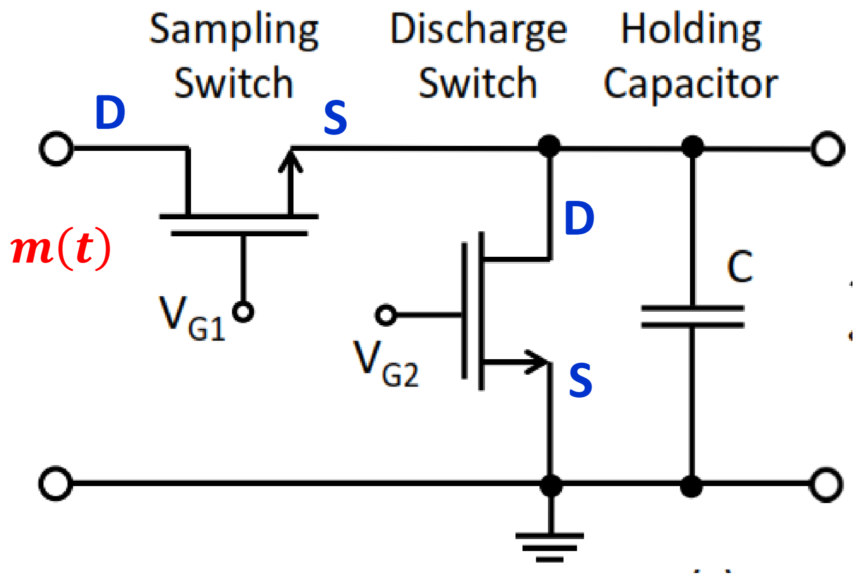
- The sample and hold circuit performs the task of sampling.
- The message m(t) is bandlimited to W Hz.
- The sample and hold circuit consists of two field effect transistor (FET) switches and a capacitor.
- The sampling switch is closed for a short duration by a short pulse applied to the gate G1 of the transistor. During this period, the capacitor C is quickly charged up to a voltage equal to the instantaneous sample value of the incoming signal m(t) .
- The sampling switch is opened and the capacitor C holds the charge.



Sampling frequency $f_s = 1/T_s$

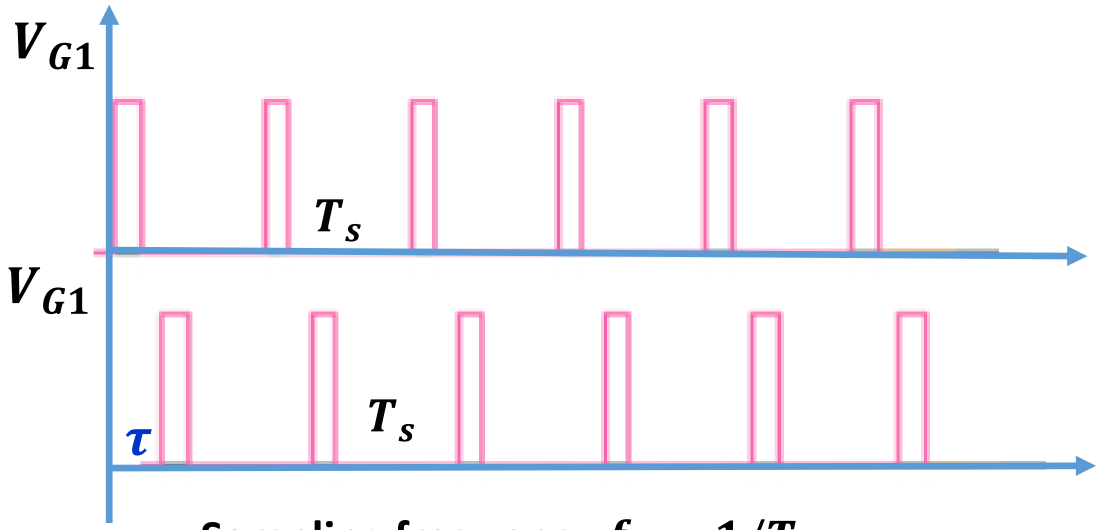


Flat-topped sampling (zero order hold sampling)

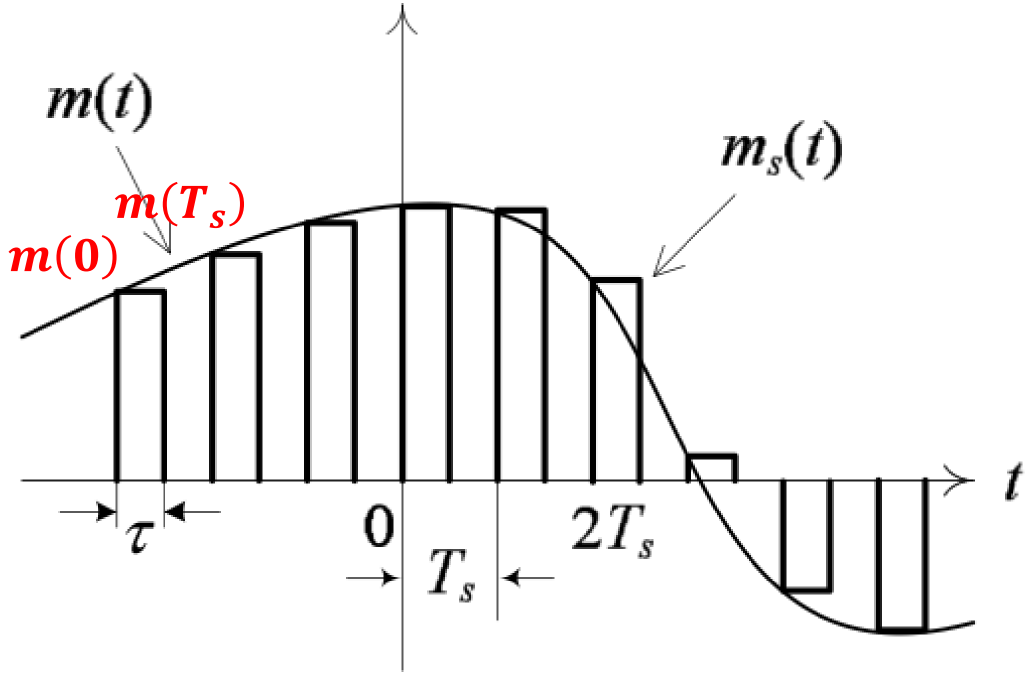


- The discharge switch is then closed by a pulse applied to its gate G2 at t = τ.
- Due to this, the capacitor C is discharged to zero volts.
- The discharge switch is then opened and thus the capacitor has no voltage.
- Hence the output of the sample and hold circuit consists of a sequence of flat topped samples

http://technical123b.blogspot.com/2016/11/pulse-amplitude-modulation-pam.html



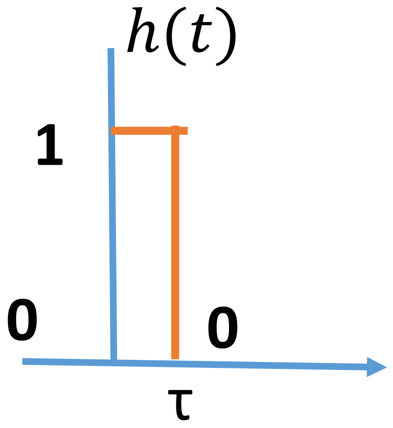
Sampling frequency $f_s = 1/T_s$

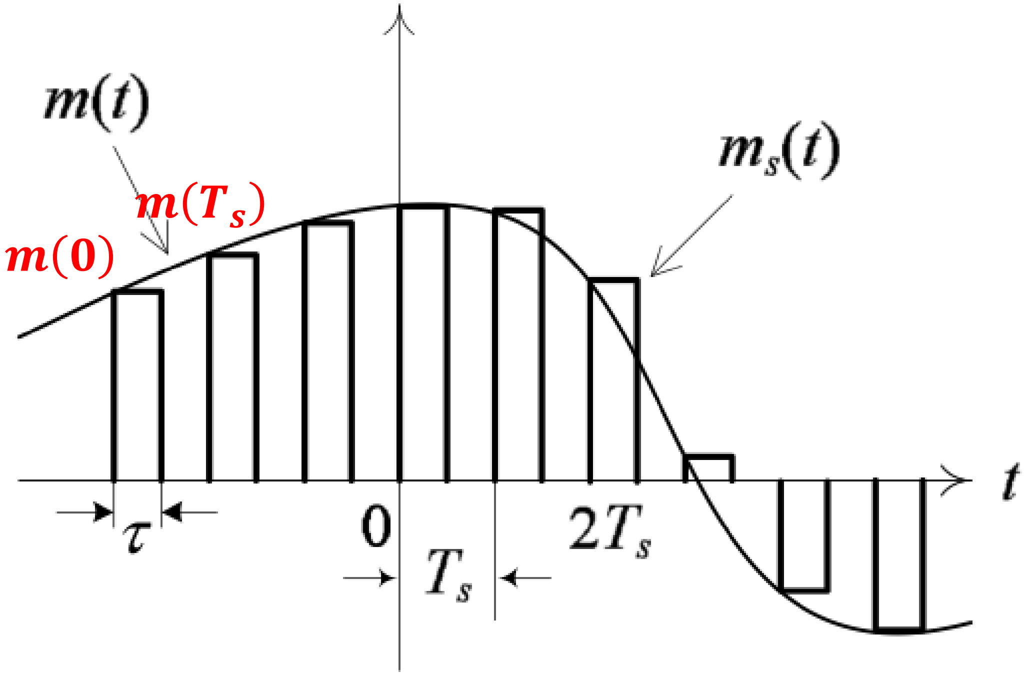


Flat-topped sampling: modeling

- Let h(t) be a basic unit amplitude pulse defined as:
- $h(t) = \begin{cases} 1 & 0 < t < \tau, \\ 0 & otherwise \end{cases}$
- In flat-topped sampling, the sampler generates a sequence of equally spaced rectangular pulses whose amplitudes are proportional to the message signal $m(t)$ at the sampling times $m(kT_s)$.
- The sampled signal is represented as

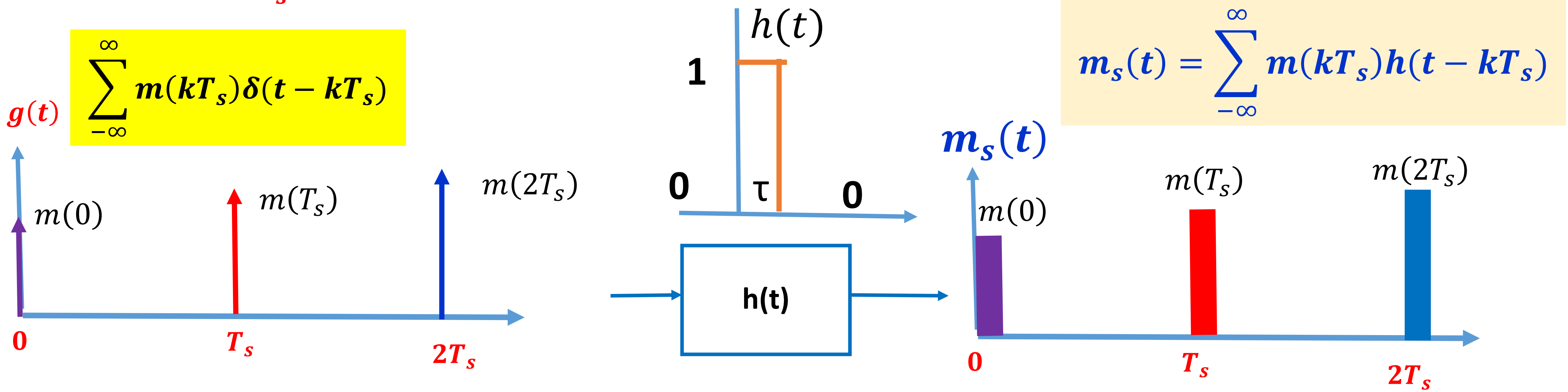
$$\boldsymbol{m_s(t) = \sum_{-\infty}^{\infty} m(kT_s)h(t - kT_s)}$$





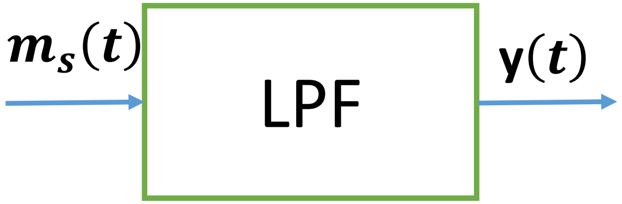
Flat-topped sampling: modeling

- The sampled signal is represented as $m_s(t) = \sum_{-\infty}^{\infty} m(kT_s)h(t - kT_s)$
- Using the identity, $\delta(t - kT_s) * h(t) = h(t - kT_s)$
- Multiplying both sides by $m(kT_s)$ we get, $m(kT_s)\delta(t - kT_s) * h(t) = m(kT_s)h(t - kT_s)$
- Therefore, $m_s(t)$ can be expressed as: $m_s(t) = h(t) * \sum_{-\infty}^{\infty} m(kT_s)\delta(t - kT_s)$
- Taking the Fourier transform, and recognizing that the second term corresponds to an ideally sampled sequence, we get
- $M_s(f) = H(f)\frac{1}{T_s}\sum_{-\infty}^{\infty} M(f - kf_s)$; where $H(f) = \tau sinc(f\tau)$.

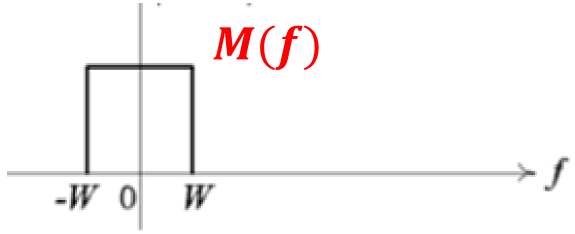


Flat-topped sampling: spectrum and message recovery

- $M_s(f) = H(f)\frac{1}{T_s}\sum_{-\infty}^{\infty} M(f - kf_s)$; $H(f) = \tau sinc(f\tau)$
- Here, we observe that the spectrum of the flat-topped sampled signal corresponds to the spectrum of the ideally sampled signal multiplied by the Fourier transform of the rectangular pulse $\tau sinc(f\tau)$.
- When $m_s(t)$ is passed through a low pass filter with bandwidth W, the output is y(t)



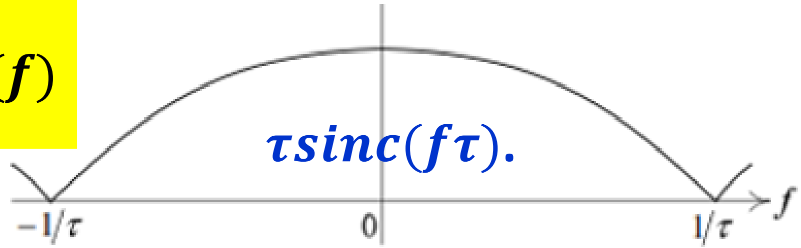
Spectrum of the message



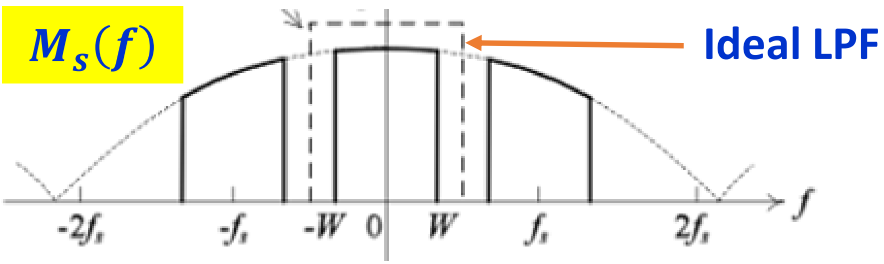
Spectrum of the ideally sampled sequence

$$Y(f) = \frac{1}{T_s}\tau sinc(f\tau)M(f)$$

Spectrum of the rectangular pulse

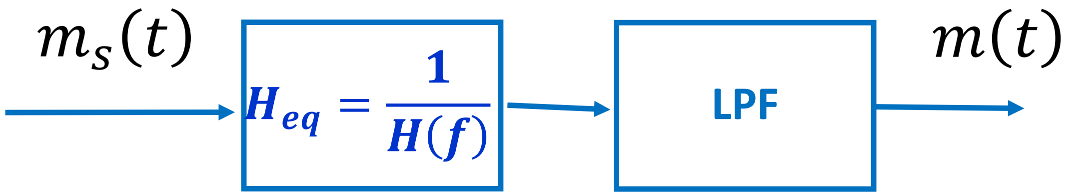


Spectrum of the sample and hold signal



Flat-topped sampling: equalization

- The LPF filter output is $Y(f) = \frac{1}{T_s}\tau sinc(f\tau)M(f)$.
- Note that $Y(f) \neq kM(f) \Rightarrow Distortion$.
- The distortion is due to the finite width τ of the sampling pulse.
- When the message B.W $W \ll 1/\tau$, the distortion is negligible. As τ increases, the distortion becomes more pronounced.
- Even though the signal is sampled at a rate $f_s > 2W$, a distortion is observed.
- A distortion-free signal can be obtained by using an equalizing filter whose transfer function is the reciprocal of the Fourier transform of the unit pulse

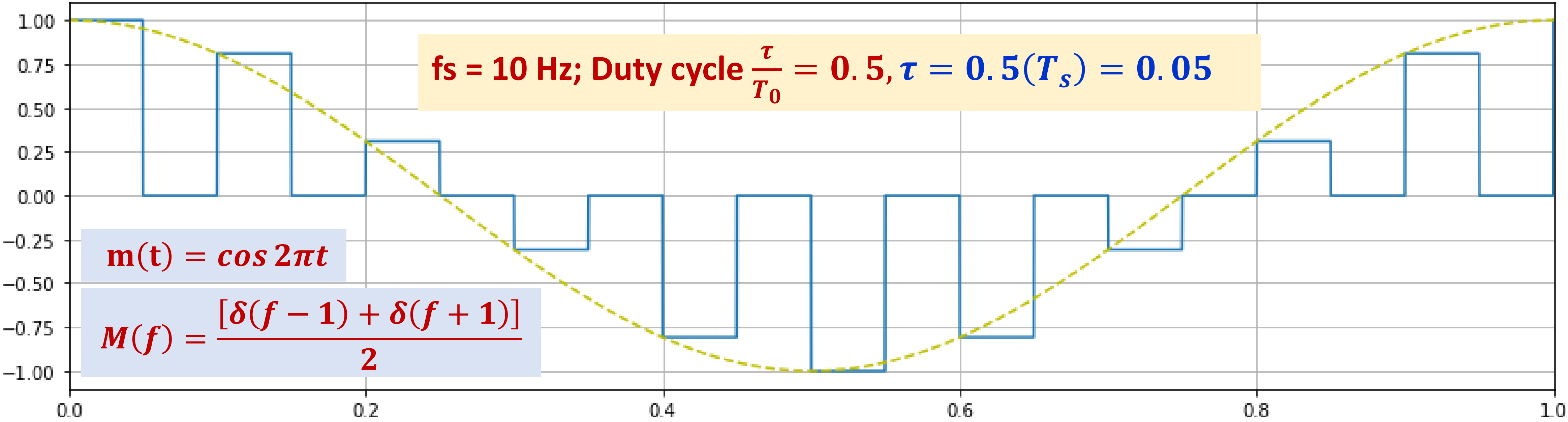


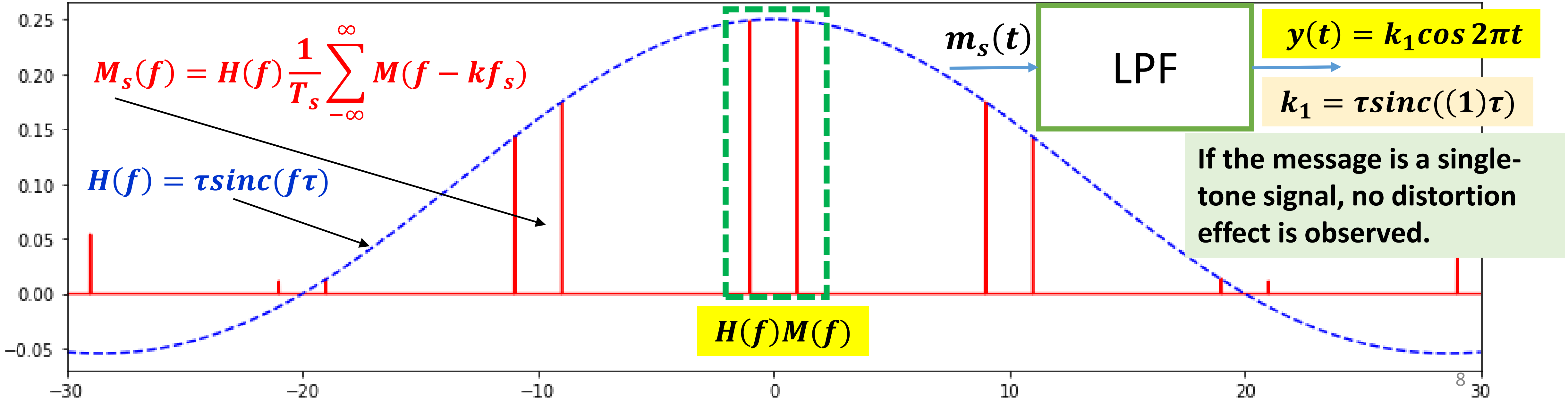
Here,

- $f_s > 2W$
- $H(f) = A\tau sinc(f\tau)$

Equalization implemented at the receiver

Flat-topped sampling: single tone example



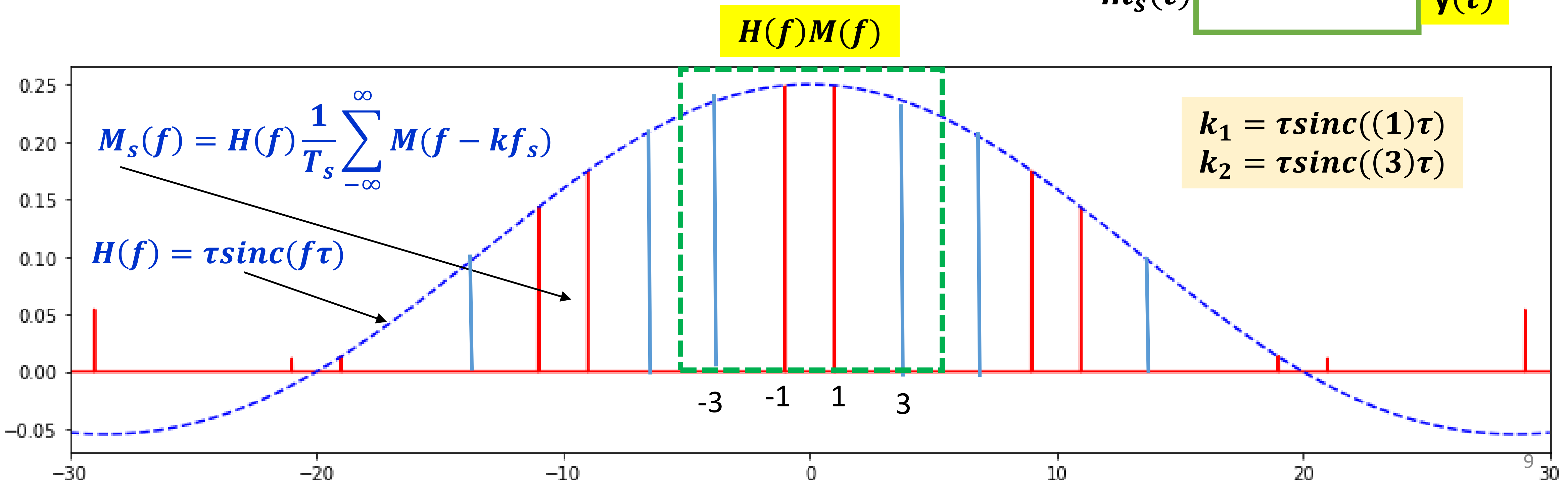


Flat-topped sampling: multi-tone example

Example: Repeat the previous example with fs = 10 Hz; Duty cycle $\frac{\tau}{T_0} = 0.5, \tau = 0.5(T_s) = 0.05$

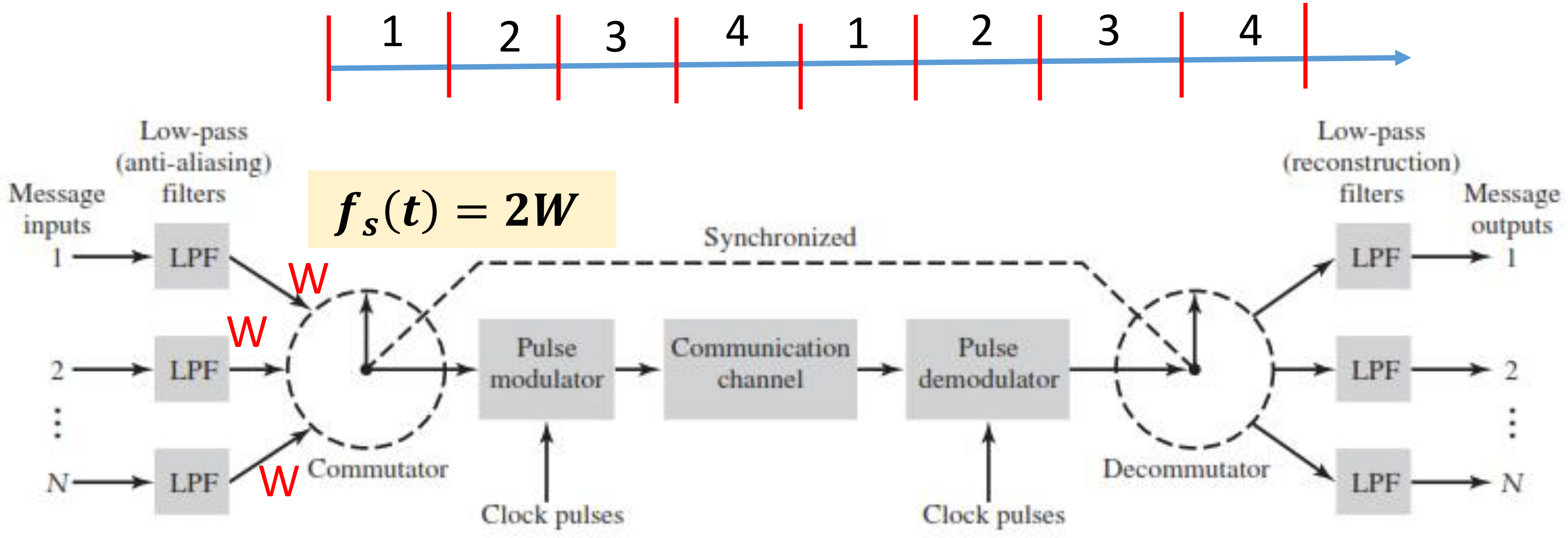
Now, let, $\mathrm{m(t)} = cos\, 2\pi t + cos\, 2\pi 3t$. $M(f) = \frac{[\delta(f-1)+\delta(f+1)]}{2} + \frac{[\delta(f-3)+\delta(f+3)]}{2}$

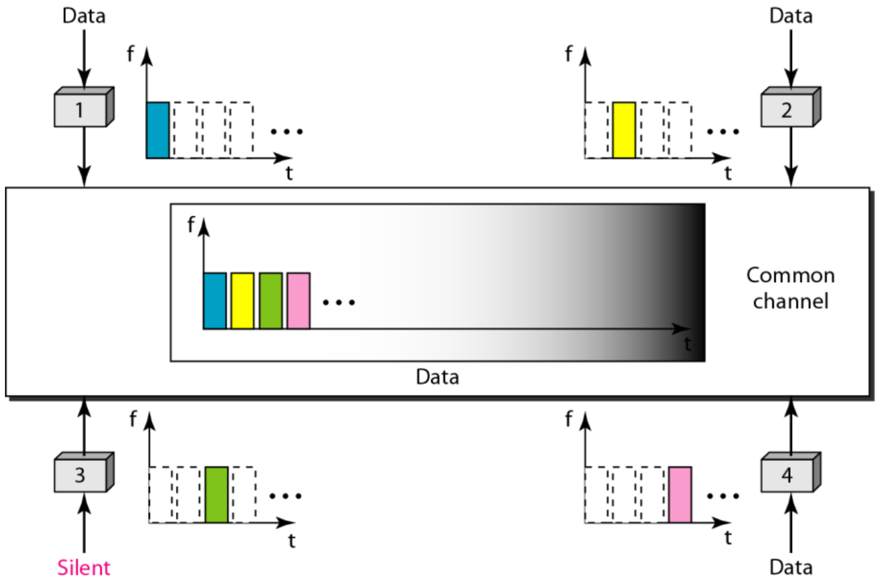
- The spectrum of the sampled signal is as shown in the figure below.
- The output of the LPF is $y(t) = k_1 cos\, 2\pi t + k_2 cos\, 2\pi 3t \neq km(t)$
- Amplitude distortion is observed



Time Division Multiplexing (TDM)

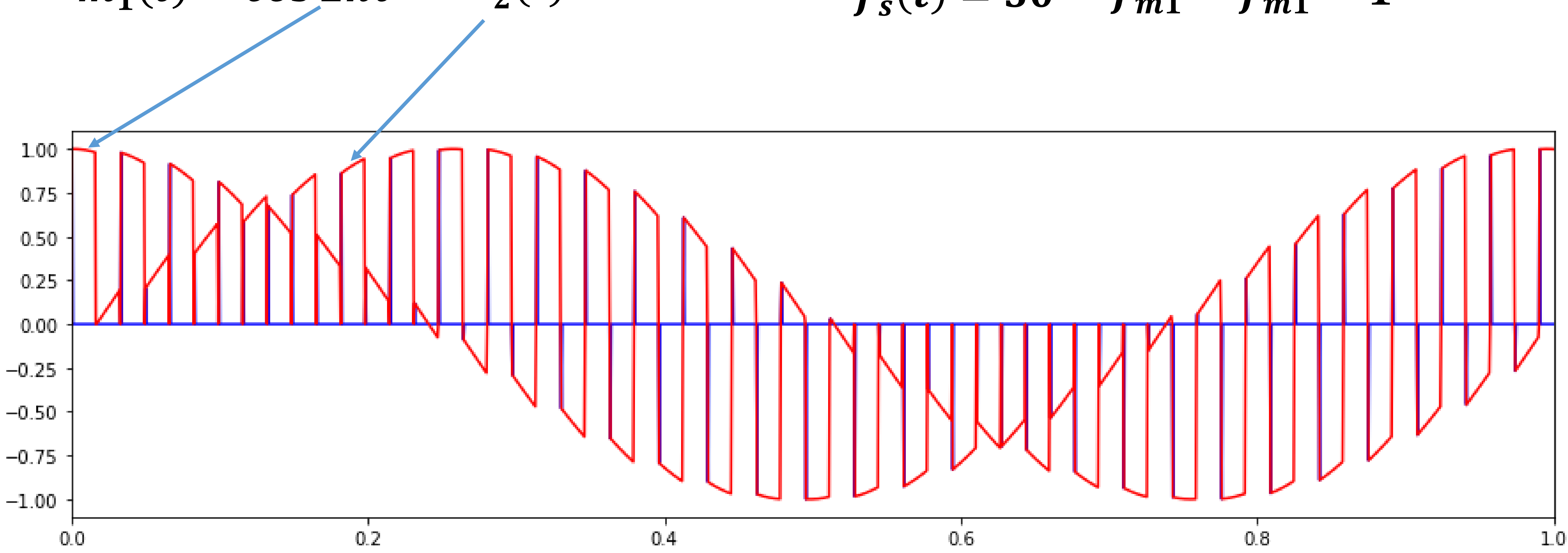
- **Time Division Multiplexing (TDM)**: A technique which allows multiple users to use the same channel by assigning each user a portion of the transmission time without interfering with other users.
- Let N be the number of sources. The time axis is divided into N slots and each slot is allocated to a source.
- Each source transmits only during its slot, avoiding the possibility of a collision.
- When a user transmits during its slot, it utilizes the entire B.W. of the channel and this B.W. will be made available to the next user during the succeeding time slot.
- The collection of the N slots is called a **cycle**.
- TDMA requires some form of synchronization.
- The number of signal samples transmitted per second should be larger than the Nyquist rate.





Time Division Multiplexing: Example

$m_1(t) = cos\, 2\pi t$ $\quad m_2(t) = \sin 2\pi t$ $\quad f_s(t) = 30$ $\quad f_{m1} = f_{m1} = 1$



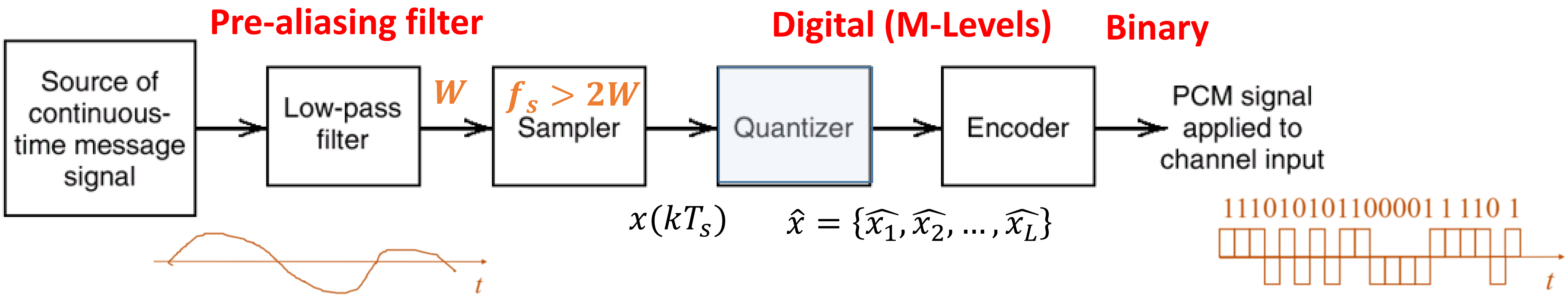
Uniform Quantization
Lecture Outline

- **Quantization** is defined as the process of converting the continuous amplitude sample $x(kT_s)$ of a message signal $x(t)$ into a discrete amplitude $\hat{x}$ taken from a finite and countable set of L possible values.

- **In this lecture**
  - We introduce the concept of quantization.
  - Define the uniform quantizer
  - Derive the average quantization noise of the uniform quantizer.
  - Find the signal to quantization noise ratio.
  - Present a number of illustrative examples

Pulse Code Modulation: Overview

- Sources are of two types: **analog and digital**.
- An analog source can be converted into digital via sampling, quantization, and binary encoding. This process is called **pulse code modulation**



➢ **Sampler:** If W is the highest frequency component in a signal, then the sampling rate required to reconstruct the message from its samples should follow the Nyquist rate where $f_s > 2W$.

➢ **Three types of sampling were discussed in previous lectures; ideal, natural, and flat-topped.**

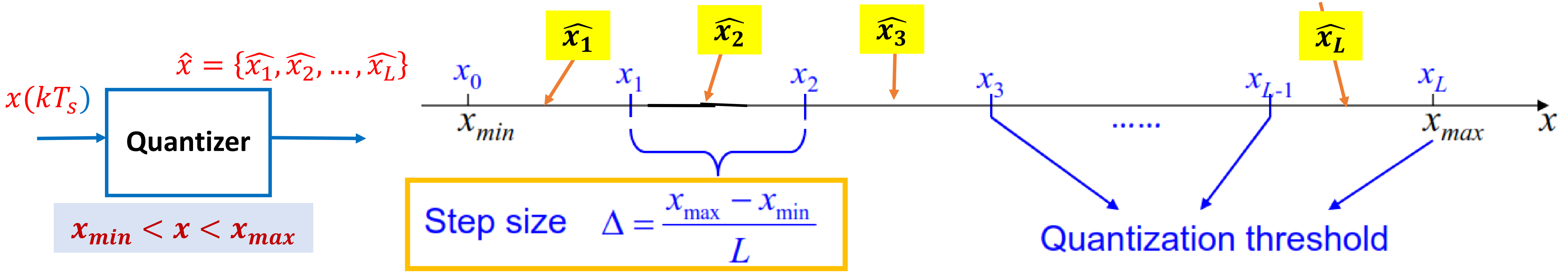
➢ The output of the sampler is a continuous amplitude discrete time signal.

➢ **Quantizer:** Converts the continuous amplitude samples $x(kT_s)$ into **discrete** level samples $\hat{x}(kT_s)$ taken from a finite set of L possible values $\hat{x} = \{\widehat{x_1}, \widehat{x_2}, \dots, \widehat{x_L}\}$.

➢ **Binary Encoder:** Each quantized level is represented by $r = log_2 L$ binary digits

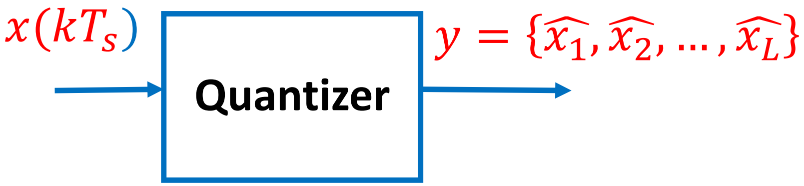
Quantization: Basic Definitions

- **Quantization**: is defined as the process of converting the continuous amplitude sample $x(kT_s)$ of a message signal into a discrete amplitude $\hat{x}(kT_s)$ taken from a finite and countable set of L possible values $\hat{x} = \{\widehat{x_1}, \widehat{x_2}, \dots, \widehat{x_L}\}$.
- The **dynamic range** of the quantizer is the range of values for which the quatizer is designed**,** $\boldsymbol{x_{min} < x < x_{max}}$
- This range is partitioned into L intervals such that if $x(kT_s) \in R_i$, the quantizer output will be a level $\widehat{x_i} = \{\widehat{x_1}, \widehat{x_2}, \dots, \widehat{x_L}\}$
- The qauntizer output is called a **representation or reconstruction level**
- The boundary points separating adjacent regions are called **decision or threshold levels**.
- The spacing between representation levels is called the **step size (Δ)**

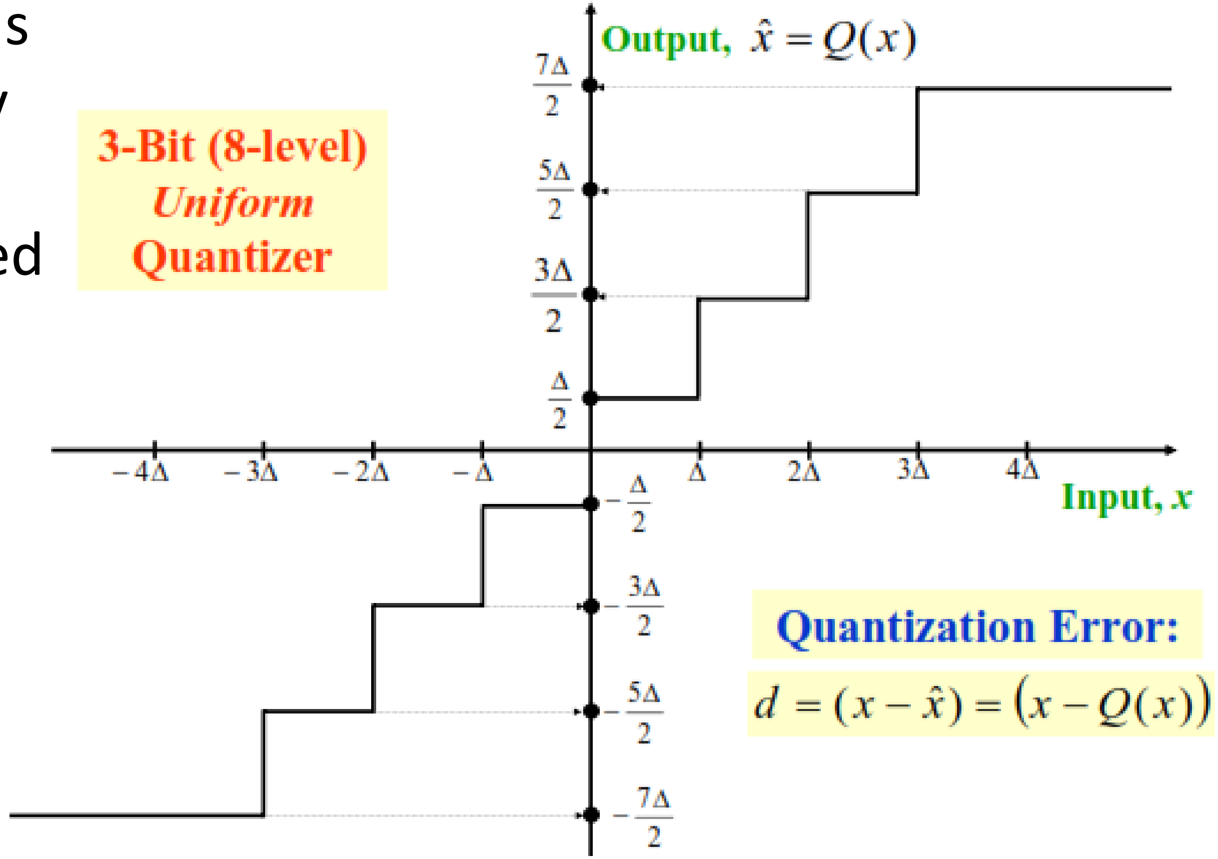


The Uniform Quantizer: Input-Output Characteristic

- A quantizer is called **uniform** when the L regions are of equal length Δ and the spacing between representation levels is uniform and equals to Δ.
- The input-output characteristic of a uniform quantizer (midrise type) is shown below for L=8.
- If the dynamic range of the quantizer varies between $x_{min} < x < x_{max}$, then $\boldsymbol{\Delta} = \frac{\boldsymbol{x_{max} - x_{min}}}{\mathbf{L}}$
- When the spacing between the adjacent levels is made small, $\hat{x}(kT_s)$ can be made practically indistinguishable from $x(kT_s)$.
- There is always a loss of information associated with the quantization process. Therefore, it is not possible to completely recover the sampled signal from the quantized signal.

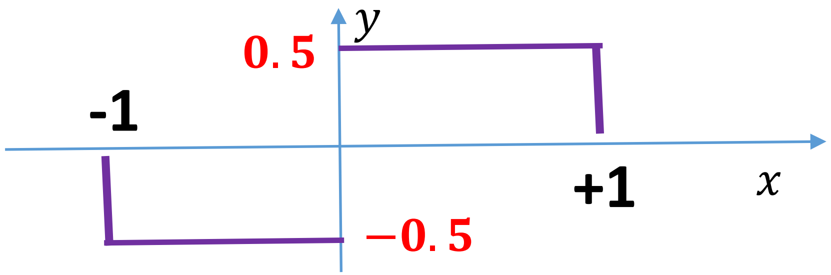


$$\boldsymbol{e = (x - y).}$$

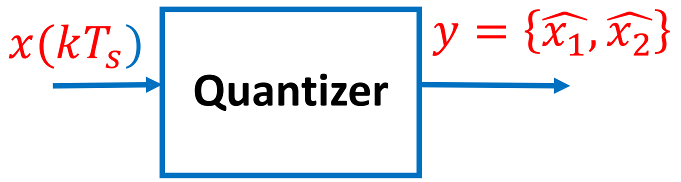


Example: the one-bit quantizer

- **Example:** The signal $x(t) = \cos(2\pi t)$ is uniformly sampled at a rate of 20 samples per second. The samples are applied to a sign detector, whose input-output characteristic is defined as:

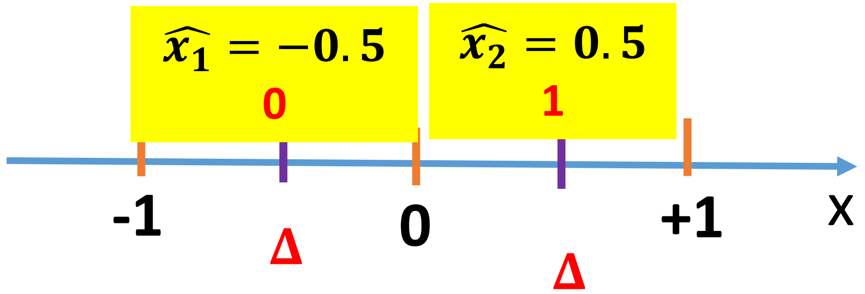


- $y(t) = \begin{cases} 0.5, & 0 < x < 1 \\ -0.5, & -1 < x < 0 \end{cases}$
- The next figures depict the input samples to the sign detector, the quantized output, and the quantization error defined as $e = (x - y)$.
- **Here,** $\Delta = \frac{1-(-1)}{2} = 1$;



t=	0	0.0500	0.1000	0.1500	0.2000	0.2500
x(t)=	1.0000	0.9511	0.8090	0.5878	0.3090	0.0000
y =	0.5	0.5	0.5	0.5	0.5	0.5
e=	0.5	0.4511	0.3090	0.0878	-0.191	-0.5

Note that $-\frac{\Delta}{2} < e < \frac{\Delta}{2} \Rightarrow -0.5 < e < 0.5$

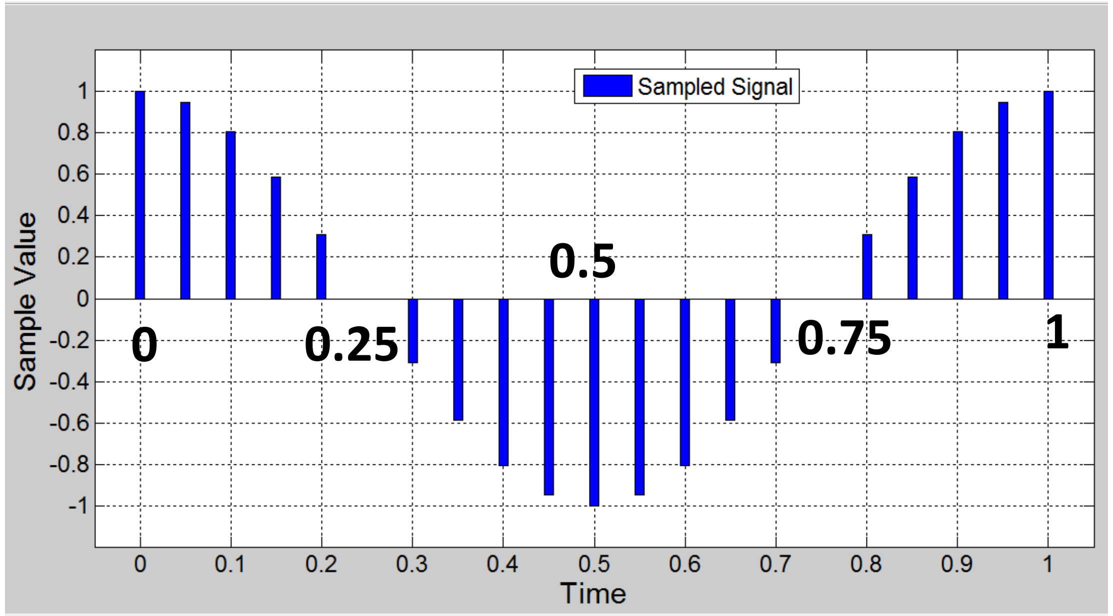


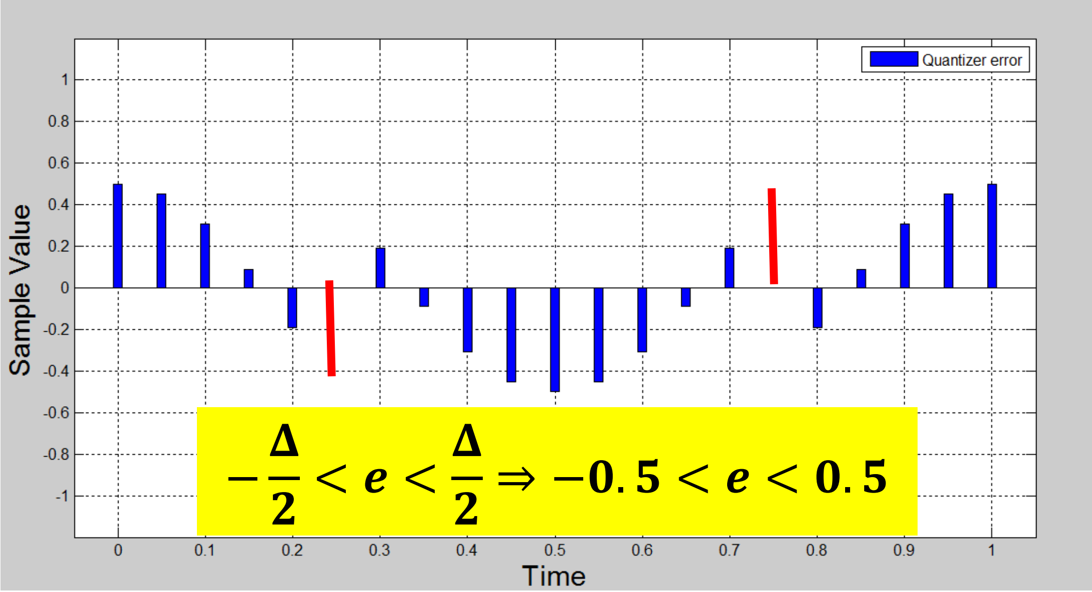
Quantization: the one-bit quantizer

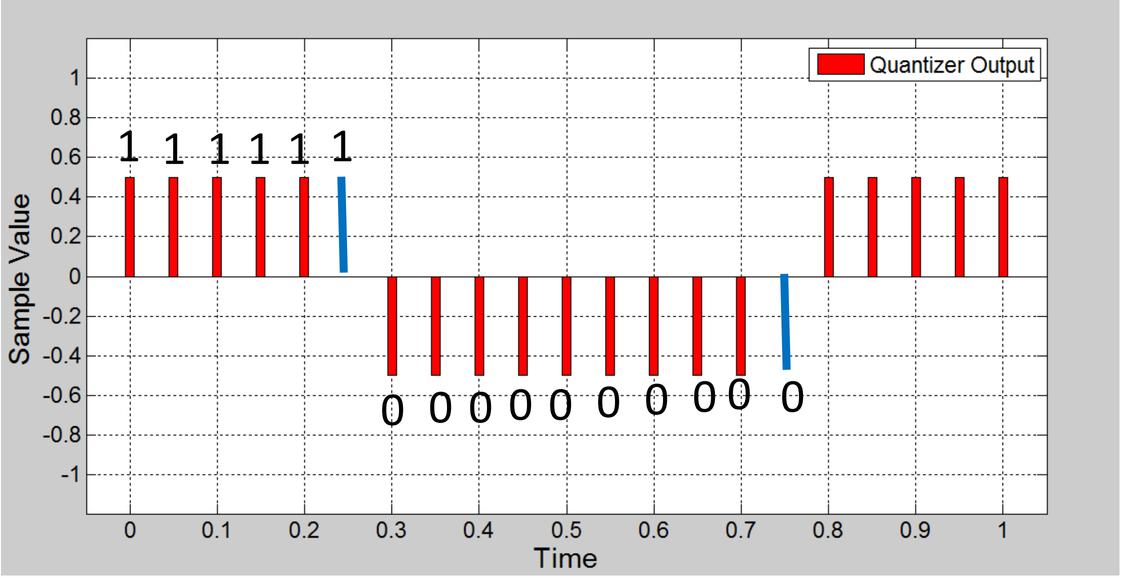
- **Example:** The signal $x(t) = \cos(2\pi t)$ is uniformly sampled at a rate of 20 samples per second.
- The samples are applied to a sign detector.
- Binary digits are assigned to the quantizer output.

t=	0	0.0500	0.1000	0.1500	0.2000	0.2500
x(t) =	1.0000	0.9511	0.8090	0.5878	0.3090	0.0000
y =	0.5	0.5	0.5	0.5	0.5	0.5
e=	0.5	0.4511	0.3090	0.0878	-0.191	-0.5

Note that $-\frac{\Delta}{2} < e < \frac{\Delta}{2} \Rightarrow -0.5 < e < 0.5$



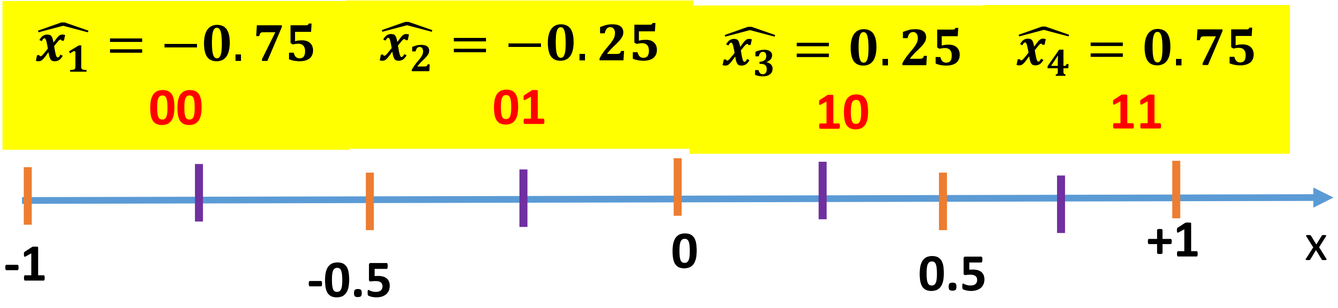




Quantization: the two-bit quantizer

- **Example**: The signal $x(t) = \cos(2\pi t)$ is sampled uniformly at a rate of 20 samples per second. The samples are applied to a four-level uniform quantizer with input-output characteristic

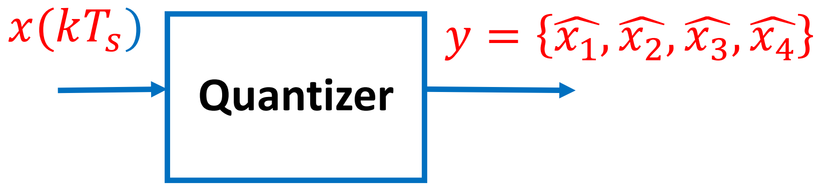
- $$y(t) = \begin{cases} 0.75, & 0.5 < x < 1 \\ 0.25, & 0 < x < 0.5 \\ -0.25, & -0.5 < x < 0 \\ -0.75, & -1 < x < -0.5 \end{cases}$$



- The next figures depict the quantizer operation, the input samples to the sign detector, the quantized output, and the quantization error defined as $e = (x - y)$.

t =	0	0.0500	0.1000	0.1500	0.2000	0.2500
x(t) =	1.0000	0.9511	0.8090	0.5878	0.3090	0.0000
y =	0.75	0.75	0.75	0.75	0.25	0.25
e =	0.25	0.2011	0.059	-0.1622	0.059	-0.25

Note that $-\frac{\Delta}{2} < e < \frac{\Delta}{2} \Rightarrow -0.25 < e < 0.25$



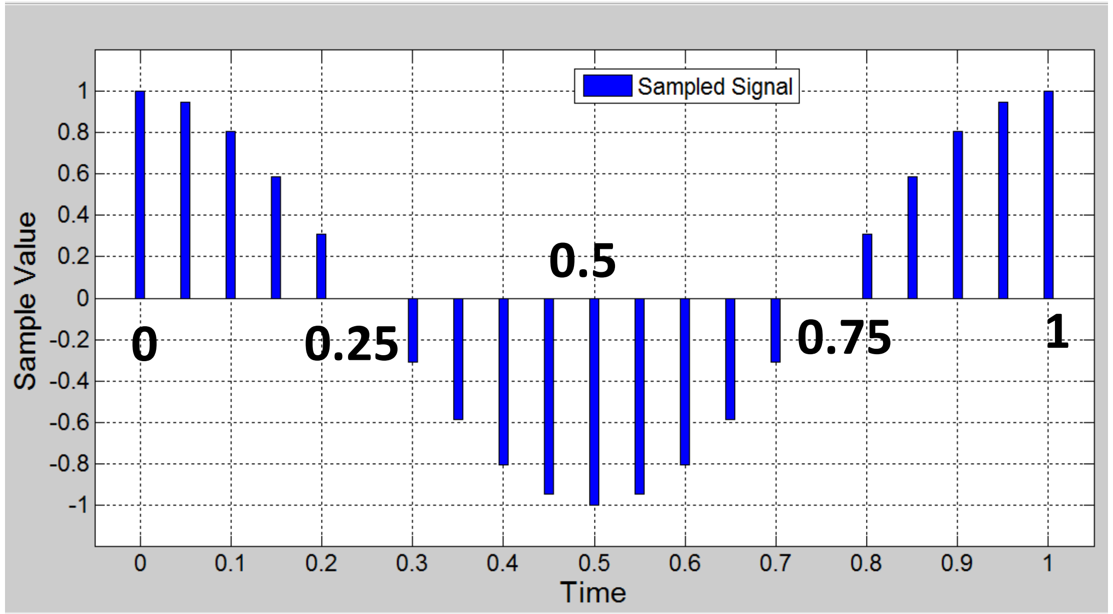
$$\Delta = \frac{1-(-1)}{4} = 0.5$$

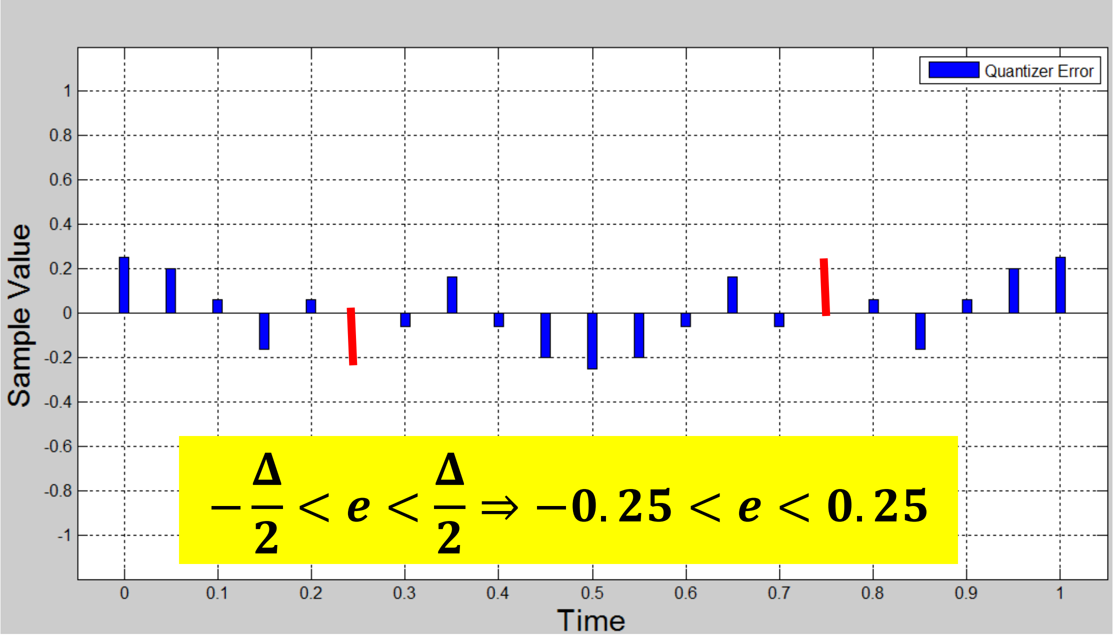
Quantization: the two-bit quantizer

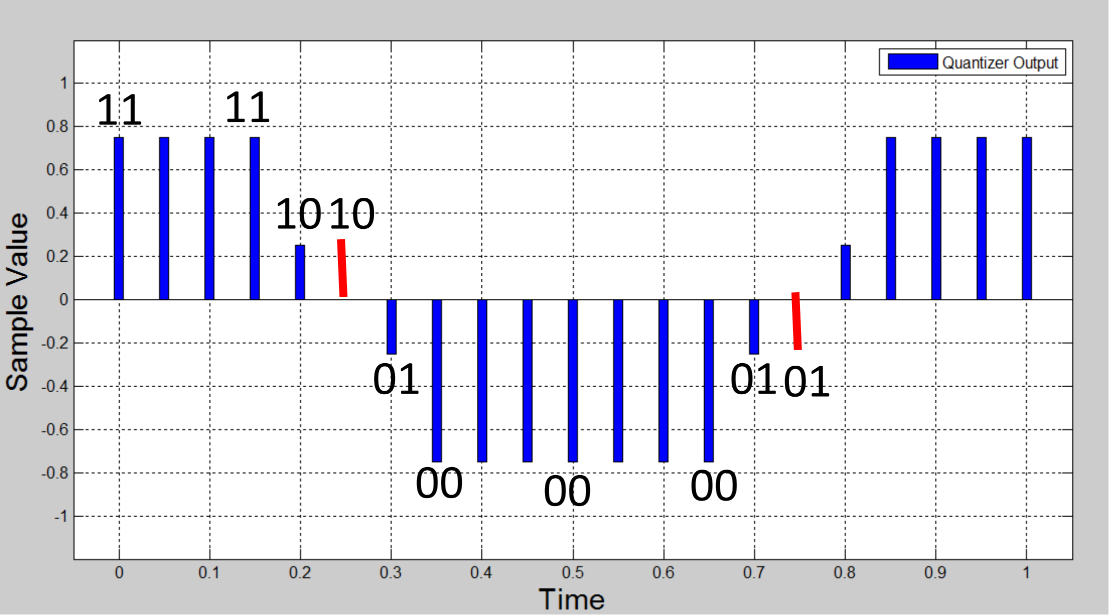
- **Example:** The signal $x(t) = \cos(2\pi t)$ is sampled uniformly at a rate of 20 samples per second. The samples are applied to a four-level uniform quantizer
- Binary digits are assigned to the quantizer output

t =	0	0.0500	0.1000	0.1500	0.2000	0.2500
x(t) =	1.0000	0.9511	0.8090	0.5878	0.3090	0.0000
y =	0.75	0.75	0.75	0.75	0.25	0.25
e =	0.25	0.2011	0.059	-0.1622	0.059	-0.25

Note that $-\frac{\Delta}{2} < e < \frac{\Delta}{2} \Rightarrow -0.25 < e < 0.25$

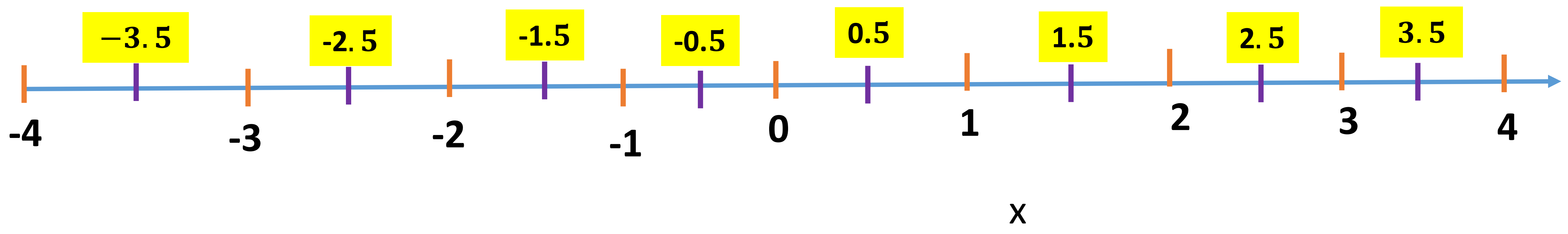






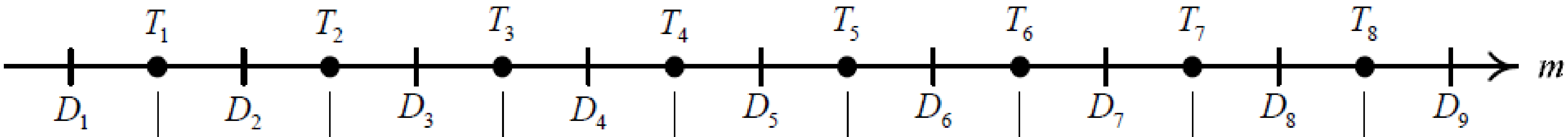
Example: Quantizer Design

- Design an 8-level uniform quantizer with a dynamic range of (-4, +4) V. Here, you need to specify the thresholds and the representation values.
- How many binary digits are needed to represent the samples? **(3 bits)**
- Find the representation value and the quantization error when a 1.64 V sample is applied to the quantizer.
- **Solution:** $\boldsymbol{\Delta = \frac{4-(-4)}{8} = 1}$
- **Since L = 8, then n=3.** $\mathbf{L = 2^n; n = \; log_2(L)}$
- **When x = 1.64, y = 1.5, and the error is:** $\boldsymbol{e = (x - \hat{x}) = (1.64 \; - 1.5) = 0.14.}$

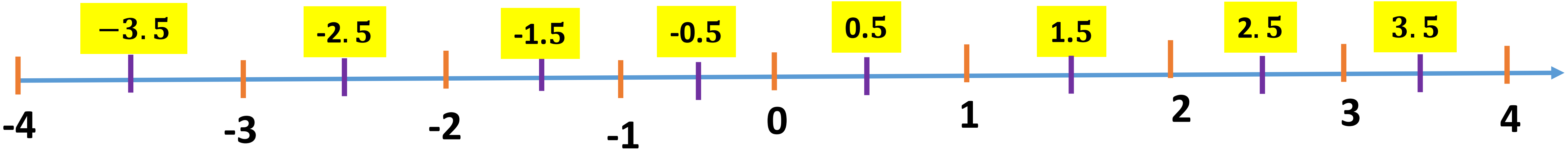


Example: continued

- Find the binary representation corresponding to the sample -2.1 V if natural binary encoding is employed **(-2.5; 001)**
- Are there other binary encoding formats?



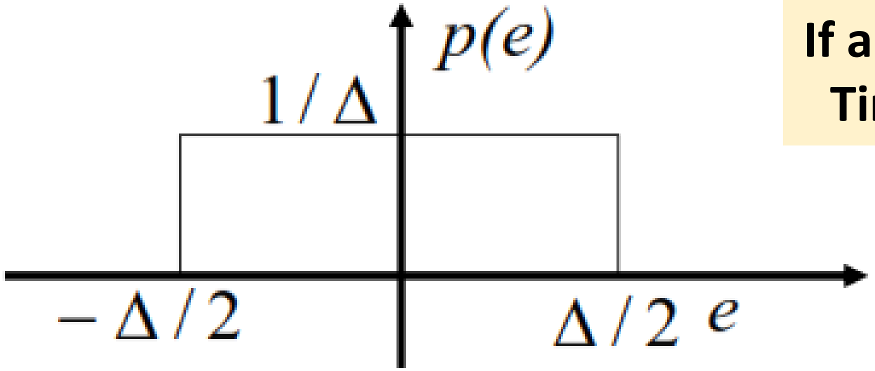
Ordinal:	0	1	2	3	4	5	6	7
NBC:	000	001	010	011	100	101	110	111
Gray:	000	001	011	010	110	111	101	100
FBC:	011	010	001	000	100	101	110	111



Quantization Noise

- The **quantization error** per sample is the difference between the input and output of the quantizer, i.e., $\boldsymbol{e = (x - \hat{x})}$
- The time average of the mean squared error $\frac{1}{T}\int (x-\hat{x})^2 dt$; **(Time Average)**
- The maximum error (also referred to as the resolution) = $|\frac{\Delta}{2}|$
- When Δ is small, the error, e, is assumed to be a uniform random variable over the interval $-\Delta|2 < e < \Delta|2$.
- The average quantization error (distortion) over all samples of the signal is
- $D = E(x-\hat{x})^2 = E(e^2) \Rightarrow D = \frac{1}{\Delta}\int_{-\Delta/2}^{+\Delta/2}(e)^2\, de \;\Rightarrow$ $\mathbf{D = \Delta^2|12}$ **(Statistical Average)**
- **Remark:** Note that D depends on the design of the quantizer and not on the signal applied to it, as we will see in the next two examples.

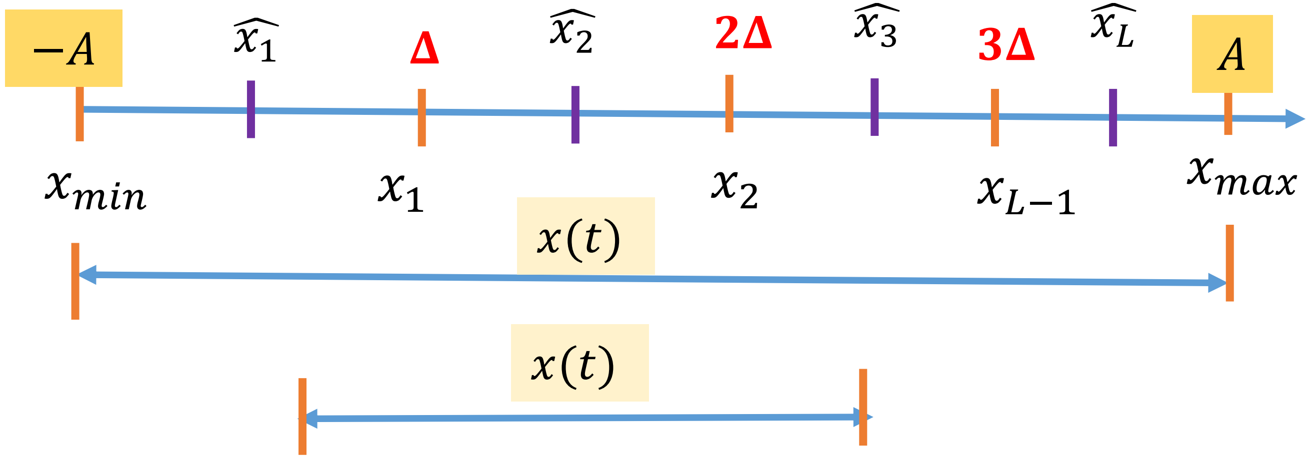
$$\left[-\frac{\Delta}{2}, \frac{\Delta}{2}\right]$$



If a process is assumed ergodic, then Time Average = Statistical Average

SQNR: Signal Matches Dynamic Range of a Uniform Quantizer

- **Example**: Let the sinusoidal signal $x(t) = Acos(2\pi f_0 t)$ be applied to a uniform quantizer with a dynamic range $(-A, A)$. We need to find the $SQNR$.
- **Solution:** The average power, P_x , in $x(t)$ is: $\boldsymbol{P_x = A^2|2}$
- The signal power to quantization noise ratio: $\boldsymbol{SQNR = \frac{A^2/2}{\Delta^2/12}}$
- Here, $\Delta = 2A/L$. If $L = 2^n$, then the $SQNR$ become: $\boldsymbol{SQNR = \frac{3}{2} L^2 = \frac{3}{2} 2^{2n}}$
- In dB, the $SQNR$, becomes: $\boldsymbol{SQNR = 10log\frac{P_x}{D} = 6.02n + 1.76}$ (dB)
- $SQNR$ increases exponentially with the number n of bits per sample.
- There is a 6-dB improvement in $SQNR$ for each bit added to represent the sample values



$$\boldsymbol{\Delta = \frac{x_{max} - x_{min}}{L}}$$

Strong signal applied to a uniform quantizer

Weak signal applied to a uniform quantizer

Example: Signal Matches Dynamic Range of a Uniform Quantizer

$x(t) = (10)cos(2\pi(10)t)$

Quantizer: (-10, 10);

n = 3 bits (8 levls)

$$\Delta = \frac{20}{8} = 2.5$$

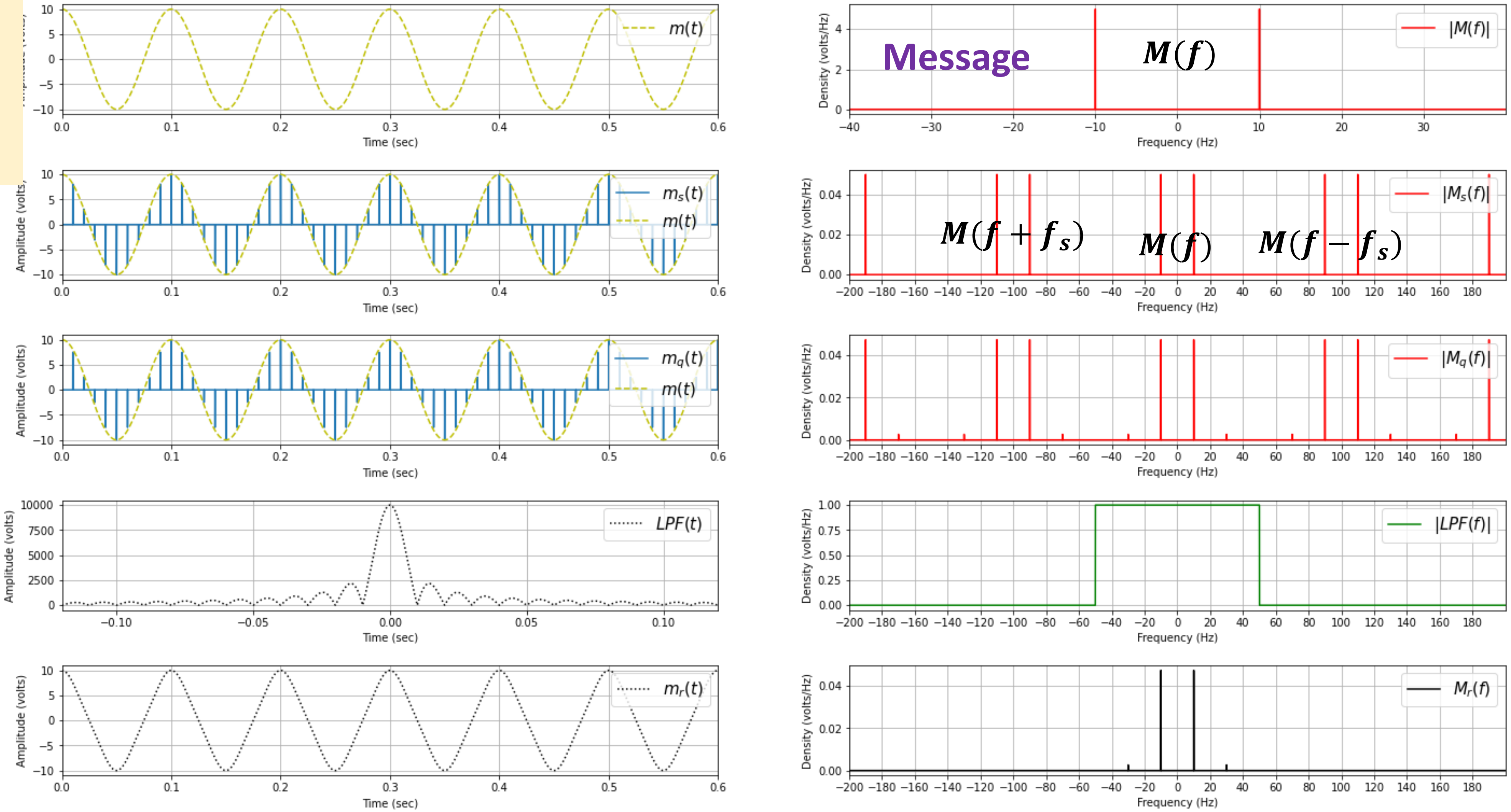
Ideally sampled Signal

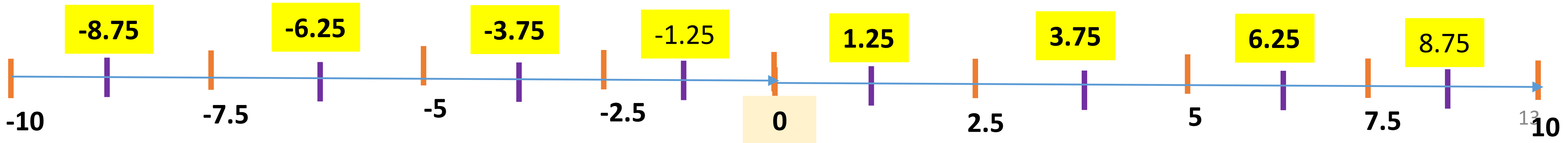
fs = 100

Quantized Signal

Filter B.W = fs/2=50

Recovered Signal





Example: Weak Signal Applied to a Uniform Quantizer

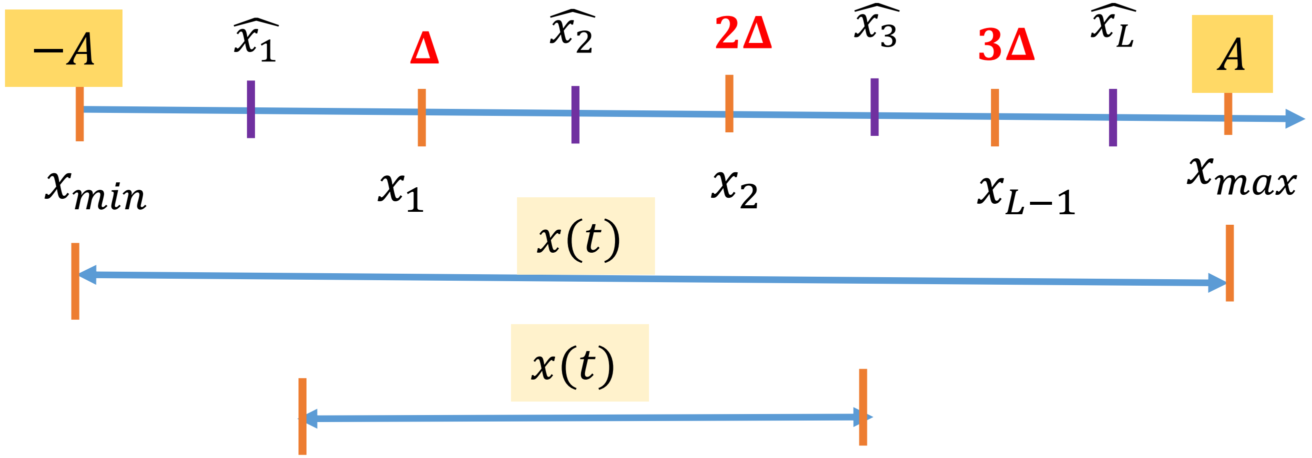
- **Example:** Now, let the sinusoidal signal $x(t) = A/2cos(2\pi f_0 t)$ be applied to the same uniform quantizer of the previous example, with a dynamic range $(-A, A)$. We need to find the $SQNR$.

- **Solution:** $\boldsymbol{SQNR = \frac{(A/2)^2/2}{\Delta^2/12} = \frac{12}{32}L^2 = \frac{12}{32}2^{2n}}$

- In dB, the $SQNR$, becomes: $SQNR = 10log\frac{P_x}{D} = 6.02n - 4.77$

- **Remark:** If the message signal $x(t)$ is a random signal, with an amplitude probability density function $f_X(x)$ and zero mean ($E(X) = 0$), then the SQNR is given as:

- $\boldsymbol{SQNR = \frac{E(X^2)}{E(x-\hat{x})^2} = \frac{\int_{-\infty}^{\infty} X^2 f_X(x)dx}{E(x-\hat{x})^2}}$; Beyond the scope of this lecture.



$$\Delta = \frac{x_{max} - x_{min}}{L}$$

Strong signal applied to a uniform quantizer

Weak signal applied to a uniform quantizer

Example: Weak Signal Applied to a Uniform Quantizer

$x(t) = (1)cos(2\pi(10)t)$

Quantizer: (-10, 10);

n = 3 bits (8 levls)

$$\Delta = \frac{20}{8} = 2.5$$

Ideally sampled Signal

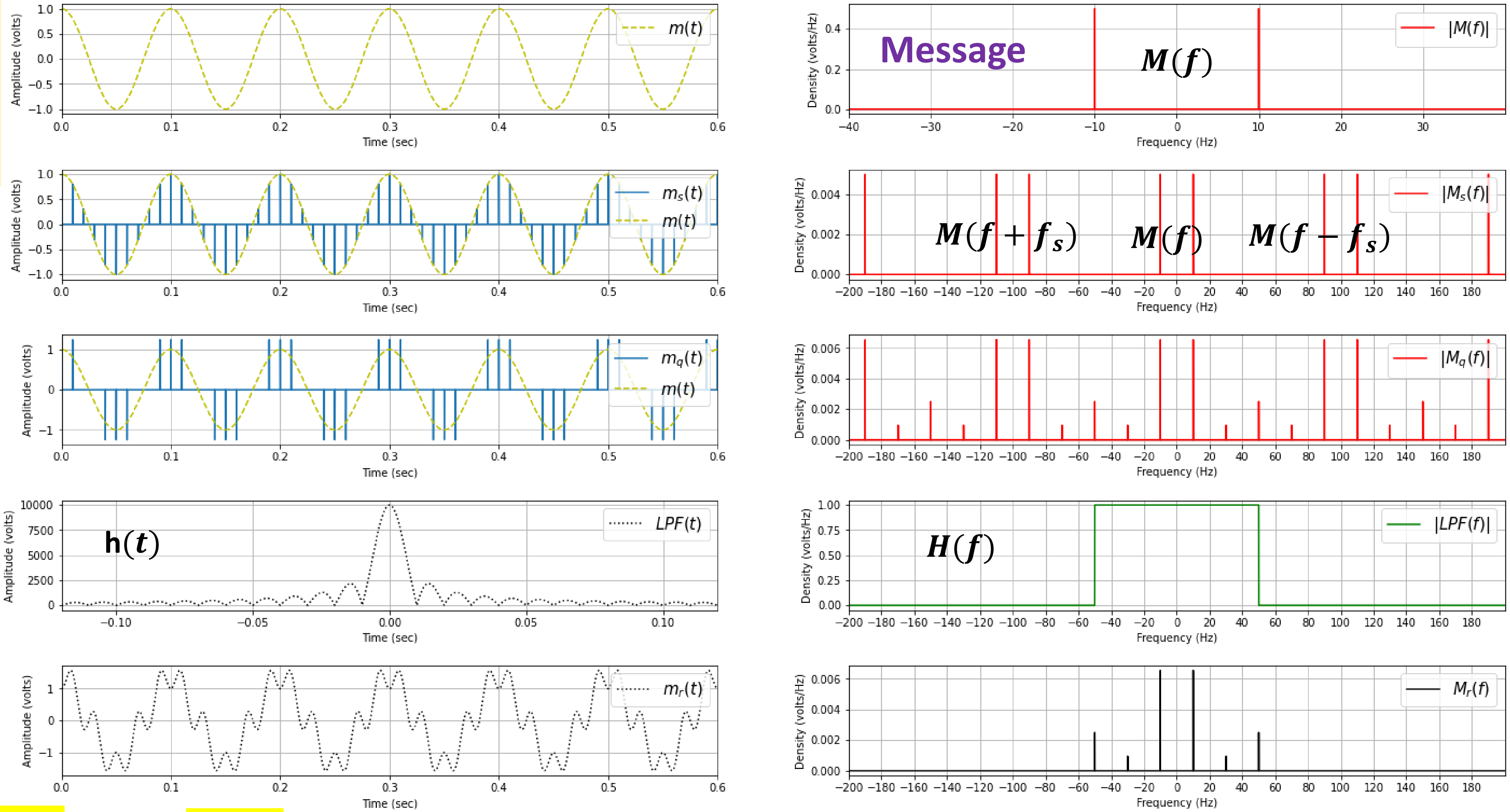
fs = 100

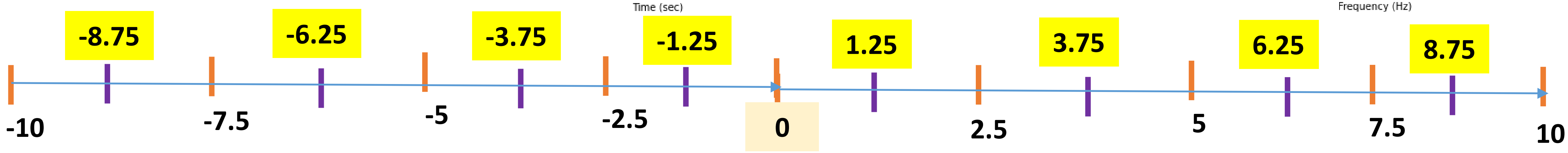
Quantized Signal

$-\frac{\Delta}{2}, +\frac{\Delta}{2}$, + 1.25, -1.25

Filter B.W = fs/2=50

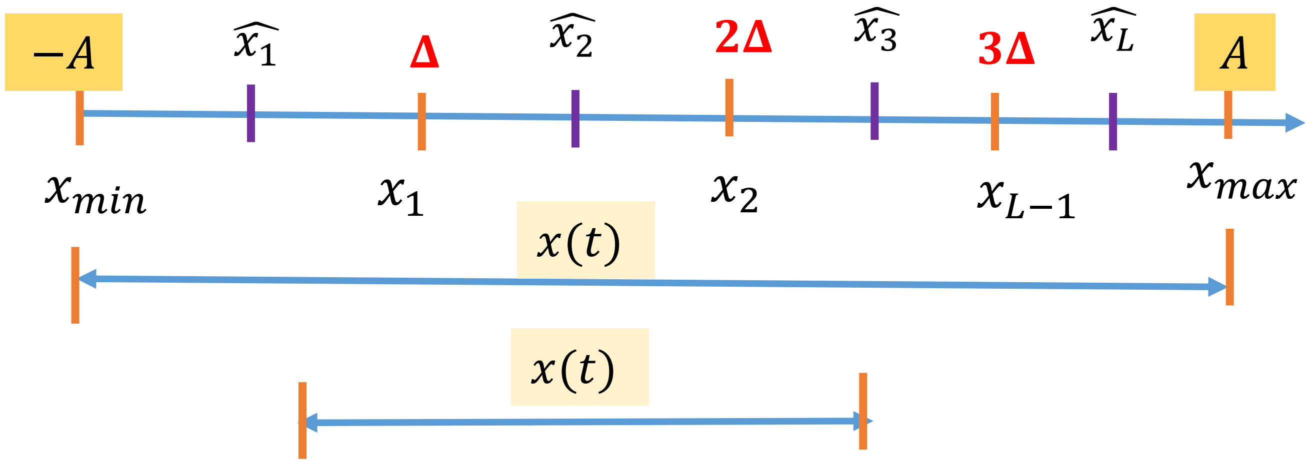
Recovered Signal





A Problem with the Uniform Quantizer

- **Problem**: Let the sinusoidal signal $x(t)$ be applied to a uniform quantizer with a dynamic range $(-A, A)$
- We have seen in the previous examples that the SQNR depends on the signal power. As the signal power decreases, the SQNR decreases (quantization noise is constant). **Quality deteriorates**
  - Case 1: $x(t) = A\cos(2\pi f_1 t)$, strong signal applied to the quantizer: $SQNR1 = \frac{3}{2}L^2$; with probability 0.2
  - Case 2: $x(t) = \frac{A}{2}\cos(2\pi f_2 t)$, weak signal applied to the quantizer: $SQNR2 = \frac{12}{32}L^2 = \frac{1}{4}SQNR1$ with probability 0.3
  - Case 3: $x(t) = \frac{A}{4}\cos(2\pi f_3 t)$, weak signal applied to the quantizer: $SQNR3 = \frac{4}{32}L^2 = \frac{1}{16}SQNR1$ with probability 0.5
- The average quantization noise is :
- $E(SQNR) = (0.2)\frac{3}{2}L^2 + (0.3)\frac{12}{32}L^2 + (0.5)\frac{1}{16}L^2 = 0.44375L^2$
- **Solution**: Use non-uniform quantization (the subject of the next lecture), the use of which will ensure an almost constant SQNR for strong as well as weak signal components.



$$\Delta = \frac{x_{max} - x_{min}}{L}$$

Strong signal applied to a uniform quantizer

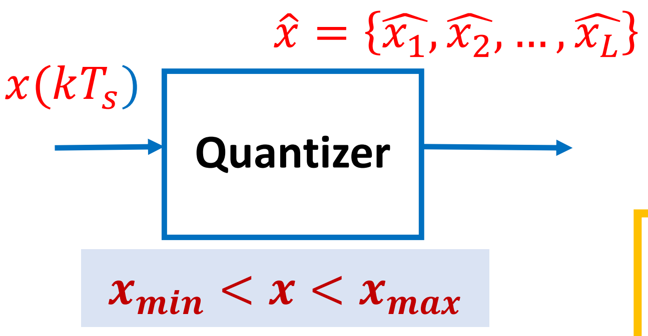
Weak signal applied to a uniform quantizer

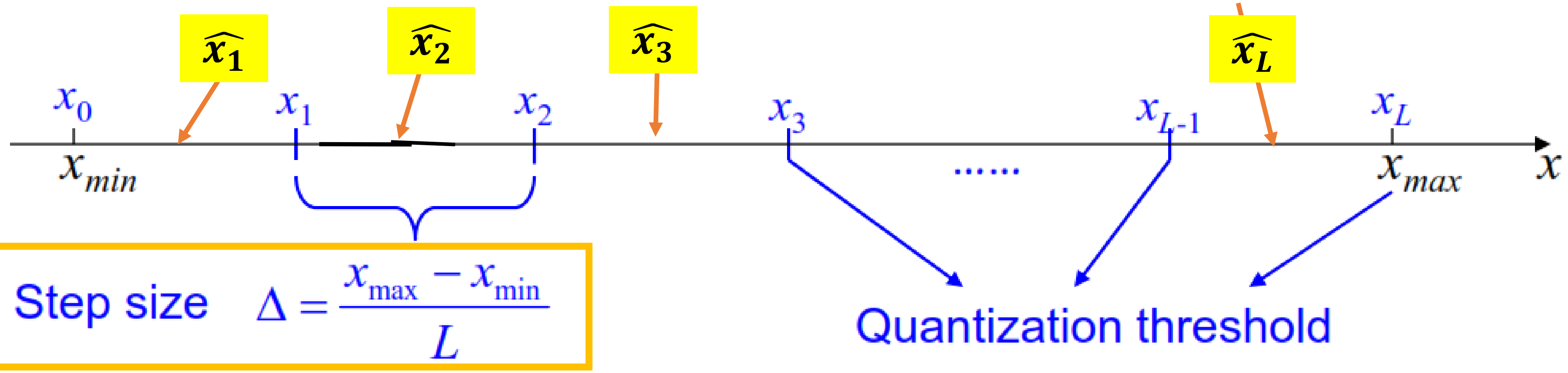
Non-uniform Robust Quantization
Lecture Outline

- **In this lecture**
  - We define the non-uniform robust quantizer
  - Define the μ-law compressor and expander characteristics
  - Show the improvement in SQNR of non-uniform quantization over uniform quantization.
  - Present a number of illustrative examples

Quantization: Basic Definitions

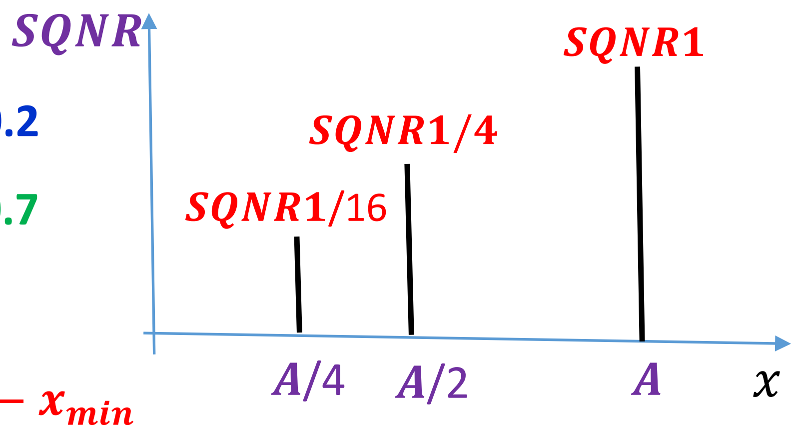
- **Quantization**: is defined as the process of converting the continuous amplitude sample $x(kT_s)$ of a message signal into a discrete amplitude $\hat{x}(kT_s)$ taken from a finite and countable set of L possible values $\{\widehat{x}|$.
- The **dynamic range** of the quantizer is the range of values for which the quatizer is designed, $x_{min} < x < x_{max}$
- This range is partitioned into L intervals such that if $x(kT_s) \in R_i$, the quantizer output will be a level $\hat{x}_i = \{\widehat{x_1}, \widehat{x_2}, \ldots, \widehat{x_L}\}$
- The qauntizer output is called a **representation or reconstruction level**
- The boundary points separating adjacent regions are called **decision or threshold levels**.
- The spacing between representation levels is called the **step size (Δ)**
- A quantizer is called **uniform** when the L regions are of equal length Δ and the spacing between representation levels is uniform and equals to Δ, where $\boldsymbol{\Delta = \frac{x_{max} - x_{min}}{L}}$.

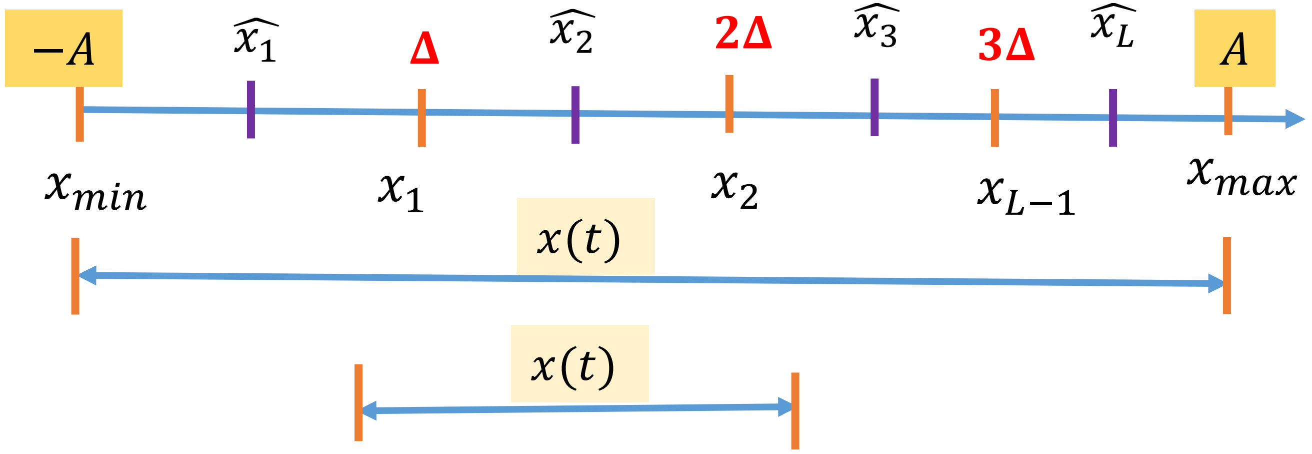




Drawback of the Uniform Quantizer

- **The Drawback of the uniform quantizer**:
  - The uniform quantizer is easy to build; however, it is not optimal when the input signal is weak most of the time. Here, the signal does not use the entire set of available quantization levels.
  - Maximum SQNR is achieved when the signal strength matches the dynamic range of the quantizer.
- **Demonstration**: Let the sinusoidal signal $x(t)$ be applied to a uniform quantizer with a dynamic range $(-A, A)$
- We have seen in the previous lecture that the SQNR depends on the signal power. As the signal power decreases, the SQNR decreases, since the quantization noise is constant ($SQNR = \frac{P_x}{\Delta^2/12}$)
  - Let : $x(t) = A\cos(2\pi f_1 t)$, $SQNR1 = \frac{3}{2}L^2$; with probability **0.1**
  - $x(t) = \frac{A}{2}\cos(2\pi f_2 t)$, $SQNR2 = \frac{12}{32}L^2 = \frac{1}{4}SQNR1$ with probability **0.2**
  - $x(t) = \frac{A}{4}\cos(2\pi f_3 t)$, $SQNR3 = \frac{4}{32}L^2 = \frac{1}{16}SQNR1$ with probability **0.7**





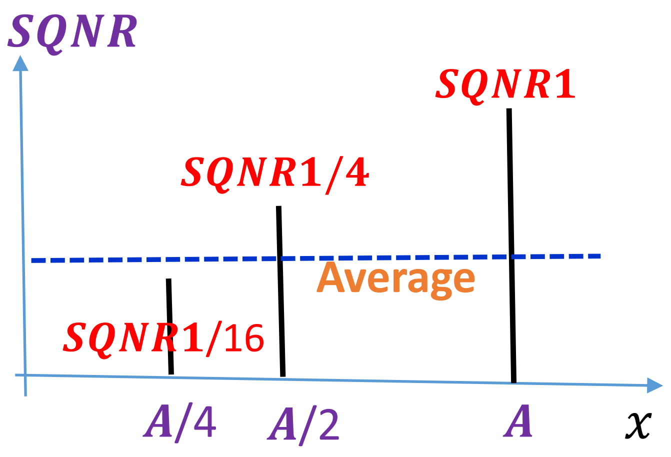
$$\Delta = \frac{x_{max} - x_{min}}{L}$$

Strong signal applied to a uniform quantizer

Weak signal applied to a uniform quantizer

Drawback of the Uniform Quantizer

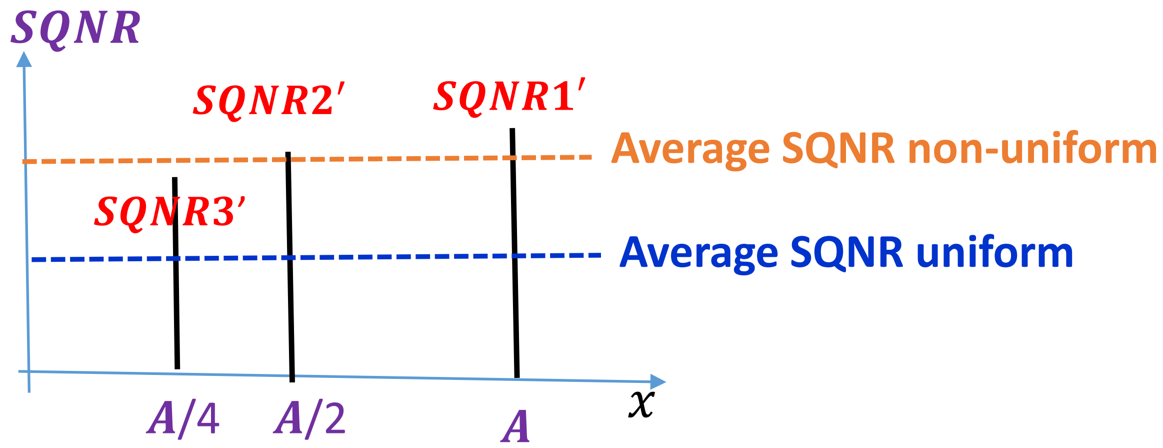
- The average SQNR is:
- $E(SQNR) = (0.1)SQNR1 + \frac{(0.2)SQNR1}{4} + \frac{(0.7)SQNR1}{16} = (0.19375)SQNR1$
- The average SQNR is closer to that of the smallest value of the SQNR.
- The overall performance is not satisfactory .
- Small amplitudes will be subjected to more distortion than large amplitudes.
- The non-uniform quantization, the subject of this lecture, will ensure an almost constant SQNR for strong as well as weak signal components.



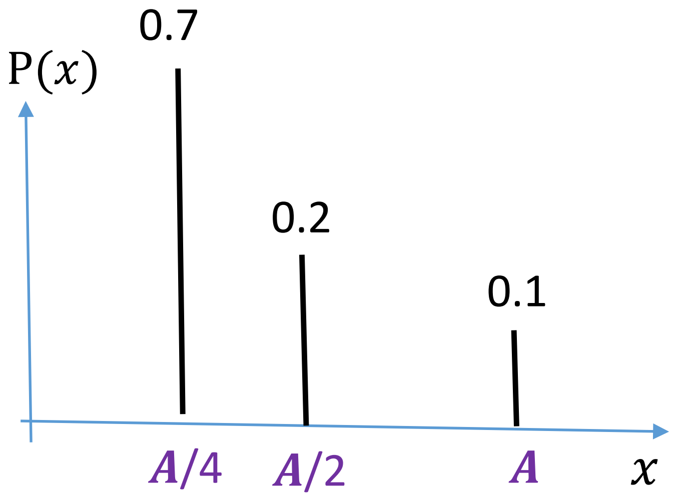
Uniform Quantizer

Two Remarks:

- **SQNR is higher than that of the uniform quanrtizer.**
- **The SQNR is almost the same for all signal levels**



Non-uniform Quantizer



Example: Signal Matches Dynamic Range of a Uniform Quantizer

$x(t) = (10)cos(2\pi(10)t)$

Quantizer: (-10, 10);

n = 3 bits (8 levls)

$$\Delta = \frac{20}{8} = 2.5$$

Ideally sampled Signal

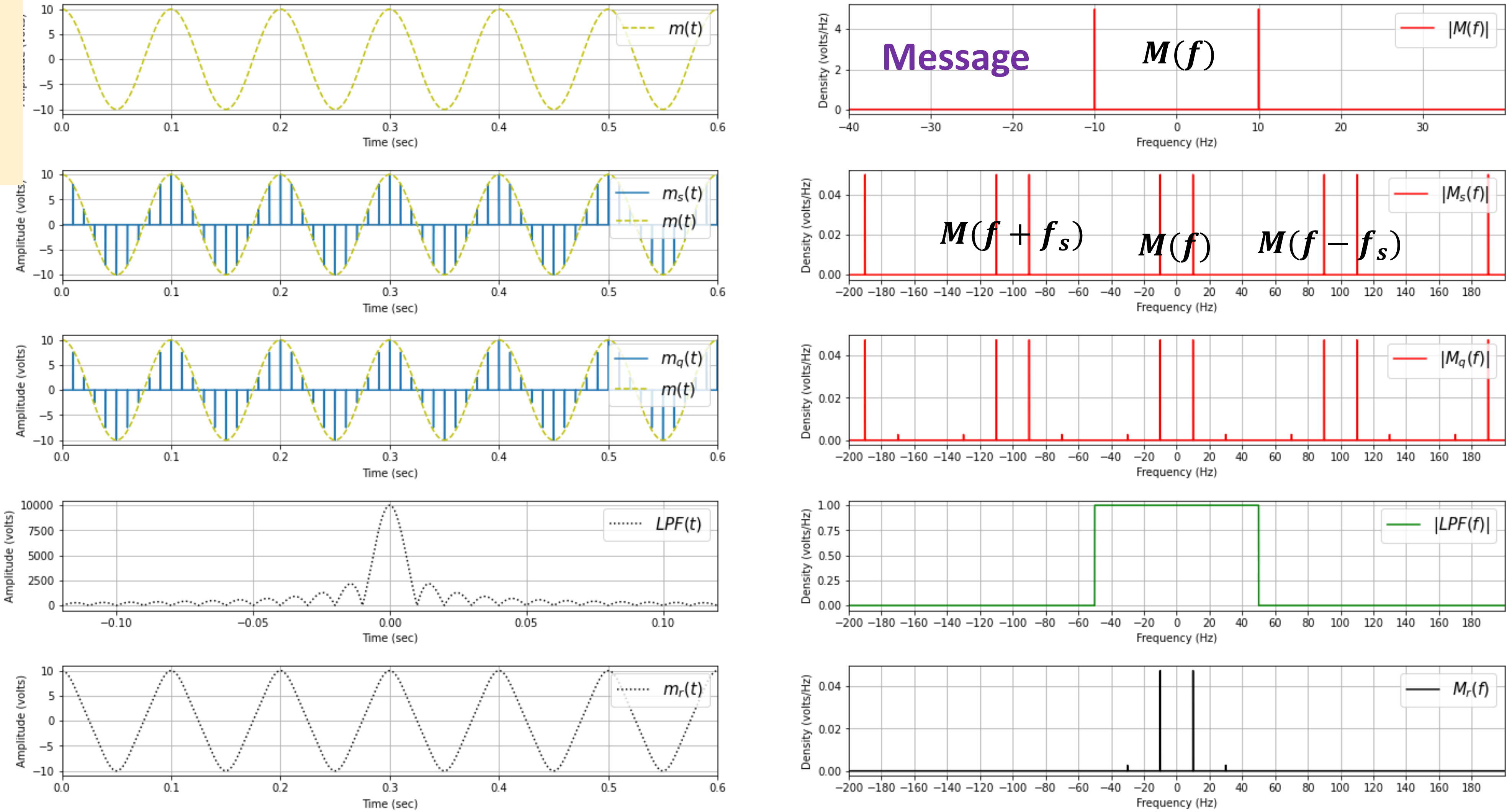
fs = 100

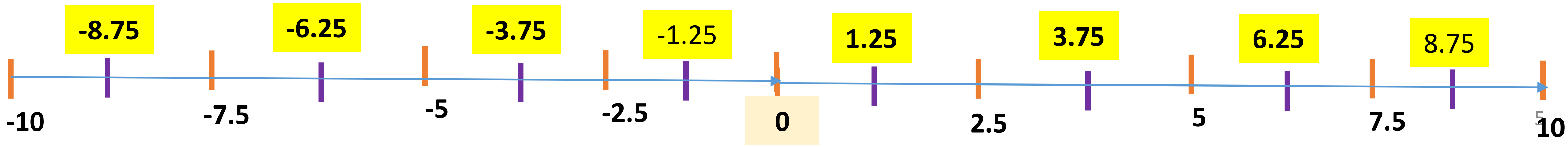
Quantized Signal

Filter B.W = fs/2=50

Recovered Signal

All levels are utilized





Example: Weak Signal Applied to a Uniform Quantizer

$x(t) = (1)cos(2\pi(10)t)$
Quantizer: (-10, 10);
n = 3 bits (8 levls)

$$\Delta = \frac{20}{8} = 2.5$$

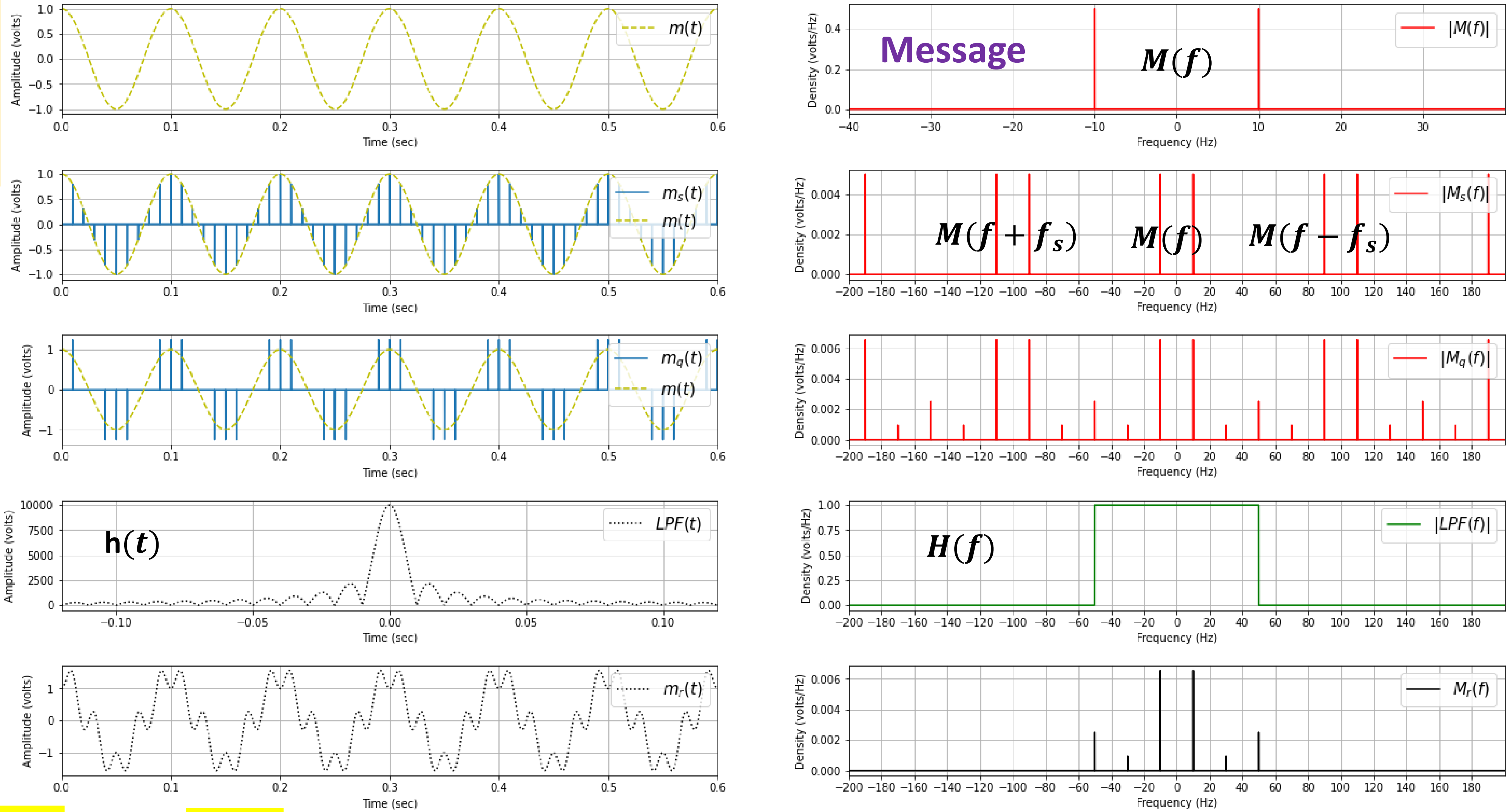
Ideally sampled Signal
fs = 100

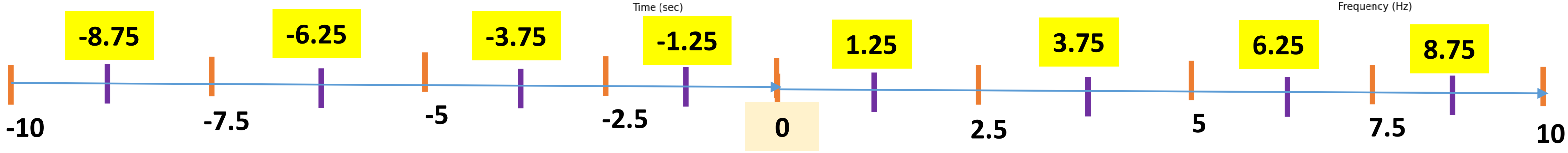
Quantized Signal
$-\frac{\Delta}{2}, +\frac{\Delta}{2}$, + 1.25, -1.25

Filter B.W = fs/2=50

Recovered Signal

Only two levels are utilized

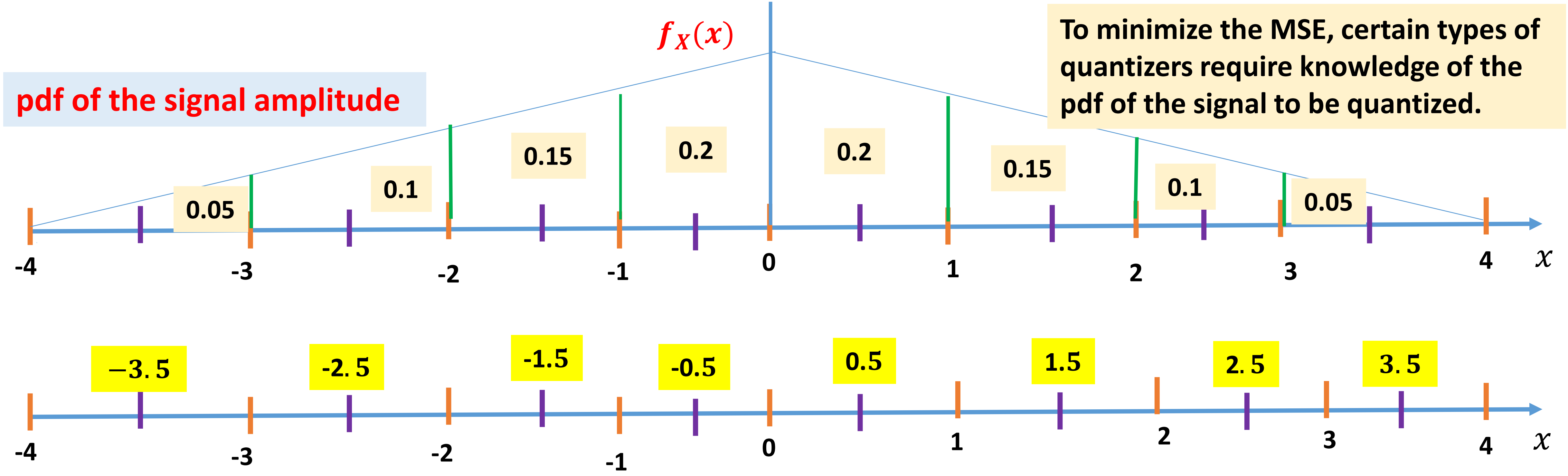




Uniform and Non-uniform Robust Quantization

- In practical applications, especially speech signals, small signal amplitudes occur more often than large signal amplitudes. This means that for the same signal, small amplitudes will be subject to distortion more than large amplitudes. That is, larger amounts of distortion have a higher probability of occurrence.
- The non-uniform robust quantizer, which employs the μ-law can
  - **Provide a SQNR that is somewhat constant and independent of the signal strength**
  - **Provide a SQNR that is also independent probability distribution of the sampled input signal**

To minimize the MSE, certain types of quantizers require knowledge of the pdf of the signal to be quantized.

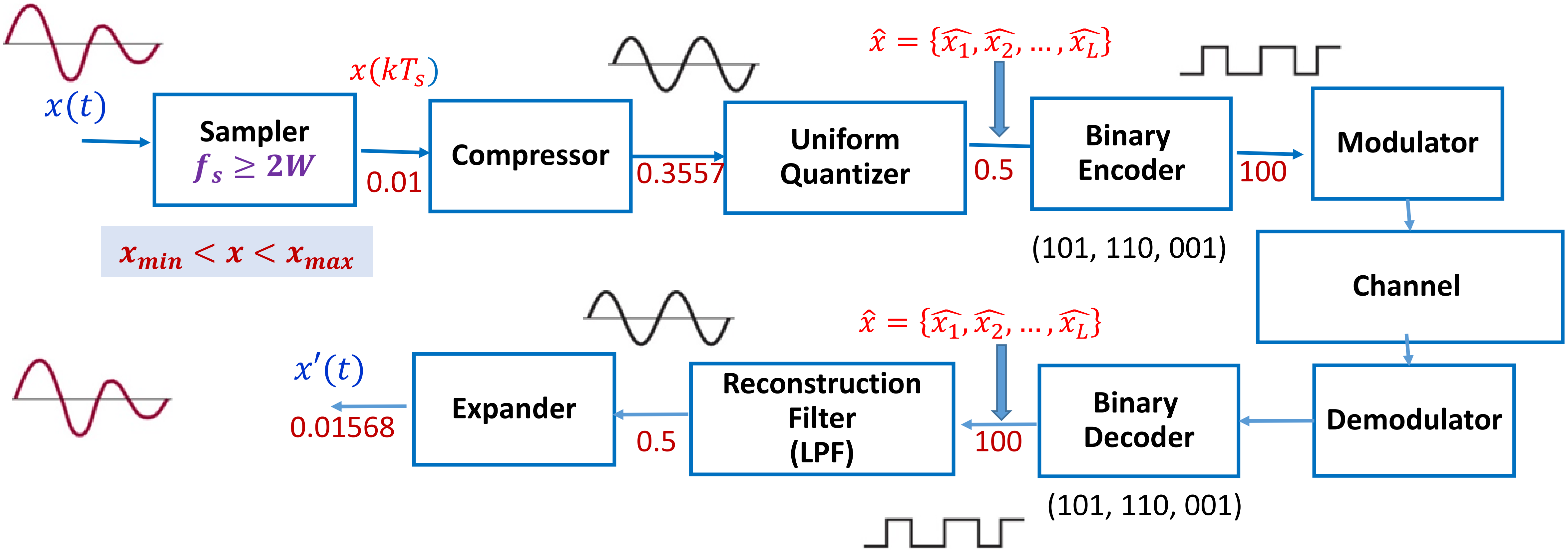


Non-uniform Robust Quantization

- We will use a type of non-uniform quantizers, called **companding, t**hat **does not require knowledge of the pdf of the signal to be quantized, and yields an almost uniform SQNR over a wide range of signal variations.**
- The process of pre-distorting the signal at the transmitter is known as (signal) **compression**. At the receiver, this process is reversed to remove distortion and is known as (signal) **expansion.** The two operations together, are typically, referred to as **companding** (or companion).
- The compressor is a nonlinear operation that amplifies weak signal values more than it amplifies large signal values, thus stretching the signal over more representation levels. This will enhance weak signal levels and improve their SQNR. Large signal levels will also suffer from distortion, but the overall effect on the signal is an improvement in SQNR.
- Since the probability of smaller amplitudes is higher than the larger amplitudes, the overall result is an improvement.
- In North America, μ -law companding (with $\mu = 255$) is the standard.

Robust (Non-uniform) Quantization

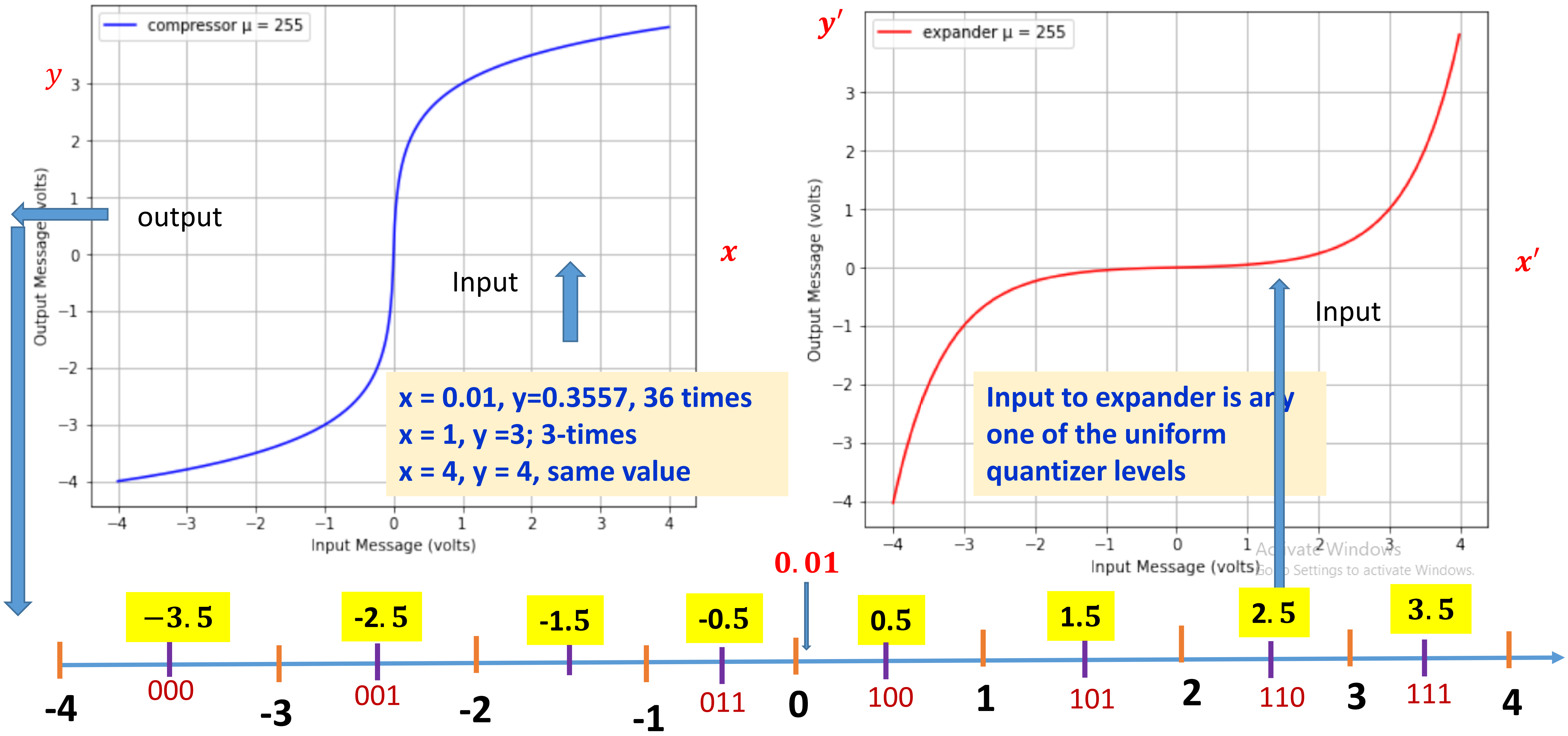
- In summary, companding is performed as follows:
  - Compress the signal using the μ-law. The output is approximately uniformly distributed.
  - Apply the compressed sample to a uniform quantizer
  - Transmit the quantized sample to the receiver.
  - Apply the received sample to the expander. The output is the desired signal value.



Non-uniform Robust Quantization

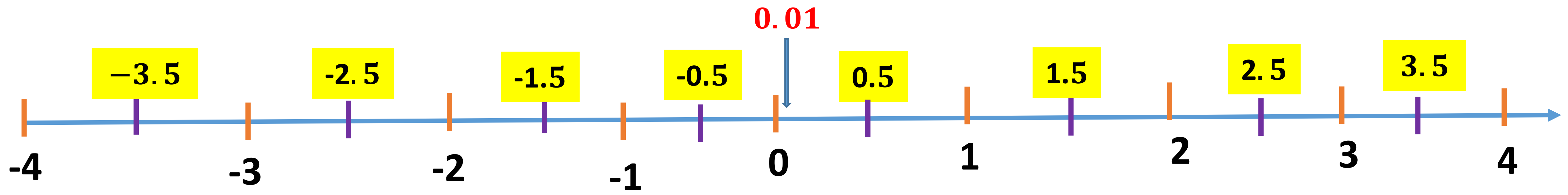
$$y = y_{max}\frac{\ln\left[1+\mu\left(\frac{|x|}{x_{max}}\right)\right]}{\ln(1+\mu)}sgn(x)$$

$$\frac{y'}{y'_{max}} = \left[\exp\left(\frac{x'}{x'_{max}}(\ln(1+\mu))\right)-1\right]/\mu\ \ sgn(x')$$



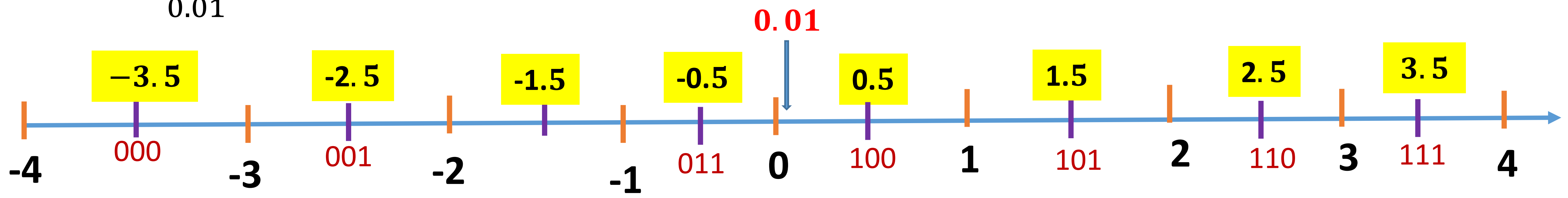
Non-uniform Robust Quantization

- **Example**: If a sample of magnitude 0.01 is applied to a 3-bit uniform quantizer with a dynamic range (-4, 4) V. Find the quantizer output, the quantization error, and the value of the sample at the receiving side.
- If the sample is applied to quantizer that employs companding, find the value of the received sample, and the quantization error.
- **Uniform Quantization**: when x = 0.01, $\hat{x} = 0.5$; This is the same voltage that is supposed to be received, assuming no transmission error.
- The error is: $|\mathbf{x} - \hat{\boldsymbol{x}}| = |\mathbf{0.01} - \mathbf{0.5}| = \mathbf{0.49}$
- The % of error is: $\hat{\boldsymbol{x}} = \frac{\mathbf{0.5 - 0.01}}{\mathbf{0.01}} \mathbf{100\%} = \mathbf{4900\%}$.
- The error is 49 times the sample value.



Non-uniform Robust Quantization

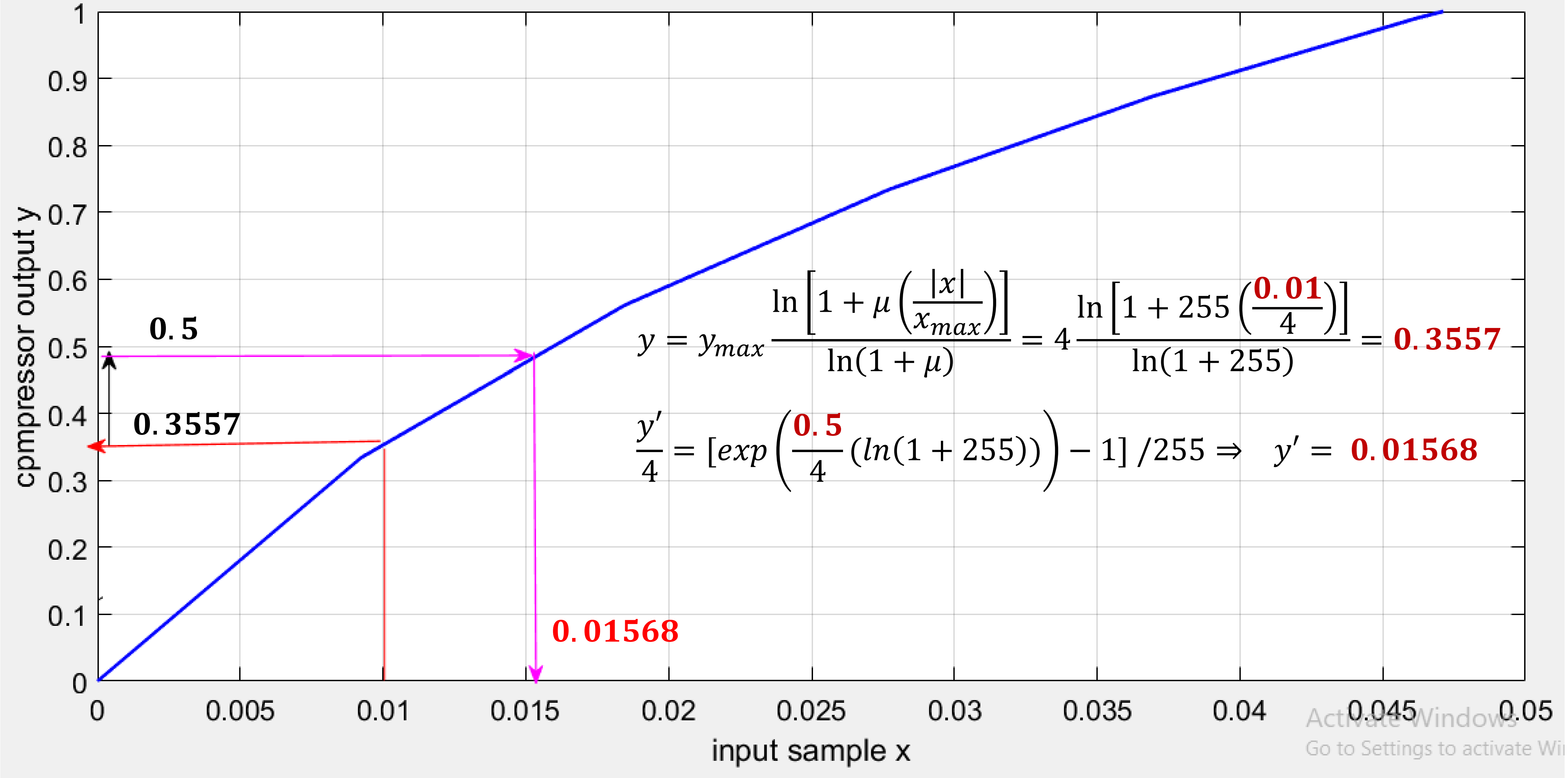
- **Non-uniform Quantization**: when x = 0.01 is applied to the μ-law,
- $y = y_{max} \frac{\ln\left[1+\mu\left(\frac{|x|}{x_{max}}\right)\right]}{\ln(1+\mu)} = 4 \frac{\ln\left[1+255\left(\frac{0.01}{4}\right)\right]}{\ln(1+255)} = \mathbf{0.3557}$
- The 0.3557 is applied to the uniform quantizer, whose output is **0.5.**
- The 0.5 V is encoded as 100 and transmitted. Assume no transmission errors.
- At the receiver, the 0.5 is applied to the expander (the inverse μ-law characteristic)
- $\frac{y\prime}{4} = \left[\exp\left(\frac{0.5}{4}(\ln(1+255))\right) - 1\right]/255 \Rightarrow y' = \mathbf{0.01568}$
- Received sample $y' = \mathbf{0.01568}$
- The error is: $|x - y'| = |0.01 - 0.0157| = 0.0057$
- Error: $\frac{0.01-0.0157}{0.01} 100\% = 57\%$



$$y = y_{max} \frac{ln\left[1+\mu\left(\frac{|x|}{x_{max}}\right)\right]}{ln(1+\mu)} sgn(x)$$

$$\frac{y'}{y'_{max}} = \left[exp\left(\frac{x'}{x'_{max}}(ln(1+\mu))\right) - 1\right]/\mu$$

mu law characteristic



$$y = y_{max}\frac{\ln\left[1+\mu\left(\frac{|x|}{x_{max}}\right)\right]}{\ln(1+\mu)} = 4\frac{\ln\left[1+255\left(\frac{0.01}{4}\right)\right]}{\ln(1+255)} = 0.3557$$

$$\frac{y'}{4} = \left[exp\left(\frac{0.5}{4}(ln(1+255))\right) - 1\right]/255 \Rightarrow \quad y' = 0.01568$$

Uniform and Non-uniform Robust Quantization: More Values

Sample Value	Uniform Quantizer		Non-uniform Quantizer	
	Output	% Error	Output	% Error
0.01	0.5	$\frac{0.01 - 0.5}{0.01} 100\% = \mathbf{4900\%}$	0.0157	$\frac{0.01 - 0.0157}{0.01} 100\% = \mathbf{57\%}$
0.07	0.5	$\frac{0.07 - 0.5}{0.07} 100\% = \mathbf{707\%}$	0.1098	$\frac{0.07 - 0.1098}{0.07} 100\% = \mathbf{56.8\%}$
1.7	1.5	$\frac{1.7 - 1.5}{1.7} 100\% = \mathbf{11.76\%}$	1.9922	$\frac{1.7 - 1.9922}{1.7} 100\% = \mathbf{17.18\%}$
2.9	2.5	$\frac{2.9 - 2.5}{2.9} 100\% = \mathbf{13.79\%}$	1.9922	$\frac{2.9 - 1.9922}{2.9} 100\% = \mathbf{31.3\%}$
3.8	3.5	$\frac{3.8 - 3.5}{3.8} 100\% = \mathbf{7.89\%}$	1.9922	$\frac{3.8 - 1.9922}{3.8} 100\% = \mathbf{47.5\%}$

Uniform and Non-uniform Robust Quantization

- This figure shows a weak and a strong signal applied to a uniform and a non-uniform quantizer. As we can see, the non-uniform quantizer represents both signals quite adequately.