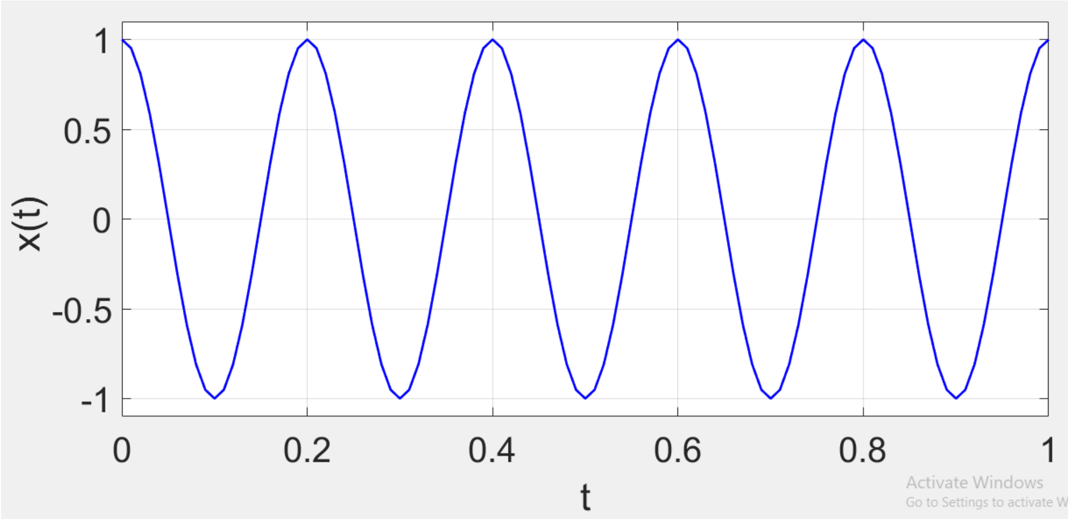
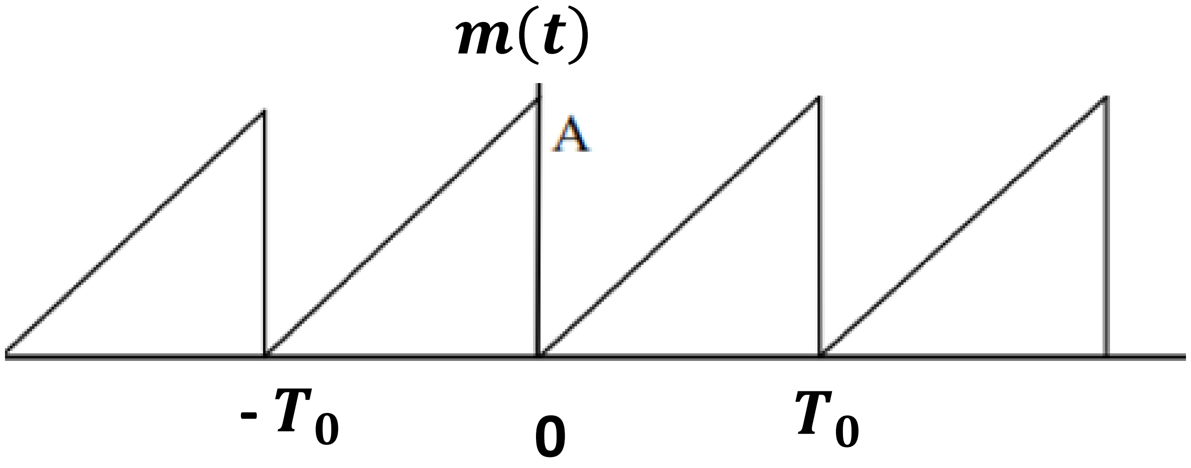
Signal Classifications

- **Definition:** A signal may be defined as a single valued function of time that conveys information.
- Depending on the feature of interest, we may distinguish four different classes of signals:
  - Periodic and Non-periodic Signals
  - Deterministic and Random Signals
  - Analog and Digital Signals
  - Energy and Power Signals

Classification of Signals: Periodic and Non-periodic

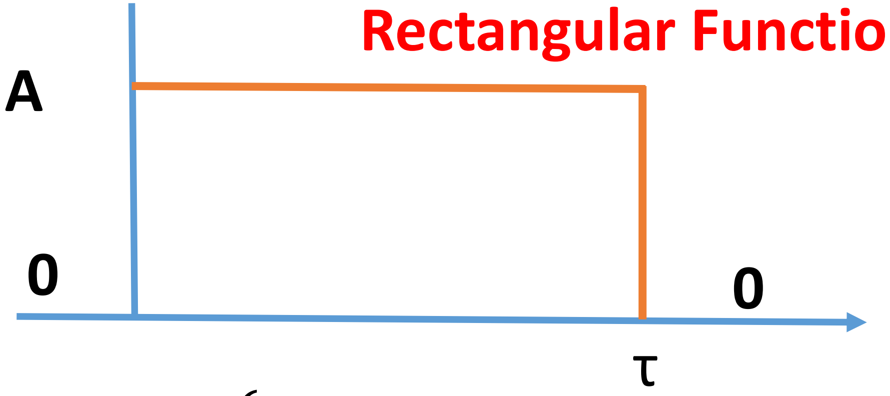
- **A periodic signal** g(t) is a function of time that satisfies the condition
$$g(t) = g(t + T_0), \forall t.$$
- The smallest value of T_0 that satisfies this condition is called the period of $g(t)$.
- **Example**: The sinusoidal signal $x(t) = \cos(2\pi(5)t)$ is periodic with period $\boldsymbol{T_0 = 1/5}$.
- The reciprocal of the period is the **fundamental frequency** $f_0 = \frac{1}{T_0}$. In this example, $\boldsymbol{f_0 = 5}$ **Hz**.
- **Example**: The saw-tooth function shown is another example of a periodic signal.
- $m(t) = \frac{A}{T_0} t, \quad 0 \le t \le T_0$
- If $T_0 = 0.001\ sec$, then the fundamental frequency $\boldsymbol{f_0 = 1000}$ **Hz**



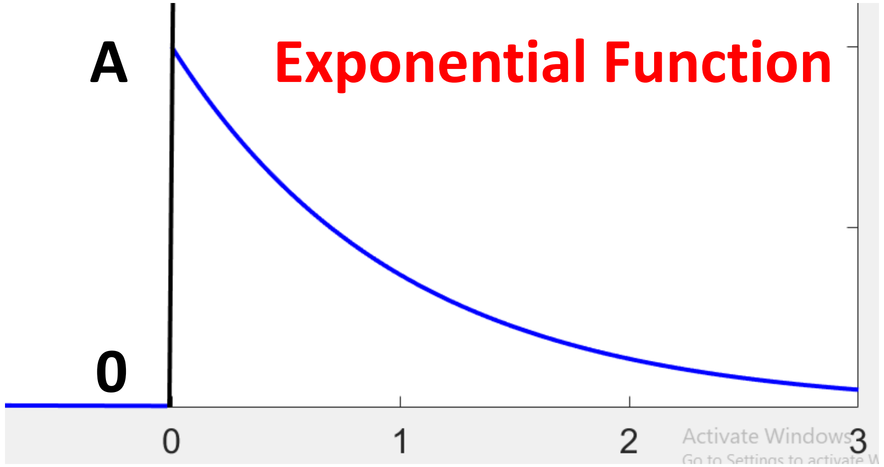


Non-periodic Signals

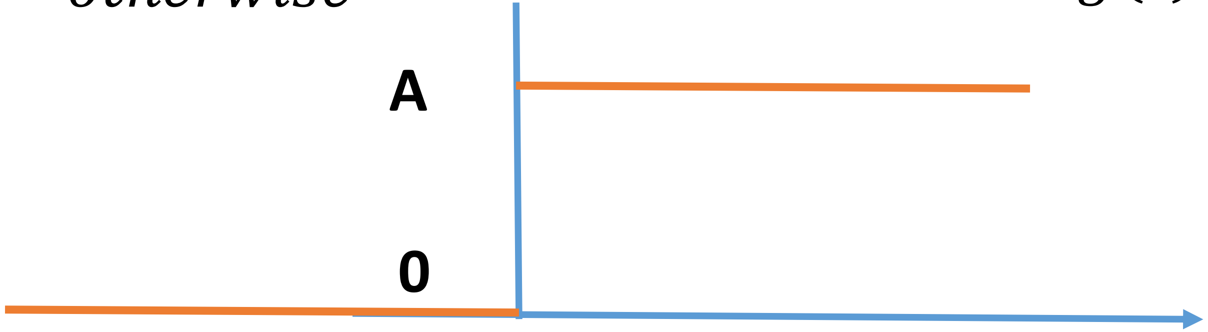
- A non-periodic signal g(t) is one for which there does not exist a T_0 for which the condition $g(t) = g(t + T_0)$ is satisfied, i.e., the signal does not repeat itself each T_0 .



$$g(t) = \begin{cases} A, & 0 \leq t \leq \tau \\ 0, & otherwise \end{cases}$$



$$g(t) = \begin{cases} A\exp(-\alpha t), & 0 \leq t < \infty \\ 0, & t < 0 \end{cases}$$



Step Function Au(t)

$$g(t) = \begin{cases} A, & t > 0 \\ 0 & t < 0 \end{cases}$$

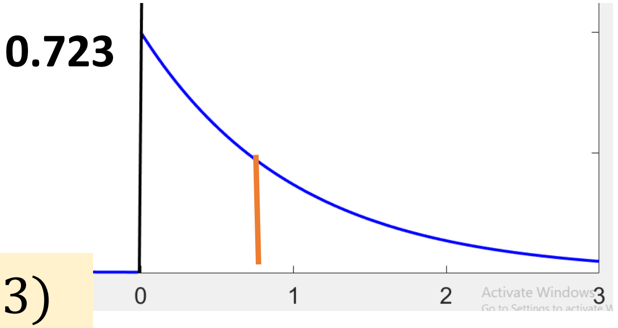
Deterministic and Random Signals

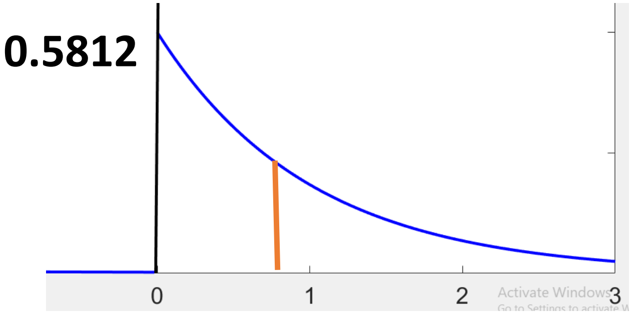
- **A deterministic signal** is one about which there is no uncertainty with respect to its value at any time. It is a completely specified function of time.
- **Deterministic Signal Example:** $x(t) = Ae^{-at}u(t)$; $A = 1$ and α is a constant.
- **A random signal** is one about which there is some degree of uncertainty before it actually occurs. (It is a function of a random variable)
- **Random Signal Example:** $x(t) = Ae^{-at}u(t)$; α is a constant and A is a random variable with the following probability density function (two possible realizations shown below)

$$f_A(a) = \begin{cases} 1 & 0 \le a \le 1 \\ 0 & otherwise \end{cases}$$

- **Random Signal Example:** $x(t) = \cos(2\pi f_c t + \Theta)$; f_c is a constant and Θ is a random variable uniformly distributed over the interval $(0, 2\pi)$ with the following probability density function (pdf).

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi} & 0 \le \theta \le 2\pi \\ 0 & otherwise \end{cases}$$



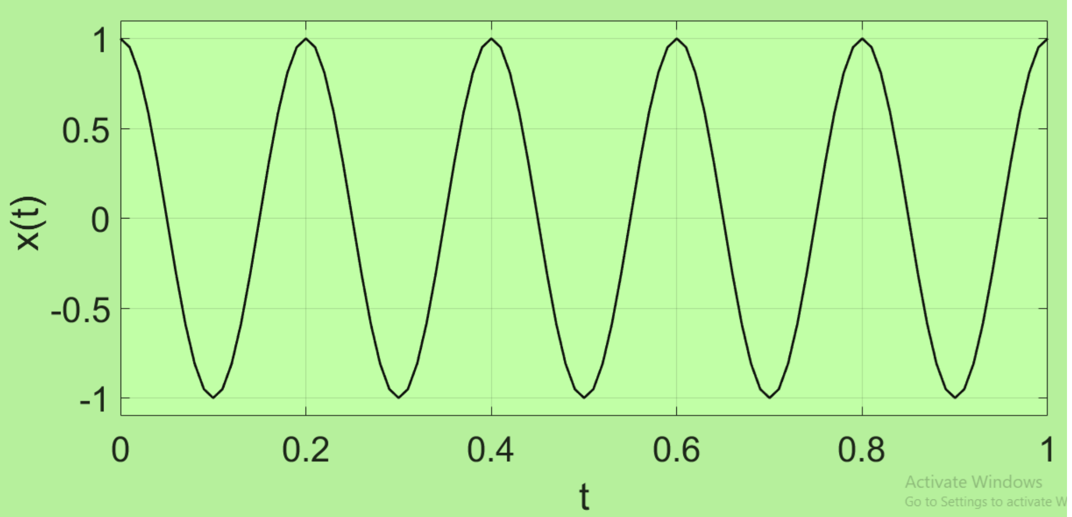


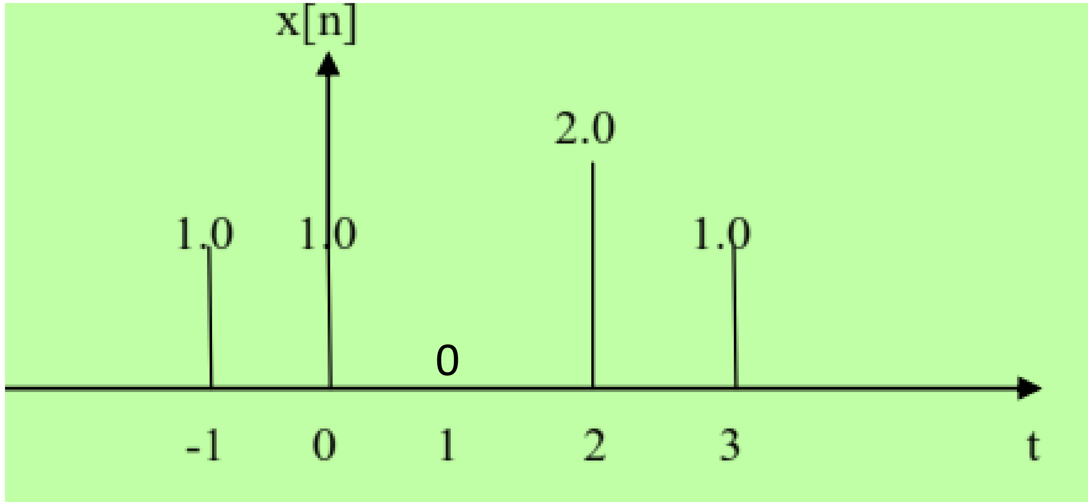
$x(t) = \cos(2\pi f_c t + 30)$

$x(t) = \cos(2\pi f_c t + 63)$

Analog and Digital Signals

- In an **analog signal** the amplitude takes on any value within a defined range of continuous values.
- **Example**: The sinusoidal signal $x(t) = A\cos 2\pi f_0 t\ , -\infty < t < \infty$, is an example of an analog signal.
- ***A digital signal*** : The values assumed by the signal belong to a finite and countable set.
- **Example**: The sequence x[n] shown below is an examples of a digital signal. The amplitudes are drawn from the finite set {1, 0, 2}.

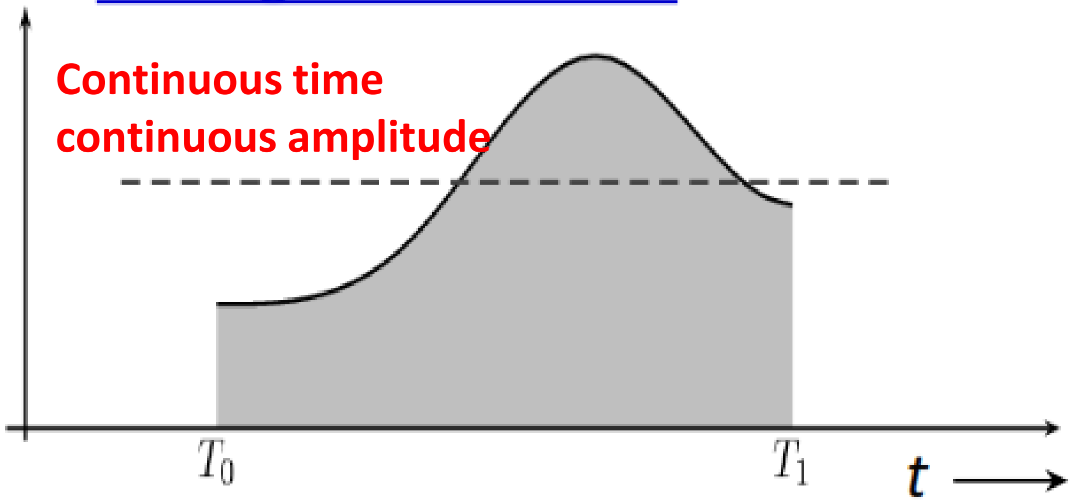




Analog and Digital Signals: Continuous Valued and Discrete Valued

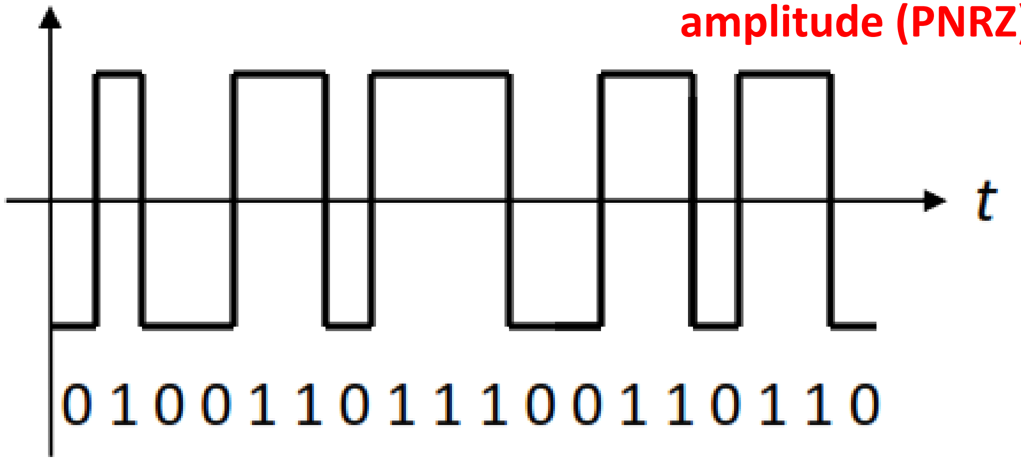
Analog & continuous

Continuous time
continuous amplitude



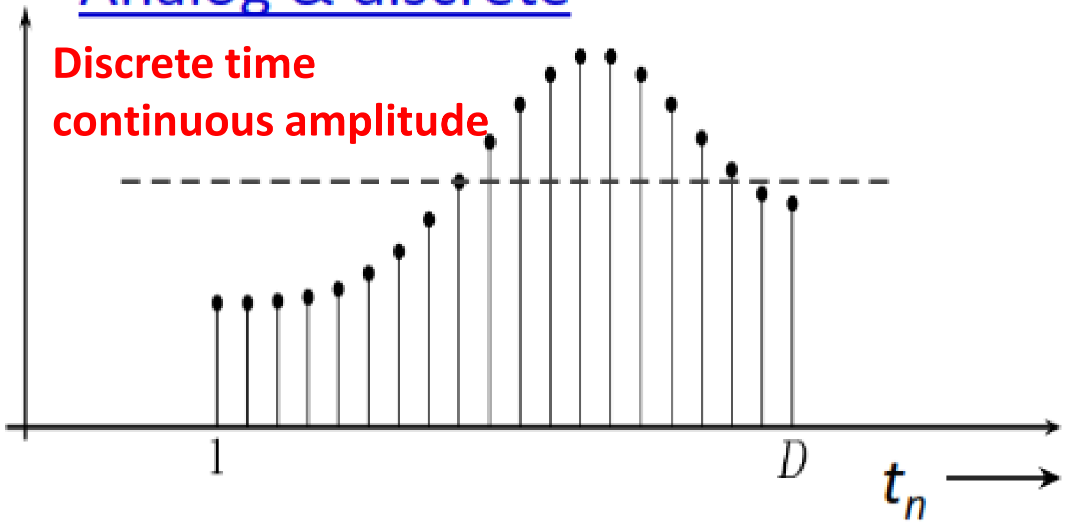
Digital & continuous

Continuous time discrete amplitude (PNRZ)



Analog & discrete

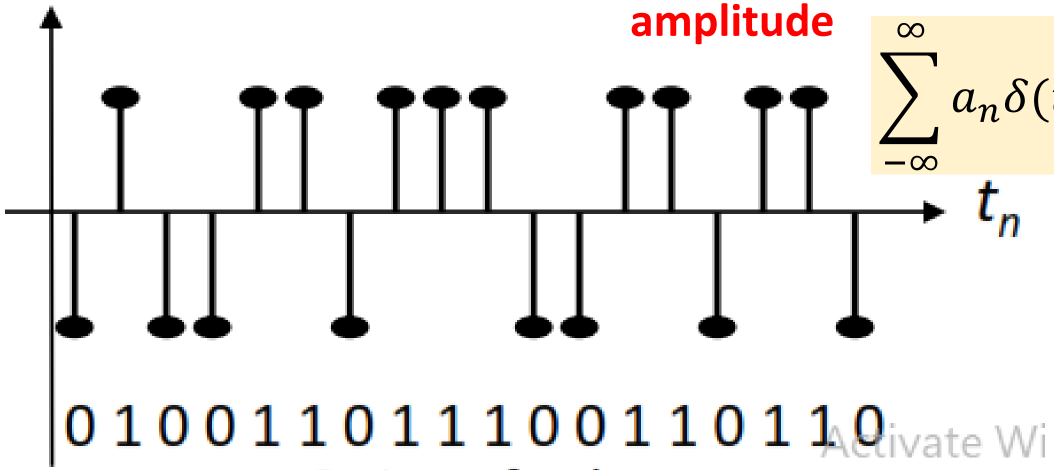
Discrete time
continuous amplitude



Digital & discrete

discrete time discrete amplitude

$$\sum_{-\infty}^{\infty} a_n \delta(t - kT_s)$$

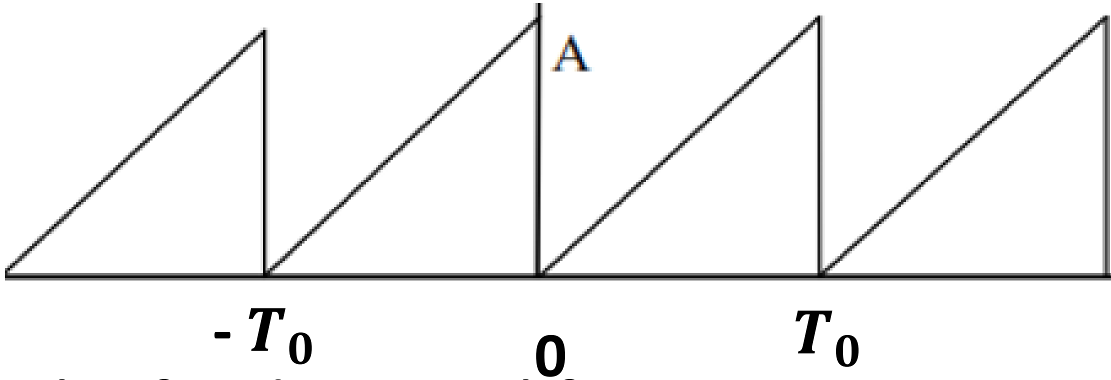


String of values

Average Value of a Signal

- ***The average value of a signal*** $g(t)$ over an observation interval of 2T centered at the origin is:

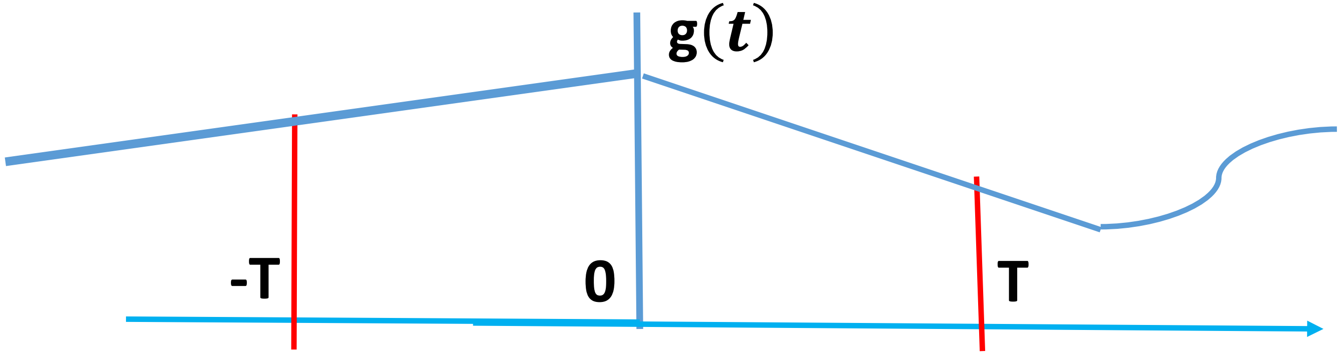
$$g_{av} = \frac{1}{2T}\int_{-T}^{T} g(t)\, dt$$

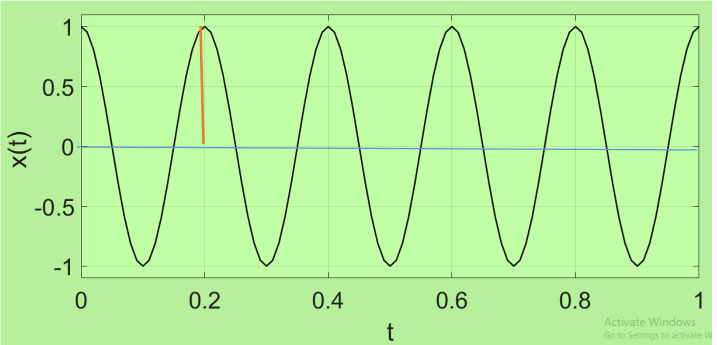


- The average value of a periodic signal $g(t)$ is

$\boldsymbol{g_{av} = \frac{1}{T_0}\int_0^{T_0} g(t)\, dt}$; T_0 is the period; f_0 is the fundamental frequency.

- **Example**: Find the average value of the sinusoidal signal
- $x(t) = A\cos 2\pi f_0 t\ , -\infty < t < \infty$
- **Solution**: $x_{av} = \frac{1}{T_0}\int_0^{T_0} A\cos 2\pi f_0 t dt = -\frac{A\sin 2\pi f_0 t}{2\pi f_0}\Big|_0^{T_0} = 0$





Energy and Power Signals

- ***The instantaneous power*** in a signal g(t) is defined as that power dissipated in a 1-Ω resistor, i.e.,

$$p(t) = |g(t)|^2$$

- ***The average power*** over an observation interval of 2T centered at the origin is:

$$P_{av} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |g(t)|^2 \, dt$$

- The total energy of a signal $g(t)$ is

$$E = \lim_{T\to\infty} \int_{-T}^{T} |g(t)|^2 \, dt$$

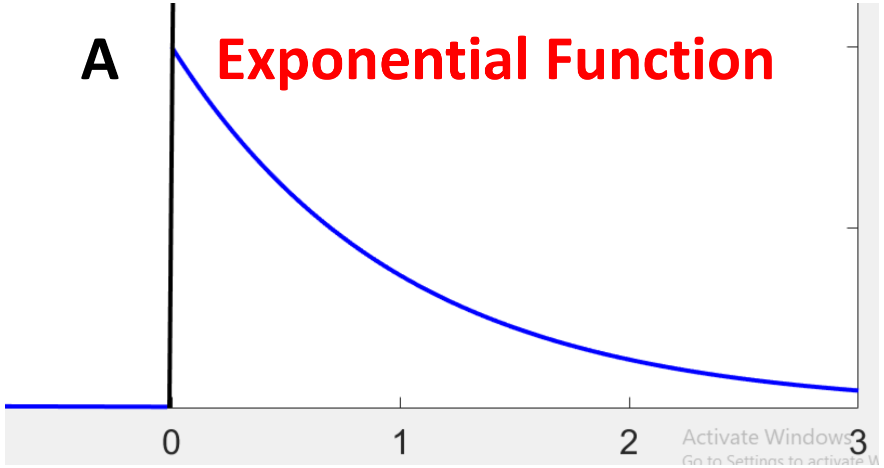
- A signal g(t) is classified as **energy signal** if it has a finite energy, i.e,$0 < E < \infty$.
- A signal g(t) is classified as **power signal** if it has a finite power, i.e, $0 < P_{av} < \infty$.
- The average power in a periodic signal $g(t)$ is

$\boldsymbol{P_{av} = \frac{1}{T_0} \int_0^{T_0} |g(t)|^2 \, dt}$; T_0 is the period; f_0 is the fundamental frequency.

Energy and Power Signals

- **Example: Consider the Exponential Pulse**

$g(t) = Ae^{-\propto t}u(t)$. **Is it an energy or a power signal?**



Solution: Let us first find the energy in the signal

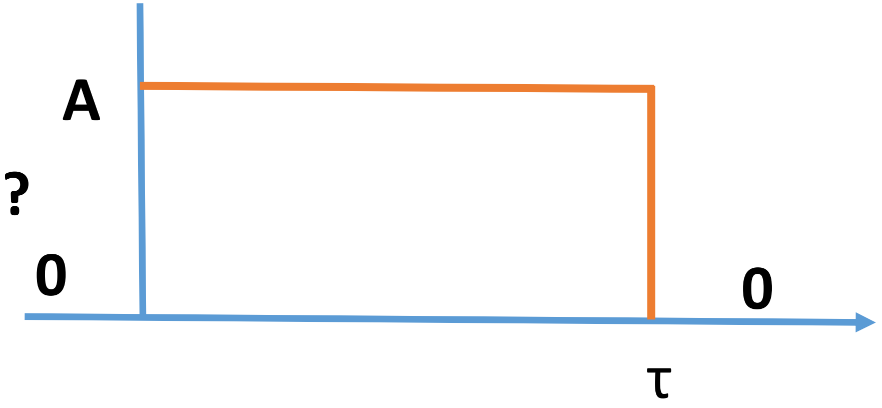
$$E = \int_0^\infty A^2\, e^{-2\alpha t} dt = A^2 \frac{-e^{-2\alpha t}}{2\alpha} \Big|_0^\infty = \frac{A^2}{2\alpha}.$$

Since E is finite, then $g(t)$ is an energy signal.

- **Example: Consider the Rectangular Pulse**

$$g(t) = \begin{cases} A, & 0 < t < \tau \\ 0, & o.w \end{cases}$$

Is it an energy or a power signal?



- **Solution**: Let us first find the energy in the signal

$$E = \int_0^\tau A^2\, dt = A^2\tau.$$ This is an energy signal since E is finite.

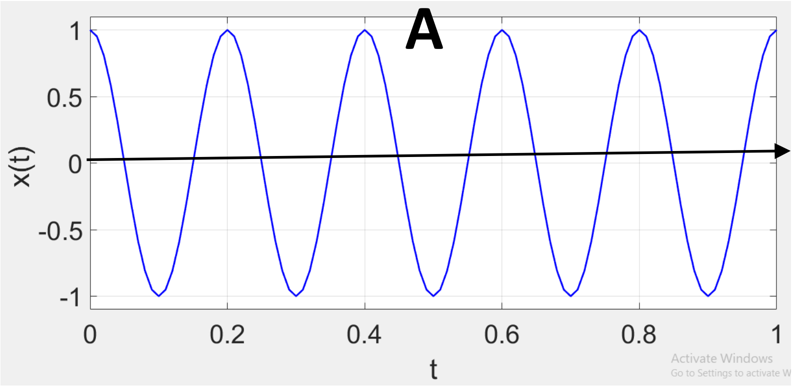
Energy and Power Signals

Example: Consider the Periodic Sinusoidal Signal

- $g(t) = Acos(2\pi f_0 t),\ -\infty < t < \infty$; **Is it an energy or a power signal**
- Since $g(t)$ is periodic, then

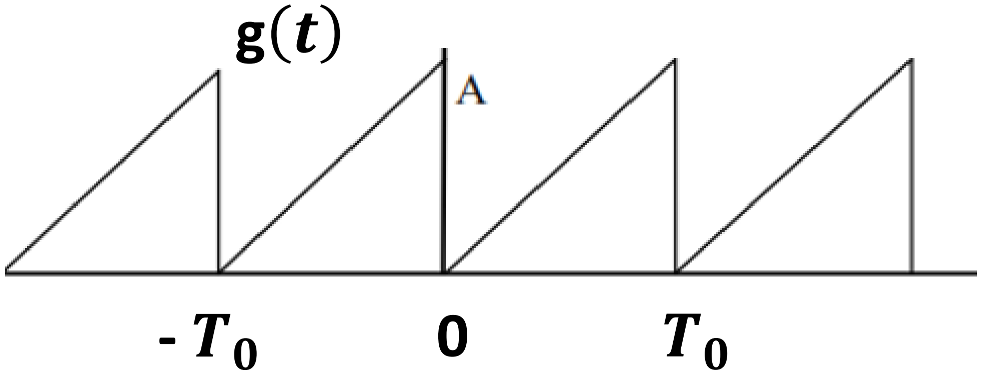
$$P_{av} = \frac{1}{T_0}\int_0^{T_0} A^2 \cos^2 \omega t\, dt = \frac{A^2}{T_0}\int_0^{T_0}\left(\frac{1+\cos 2\omega t}{2}\right) = \left(\frac{A^2}{T_0}\right).\left(\frac{T_0}{2}\right) \Rightarrow \boldsymbol{P_{av} = \frac{A^2}{2}}.$$

- Here, P_{av} is finite. Therefore, $g(t)$ is a power signal.



Example: Consider the Periodic Saw-tooth Signal

- $g(t) = \frac{A}{T_0}t,\ 0 \le t \le T_0$. **Is it an energy or a power signal?**
- Let us evaluate the average power in $g(t)$
- $P_{av} = \frac{1}{T_0}\int_0^{T_0}\frac{A^2}{T_0{}^2}t^2 dt = \frac{1}{T_0}\frac{A^2}{T_0{}^2}\frac{t^3}{3}\Big|_0^{T_0} = \frac{A^2 T_0{}^3}{3\,T_0{}^3} = \frac{A^2}{3}.$
- Here, P_{av} is finite. Therefore, $g(t)$ is a power signal.



Energy and Power Signals

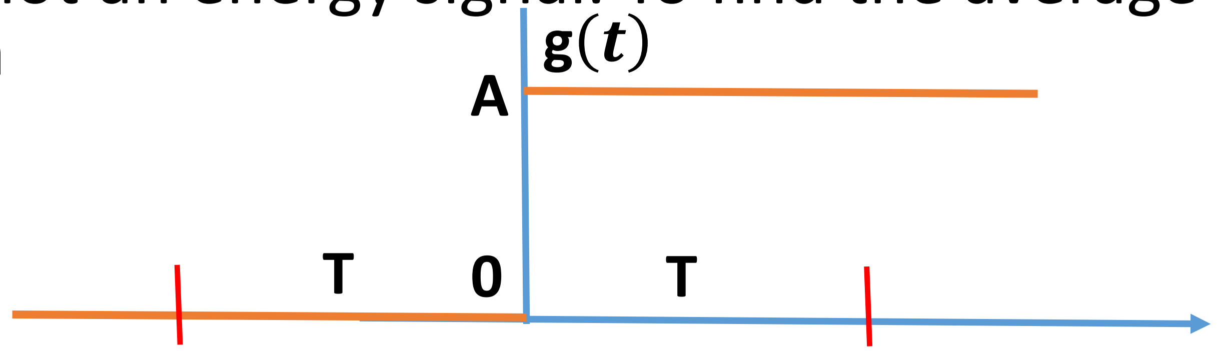
Example: The Unit Step Function $g(t) = Au(t)$

- This is a non-periodic signal. Let us first try to find its energy

$$E = \int_0^{\infty} A^2 \, dt \to \infty$$

- Sine E is not finite, then $g(t)$ is not an energy signal. To find the average power, we employ the definition

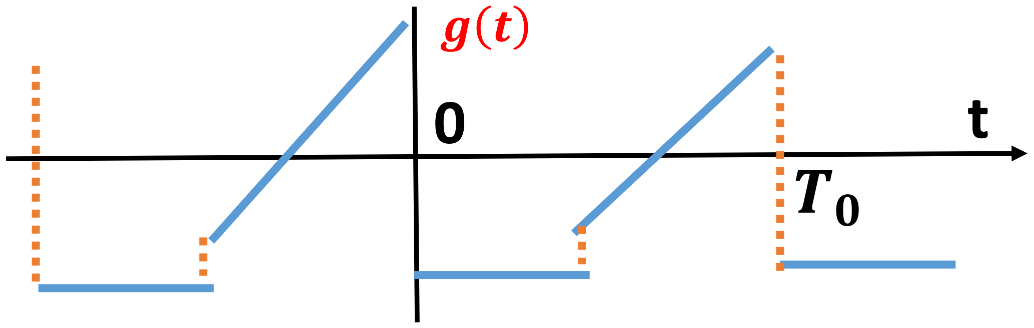
$$P_{av} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |g(t)|^2 \, dt,$$



- where 2T is chosen to be a symmetrical interval about the origin.
- $$P_{av} = \lim_{T\to\infty} \frac{1}{2T} \int_0^{T} A^2 \, dt = \lim_{T\to\infty} \frac{A^2 \, T}{2T} = \frac{A^2}{2}.$$
- So, even-though $g(t)$ is non-periodic, it turns out that it is a power signal.
- **Remark**: This is an example where the general rule (periodic signals are power signals and non-periodic signals are energy signals) fails to hold.

Fourier Series

- Let $g(t)$ be a periodic function of time with period $T_0 = \frac{1}{f_0}$ such that
  - The function g(t) is absolutely integrable over one period, i.e.,, $\int_0^{T_0} |g(t)|\, dt < \infty$
  - Any discontinuities in g(t) are finite (the amount of jump at points of discontinuity is finite).
  - g(t) has only a finite number of discontinuities and only a finite number of maxima and minima in the period
- When these conditions (called the **Dirichlet's conditions**) apply, g(t) may be expanded in a trigonometric Fourier series of the form
- $\boldsymbol{g(t) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)}$, where,
  - $a_0 = \frac{1}{T_0}\int_0^{T_0} g(t)\, dt$; (dc or average value)
  - $a_n = \frac{2}{T_0}\int_0^{T_0} g(t) \cos n\omega_0 t\, dt$
  - $b_n = \frac{2}{T_0}\int_0^{T_0} g(t) \sin n\omega_0 t\, dt$



- These conditions are sufficient (but not necessary)
- In this representation, we can associate with g(t) a FS. This does not mean equality.
- At points where g(t) is continuous, the FS converges to the function g(t)
- At a point of discontinuity t_0, the FS converges to $\frac{1}{2}\left(g(t_0 -) + g(t_0 +)\right)$

Coefficients of the Fourier Series

- $\boldsymbol{g(t) \sim a_0 + \sum_{n=1}^{\infty}(a_n\ \cos n\omega_0 t + b_n \sin n\omega_0 t)}$, (1)
- Orthogonality Relations: You can easily verify the following relations:
- $\int_0^{T_0} \cos n\omega_0 t \cos m\omega_0 t = \begin{cases} \frac{T_0}{2} & , n = m \\ 0 & n \neq m \end{cases}$,
- $\int_0^{T_0} \sin n\omega_0 t \sin m\omega_0 t = \begin{cases} \frac{T_0}{2} & , n = m \\ 0 & n \neq m \end{cases}$
- $\int_0^{T_0} \sin n\omega_0 t \cos m\omega_0 t = 0$ for all n and m.

- To get a_0, we integrate both sides of (1) with respect to t over one period.
- $\int_0^{T_0} g(t)\, dt \sim \int_0^{T_0} a_0\, dt + \int_0^{T_0} [\sum_{n=1}^{\infty}(a_n\ cos\, n\omega_0 t + b_n\, sin\, n\omega_0 t)]\, dt$
- **Result:** $a_0 = \frac{1}{T_0}\int_0^{T_0} g(t)\, dt$; (dc or average value)

Coefficients of the Fourier Series

- $\boldsymbol{g(t) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)}$, (1)
- To get a_n, we multiply both sides of (1) by $\cos m\omega_0 t$, integrate over one period and use the orthogonality relations.
- $\int_0^{T_0} g(t) \boldsymbol{\cos m\omega_0 t}\, dt \sim \int_0^{T_0} a_0 \boldsymbol{\cos m\omega_0 t}\, dt$

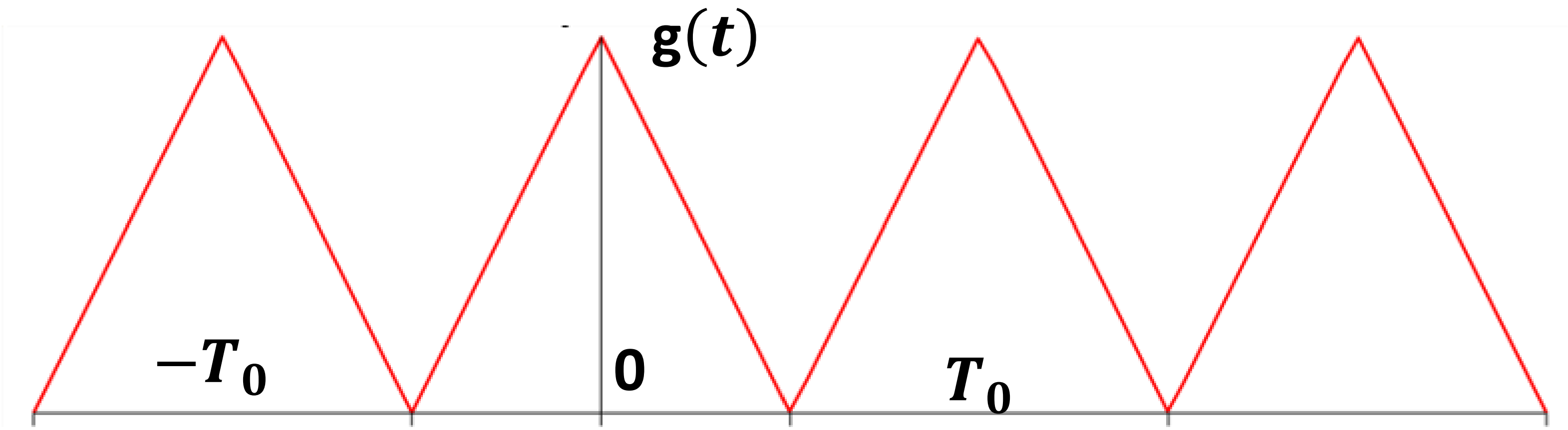
  $+ \int_0^{T_0} [\sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \boldsymbol{\cos m\omega_0 t}]\, dt$
- **Result**: $a_n = \frac{2}{T_0}\int_0^{T_0} g(t) \cos n\omega_0 t\, dt$
- To get b_n, we multiply both sides of (1) by $\sin m\omega_0 t$, integrate over one period and use the orthogonality relations.
- $\int_0^{T_0} g(t) \boldsymbol{\sin m\omega_0 t}\, dt \sim \int_0^{T_0} a_0 \boldsymbol{\sin m\omega_0 t}\, dt$

  $+ \int_0^{T_0} [\sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \boldsymbol{\sin m\omega_0 t}]\, dt$
- **Result:** $b_n = \frac{2}{T_0}\int_0^{T_0} g(t) \sin n\omega_0 t\, dt$

Example: Existence of Fourier Series

- The Dirichlet conditions apply to the waveform given below.
- The function g(t) is absolutely integrable, i.e., $\int_0^{T_0} |g(t)|\, dt < \infty$.
- The function g(t) is continuous over the period (no discontinuities)
- Has one maximum and one minimum within one period.
- Therefore, **the FS exists**. Moreover, the **FS converges to g(t) at all points**. That is,

- $\boldsymbol{g(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)}$; Note the equality sign

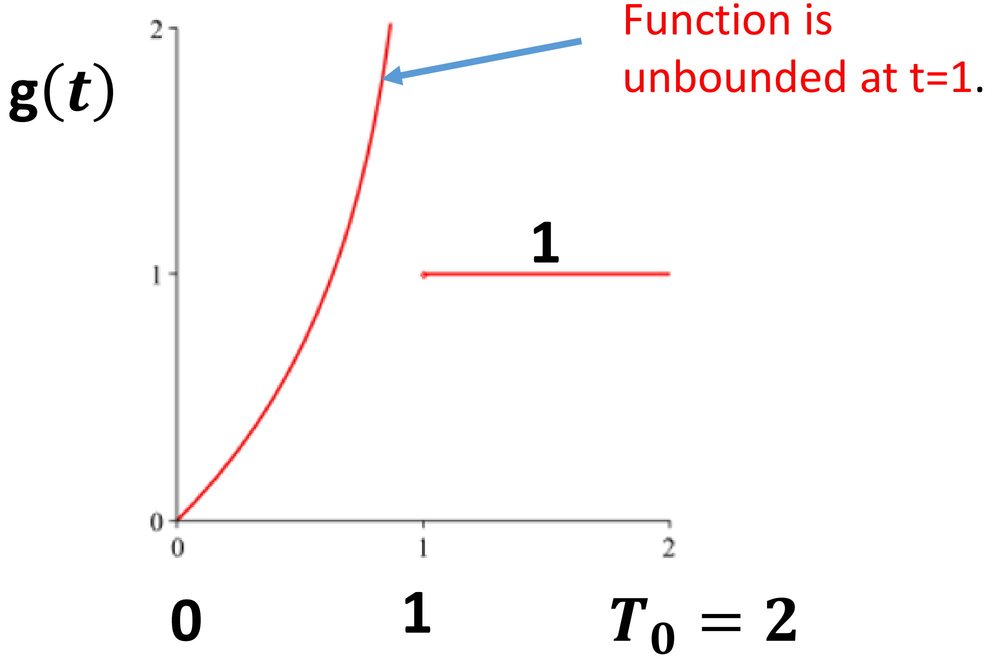


Example: Existence of Fourier Series

- Let g(t), defined over one period, be given by

$$g(t) = \begin{cases} -\ln(1-t), & 0 < t < 1 \\ 1, & 1 < t < 2 \end{cases}$$

- $\lim_{t \to 1} \big(g(t)\big) = -\ln(1-t) \to \infty$
- the function g(t) has a discontinuity. However, this discontinuity is infinite.
- **Therefore, the FS does not exists**



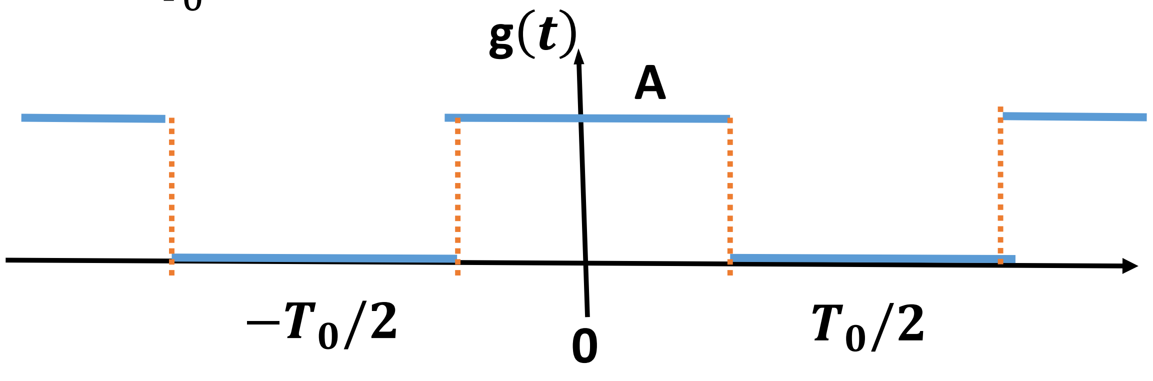
Example: Fourier Series Coefficient Evaluation

- **Example**: Find the trigonometric Fourier series of the periodic rectangular signal defined over one period T_0 as:

$$g(t) = \begin{cases} +A, & -T_0/4 \le t \le T_0/4 \\ 0, & otherwise \end{cases}$$

- **Solution**: The FS is given as $g(t) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$
- $a_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\, dt = \frac{1}{T_0}\int_{-T_0/4}^{T_0/4} A\, dt = A/2$
- $b_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} g(t) \sin(\frac{2\pi n}{T_0} t)\, dt = \frac{2}{T_0}\int_{-T_0/4}^{T_0/4} A \sin(\frac{2\pi n}{T_0} t)\, dt = 0$
- $a_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} g(t) \cos(\frac{2\pi n}{T_0} t)\, dt = \frac{2}{T_0}\int_{-T_0/4}^{T_0/4} A \cos(\frac{2\pi n}{T_0} t)\, dt$
- $a_n = \begin{cases} \frac{2A}{n\pi}, & n = 1, 5, 9, \ldots \\ \frac{-2A}{n\pi}, & n = 3, 7, 11, \ldots \\ 0, & n = 2, 4, 6 \ldots \end{cases}$

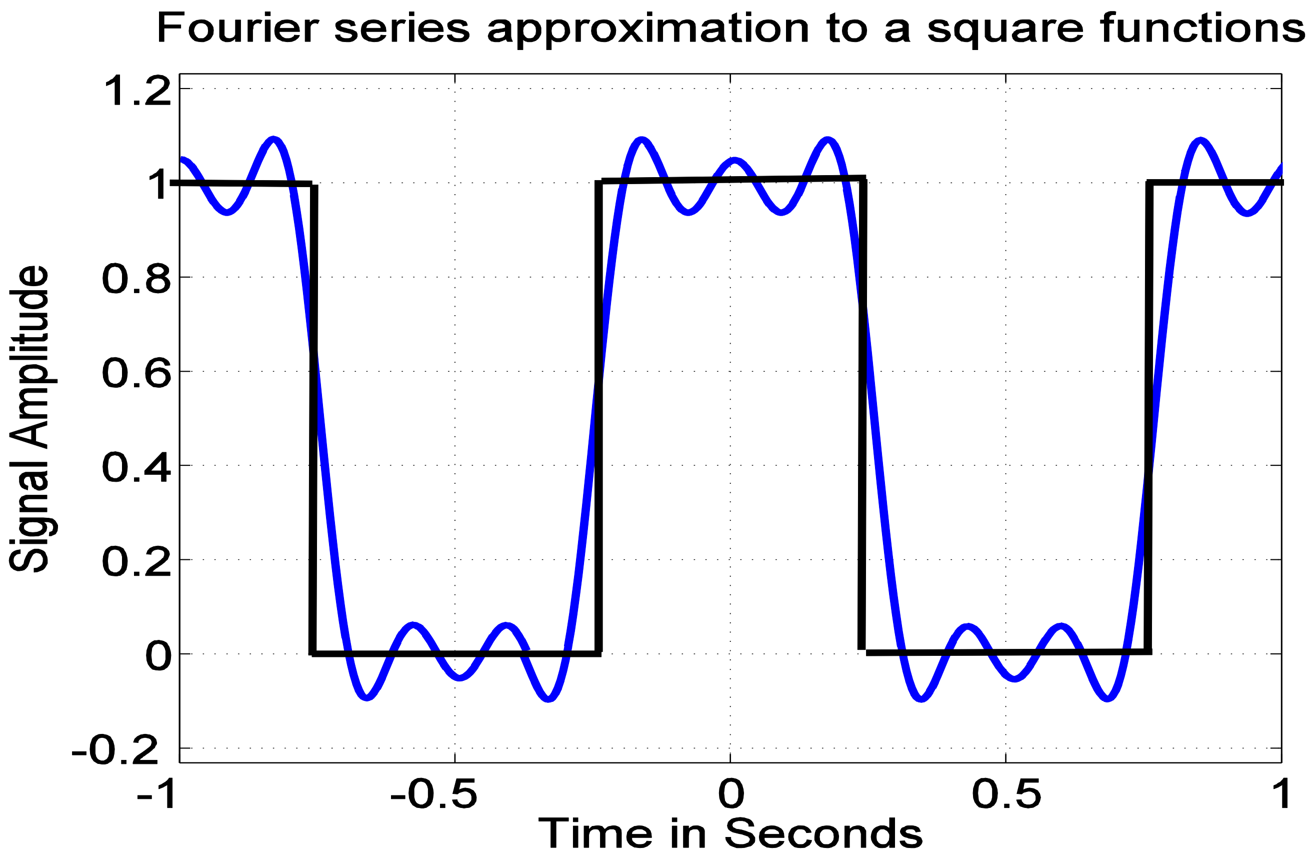
Dirichlet conditions apply.
Therefore, a FS exists



Example: Convergence of Fourier Series

- The first four terms in the expansion of $g(t)$ are:
- $\tilde{g}(t) = \frac{A}{2} + \frac{2A}{\pi}\{\cos(2\pi f_0 t) - \frac{1}{3}\cos(2\pi 3 f_0 t) + \frac{1}{5}\cos(2\pi 5 f_0 t)\}$
- The function $\tilde{g}(t)$ along with $g(t)$ are plotted in the figure for $-1 \leq t \leq 1$ assuming $A = 1$ and $f_0 = 1$

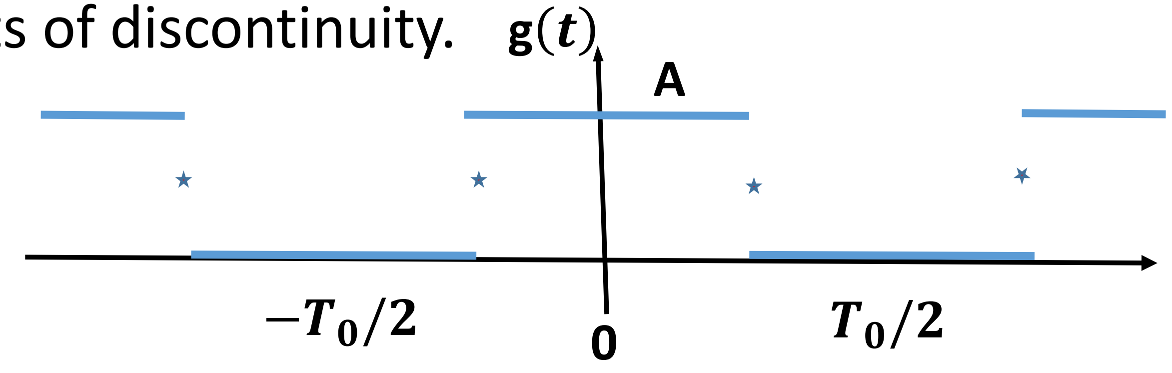
Comments: As more terms are added to $\tilde{g}(t)$, $\tilde{g}(t)$ becomes closer to $g(t)$ and in the limit as $n \rightarrow \infty$, $\tilde{g}(t)$ becomes equal to $g(t)$ at all points except at the points of discontinuity.



Convergence of the Fourier Series

The Fourier series of the signal $g(t) = \begin{cases} +A, & -T_0/4 \le t \le T_0/4 \\ 0, & otherwise \end{cases}$ is given by

- $g(t) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$. The FS is shown in the figure below
- $a_0 = A/2$, $b_n = 0$,
- $a_n = \begin{cases} \frac{2A}{n\pi}, & n = 1, 5, 9, \dots \\ \frac{-2A}{n\pi}, & n = 3, 7, 11, \dots \\ 0, & n = 2, 4, 6 \dots \end{cases}$
- The FS converges to g(t) at all points where g(t) is continuous
- Converges to A/2, the average value at points of discontinuity.



Fourier Cosine and Sine Series

- Let $g(t)$ be a periodic function of time with period $T_0 = \frac{1}{f_0}$ such that its FS exists.
- **Fourier Cosine Series**:
- Let g(t) be an even function of t, then
- $b_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\sin(\frac{2\pi n}{T_0}t)\,dt = 0$
- $a_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\cos(\frac{2\pi n}{T_0}t)\,dt = \frac{4}{T_0}\int_{0}^{T_0/2} g(t)\cos(\frac{2\pi n}{T_0}t)\,dt$
- The FS becomes a Fourier cosine series $\boldsymbol{g(t) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t}$
- **Fourier Sine Series**:
- Let g(t) be an odd function of t, then
- $a_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\,dt = 0,\ \ a_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\cos(\frac{2\pi n}{T_0}t)\,dt = 0$
- $b_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\sin(\frac{2\pi n}{T_0}t)\,dt = \frac{4}{T_0}\int_{0}^{T_0/2} g(t)\sin(\frac{2\pi n}{T_0}t)\,dt$
- The FS becomes a Fourier sine series $\boldsymbol{g(t) \sim \sum_{n=1}^{\infty} b_n \sin n\omega_0 t}$

Complex Form of the Fourier Series

The Fourier series can also be expressed in the complex form:

$$g(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

where, $$C_n = \frac{1}{T_0}\int_0^{T_0} g(t)\, e^{-jn\omega_0 t} dt.$$

- Note that C_n is a complex valued quantity, which can be written as
- $C_n = |C_n| e^{j\theta n}$
- The plot of $|C_n|$ versus frequency is called the ***Discrete Amplitude Spectrum***.
- The plot of θ_n versus frequency is called the ***Discrete Phase Spectrum***.
- The term at f_0 is referred to as the fundamental frequency. The term at $2f_0$ is referred to as the second order harmonic, the term at $3f_0$ is referred to as the third order harmonic and so on.

Parseval's Power Theorem

- The average power of a periodic signal g(t) is given by:

$$\boldsymbol{P_{av} = \frac{1}{T_0}\int_0^{T_0}|g(t)|^2\,dt = \sum_{n=-\infty}^{\infty}|C_n|^2 = |C_0|^2 + 2\sum_{n=1}^{\infty}|C_n|^2}$$

- $$\boldsymbol{= |a_0|^2 + \frac{1}{2}\sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2}$$

- **Proof:** $g(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$, where, $C_n = \frac{1}{T_0}\int_0^{T_0} g(t)\, e^{-jn\omega_0 t}dt.$
- $|\boldsymbol{g(t)}|^2 = g(t)g^*(t) = \left(\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}\right)\left(\sum_{m=-\infty}^{\infty} C_m^* e^{-jm\omega_0 t}\right)$
- $\boldsymbol{\frac{1}{T_0}\int_0^{T_0}|g(t)|^2 dt} = \boldsymbol{\frac{1}{T_0}\int_0^{T_0}}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} C_n\, C_m^*\, e^{j(n-m)\omega_0 t}dt$
- Orthogonality: $\int_0^{T_0} e^{j(n-m)\omega_0 t}dt = \begin{cases} T_0 & , n = m \\ 0 & n \neq m \end{cases}$
- $\boldsymbol{\frac{1}{T_0}\int_0^{T_0}|g(t)|^2 dt} = \sum_{n=-\infty}^{\infty}|C_n|^2 = |C_0|^2 + 2\sum_{n=1}^{\infty}|C_n|^2$

Power Spectral Density

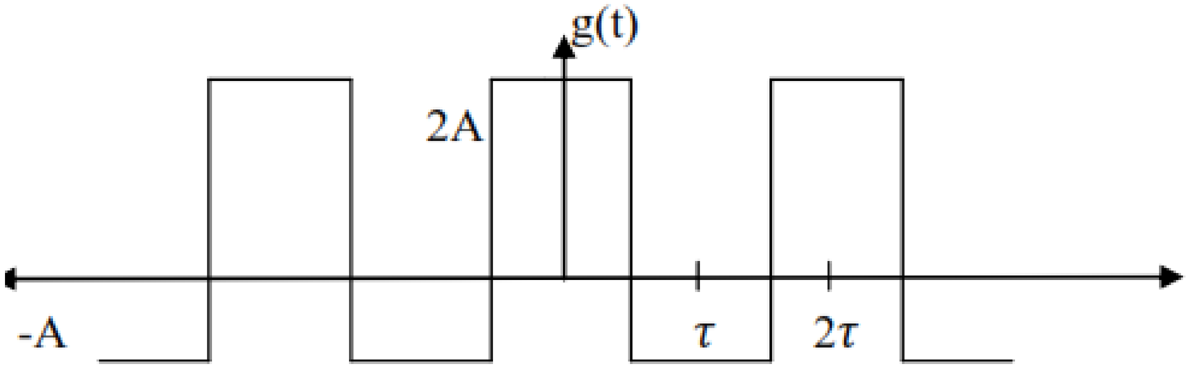
- The plot of $|C_n|^2$ versus frequency is called the ***power spectral density*** (PSD).
- It displays the power content of each frequency (spectral) component of a signal.
- For a periodic signal, the PSD consists of discrete terms at multiples of the fundamental frequency.
- The next example demonstrate these properties

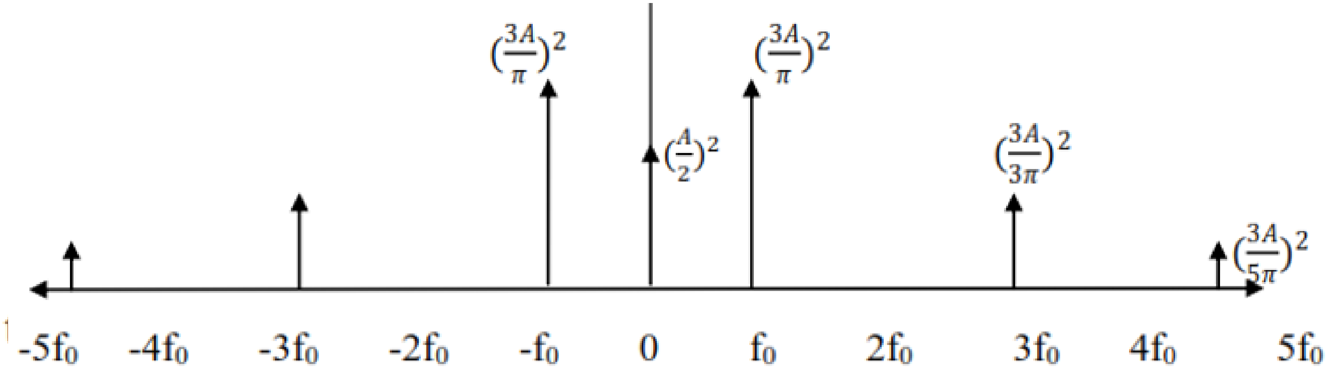
Power Spectral Density

- **Example:** Find the power spectral density of the $g(t)$ shown in the figure.
- Here, we need to find the complex Fourier series expansion, where the period $T_0 = 2\tau$
- $g(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$; $\quad C_n = \frac{1}{T_0}\int_0^{T_0} g(t)\, e^{-jn\omega_0 t} dt$
- $C_n = \begin{cases} \frac{A}{2}, & n = 0 \\ \frac{3A}{|n|\pi}, & n = \pm 1, \pm 5, \pm 9, \ldots \\ \frac{-3A}{|n|\pi}, & n = \pm 3, \pm 7, \pm 11, \ldots \\ 0, & n = \pm 2, \pm 4, \ldots \end{cases}$

$$|C_n|^2 = \begin{cases} (\frac{A}{2})^2, & n = 0 \\ (\frac{3A}{n\pi})^2, & n: odd \\ 0, & n: even \end{cases}$$

$$S_g(f) = \sum_{n=-\infty}^{\infty} |C_n|^2\, \delta(f - nf_0)$$





Fourier Transform

- Let $g(t)$ be a function of time t. The Fourier transform maps the function g(t) into another function G(f) defined into the frequency domain. The Fourier transform is defined as:

$$\boldsymbol{G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t}\, dt}$$

- The inverse Fourier transform is defined as

$$\boldsymbol{g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi f t}\, df}$$

- Conditions for existence (Dirichlet conditions, which are the same as those for the FS)
  - The function g(t) is absolutely integrable, i.e.,, $\int_0^{T_0} |g(t)|\, dt < \infty$.
  - Any discontinuities in g(t) are finite
  - g(t) has only a finite number of discontinuities and only a finite number of maxima and minima in any finite interval.
- **Remarks**:
  - These conditions are sufficient but not necessary
  - A **weaker sufficient condition** for existence is $\boldsymbol{\int_{-\infty}^{\infty} |g(t)|^2\, dt < \infty}$ (g(t) is an energy signal). This is the finite-energy condition that is satisfied by all physically realizable waveforms.
  - Generally, physical waveforms encountered in engineering practice are Fourier transformable.
  - The Fourier transform can be derived from the Fourier series by allowing the period T_0 to go to infinity, but this will not be covered in this presentation.

Fourier Transform: Amplitude and Phase Spectrum

Observations: $\boldsymbol{G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t}\ dt}$

- G(f) is a complex function of frequency f, which can be expressed as:

  $$G(f) = |G(f)| j^{\theta(f)}$$

- The function $G(f)$ is often referred to as the spectrum of $g(t)$.
  - $|G(f)|$: is the ***continuous amplitude spectrum*** of $g(t)$, (an even function of f).
  - $\theta(f)$: is the ***continuous phase spectrum*** of $g(t)$, (an odd function of f).

- **Notation**:
  - To denote that $G(f)$ is the Fourier transform of $g(t)$, we write $G(f) = \mathfrak{I}(g(t))$
  - To denote that $g(t)$, *is the inverse Fourier transform of* $G(f)$, we write $g(t) = \mathfrak{I}^{-1}(G(f))$
  - Sometimes, the following notation is used for a Fourier transform pair g$(t) \leftrightarrow G(f)$.

Rayleigh Energy Theorem

Rayleigh Energy Theorem: The energy in a signal g(t) is given by:

$$\boldsymbol{E = \int_{-\infty}^{\infty} |g(t)|^2\, dt = \int_{-\infty}^{\infty} |G(f)|^2 df}$$

- The proof of this result is the same as that for Parseval's power theorem
- The function $|G(f)|^2$ is called the ***energy spectral density***. It depicts the range of frequencies over which the signal energy extends and the frequency bands which are significant in terms of their energy contents.
- For a non-period signal energy signal, the **energy spectral density is a continuous function of f.**

A General Form of the Rayleigh Energy Theorem

- For two energy functions $g(t)$ and $v(t)$, the following result holds:

$$\boldsymbol{\int_{-\infty}^{\infty} g(t)v(t)^*\, dt = \int_{-\infty}^{\infty} G(f)V(f)^*\, df}$$

Example: Exponential Pulse

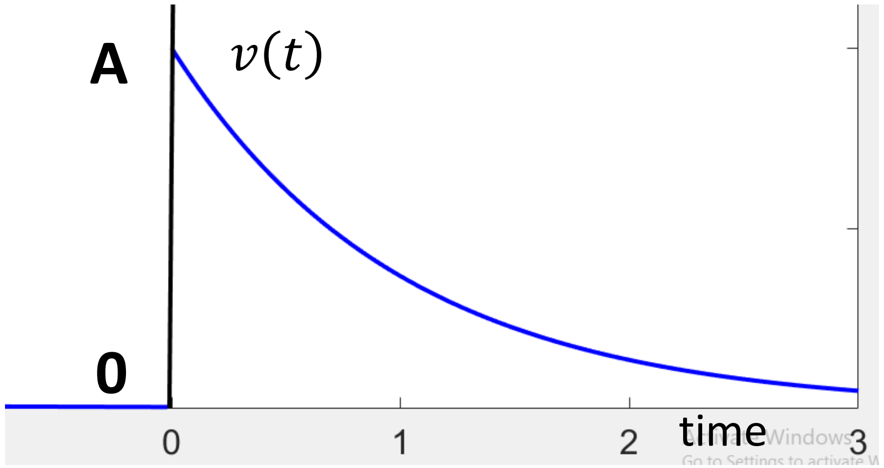
- $v(t) = \begin{cases} A\,e^{-bt} & t > 0 \\ 0 & t < 0 \end{cases}$
- $\boldsymbol{E = \int_{-\infty}^{\infty} |g(t)|^2\, dt = \int_0^{\infty} A^2\, e^{-2bt} dt = (A^2/\,2b)}$ **, F.T exists**
- $V(f) = \int_0^{\infty} v(t) e^{-j2\pi f t}\; dt = \int_0^{\infty} A e^{-bt}\, e^{-j2\pi f t} dt$

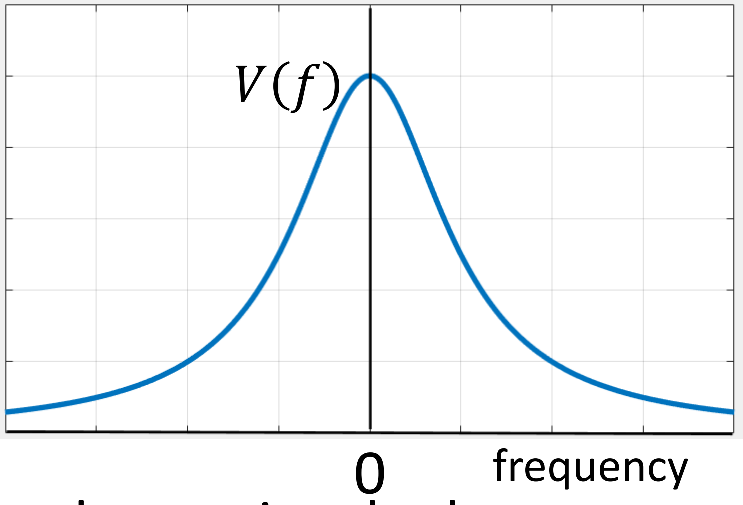
$$V(f) = A \int_0^{\infty} e^{-(b+j2\pi f)t}\; dt = A \frac{e^{-(b+j2\pi f)t}}{-(b+j2\pi f)} \Big|_0^{\infty} = \frac{A}{b+j2\pi f}.$$

- $V(f) = \frac{A}{b+j2\pi f},$
- $|V(f)| = \frac{A}{(b^2+(2\pi f)^2)^{1/2}}$
- The energy spectral density is: $S_v(f) = |V(f)|^2 = \frac{A^2}{b^2+(2\pi f)^2}$

Exercise: For the given v(t), verify Rayleigh Energy Theorem:

$$\boldsymbol{E = \int_{-\infty}^{\infty} |v(t)|^2\, dt = \int_{-\infty}^{\infty} |V(f)|^2 df}$$

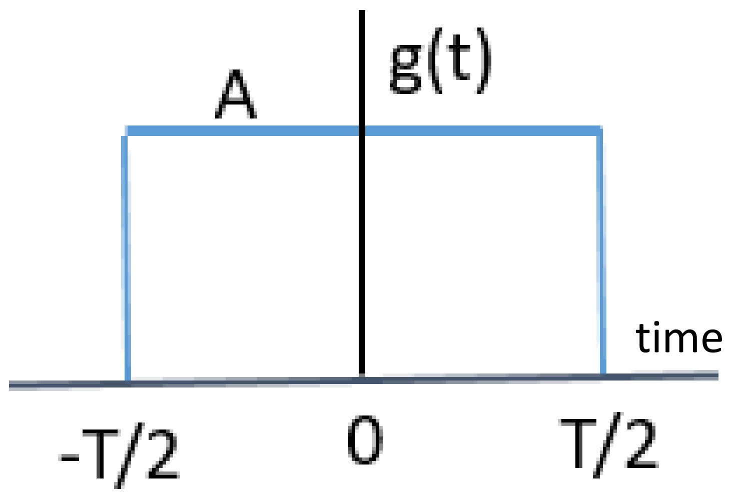


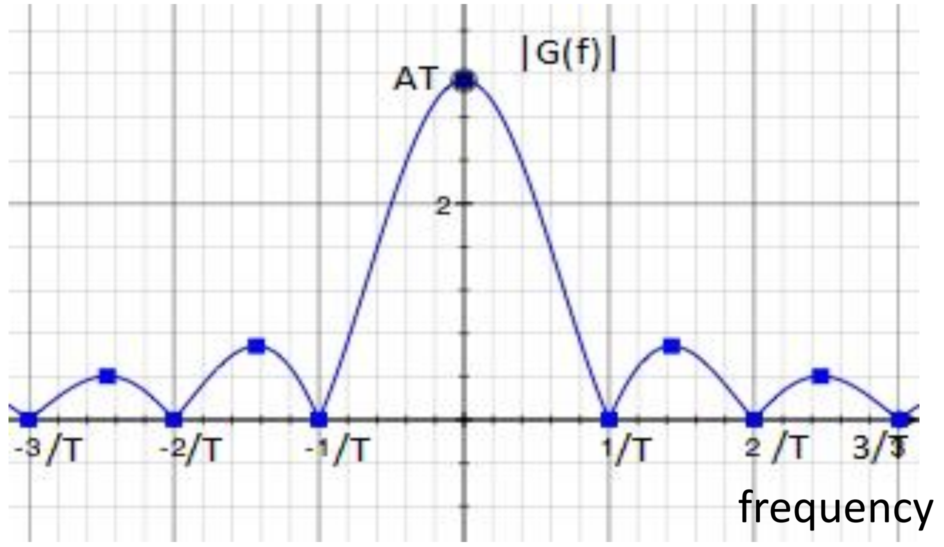


- **Remark**: The signal $v(t)$ is called a ***baseband signal*** since the signal occupies the low frequency part of the spectrum. That is, the energy in the signal is found around the zero frequency. When the signal is multiplied by a high frequency carrier, the spectrum becomes centered around the carrier and the modulated signal is called a ***bandpass signal***.

Example: The Rectangular Pulse $g(t) = Arect(\frac{t}{T})$

- $G(f) = \int_{-T/2}^{T/2} Ae^{-j2\pi ft}dt = \frac{A}{\pi f}\sin \pi fT$, $AT\frac{\sin \pi fT}{\pi fT} \triangleq AT\ sincfT$
- $|\boldsymbol{G(f)}| = \boldsymbol{AT}\ |\boldsymbol{sincfT}|$
- The maximum of $|G(f)|$ occurs at $f = 0$ since $\lim_{x\to 0}\frac{\sin x}{x} = 1$. Also, $G(f) = 0$ when $sin(\pi fT) = 0$, which occurs at the points that satisfy $\pi fT = n\pi, \Rightarrow fT = n, or\ f = \frac{n}{T}, n = \pm1, \pm2, \pm3, \ldots$





Properties of the Fourier Transform

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}\, dt$$

- **Linearity (superposition)**

  Let $g_1(t) \leftrightarrow G_1(\mathrm{f})$ and $g_2(t) \leftrightarrow G_2(\mathrm{f})$, then

  $c_1g_1(t)+c_2g_2(t) \leftrightarrow c_1G_1(f)+c_2G_2(f)$; c_1, c_2 are constants

- **Time Scaling**

$g(t) \leftrightarrow G(f)$	$g(at) \leftrightarrow \dfrac{1}{\lvert a\rvert} G(f/a)$

- **Duality**

$g(t) \leftrightarrow G(f)$	$G(t) \leftrightarrow g(-f)$

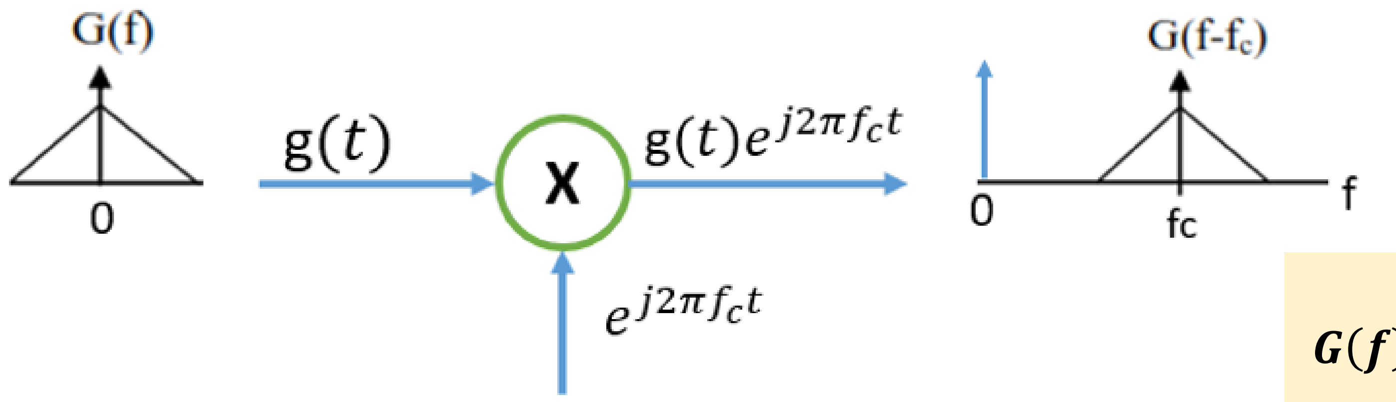
- **Time Shifting**

$g(t) \leftrightarrow G(f)$	$g(t-t_0) \leftrightarrow G(f)e^{-j2\pi ft_0}$
Delay in time domain corresponds to a phase shift in frequency domain	

Properties of the Fourier Transform

- **Frequency Shifting**

$g(t) \leftrightarrow G(f)$	$g(t)e^{j2\pi f_c t} \leftrightarrow G(f - f_c)$; f_c is constant



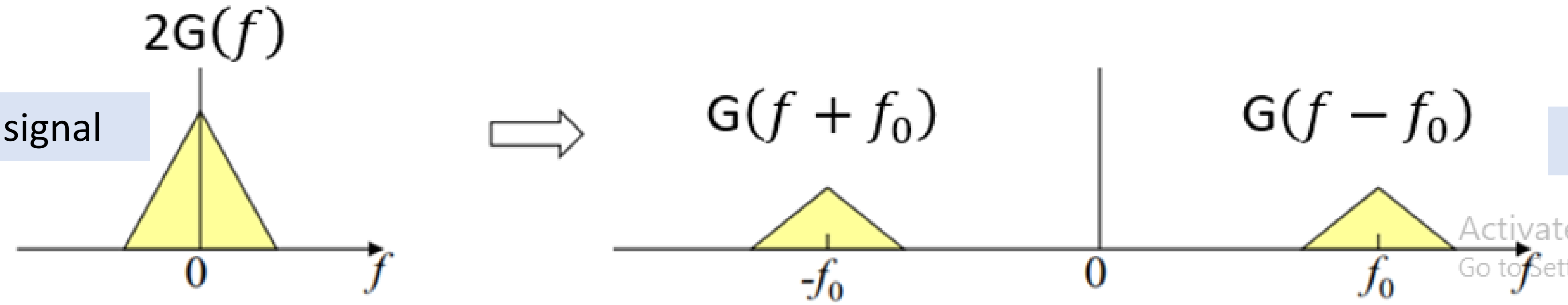
Frequency Shifting Property of the Fourier Transfer

$$\boldsymbol{G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi f t}\, dt}$$

- **Modulation Property**

$g(t) \leftrightarrow G(f)$	$2g(t)\cos(2\pi f_0 t) \leftrightarrow G(f - f_0) + G(f + f_0)$; f_0 is a constant

Low-pass signal



Band-pass signal

Properties of the Fourier Transform

- **Area under $G(f)$**

$g(t) \leftrightarrow G(f)$	$g(t=0) = \int_{-\infty}^{\infty} G(f)df$
The value $g(t=0)$ is equal to the area under its Fourier transform function	

- **Area under $g(t)$**

$g(t) \leftrightarrow G(f)$	$G(0) = \int_{-\infty}^{\infty} g(t)dt$
The area under a function g(t) is equal to the value of its Fourier transform G(f) at f = 0, where $G(0)$ implies the presence of a dc component.	

$$\boldsymbol{g(t) = \int_{-\infty}^{\infty} G(f)e^{j2\pi ft}\, df}$$

$$\boldsymbol{G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}}$$

Properties of the Fourier Transform

- **Differentiation in the Time Domain**

If $g(t)$ and its derivative $g'(t)$ are Fourier transformable, then,

$$g'(t) \leftrightarrow (j2\pi f)G(f)$$

i.e., differentiation in the time domain $\Longrightarrow$ multiplication by j$2\pi f$ in the frequency domain.

(Differentiation in the time domain enhances high frequency components of a signal)

Also, $$\frac{d^n g(t)}{dt^n} \leftrightarrow (j2\pi f)^n G(f)$$

$$\boldsymbol{g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi f t}\, df}$$

- **Integration in the Time Domain**

$$\int_{-\infty}^{t} g(\tau)d\tau \leftrightarrow \frac{1}{j2\pi f}G(f);\ \text{assuming } G(0) = 0$$

i.e., integration in the time domain corresponds to division by (j$2\pi f$) in the frequency domain. This amounts to low pass filtering, where high frequency components are attenuated due to filtering.

When $G(0) \neq 0$, the above result becomes:

$$\int_{-\infty}^{t} g(\tau)d\tau \leftrightarrow \frac{1}{j2\pi f}G(f) + \frac{1}{2}\, G(0)\delta(f).$$

Properties of the Fourier Transform

- **Multiplication of two signals in the time domain**

$$g_1(t)\, g_2(t) \leftrightarrow \int_{-\infty}^{\infty} G_1(\lambda)\, G_2(f-\lambda) d\lambda = G_1(f) * G_2(f)$$

Multiplication of two signals in the time domain is transformed into the convolution of their Fourier transforms in the frequency domain.

- **Convolution of two signals in the time domain**

$$g_1(t) * \ g_2(t) \leftrightarrow G_1(f) G_2(f)$$

Convolution of two signals in the time domain is transformed into a multiplication of their Fourier transforms in the frequency domain

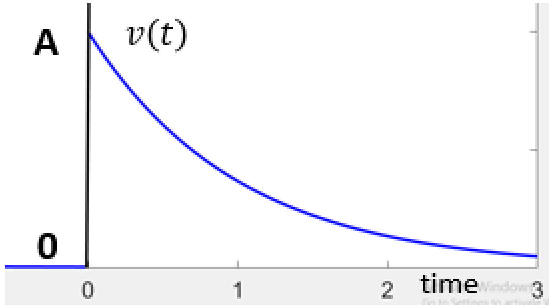
- **Multiplication by t in the time domain corresponds to differentiation in the** frequency domain

$$\Im\{\mathbf{tg}(t)\} = \frac{j}{2\pi}\frac{dG(f)}{df}$$

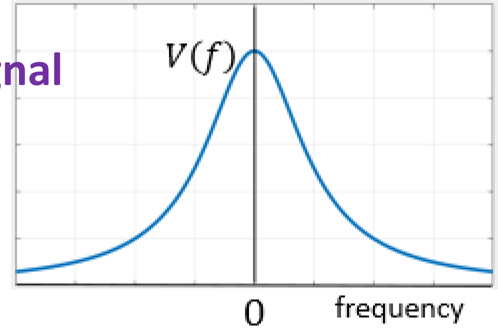
$$\boldsymbol{G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t}}$$

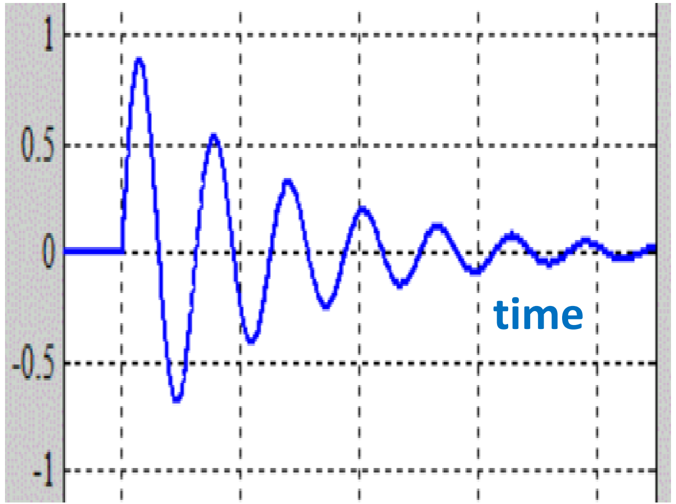
Examples: The RF Negative Exponential Pulse

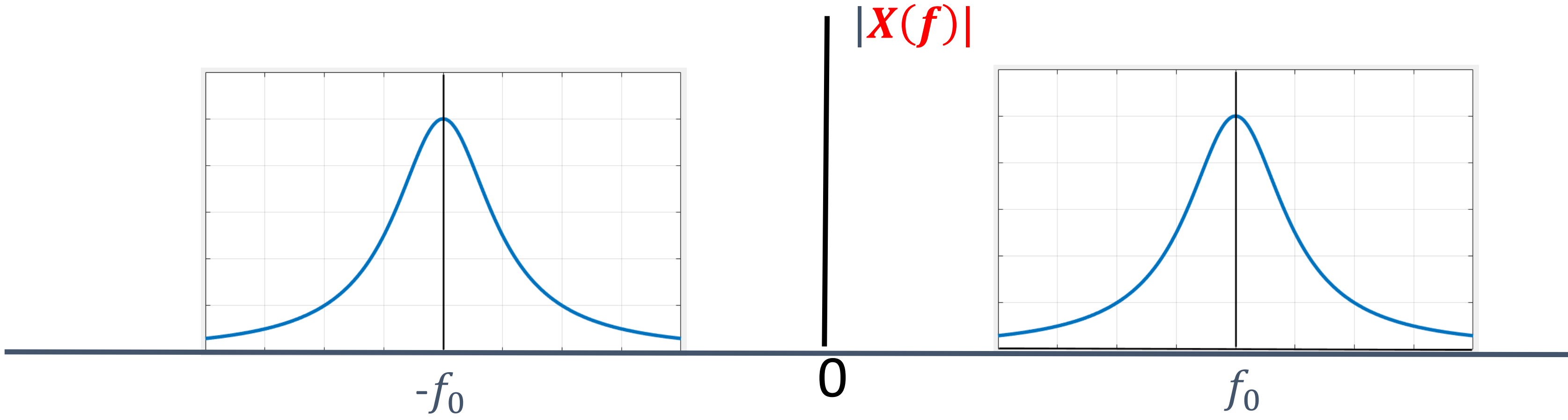
- **Example**: Find the Fourier transform of $x(t) = A\,e^{-bt}cos(2\pi f_0 t)$, t > 0
- **Solution**: Note that x(t) can be expressed as
- $\mathbf{x(t)} = \mathbf{g}(t)cos(2\pi f_0 t),\ g(t) = \begin{cases} A\,e^{-bt} & t > 0 \\ 0 & t < 0 \end{cases}$
- $G(f) = \left(\frac{A}{b+j2\pi f}\right)$
- Use the modulation property
- $X(f) = \frac{1}{2}\{G(f - f_0) + G(f + f_0)\}$
- $X(f) = \frac{1}{2}\left\{\frac{A}{b+j2\pi(f-f_0)} + \frac{A}{b+j2\pi(f+f_0)}\right\}$; **Band-pass signal**



Baseband signal





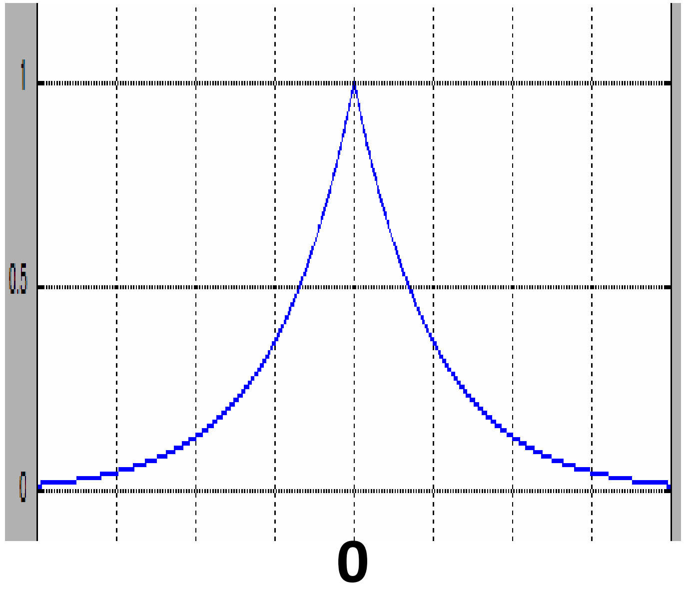


Example: double-sided exponential pulse

- **Example**: Find the Fourier transform of the double-sided exponential pulse

$$\mathbf{g}(t) = Ae^{-b|t|}, -\infty < t < \infty$$

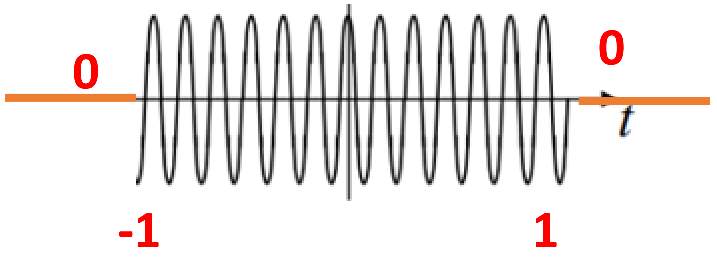
- **Solution**: You can easily find that the energy in g(t) is finite, and hence the F.T. exists.
- $$\mathrm{G}(f) = \int_{-\infty}^{0} Ae^{bt}e^{-j2\pi ft}\, dt + \int_{0}^{\infty} Ae^{-bt}\, e^{-j2\pi ft} dt$$
- $$\mathrm{G}(f) = \frac{A}{b-j2\pi f} + \frac{A}{b+j2\pi f} = \frac{2bA}{b^2+(2\pi f)^2}$$



Examples: Fourier Transform of an RF Pulse

Find the Fourier transform of the RF pulse $x(t) = \cos(2\pi f_0 t)\,; 1 \le t \le 1$,

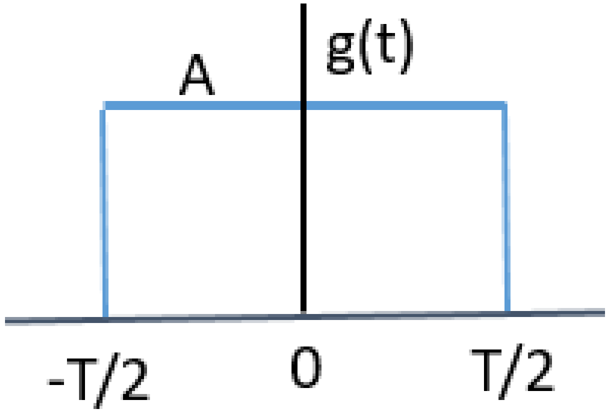
Solution: x(t) can be viewed as a product of the rectangular pulse and the cosine function $x(t) = \mathrm{g(t)}\cos(2\pi f_0 t)$, where

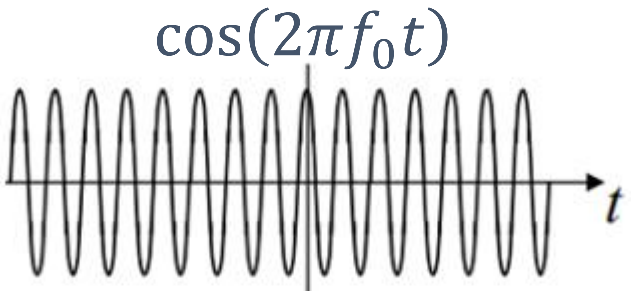


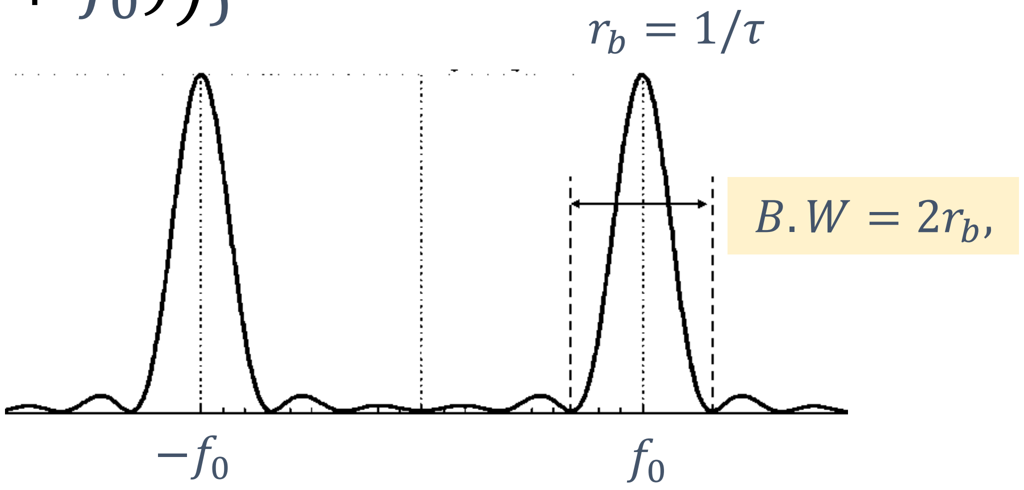
$$g(t) = u(t+1) - u(t+1) = rect(\frac{t}{2})$$

$$\mathrm{G}(f) = AT\ sincfT = 2sinc(2f)$$

- $X(f) = \frac{1}{2}\{X(f - f_0) + X(f + f_0)\}$
- $X(f) = \frac{1}{2}\{2sinc\big(2(f - f_0)\big) + 2sinc\big(2(f + f_0)\big)\}$







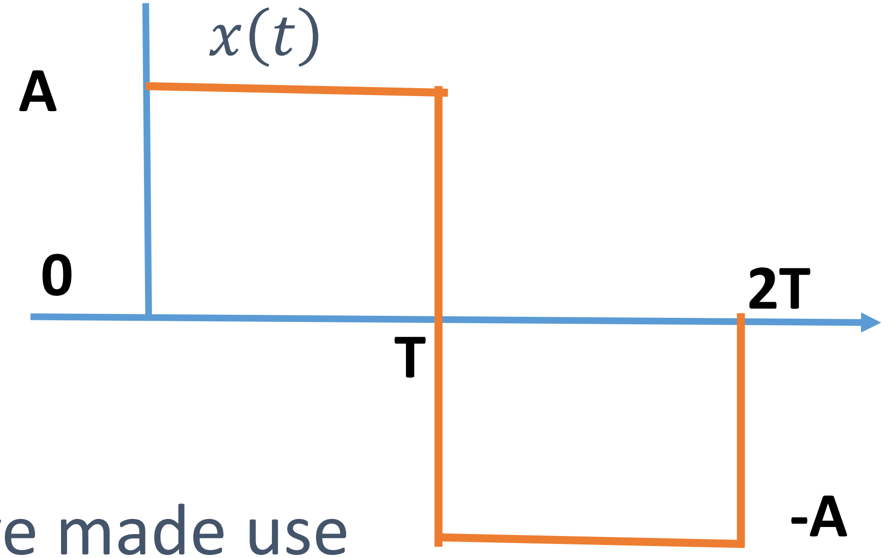
Examples: Fourier Transform of the doublet pulse

Find the Fourier transform of the pulse $x(t)$ shown in the figure

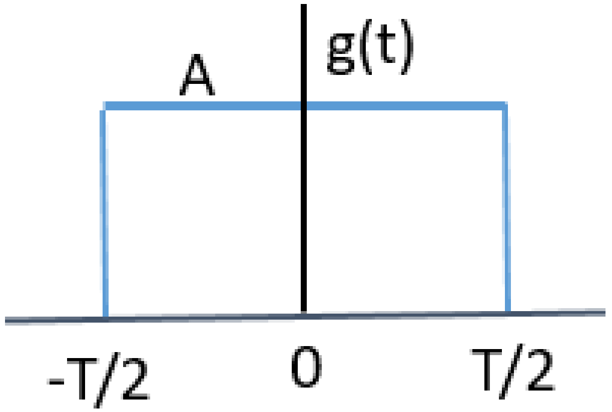
Solution: x(t) can be expressed in terms of the rectangular pulse g(t) as

$$x(t) = g(t - T/2) - g(t - 3T/2)$$

$$X(f) = G(f)e^{-\frac{j2\pi fT}{2}} - G(f)e^{-\frac{j2\pi f3T}{2}}$$



- $X(f) = G(f)e^{-j2\pi fT}(e^{\frac{j2\pi fT}{2}} - e^{-\frac{j2\pi fT}{2}})$
- $X(f) = G(f)e^{-j2\pi fT}(j2)sin(\frac{2\pi fT}{2})$
- Remark: Note that in this example, we have made use of the linearity and time shifting properties.



Examples: Fourier Transform of the triangular pulse

Find the Fourier transform of the pulse $y(t)$ shown in the figure

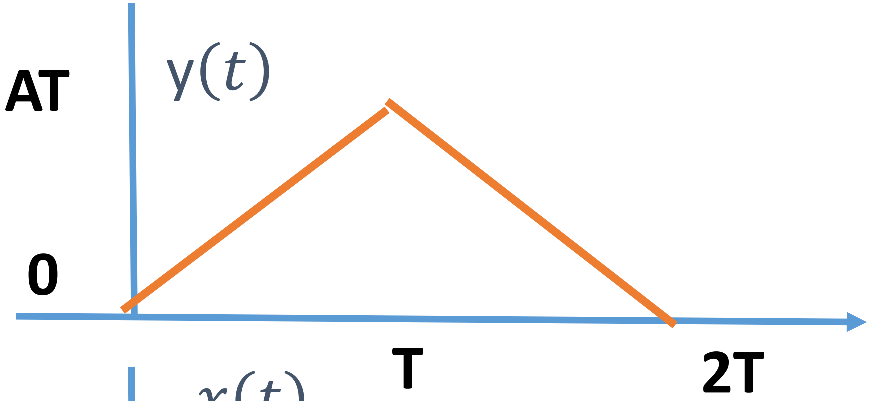
Solution: If we differentiate y(t), we get x(t) of the previous example

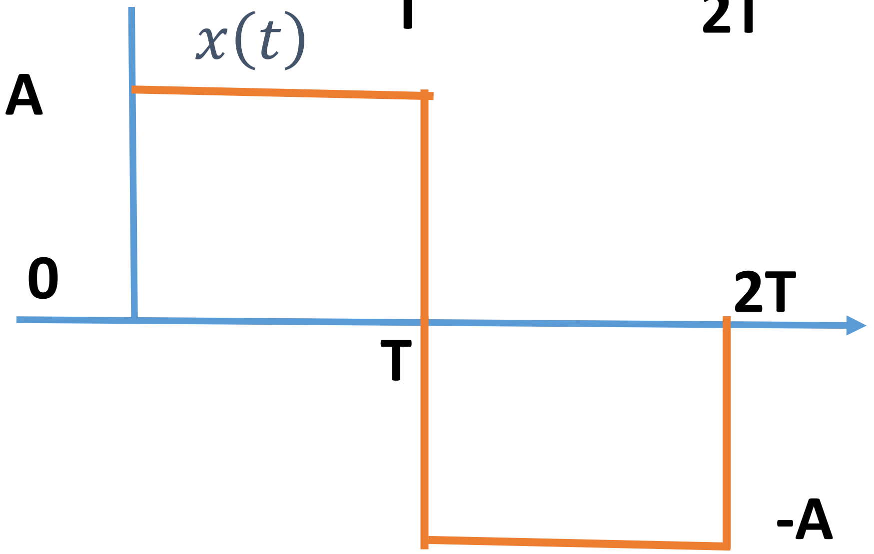
$$\frac{dy(t)}{dt} = x(t). \quad \text{Taking the F.T of both sides,}$$

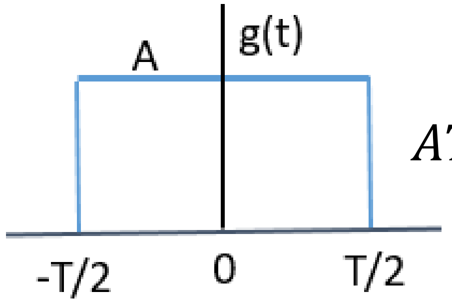
$$j2\pi f Y(f) = X(f)$$

$$Y(f) = \frac{X(f)}{j2\pi f} = \frac{G(f)e^{-j2\pi fT}(j2)\sin(\frac{2\pi fT}{2})}{j2\pi f}$$

$$G(f) = \frac{TG(f)e^{-j2\pi fT}\sin(2\pi fT)}{\pi fT} = AT^2(\mathrm{sinc} fT)^2 e^{-j2\pi fT}$$







$$AT\frac{\sin \pi fT}{\pi fT} \triangleq AT\,\mathrm{sinc} fT$$

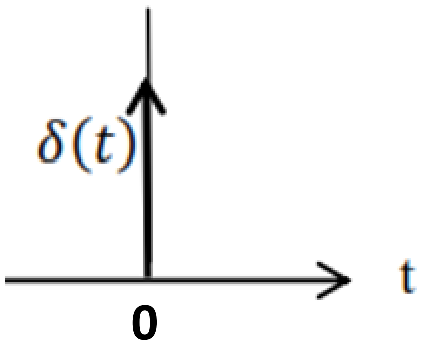
Same result can be obtained by realizing that y(t)=g(t)*g(t) and using the convolution property Y(f)=G(f).G(f) and then using the time shifting property

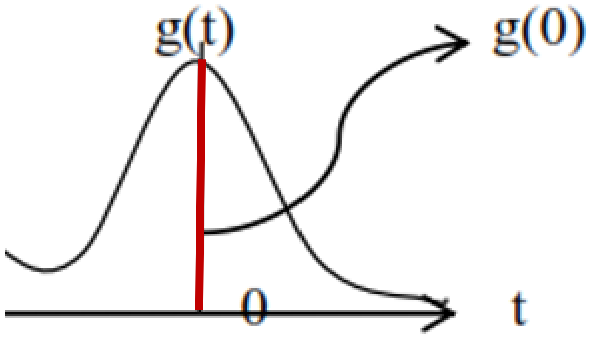
Fourier Transform of Power Signals

- For a **non-periodic (energy) signal** g(t), the Fourier transform exists when
- $$E = \int_{-\infty}^{\infty} |g(t)|^2 \, dt < \infty$$ (sufficient condition for existence)
- so that $G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t} dt$ **exists**
- **For power signals**, the integral $\int_{-\infty}^{\infty} g(t) e^{-j2\pi f t} dt$ **does not exist**.
- However, one can still finds the Fourier transform of power signals by employing the delta function. This function is defined next.
- **Dirac – Delta Function (Impulse Function)**

This function is defined as

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$

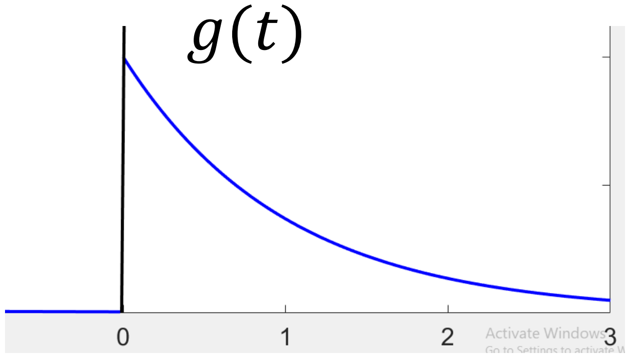


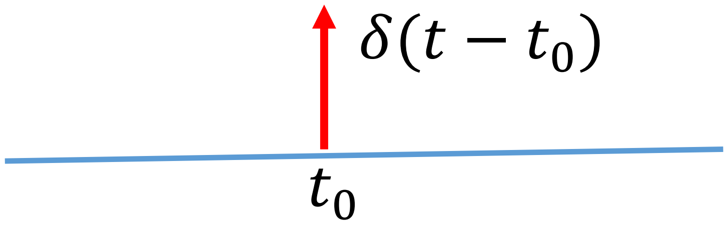


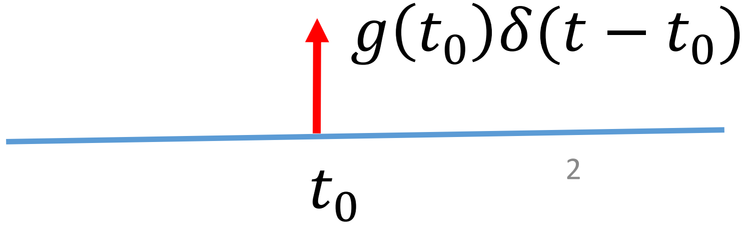
- such that: $\int_{-\infty}^{\infty} \delta(t) dt = 1$ and $\int_{-\infty}^{\infty} g(t)\delta(t) dt = g(0)$
- Here, g(t) is a continuous function of time. The second property, known as the sifting property, shows that the delta function samples the function g(t) at the time of its occurrence.

Some Properties of the Delta Function

- $g(t)\delta(t-t_0) = g(t_0)\delta(t-t_0)$; (Multiplication)
- $\int_{-\infty}^{\infty} g(t)\delta(t-t_0)dt = g(t_0)$; (Sifting or sampling property)
- $\delta(\alpha t) = \frac{1}{|\alpha|}\delta(t)$
- $\delta(t) * g(t) = g(t)$
- $\delta(t) = \frac{du(t)}{dt} \quad \Rightarrow u(t) = \int_{-\infty}^{t} \delta(t)dt$
- $\delta(t) = \delta(-t)$; an even function of its argument.
- Fourier transform: $\Im\{\delta(t)\} = 1$
- $\Im\{\delta(t-t_0)\} = e^{-j2\pi f t_0}$







Applications of the Delta Function

- **Fourier transform of the delta function**
- $\Im\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi f t} dt = 1$. This follows from the sifting property

$$\int_{-\infty}^{\infty} g(t)\delta(t)dt = g(0) = 1$$

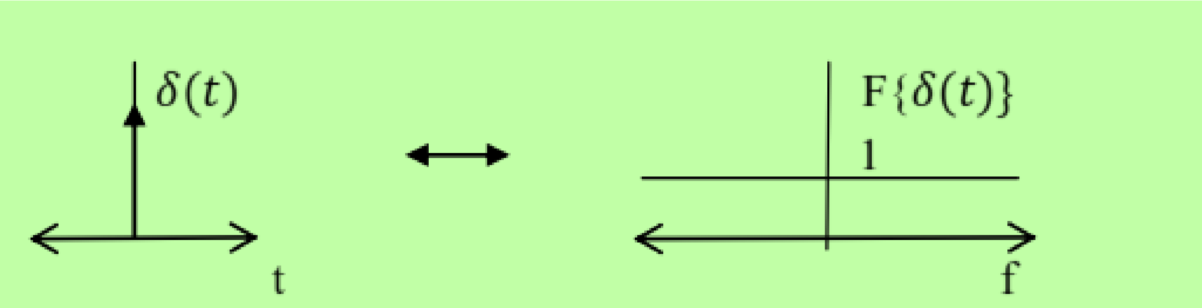
- $\Im\{\delta(t - t_0)\} = e^{-j2\pi f t_0}$; (using the time delay property $\Im\{g(t - t_0)\} = G(f)e^{-j2\pi f t_0}$
- **DC or a Constant Signal**

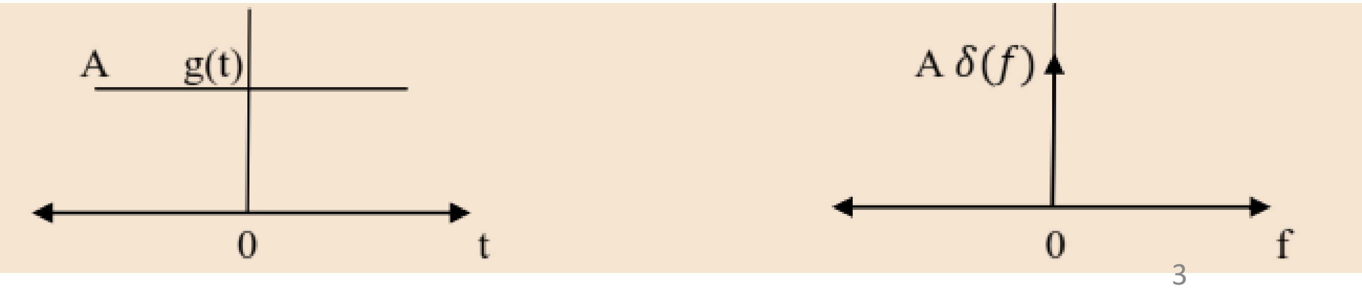
$g(t) \leftrightarrow G(f)$

$G(t) \leftrightarrow g(-f)$

- Since $\Im\{\delta(t)\} = 1$, then by the duality property $\Im\{1\} = \delta(f)$

- Note how the time-bandwidth relationship holds for this pair. A narrow pulse in time extends over a large frequency spectrum).
- Also, the transform of a dc signal is an impulse at f =0.





Applications of the Delta Function

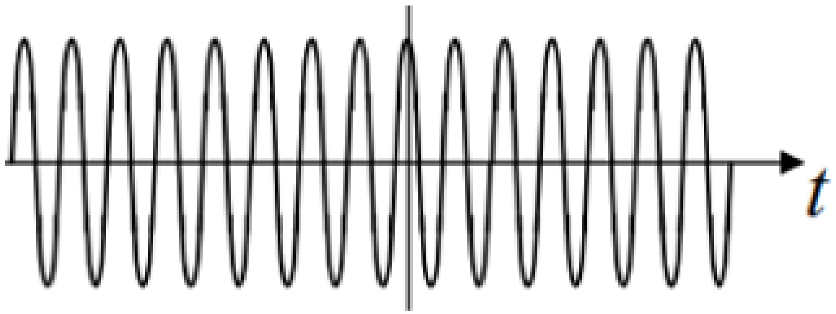
- **Complex Exponential Function**
- $\Im\{Ae^{j2\pi f_c t}\} = A\delta(f - f_c)$;
- follows from the duality property, since $\Im\{\delta(t - t_0)\} = e^{-j2\pi f t_0}$

$\boldsymbol{g(t) \leftrightarrow G(f)}$

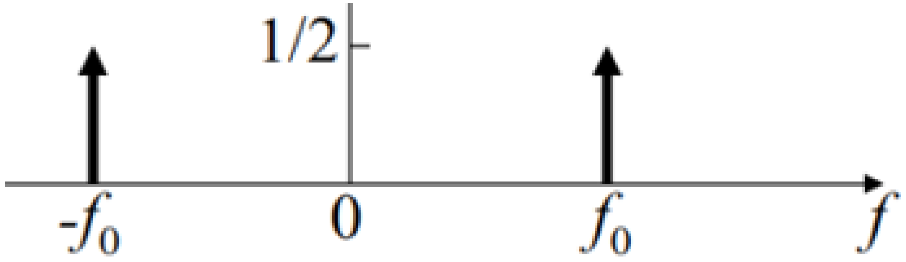
$\boldsymbol{G(t) \leftrightarrow g(-f)}$

- **Sinusoidal Functions**
- $\Im\{\cos 2\pi f_0 t\} = \Im \frac{1}{2}\{Ae^{j2\pi f_c t} + Ae^{-j2\pi f_c t}\} = \frac{1}{2}\{\delta(f - f_0) + \delta(f + f_0)\}$
- $\Im\{\sin 2\pi f_0 t\} = \Im \frac{1}{j2}\{Ae^{j2\pi f_c t} - Ae^{-j2\pi f_c t}\} = \frac{1}{2j}\{\delta(f - f_0) - \delta(f + f_0)\}$

$g(t) = \cos(2\pi f_0 t)$



G(f)



Applications of the Delta Function

- **Signum Function**
-

$$sgn(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$

$$\Im\{sgn(t)\} = \frac{1}{j\pi f}$$

$$v(t) = \begin{cases} e^{-bt} & t > 0 \\ -e^{bt} & t < 0 \end{cases}$$

$$G(f) = \frac{1}{b+j2\pi f} - \frac{1}{b-j2\pi f} = \frac{-j(2)2\pi f}{b^2+(2\pi f)^2}$$

$$\log_{b\to 0} G(f) = \frac{1}{j\pi f}$$

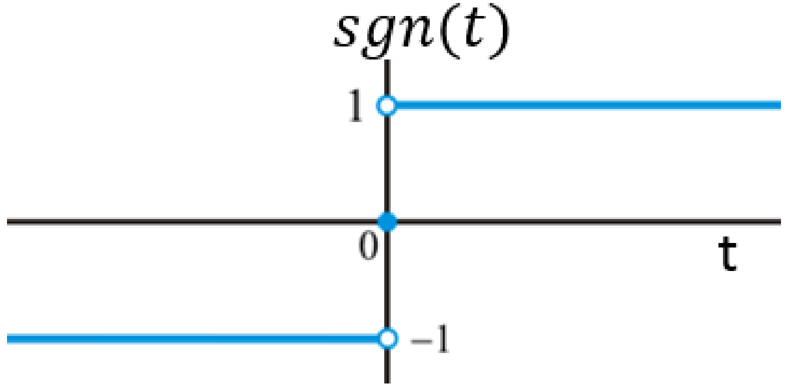
- **Unit Step Function**

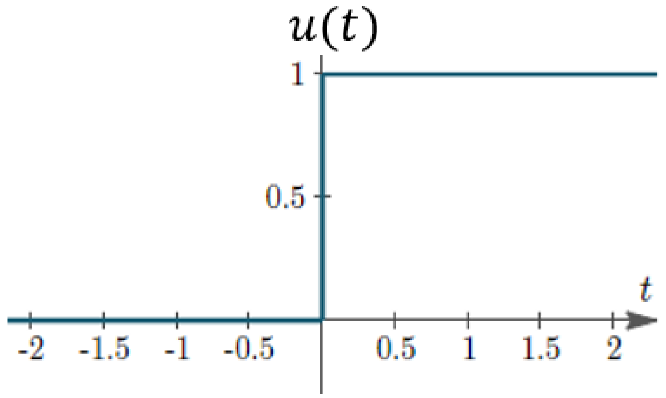
$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases}$$

$$sgn(t) = 2u(t) - 1$$

$$u(t) = \frac{1}{2}\{sgn(t) + 1\}$$

$$\Im\{u(t)\} = \frac{1}{j2\pi f} + \frac{1}{2}\,\delta(f)$$





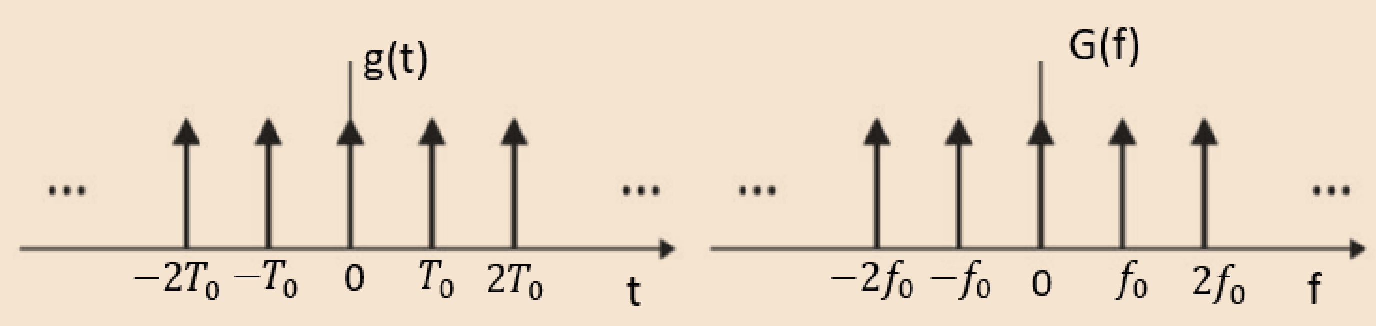
Applications of the Delta Function

- **Periodic Signals:** A periodic signal g(t) is expanded in the complex Fourier Series form as:
- $\boldsymbol{g(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \Rightarrow \mathfrak{I}\{g(t) = \sum_{n=-\infty}^{\infty} C_n\, \delta(f - nf_0)}$

Example: Consider the following train of impulses $g(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT_0)$

Solution: The Fourier coefficients are obtained by integrating over one period of $g(t)$.

- $C_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\, e^{-jn\omega_0 t} dt = \frac{1}{T_0} = f_0$; Note that the sifting property has been used.
- Therefore, the complex Fourier series of $g(t)$ is
- $g(t) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}; \Rightarrow \mathfrak{I}\{g(t)\} = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} \mathfrak{I}\{e^{jn\omega_0 t}\} = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} \delta(f - nf_0)$
- $\boldsymbol{\mathfrak{I} \sum_{m=-\infty}^{\infty} \delta(t - mT_0) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} \delta(f - nf_0)}$.

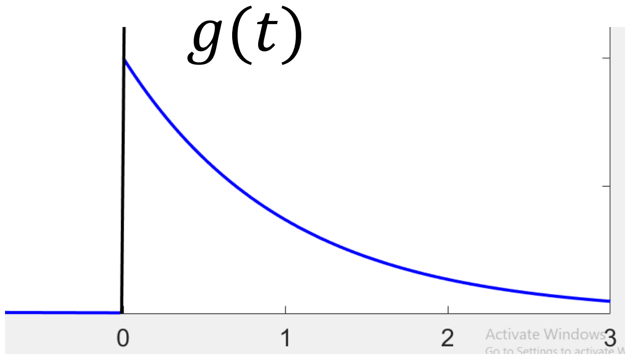


Remark 1: Note that the signal is periodic in the time domain and its Fourier transform is periodic in the frequency domain.

Remark 2: This sequence will be found useful when the sampling theorem is considered later in the course.

Examples

- Let g(t) be given as: $\mathbf{g}(t) = \begin{cases} A\, e^{-bt} & t > 0 \\ 0 & t < 0 \end{cases}.$
- The Fourier transform of g(t) is: $G(f) = \left(\frac{A}{b+j2\pi f}\right)$



- **Evaluate the following**

➢ $\mathbf{g}(t)\delta(t-0.5) = \mathbf{g}(t=0.5)\delta(t-0.5) = A\, e^{-0.5b}\delta(t-0.5).$

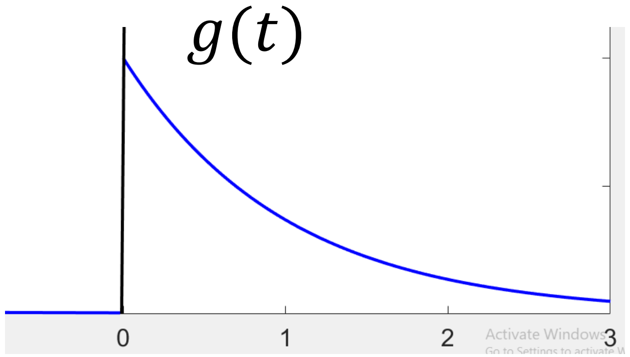
➢ $\mathbf{g}(t)\delta(t+1) = \mathbf{g}(t=-1)\delta(t+1) = (0)\delta(t+1) = 0.$

➢ $\mathbf{g}(t) * \delta(t-1) = g(t-1) = \begin{cases} A\, e^{-b(t-1)} & t > 1 \\ 0 & t < 1 \end{cases}$

➢ $\mathfrak{I}\{\mathbf{g}(t) * \mathbf{g}(t)\} = G(f)G(f) = \left(\frac{A}{b+j2\pi f}\right)\left(\frac{A}{b+j2\pi f}\right) = \left(\frac{A}{b+j2\pi f}\right)^2$

Examples

- Let g(t) be given as: $\mathbf{g}(t) = \begin{cases} A\, e^{-bt} & t > 0 \\ 0 & t < 0 \end{cases}.$



- The Fourier transform of g(t) is: $G(f) = \left(\frac{A}{b+j2\pi f}\right)$
- **Evaluate the following**

➢ $\int_{-\infty}^{\infty} g(t)\delta(t-1)dt = g(t=1) = A\, e^{-b}$; (sifting property)

➢ $\mathfrak{I}\{\mathbf{g}(t) - \mathbf{g}(t-1)\} = \boldsymbol{G}(\boldsymbol{f}) - \boldsymbol{G}(\boldsymbol{f})\boldsymbol{e}^{-j2\pi f} = \frac{A}{b+j2\pi f}(1 - \boldsymbol{e}^{-j2\pi f})$

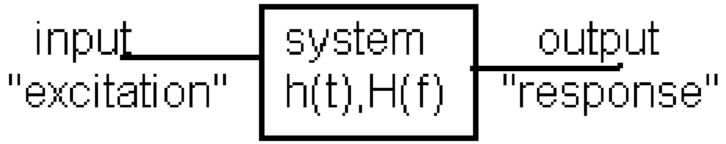
➢ $\mathfrak{I}\{\mathbf{tg}(t)\} = \begin{cases} At\, e^{-bt} & t > 0 \\ 0 & t < 0 \end{cases}$

➢ $\mathfrak{I}\{\mathbf{tg}(t)\} = \frac{\boldsymbol{j}}{\mathbf{2}\boldsymbol{\pi}} \frac{\boldsymbol{dG(f)}}{\boldsymbol{df}} = \frac{\boldsymbol{j}}{\mathbf{2}\boldsymbol{\pi}} \frac{(-)j2\pi}{(b+j2\pi f)^2} = \frac{1}{(b+j2\pi f)^2}$

➢ Note: Prove that $\mathfrak{I}\{\text{tg}(t)\} = \left(\frac{j}{2\pi}\right)\frac{dG(f)}{df}$ and $\mathfrak{I}\left\{\frac{dg(t)}{dt}\right\} = (j2\pi f)G(f)$

Transmission of Signals through Linear Systems

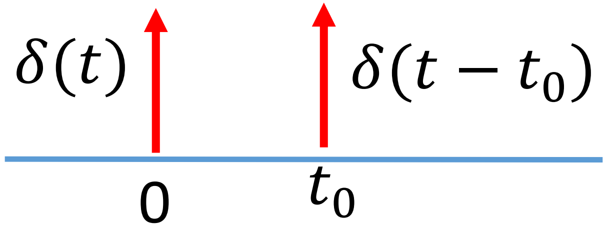
- ***Definition:*** A **system** refers to any physical device that produces an output signal in response to an input signal.
- ***Definition:*** A system is **linear** if the principle of superposition applies.

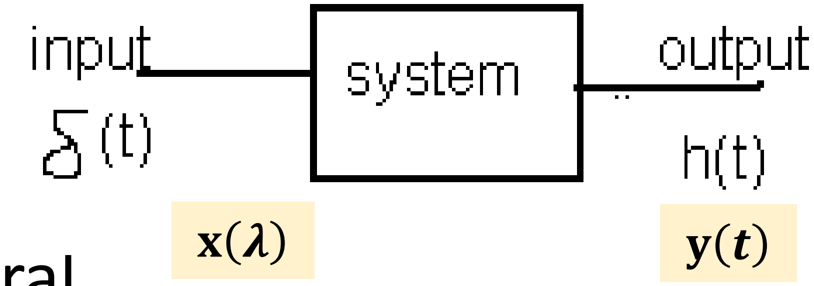


- If $x_1(t)$ produces output $y_1(t)$
- $x_2(t)$ produces output $y_2(t)$
- then $a_1x_1(t)+ a_2x_2(t)$ produces an output $a_1y_1(t)+ a_2y_2(t)$
- Also, a zero input should produce a zero output.
- Examples of linear systems include **filters and communication channels**.
- ***Definition:*** A **filter** refers to a frequency selective device that is used to limit the spectrum of a signal to some band of frequencies (will be discussed in detail in a later lecture)
- ***Definition:*** A **channel** refers to a transmission medium that connects the transmitter and receiver of a communication system.
- Time domain and frequency domain may be used to evaluate system performance.

Basic Time-domain Definitions

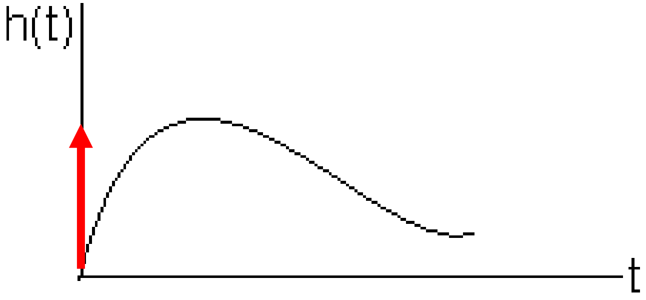
- ***Definition***: The **impulse response h(t)** is defined as the response of a system to an impulse $\delta(t)$ applied to the input at t=0 .
- ***Definition***: A system is **time-invariant** when the shape of the impulse response is the same no matter when the impulse is applied to the system.
- $\delta(t) \rightarrow \mathrm{h}(t)$, then $\delta(\mathrm{t} - t_0) \rightarrow \mathrm{h}(\mathrm{t} - \mathrm{t}_0)$
- When the input to a linear time-invariant system in a signal x(t) , then the output is given by

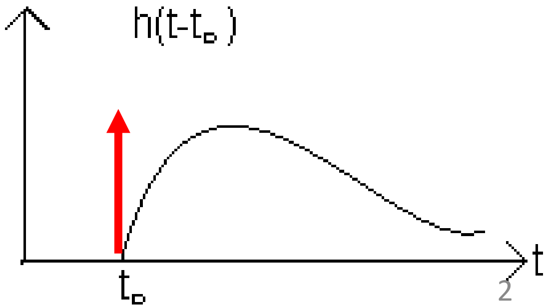




- $\mathbf{y}(\boldsymbol{t}) = \int_{-\infty}^{\infty} \mathbf{x}(\boldsymbol{\lambda})\mathbf{h}(\boldsymbol{t} - \boldsymbol{\lambda})\mathbf{d}\boldsymbol{\lambda}$

  $= \int_{-\infty}^{\infty} \mathbf{h}(\boldsymbol{\lambda})\mathbf{x}(\boldsymbol{t} - \boldsymbol{\lambda})\boldsymbol{d\lambda}$; convolution integral



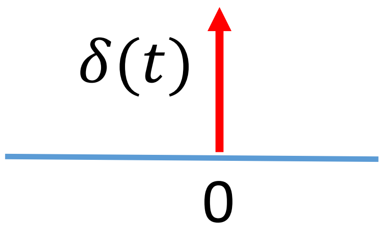


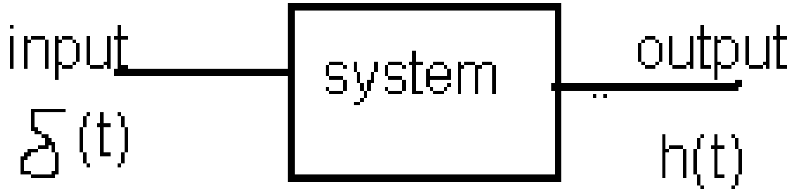
Basic Time-domain Definitions

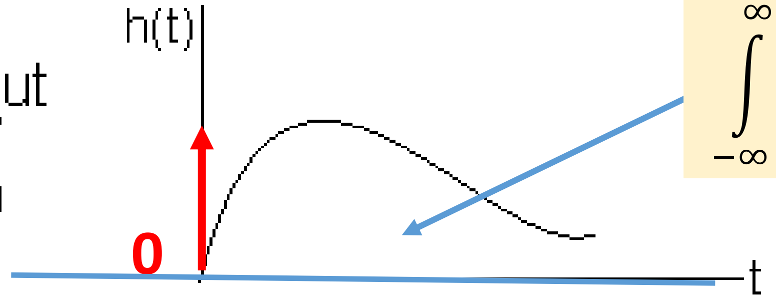
- ***Definition***: A system is said to be **causal** if it does not respond before the excitation is applied, i.e.,
- h(t) = 0 for t < 0 ; the causal system is physically realizable.
- ***Definition***: A system is said to be **stable** if the output signal is bounded for all bounded input signals.

$$\mathbf{y}(t) = \int_{-\infty}^{\infty} \mathbf{h}(\tau)\mathbf{x}(t-\tau)\ \mathbf{d\tau}$$

- If $|\, x(t)\, | \leq M$; M is the maximum value of the input
- then $|\, y(t)\, | \leq \int_{-\infty}^{\infty} |h(\tau)|\,|x(t-\tau)|\, d\tau = M \int_{-\infty}^{\infty} |\, h(\tau)\, |\, d\tau$
- Therefore, a necessary and sufficient condition for stability (a bounded output) is
- $\int_{-\infty}^{\infty} |\mathbf{h}(t)|\ \mathbf{dt} < \infty$; h(t) is absolutely integrable (zero initial conditions assumed)







$$\int_{-\infty}^{\infty} |\mathbf{h}(t)|\ \mathbf{dt} < \infty$$

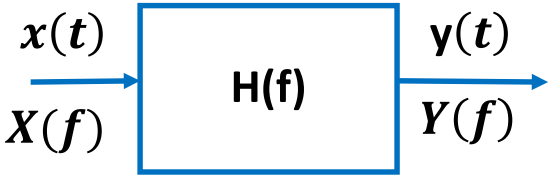
Basic Frequency-domain Definitions

- ***Definition***: The **transfer function** of a linear time invariant system is defined as the Fourier transform of the impulse response h(t)

$$\boldsymbol{H(f) = \mathfrak{I}\{h(t)\}}$$

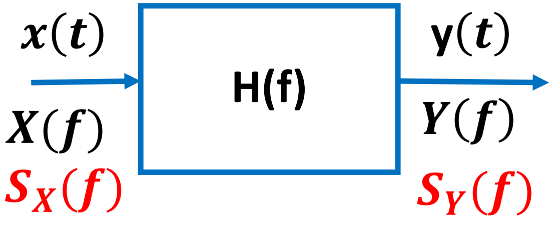
- Since $\boldsymbol{y(t) = x(t) * h(t)}$, then $\boldsymbol{Y(f) = H(f)X(f)}$.
- The system transfer function is thus the ratio of the Fourier transform of the output to that of the input $\boldsymbol{H(f) = \frac{Y(f)}{X(f)}}$
- The transfer function $H(f)$ is a complex function of frequency, which can be expressed as
- $H(f) = |H(f)|e^{j\,\theta(f)}$
- where, $|H(f)|$: Amplitude spectrum

  $\theta(f)$: Phase spectrum.



System input–output energy spectral density

- Let $x(t)$ be applied to a LTI system, then the Fourier transform of the output is related to the Fourier transform of the input through the relation
- $Y(f) = H(f)X(f)$.
- Taking the absolute value and squaring both sides, we get
- $|Y(f)|^2 = |H(f)|^2|X(f)|^2$



$$\boldsymbol{S_Y(f) = |H(f)|^2 S_X(f)}$$

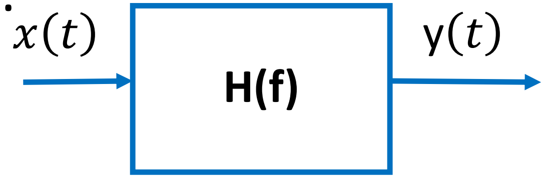
- $S_X(f)$, $S_Y(f)$: Input and output Energy Spectral Density

$$output\ energy\ spectral\ densit = |H(f|^2\ (input\ energy\ spectral\ density)$$

- Total input and output energies
- $E_x = \int_{-\infty}^{+\infty} S_x(f)df = \int_{-\infty}^{+\infty} |X(f)|^2 df$; **Recall Rayleigh Energy Theorem**
- $E_y = \int_{-\infty}^{+\infty} S_Y(f)df = \int_{-\infty}^{+\infty} |\mathrm{H(f)}|^2 S_X(f)df$

Example: Response of a LPF filter to a sinusoidal input

- **Example**: The signal $x(t) = \cos(2\pi f_0 t), -\infty < t < \infty$, is applied to a filter described by the transfer function $H(f) = \frac{1}{1+jf/B}$, B is the 3-dB bandwidth. Find the filter output $y(t)$.
- **Solution**: Here, we will find the output using the frequency domain approach.



- $Y(f) = H(f)X(f), \;\; H(f) = \frac{1}{\sqrt{1+(\frac{f}{B})^2}} e^{-j\theta}; \;\; \theta = \tan^{-1}\frac{f}{B}; \;\; \theta_0 = \tan^{-1}\frac{f_0}{B}$
- $Y(f) = H(f)[\frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0), \;\Rightarrow Y(f) = \frac{1}{2}H(f_0)\delta(f - f_0) + \frac{1}{2}H(-f_0)\delta(f + f_0)$
- $Y(f) = \frac{1}{2}\frac{1}{\sqrt{1+(\frac{f_0}{B})^2}} e^{-j\theta_0}\delta(f - f_0) + \frac{1}{2}\frac{1}{\sqrt{1+(\frac{f_0}{B})^2}} e^{j\theta_0}\delta(f + f_0)$

$\boldsymbol{g(t)\delta(t - t_0) = g(t_0)\delta(t - t_0);}$

- Taking the inverse Fourier transform, we get
- $y(t) = \frac{1}{\sqrt{1+(\frac{f_0}{B})^2}} \frac{1}{2}[e^{j(2\pi f_0 t - \theta_0)} + e^{-j(2\pi f_0 t - \theta_0)}]$, $\quad \boldsymbol{y(t) = \frac{1}{\sqrt{1+(\frac{f_0}{B})^2}} cos(2\pi f_0 t - tan^{-1}\frac{f_0}{B})}$
- Note that in the last step we have made use of the Fourier transform pair $e^{j2\pi f_0 t} \leftrightarrow \delta(f - f_0)$
- **Remark**: Note that the amplitude of the output as well as its phase depend on the frequency of the input, f_0, and the bandwidth of the filter, B.

Response of a LPF to a sum of two sinusoidal signals

- **Example**: The signal $x(t) = \, cos\, w_0 t - \frac{1}{\pi} cos\, 3w_0 t$ is applied to a filter described by the transfer function $H(f) = \frac{1}{1+jf/B}$. Use the result of the previous example to find the filter output $y(t)$.
- **Solution**: From the previous example, we have
- $\cos(2\pi f_0 t) \;\rightarrow \frac{1}{\sqrt{1+(\frac{f_0}{B})^2}} \cos(2\pi f_0 t - \tan^{-1}\frac{f_0}{B})$
- Therefore, using linearity property
- $cos\, w_0 t - \frac{1}{\pi} cos\, 3w_0 t \rightarrow$
- $\frac{1}{\sqrt{1+\left(\frac{f_0}{B}\right)^2}} \cos\left(2\pi f_0 t - \tan^{-1}\frac{f_0}{B}\right) \quad - \frac{1}{\pi} \frac{1}{\sqrt{1+\left(\frac{3f_0}{B}\right)^2}} \cos\left(2\pi 3 f_0 t - \tan^{-1}\frac{3f_0}{B}\right)$

Example: Response of a LPF to a periodic square pulse

- **Example**: Consider the periodic rectangular signal $g(t)$ defined over one period T_0 as

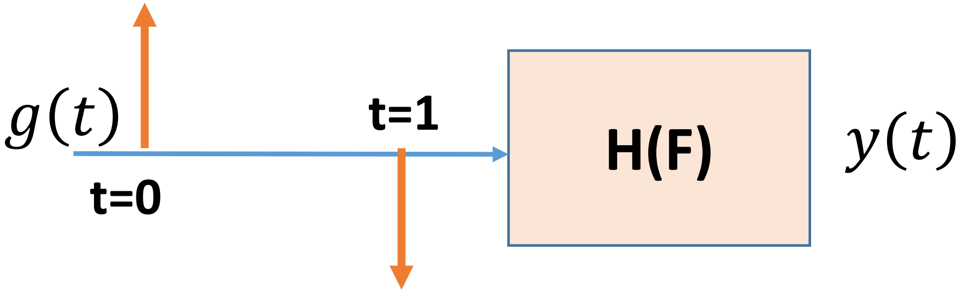
$$g(t) = \begin{cases} +A, & -T_0/4 \le t \le T_0/4 \\ 0, & otherwise \end{cases}.$$

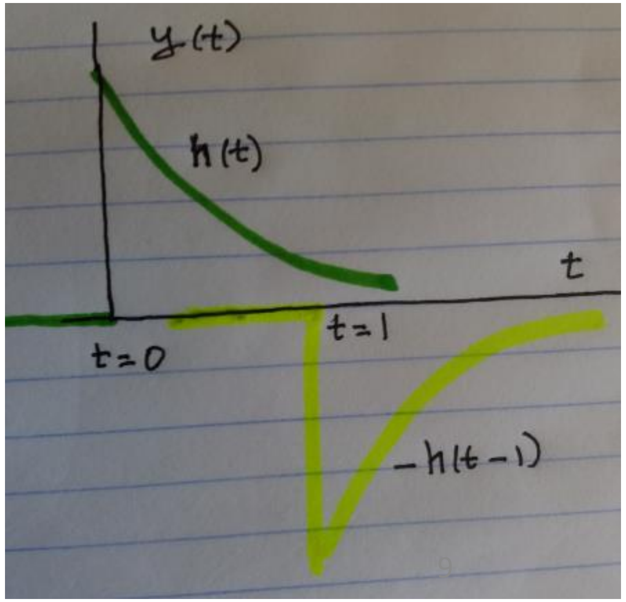
- If $g(t)$ is applied to a filter described by the transfer function $H(f) = \frac{1}{1+jf/B}$. use the result of the previous example to find the filter output $y(t)$.
- **Solution**: The Fourier series of g(t) is:
- $g(t) = \frac{A}{2} + \frac{2A}{\pi}\{\cos(2\pi f_0 t) - \frac{1}{3}\cos(2\pi 3 f_0 t) + \frac{1}{5}\cos(2\pi 5 f_0 t) - \frac{1}{7}\cos(2\pi 7 f_0 t)$
- Using the result of the previous example:
- $y(t) = \frac{A}{2} + \frac{2A}{\pi}\frac{1}{\sqrt{1+\left(\frac{f_0}{B}\right)^2}}\cos\left(2\pi f_0 t - \tan^{-1}\frac{f_0}{B}\right)$

$$-\frac{2A}{\pi}\frac{1}{3}\frac{1}{\sqrt{1+\left(\frac{3f_0}{B}\right)^2}}\cos\left(2\pi 3 f_0 t - \tan^{-1}\frac{3f_0}{B}\right) \quad +\ldots$$

Transmission of Signals through Linear Systems: A Convolution Example

- **Example:** The signal $g(t) = \delta(t) - \delta(t-1)$ is applied to a channel described by the transfer function $H(f) = \frac{1}{1+jf/B}$. Use the convolution integral to find the channel output.
- **Solution:** The impulse response of the channel is obtained by taking the inverse Fourier transform of $H(f)$, which is $h(t) = 2\pi B e^{-2\pi B t} u(t)$
- Using the linearity and time invariance property, the output can be obtained as
- $y(t) = h(t) * [\delta(t) - \delta(t-1)]$; $\quad y(t) = h(t) - h(t-1)$
- $y(t) = 2\pi B[e^{-2\pi B t}u(t) - e^{-2\pi B(t-1)}u(t-1)]$



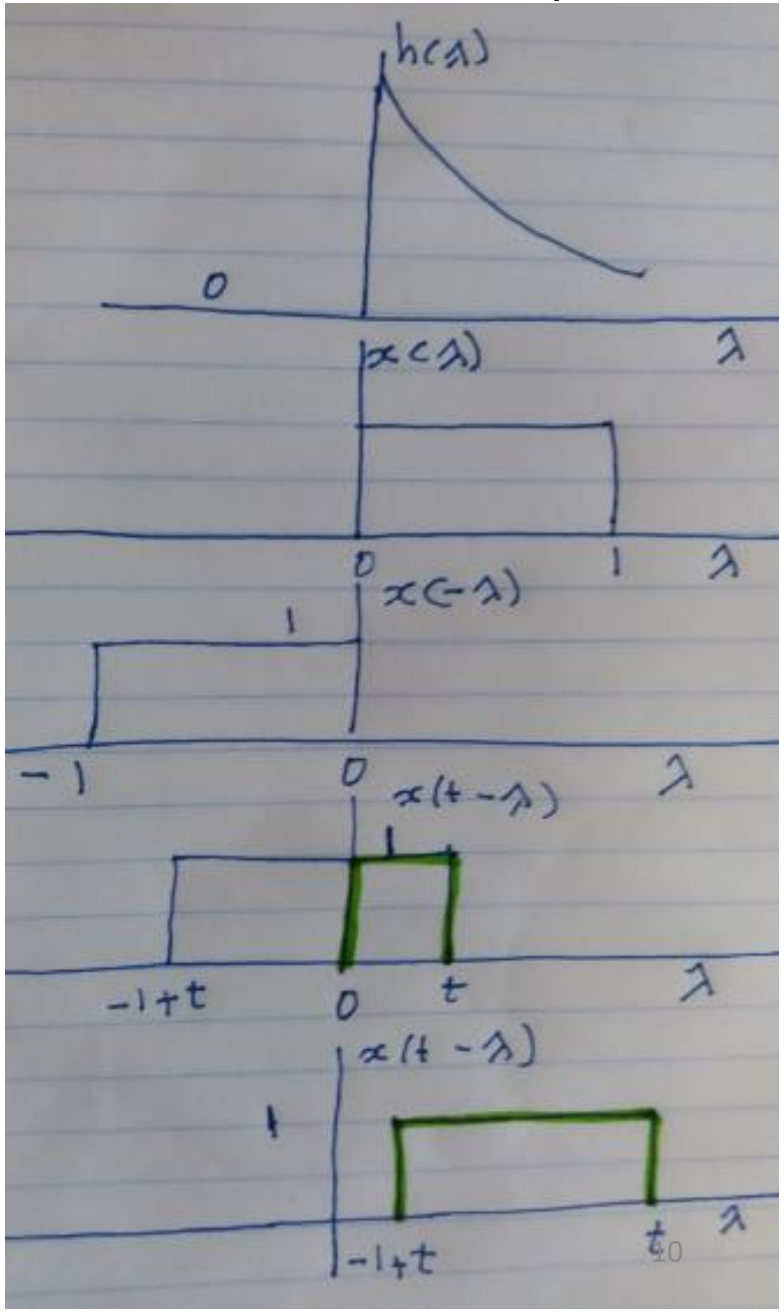


Transmission of Signals through Linear Systems: A Convolution Example

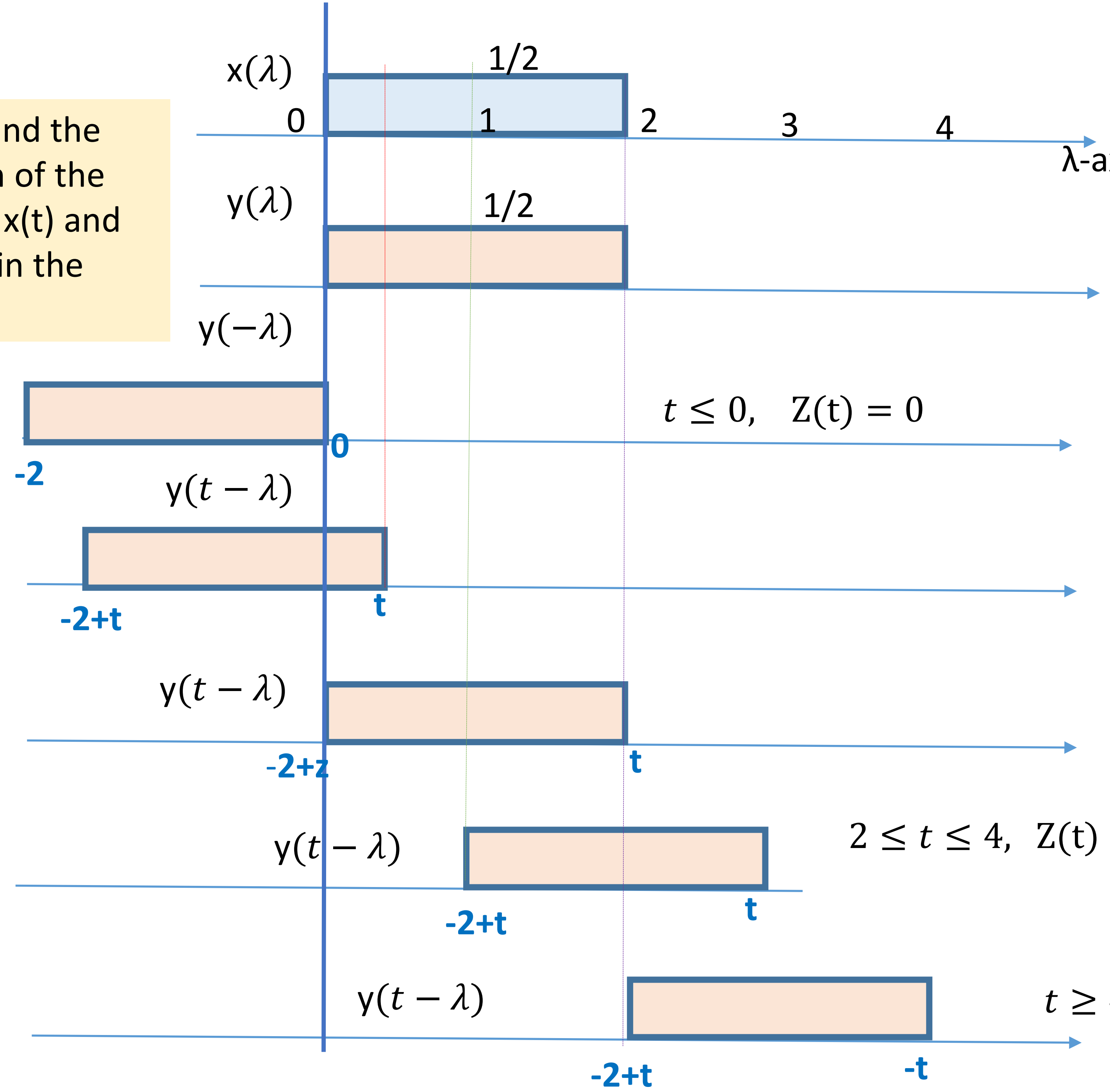
- **Example: channel response due to a rectangular pulse**
- The signal $x(t) = u(t) - u(t-1)$ is applied to a channel described by the transfer function $H(f) = \frac{1}{1+jf/B}$. Find the channel output $y(t)$.
- **Solution:** The impulse response of the channel is:
- $h(t) = 2\pi B e^{-2\pi B t} u(t)$
- The output is the convolution

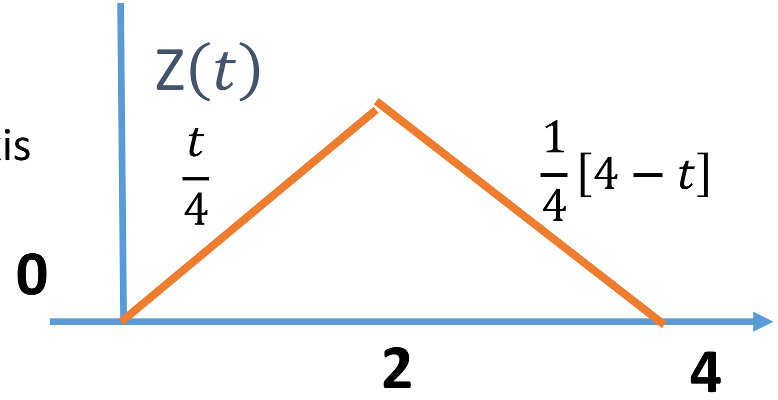
$$\mathbf{y}(\boldsymbol{t}) = \int_{-\infty}^{\infty} \mathbf{h}(\boldsymbol{\lambda})\mathbf{x}(\boldsymbol{t}-\boldsymbol{\lambda})\ \mathbf{d}\boldsymbol{\lambda}$$

- $y(t) = h(t) * [u(t) - u(t-1)]$. The answer is
- $\mathrm{y(t)} = \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)\,\mathrm{d}\lambda$
- $y(t) = 0\ for\ t < 0$
- $\mathrm{y(t)} = \int_0^t 2\pi B e^{-2\pi B\lambda}\,\mathrm{d}\lambda = 1 - e^{-2\pi B t}$, for $0 \le t < 1$
- $\mathrm{y(t)} = \int_{-1+t}^{t} 2\pi B e^{-2\pi B\lambda}\,\mathrm{d}\lambda = (e^{2\pi B} - 1)e^{-2\pi B t}$, for $t \ge 1$



Example: Find the convolution of the two signals x(t) and y(t) shown in the figure.





$$\mathbf{Z(t)} = \int_{-\infty}^{\infty} \boldsymbol{x}\,(\boldsymbol{\lambda})\,\mathbf{y}(\mathbf{t}-\boldsymbol{\lambda})\,\boldsymbol{d\lambda}$$

$$t \le 0, \quad \mathrm{Z(t)} = 0$$

$$0 \le t \le 2, \qquad \mathrm{Z(t)} = \int_0^t \frac{1}{2} \times \frac{1}{2}\ d\lambda = \frac{t}{4}$$

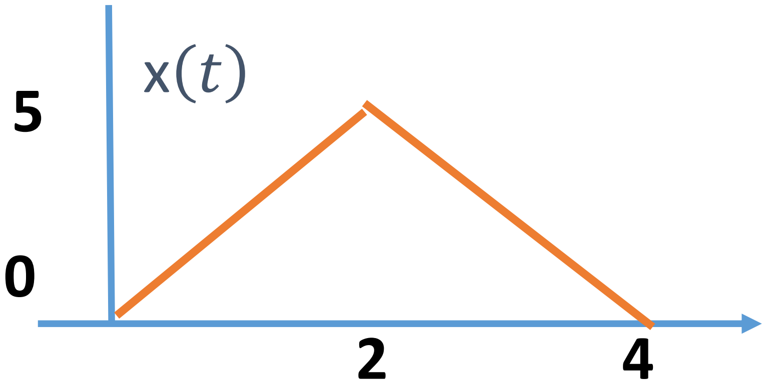
$$t = 2, \quad \mathrm{Z(t)} = \int_0^2 \frac{1}{2} \times \frac{1}{2}\ d\lambda = \frac{1}{2}$$

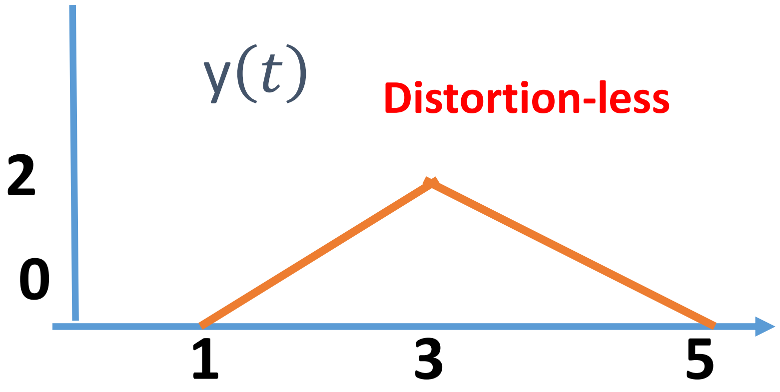
$$2 \le t \le 4, \quad \mathrm{Z(t)} = \int_{-2+t}^{2} \frac{1}{2} \times \frac{1}{2}\ d\lambda = \frac{\lambda}{4}\Big|_{-2+t}^{2} = \frac{1}{4}[4-t]$$

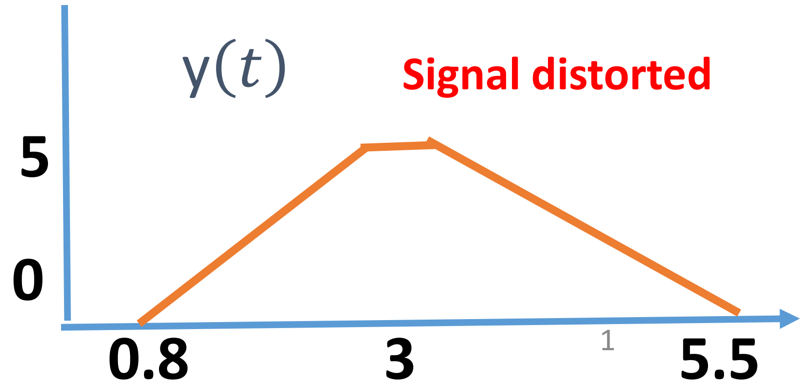
$$t \ge 4, \quad \mathrm{Z(t)} = 0$$

Signal Distortion in Transmission

- The objective of a communication system is to deliver to the receiver almost an exact copy of what the source generates.
- However, communication channels are not perfect in the sense that impairments on the channel will cause the received signal to differ from the transmitted one. During the course of transmission, the signal undergoes **attenuation**, **phase delay**, **interference** from other transmissions, **Doppler shift** in the carrier frequency, **AWGN**, and many other effects.
- **In this lecture, we consider the conditions for a distortion-less transmission over a channel. In addition, we consider linear and non-linear distortion**
- **Distortion-less Transmission**: A signal transmission is said to be *distortion-less* if the output signal y(t) is an exact replica of the input signal x(t) , i.e., y(t) has the same shape as the input, except for a constant amplification (or attenuation) and a constant time delay.







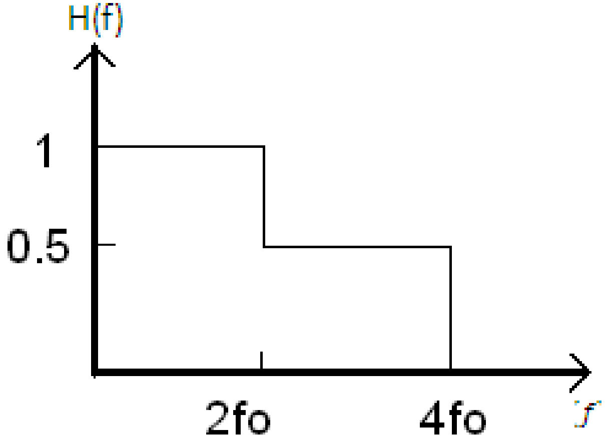
Signal Distortion in Transmission

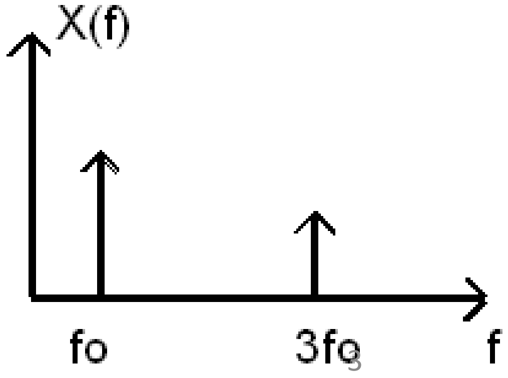
- Condition for **distortion-less transmission in the time-domain:**
- $\boldsymbol{y(t) = kx(t - t_d)}$; where k is a constant amplitude scaling, t_d is a constant time delay.
- **In the frequency domain**, the condition for a distortion-less transmission becomes
- $Y(f) = kX(f)e^{-j2\pi f t_d}$ or $\boldsymbol{H(f) = \frac{Y(f)}{X(f)} = ke^{-j2\pi f t_d} = ke^{-j\theta(f)}}$
- That is, for a distortion-less transmission, the transfer function should satisfy two conditions:
- **|H(f)| = k** ; The magnitude of the transfer function is constant (gain or attenuation) over the frequency range of interest.
- $\boldsymbol{\theta(f) = -2\pi f t_d = -(2\pi t_d)f}$; The phase function is linear in frequency with a negative slope that passes through the origin (or multiples of π).
- When |H(f)| is not constant for all frequencies of interest, *amplitude distortion* results.
- When $\theta(f) \neq -2\pi f t_d \pm 180^0$, then we have *phase distortion* (or delay distortion).
- The following examples demonstrate the two types of distortion mentioned above.

Example: amplitude distortion

- Consider the signal $x(t) = \cos w_0 t - \frac{1}{3} \cos 3w_0 t$. If this signal passes through a channel with zero time delay (i.e., $t_d = 0$) and amplitude spectrum as shown in the figure
- Find y(t)
- Is this a distortion-less transmission?
- <u>**Solution:**</u> x(t) consists of two frequency components, f_0 and $3f_0$. Upon passing through the channel, each component will be scaled by a different factor.
- $y(t) = (\mathbf{1})\cos w_0 t - (\frac{\mathbf{1}}{\mathbf{2}}) \cdot \frac{1}{3} \cos 3w_0 t$
- Since $y(t) = \left(\cos w_0 t - \frac{1}{2} \cdot \frac{1}{3} \cos 3w_0 t\right) \neq k\left(\cos w_0 t - \frac{1}{3} \cos 3w_0 t\right)$
- then this is not a distortion-less transmission.

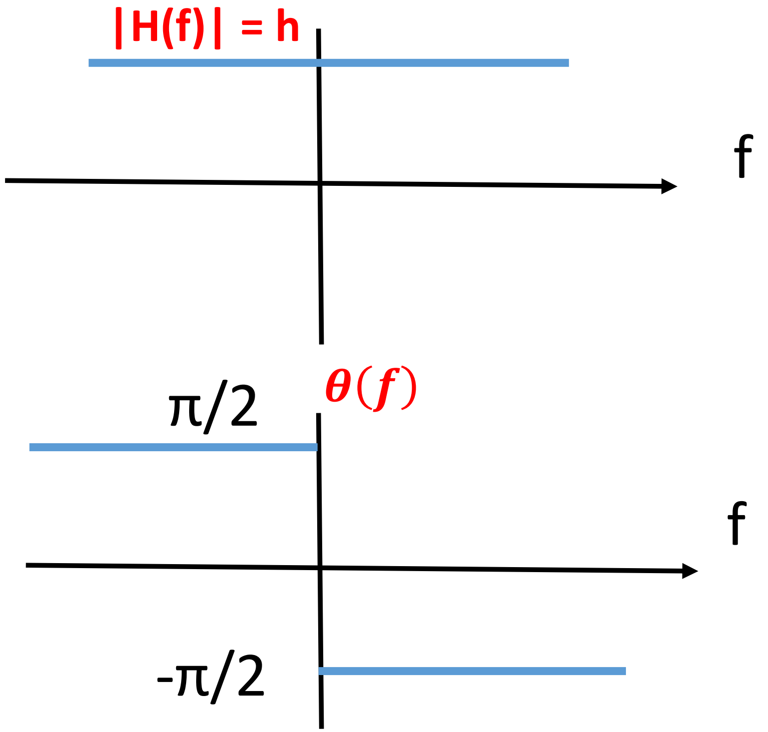
In this figure, only the positive part of the spectrum is shown





Example: phase distortion

- Consider the signal $x(t) = \ cos\ w_0 t\ - \frac{1}{3}\ cos\ 3w_0 t$. If x(t) passes through a channel whose amplitude spectrum is a constant h. Each component in x(t) suffers a $-\frac{\pi}{2}$ phase shift.
- Find y(t).
- Is this a distortion-less transmission?
- <u>**Solution:**</u>
- $x(t) = \cos w_o t - \frac{1}{3} \cos 3w_o t$
- $y(t) = \text{h}\ \cos(w_o t - \frac{\pi}{2}) - \frac{1}{3}\ h\ \cos\left(3w_o t - \frac{\pi}{2}\right)$
- $y(t) = \text{h}\ \cos w_o (t - \frac{\pi}{2w_o}) - \frac{1}{3}\ h\ \cos\left(3w_o (t - \frac{\pi}{2x3w_o})\right)$
- $y(t) = \text{h}\ \cos w_o (t - t_{d1}) - \frac{1}{3}\ h\ \cos(3w_o (t - t_{d2}))$
- Since $t_{d1} \neq t_{d2}$, we cannot write $y(t) = kx(t - t_d)$. Here, each component in $x(t)$ suffers from a different time delay. Hence, this transmission introduces phase (delay) distortion.



Example: Amplitude and Phase Distortion

- **Example**: The signal $x(t) = \ cos\ w_0 t - \frac{1}{\pi} cos\ 3w_0 t$ is applied to a filter described by the transfer function $H(f) = \frac{1}{1+jf/B}$. Use the result of the previous example to find the filter output $y(t)$.
- **Solution**: From the previous example, we have
- $\cos(2\pi f_0 t) \ \rightarrow \frac{1}{\sqrt{1+(\frac{f_0}{B})^2}} \cos(2\pi f_0 t - \tan^{-1}\frac{f_0}{B})$
- Therefore, using linearity property
- $cos\ w_0 t - \frac{1}{\pi} cos\ 3w_0 t \rightarrow$
- $\frac{1}{\sqrt{1+\left(\frac{f_0}{B}\right)^2}} \cos\left(2\pi f_0 t - \tan^{-1}\frac{f_0}{B}\right) \quad - \frac{1}{\pi} \frac{1}{\sqrt{1+\left(\frac{3f_0}{B}\right)^2}} \cos\left(2\pi 3 f_0 t - \tan^{-1}\frac{3f_0}{B}\right)$
- Note that we cannot write $y(t) = kx(t - t_d)$. Here, each component in $x(t)$ suffers from a different amplitude attenuation and a different time delay. Hence, this transmission introduces both amplitude and phase distortion.

Nonlinear distortion

- When a system contains nonlinear elements, it is **not** described by a transfer function $H(f)$, but rather by a transfer characteristic of the form
- $y(t) = a_1 x(t) + a_2 x^2(t) + a_3 x^3(t) + \dots$ (time domain)
- In the frequency domain,
- $Y(f) = a_1 X(f) + a_2 X(f)*X(f) + a_3 X(f)*X(f)*X(f) + \dots$
- Here, the output contains new frequencies not originally present in the original signal. The nonlinearity produces undesirable frequency component for $|f| \leq W$, in which W is the signal bandwidth.

Harmonic distortion in nonlinear systems

- Let the input to a nonlinear system be the single tone signal $\mathrm{x(t)} = \cos(2\pi f_0 t)$.
- This signal is applied to a channel with characteristic $y(t) = a_1x(t) + a_2x(t)^2 + a_3x(t)^3$;
- $y(t) = a_1 \cos(2\pi f_0 t) + a_2(\cos(2\pi f_0 t))^2 + a_3(\cos(2\pi f_0 t))^3$;
- upon substituting x(t) and arranging terms, we get
- $y(t) = \frac{1}{2}a_2 + \left(a_1 + \frac{3}{4}a_3\right)\cos 2\pi f_0 t + \frac{1}{2}\ a_2 \cos 4\pi f_0 t + \frac{1}{4}a_3 cos6\pi\ f_0 t$
- Note that the output contains a component proportional to x(t), which is
- $\left(a_1 + \frac{3}{4}a_3\right)\cos 2\pi f_0 t$, in addition to a second and a third harmonic terms (terms at twice and three times the frequency of the input).
- These new terms are the result of the nonlinear characteristic and are, therefore, considered as harmonic distortion. The DC term does not constitute a distortion, for it can be removed using a blocking capacitor.
- Note: Use was made of the inequalities $Cos^2x = \frac{1}{2}\{1 + cos2x\}$; $Cos^3x = \frac{1}{4}\{3cosx + cos3x\}$.

Harmonic distortion in nonlinear systems

- Let the input to a nonlinear system be the single tone signal
- $y(t) = a_1x(t) + a_2x(t)^2 + a_3x(t)^3$; $x(t) = \cos(2\pi f_0 t)$;
- $y(t) = \frac{1}{2}a_2 + \left(a_1 + \frac{3}{4}a_3\right)\cos 2\pi f_0 t + \frac{1}{2}a_2 \cos 2(2\pi f_0 t) + \frac{1}{4}a_3 cos3(2\pi f_0 t)$

- Define the second harmonic distortion
- $D_2 = \frac{|amplitude\ of\ second\ harmonic\ |}{|amplitude\ of\ fundamental\ term|}$; $D_2 = \frac{|\frac{1}{2}a_2\ |}{|\left(a_1+\frac{3}{4}a_3\right)|}\ x\ 100$
- In a similar way, we can define the third harmonic distortion as:
- $D_3 = \frac{|amplitude\ of\ third\ harmonic\ |}{|amplitude\ of\ fundamental\ term|}$; $D_3 = \frac{|\frac{1}{4}a_3\ |}{|\left(a_1+\frac{3}{4}a_3\right)|}\ x\ 100\%.$