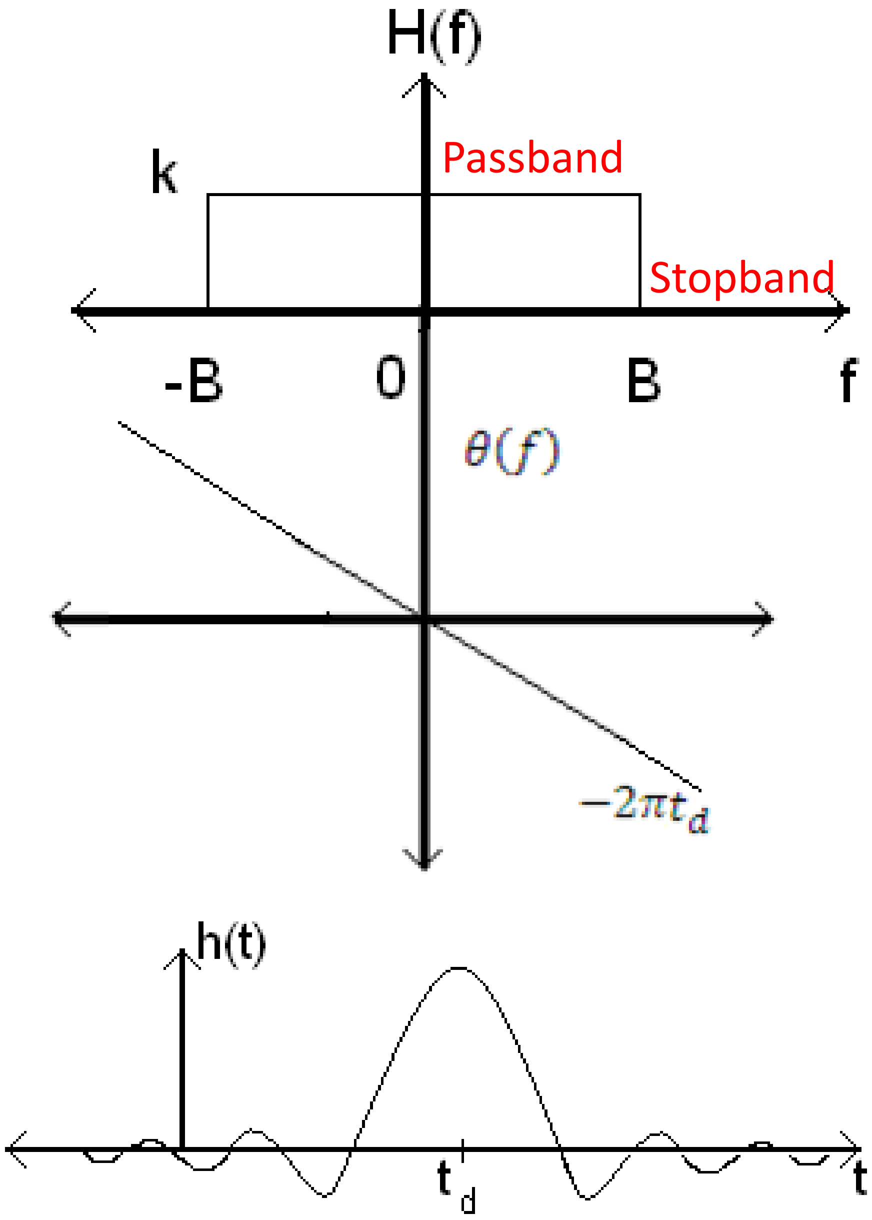
Introduction to Filters

- A filter is a frequency selective device. It allows certain frequencies to pass almost without attenuation while it suppresses other frequencies.
- Filters are an integral part of any communication system

Ideal Filters

Ideal low pass filter

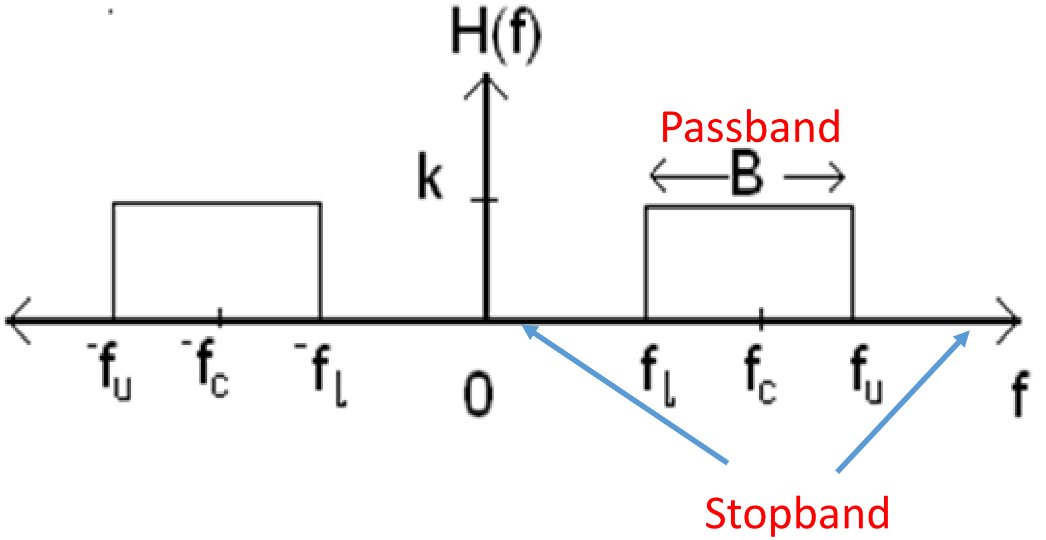
- $H(f) = \begin{cases} k\, e^{-j2\pi f t_d} & |f| < B \\ 0 & o.w \end{cases}$; B is the bandwidth
- The transfer function satisfies the condition for the distortion-less transmission (constant channel gain and linear phase shift with negative slope)
- $h(t) = 2Bk\ sinc 2B(t - t_d)$
- Since h(t) is the response to an impulse applied at t=0, and because h(t) has nonzero values for t < 0 , the filter is *non-causal* (physically non realizable).



Filters and Filtering

Ideal band-pass filter

- $H(f) = \begin{cases} k\, e^{-j2\pi f t_d} & f_l < |f| < f_u \\ 0 & o.w \end{cases}$
- Filer bandwidth B = $f_u - f_l$; difference between upper and lower positive frequencies
- $f_c = \frac{f_u + f_l}{2}$; Center frequency of the filter



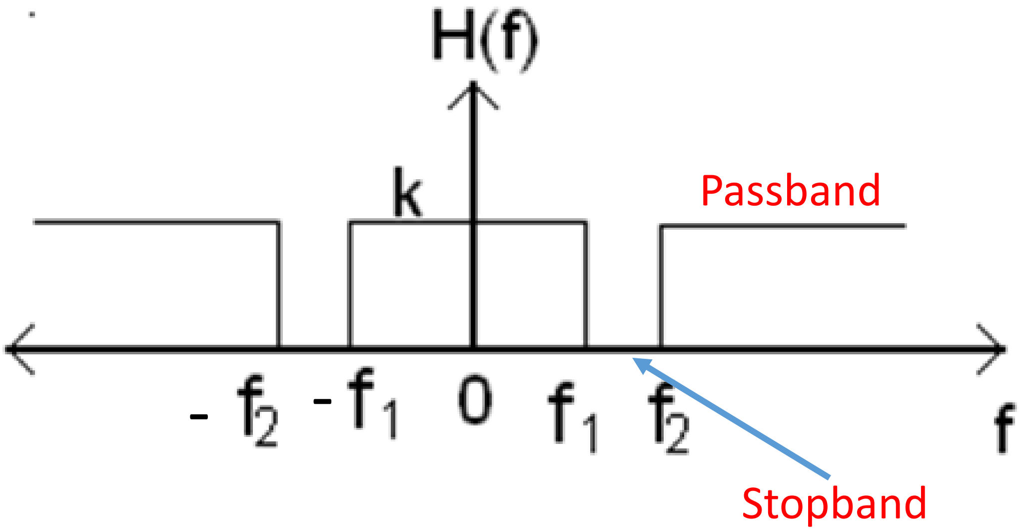
impulse response:

- $h(t) = 2Bk\, sincB(t - t_d) cos\, w_c(t - t_d)$

Filters and Filtering

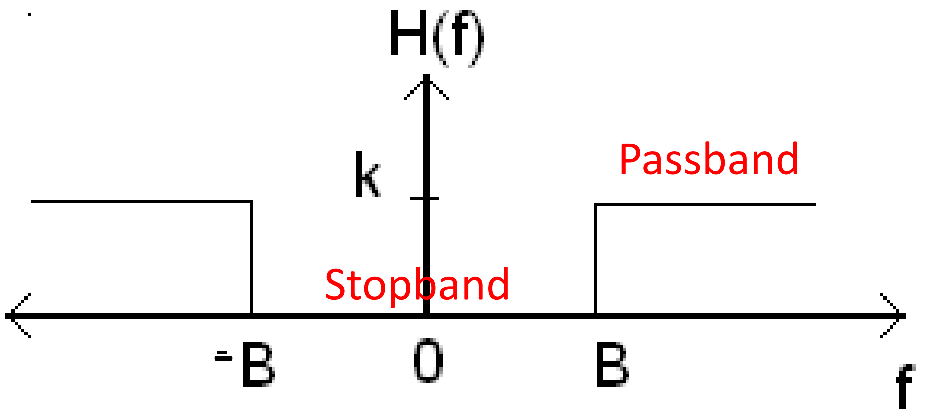
Band rejection or notch filter

- $$H(f) = \begin{cases} k\, e^{-j2\pi f t_d} & o.w \\ 0 & f_1 < |f| < f_2 \end{cases}.$$



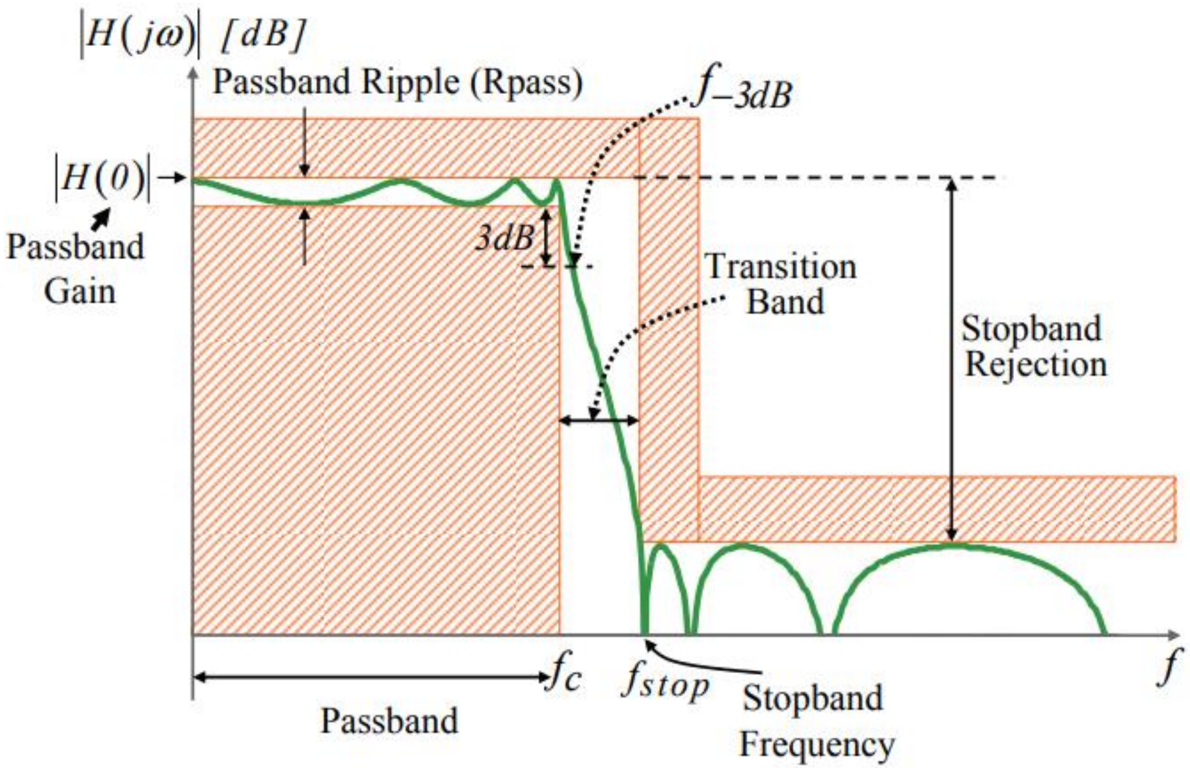
High-pass filter

- $$H(f) = \begin{cases} k\, e^{-j2\pi f t_d} & |f| > B \\ 0 & o.w \end{cases}$$



Real filters

- Ideal filter do not exist in practice, but are used to simplify the analysis of the system
- For a real filter, there are three frequency bands
  - Passband
  - Transition band
  - Stopband (rejection)
- There are several specifications that dictate the filter order
  - The passband edge frequency and the maximum allowable attenuation (ripple) within the passband
  - The 3-dB cutoff frequency.
  - The minimum required attenuation at the edge of the stopband and the desired stopband frequency.



Example: Let B be the 3-dB bandwidth
- **- 1dB at f = 0.9 B,**
- **-30 dB at f = 1.6 B**
- **The designer then finds the order of the filter that meets these specifications and then realizes that filter.**

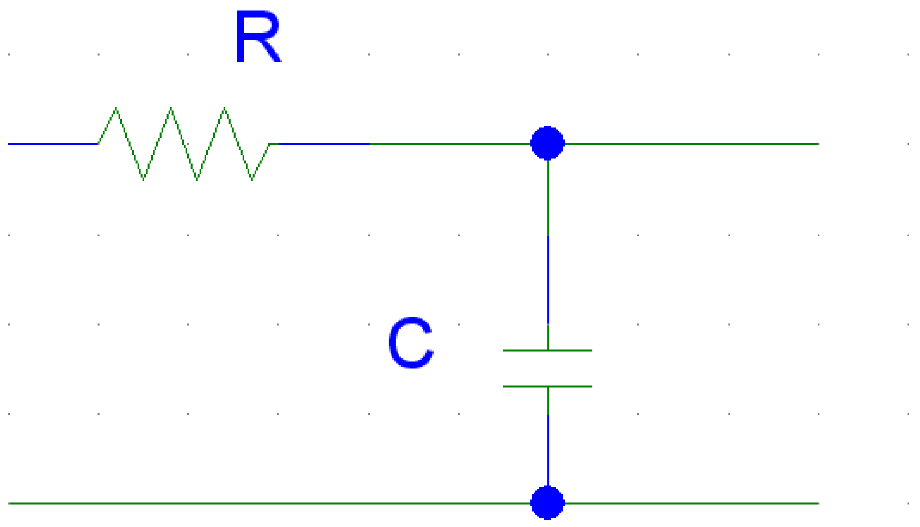
Real filters

- Here, we consider Butterworth low pass filters. The transfer function of a low pass Butterworth filter is of the form:
- $H(f) = \frac{1}{P_n\left(\frac{jf}{B}\right)}$
- B is the 3-dB bandwidth of the filter and $P_n\left(\frac{jf}{B}\right)$ is a complex polynomial of order n. The family of Butterworth polynomials is defined by the property
- $\left(\left|P_n\left(\frac{jf}{B}\right)\right|\right)^2 = 1 + \left(\frac{f}{B}\right)^{2n}$.
- Therefore, $|H(f)| = \frac{1}{\sqrt{1+\left(\frac{f}{B}\right)^{2n}}}$
- The first few polynomials are:
- $P_1(x) = 1 + x;\ P_2(x) = 1 + \sqrt{2}x + x^2;\ \ P_3(x) = (1 + x)(1 + x + x^2)$

Filters and Filtering

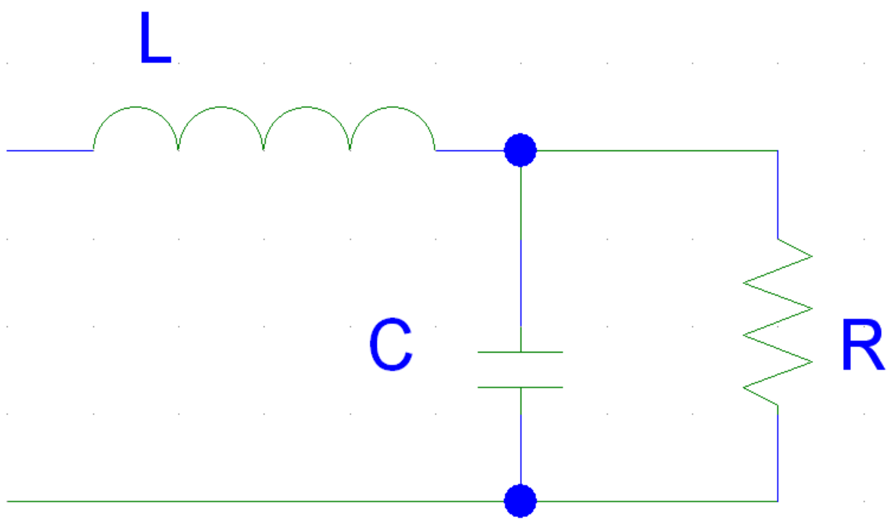
A first order LPF

- $H(f) = \frac{1}{P_1\left(\frac{jf}{B}\right)} = \frac{1}{1+jf/B} = \frac{1}{P_1(jf/B)};\quad P_1(x) = 1 + x$
- $H(f) = \frac{\frac{1}{j2\pi f_C}}{R+\frac{1}{j2\pi f_C}}$
- Let $B = \frac{1}{2\pi RC}$; $H(f) = \frac{1}{1+jf/B}$
- Note: In this filter, there is only one energy storage element



Filters and Filtering

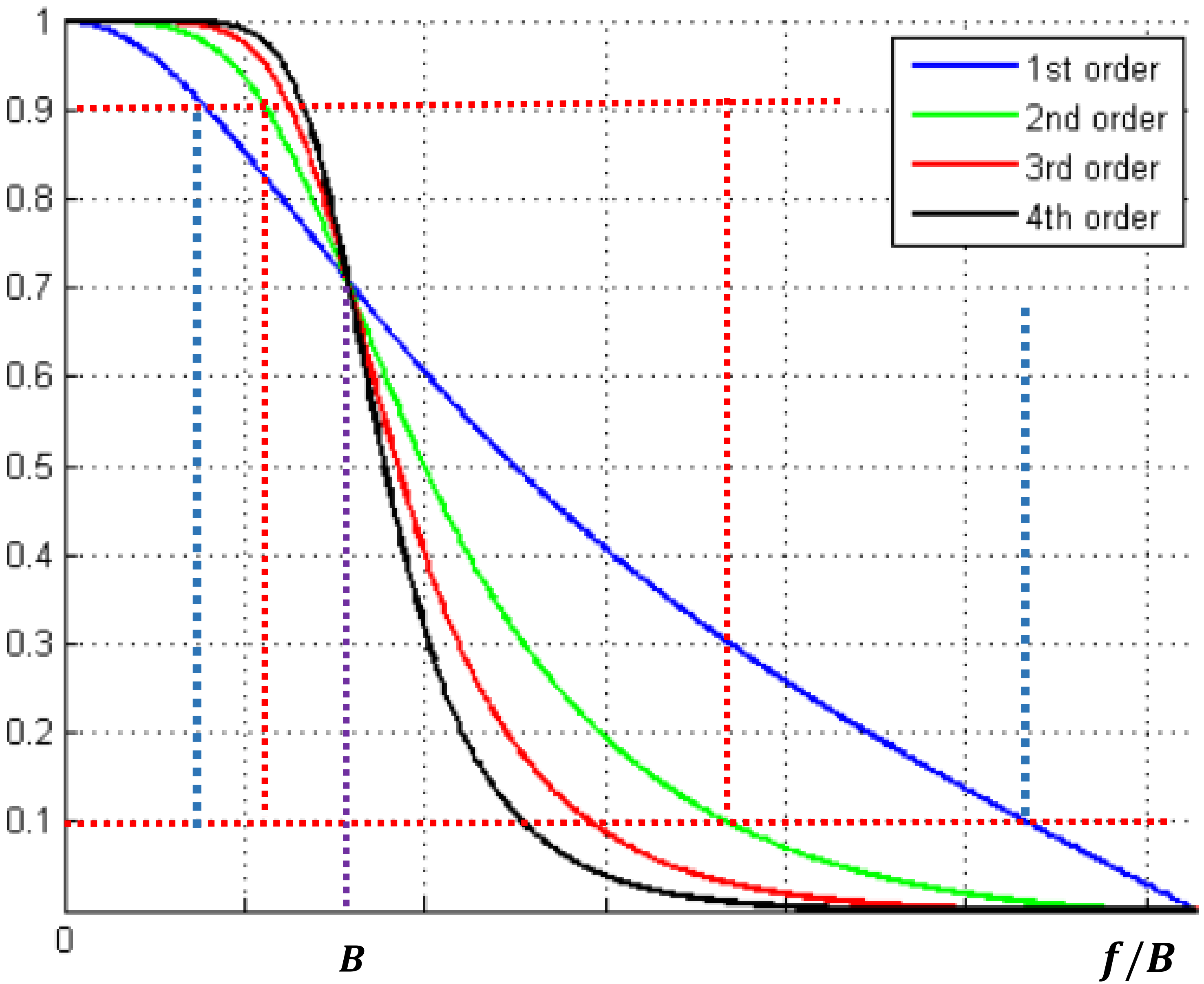
A second order LPF



- $H(f) = \frac{1}{1+\frac{jwL}{R}-\left(2\pi\sqrt{LC}f\right)^2}$
- Let $R = \sqrt{\frac{L}{2C}}$; $\boldsymbol{B = \frac{1}{2\pi\sqrt{LC}}}$
- $H(f) = \frac{1}{1+\sqrt{2}(\frac{jf}{B})-(f/B)^2}$
- $H(f) = \frac{1}{P_2(jf/B)}$; $P_2(x) = 1 + \sqrt{2}x + x^2$
- Note: In this filter, there are two energy storage element

Butterworth Low-pass Filters: Frequency Response and Filter Order

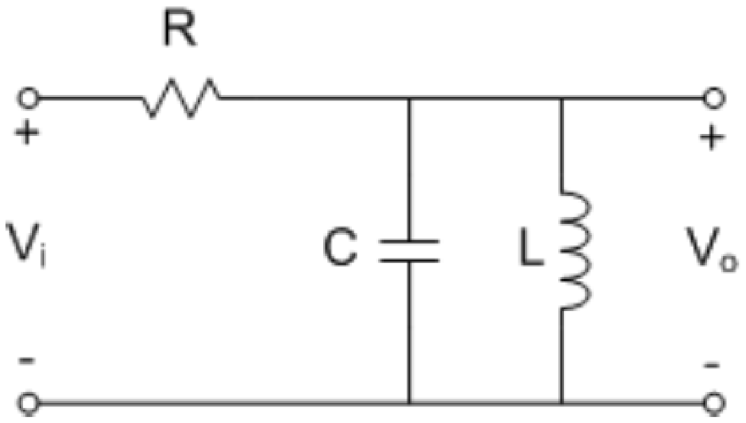
- B: is the 3-dB frequency at which the magnitude drops to 0.707 of the maximum value.
- Let the maximum allowable attenuation in the passband be 0.1 and the maximum gain within the stopband be 0.1.
- $H_1(f) = \frac{1}{1+jf/B}$
  - Passband:
  - Transition band:
- $H_2(f) = \frac{1}{1+j\sqrt{2}f/B-(f/B)^2}$
  - Passband:
  - Transition band:
- As the filter order increases, both of its pass-band and stop-band capabilities improve.



A second order BPF

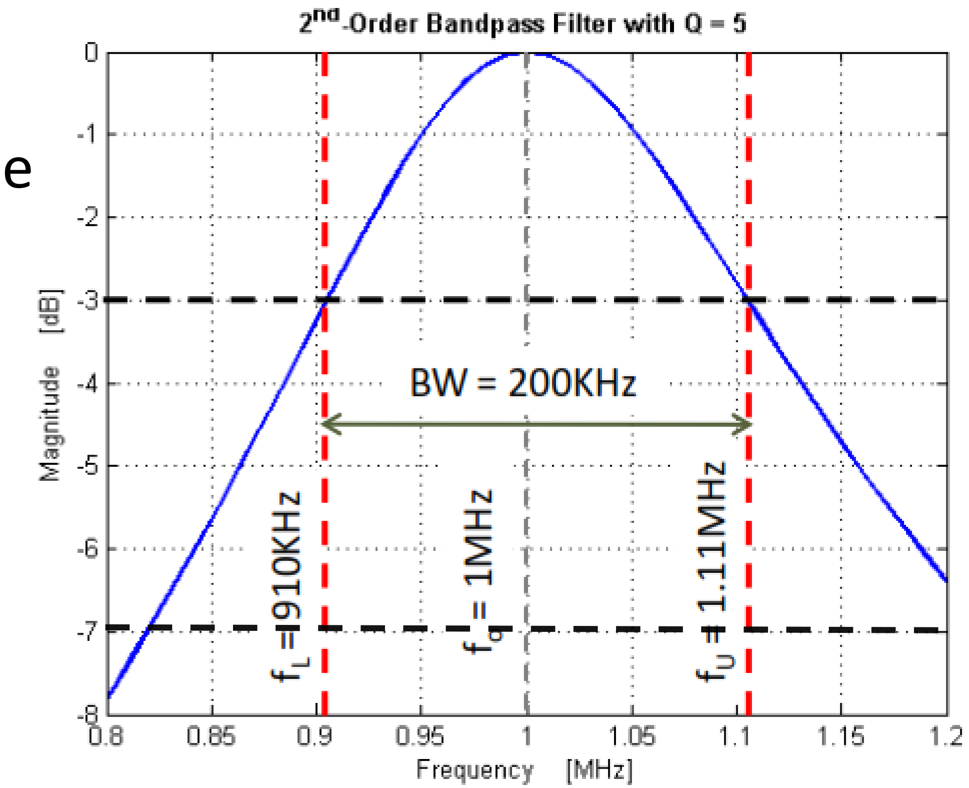
- The figure shows a band-pass filter. Its transfer function is

$$H(f) = \frac{\frac{j\omega}{RC}}{(j\omega)^2 + \frac{j\omega}{RC} + 1/LC} = \frac{\frac{\omega_0}{Q}(j\omega)}{(j\omega)^2 + \frac{\omega_0}{Q}(j\omega) + (\omega_0)^2}$$

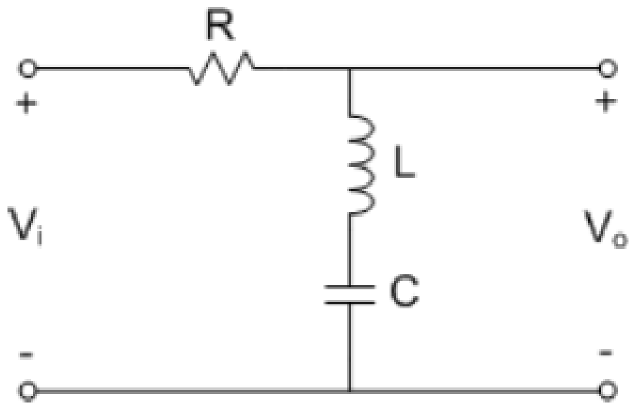


- $\omega_0 = 2\pi(\frac{1}{2\pi\sqrt{LC}})$; f_0: Resonance frequency
- $Q = \omega_0 RC = \frac{R}{\omega_0 C}$; Quality factor which determines the sharpness of the resonance.

- Bandwidth is inversely proportional to Q
- $\mathbf{Q = \frac{f_0}{B.W}}$; ***Higher Q*** provides ***higher selectivity***
- For the shown characteristic, Q = 1MHz/200KHz = 5

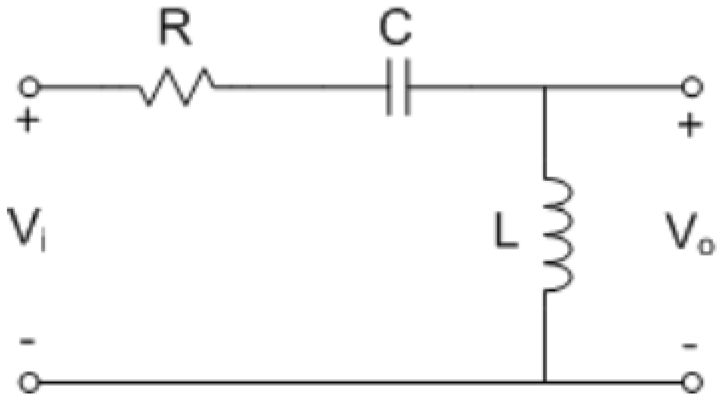


Other practical second order filters



$$H(\omega) = \frac{(j\omega)^2 + 1/LC}{(j\omega)^2 + j\omega R/L + 1/LC}$$

Second order band-stop filter



$$H(\omega) = \frac{(j\omega)^2}{(j\omega)^2 + j\omega R/L + 1/LC}$$

Second order high-pass filter

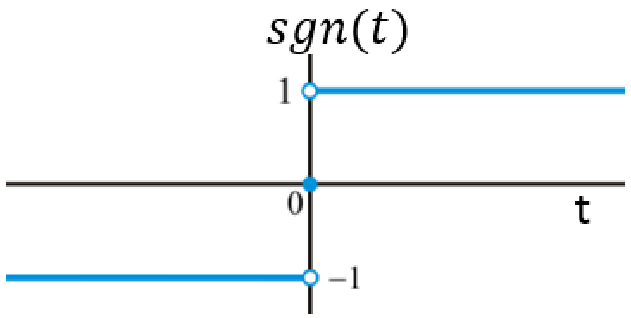
Hilbert Transform

- **The quadrature filter** is an all pass filter that shifts the phase of positive frequency by (-90°) and negative frequency by ($+90^\circ$).
- **The transfer function** of such a filter is
- $H(f) = \begin{cases} -j & f > 0 \\ j & f < 0 \end{cases} = \boldsymbol{-jsgn(f)}$
- Note that $|H(f)| = 1$ for all f.
- Using the duality property of Fourier transform, the impulse response of the filter is $\boldsymbol{h(t) = \frac{1}{\pi t}}$ $(\Im\{sgn(t)\} = \frac{1}{j\pi f})$
- The Hilbert transform is the output of the quadrature filter to the signal $g(t)$
- $\boldsymbol{\hat{g}(t) = \frac{1}{\pi t} * g(t)} = \int_{-\infty}^{\infty} \frac{g(\lambda)}{\pi(t-\lambda)} d\lambda$
- Note that the Hilbert transform of a signal is a function of time (not frequency as in the case of the Fourier transform). The Fourier transform of $\hat{g}(t)$
- $\boldsymbol{\hat{G}(f) = -jsgn(f)G(f)}$
- Hilbert transform can be found using either the time domain approach or the frequency domain approach depending on the given problem. That is
- **Time-domain**: Perform the convolution

  $\frac{1}{\pi t} * g(t)$.
- **Frequency-domain**: Find the Fourier transform $\hat{G}(f)$, then find the inverse Fourier transform
- $\hat{g}(t) = \int_{-\infty}^{\infty} \hat{G}(f)\, e^{j2\pi f t}\, df$

Some properties of the Hilbert transform

- A signal $g(t)$ and its Hilbert transform $\hat{g}(t)$ have the same energy spectral density
- $\left|\hat{G}(f)\right|^2 = |-j\, sgn(f)|G(f)||^2 = |-j\, sgn(f)|^2|G(f)|^2$
- $= |G(f)|^2$



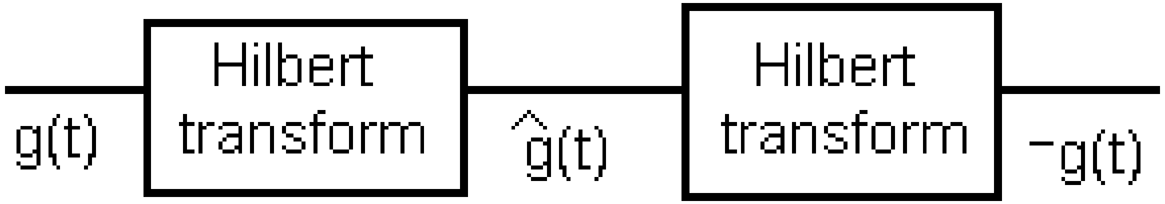
The consequences of this property are:

- If a signal g(t) is bandlimited to a bandwidth W Hz, then $\hat{g}(t)$ is bandlimited to the same bandwidth (note that $\left|\hat{G}(f)\right| = |G(f)|$)
- $\hat{g}(t)$ and $g(t)$ have the same total energy (or power). $E = \int_{-\infty}^{\infty} |G(f)|^2 df$
- $\hat{g}(t)$ and $g(t)$ have the same autocorrelation function (in the next lecture, we will see that the autocorrelation function and the energy spectral density form a Fourier transform pair $R_g(\tau) \leftrightarrow |G(f)|^2$)

Some properties of the Hilbert transform

- A signal g(t) and $\hat{g}(t)$ are orthogonal, i.e., $\int_{-\infty}^{\infty} \mathrm{g(t)}\, \hat{g}(t)dt = 0$
- This property can be verified using the general formula of Rayleigh energy theorem
- $\int_{-\infty}^{\infty} \mathrm{g(t)}\, \hat{g}(t)dt = \int_{-\infty}^{\infty} \mathrm{G(f)}\, \hat{G}^{*}(f)df = \int_{-\infty}^{\infty} \mathrm{G(f)}\, \{-jsgn(f)\, G(f)\}^{*}df$

  $= \int_{-\infty}^{\infty} jsgn(f)\, |G(f)|^{2}df = 0.$
- The result above follows from the fact that $|G(f)|^2$ is an even function of f while $sgn(f)$ is an odd function of f. Their product is odd. The integration of an odd function over a symmetrical interval is zero.
- If $\hat{g}(t)$ is a Hilbert transform of $g(t)$, then the Hilbert transform of $\hat{g}(t)$ is $-g(t)$ (each Hilbert transform introduces 90 degrees phase shift).



Example on Hilbert transform

Example: Find the Hilbert transform of the impulse function $g(t) = \delta(t)$

Solution: Here, we use the convolution in the time domain

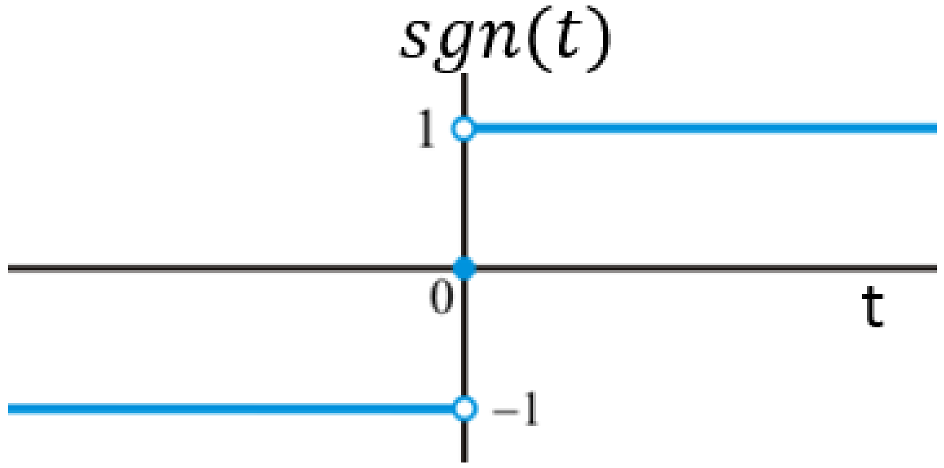
- $$\hat{g}(t) = \frac{1}{\pi t} * \delta(t)$$
- As we know, the convolution of the delta function with a continuous function is the function itself. Therefore,
- $$\hat{g}(t) = \frac{1}{\pi t}.$$

Example on Hilbert transform

Example: Find the Hilbert transform of $g(t) = \cos(2\pi f_0 t)$

Solution: Here, we use the frequency domain approach

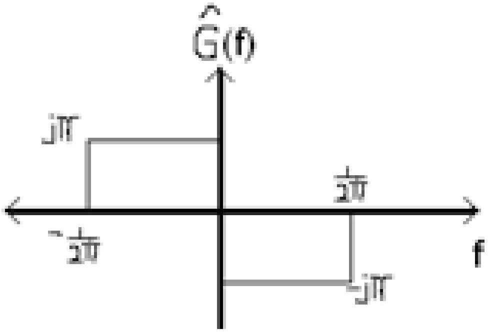
- $\hat{G}(f) = -jsgn(f)G(f) = -\frac{jsgn(f)\{\delta(f-f_0)+\delta(f+f_0)\}}{2}$
- $\hat{G}(f) = -jsgn(f)G(f) = \frac{sgn(f)\{\delta(f-f_0)+\delta(f+f_0)\}}{j2} = \frac{\{\delta(f-f_0)-\delta(f+f_0)\}}{j2}$
- $\hat{g}(t) = \sin(2\pi f_0 t)$

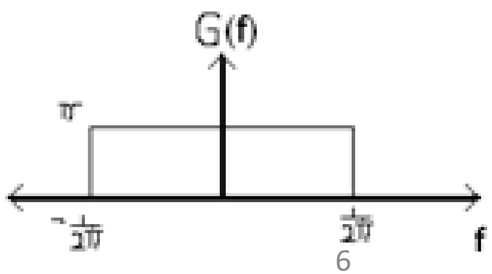


Example on Hilbert transform

- Find the Hilbert transform of $g(t) = \frac{\sin t}{t}$
- **Solution:** Here, we will first find the Fourier transform of $g(t)$, find $\hat{G}(f)$, and then find $\hat{g}(t)$:
- $A\, rect\left(\frac{t}{\tau}\right) \leftrightarrow A\tau\, sinc\, f\tau\,;\ \tau = \frac{1}{\pi}$
- $A \text{ rect}\left(\frac{t}{1/\pi}\right) \leftrightarrow A\,\frac{1}{\pi}\,\frac{\sin \pi f\tau}{\pi f\tau} = \frac{1}{\pi}\frac{\sin f}{f}$
- $\pi\, rect\left(\frac{t}{1/\pi}\right) \leftrightarrow \frac{\sin f}{f}$
- So, by the duality property, we get the pair
- $\pi\, rect\left(\frac{f}{1/\pi}\right) \leftrightarrow \frac{\sin t}{t}$
- i.e. $G(f) = \pi\, rect(\frac{f}{1/\pi})$, (See figure next)

- $\hat{G}(f) = -jsgn(f)G(f) = \begin{cases} -j\pi & 0 < f < 1/2\pi \\ j\pi & -1/2\pi < f < 0 \end{cases}$
- $\hat{g}(t) \;=\; \int_{-\infty}^{\infty} \hat{G}(f)\, e^{j2\pi f t}\, df$
- $= \int_{-1/2\pi}^{0} j\pi\;\; e^{j2\pi f t}\, df \;-\; \int_{0}^{1/2\pi} j\pi\;\; e^{j2\pi f t}\, df$
- $= \frac{1}{2t}\left(1 - e^{-jt}\right) - \frac{1}{2t}\left(e^{jt} - 1\right)$
- $= \frac{1}{t} - \frac{1}{t}\,\frac{\left(e^{jt} + e^{-jt}\right)}{2} = \frac{1 - \cos t}{t}$





Bandwidth of Signals and Systems: Lecture Outline

- Bandwidth Definitions
  - Absolute Bandwidth
  - 3-dB (half power points) Bandwidth
  - The 95 % (energy or power) Bandwidth
  - Equivalent Rectangular Bandwidth
  - Null – to – Null Bandwidth
  - Bounded Spectrum Bandwidth
  - RMS Bandwidth
- The Definition of Decibel
- Bandwidth of Periodic Signals
- Time-Bandwidth Product

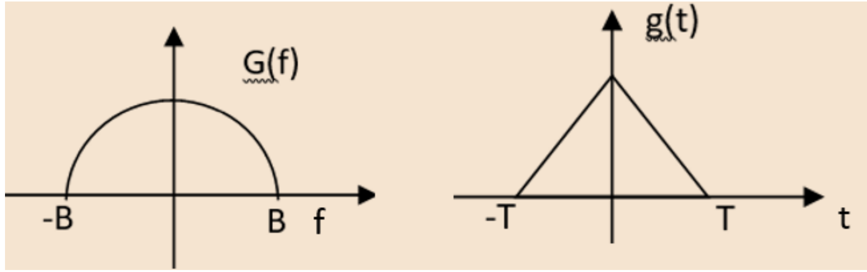
Bandwidth of Signals and Systems

- **Definition:** The amount of **positive** frequency band that a signal $g(t)$ occupies is called the bandwidth of the signal. It provides a **measure of the extent of significant frequency content** of the signal.
- **Definition:** A signal g(t) is said to be (absolutely) band-limited to B Hz if

  G(f) = 0 for |f| >B

- **Definition:** A signal g(t) is said to be (absolutely) time-limited if

  g(t) = 0 for |t| >T



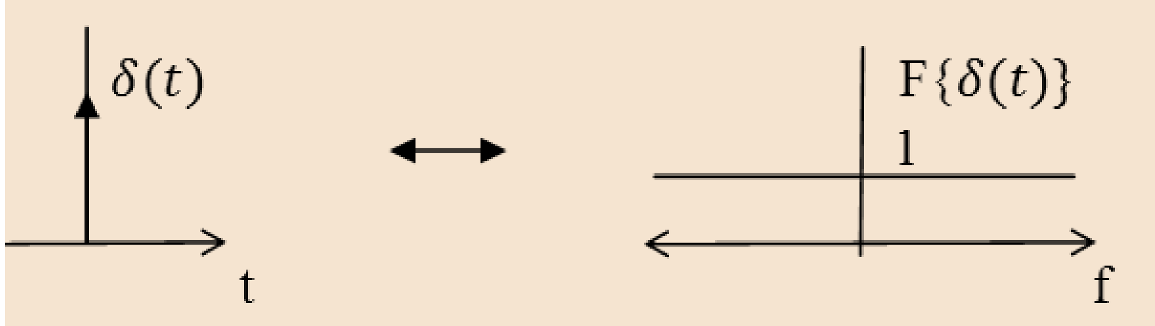
- **Theorem:** An absolutely band-limited waveform **cannot** be absolutely time-limited and vice versa, i.e., a signal g(t) **cannot** be both time-limited and bandlimited.
- In general, there is an inverse relationship between the signal bandwidth and the time duration. The bandwidth and the time duration are related through a relation, called the *time bandwidth product*, of the form (will investigate this more in the next lecture)

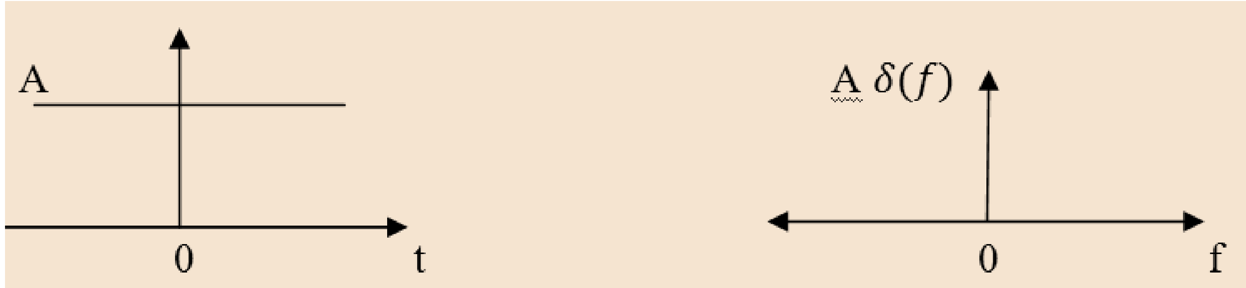
$$(\boldsymbol{Bandwidth})(\boldsymbol{Time\ Duration}) \geq \boldsymbol{Constant}$$

- The value of the constant depends on the way we define the bandwidth and the time duration. Two possible values of the constant, that we will encounter in this chapter, are ½ (for the equivalent rectangular bandwidth) and $\frac{1}{4\pi}$ (for the root mean square bandwidth).

Bandwidth of Signals and Systems

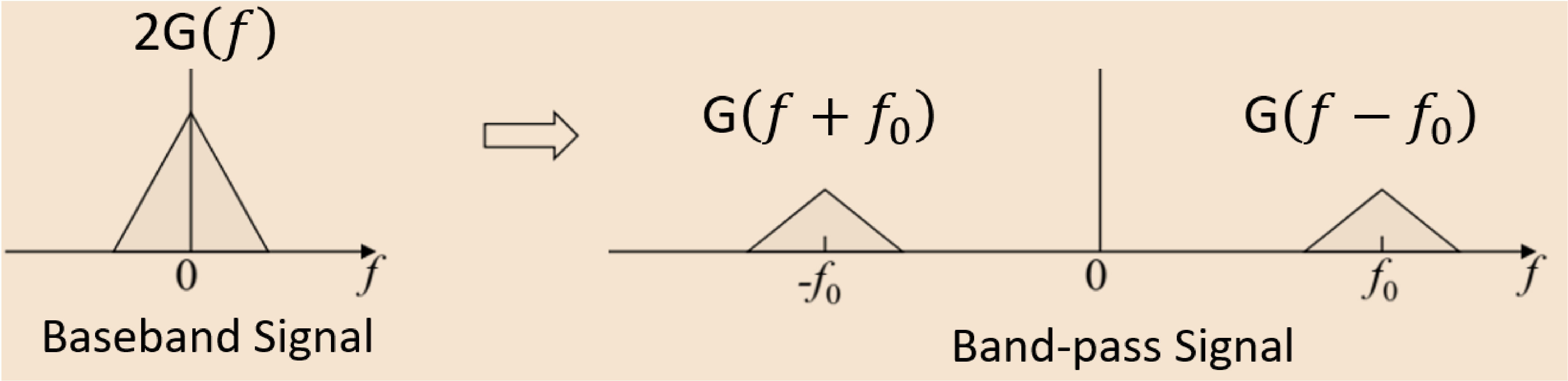
- **Theorem**: An absolutely band-limited waveform **cannot** be absolutely time-limited and vice versa, i.e., a signal g(t) **cannot** be both time-limited and bandlimited.
- We have earlier seen examples that support this theorem. For example, the delta function, which has an almost zero time duration, has a Fourier transform which extends uniformly over all frequencies (infinite bandwidth).
- Also, a constant value in the time domain (a dc) has a Fourier transform, which is an impulse in the frequency domain at the origin. These are shown below.





Bandwidth of Signals and Systems

- **Definition of a baseband signal:** A baseband signal is one for which most of the energy is contained within a band centered around the zero frequency and negligible elsewhere. Another term synonymous with baseband is ***low-pass***. In the communication systems, the message to be transmitted is a baseband signal.
- **Definition of a band-pass signal:** A band-pass signal is one for which the energy is concentrated around some high frequency carrier f_0 and negligible elsewhere. This type of signal will arise in this course when the baseband message signal $m(t)$ modulates a high frequency carrier $c(t)$ to produce the modulated signal $s(t)$.



The Definition of Decibel

- Consider a system with input voltage v_i and output voltage v_o
- The power gain of the system is defined as:

$$G = \left(\frac{P_o}{P_i}\right)$$

- In a logarithmic scale, the gain is defined as

$$G = 10log_{10}\left(\frac{P_o}{P_i}\right) \text{ dB.}$$

- If $P_o > P_i$, G > 0. Hence, the output signal possesses more power than the input. However, if $P_o < P_i$, the system introduces attenuation or loss. In this case G < 0.
- If the input and output powers are taken relative to the same reference resistance R, then

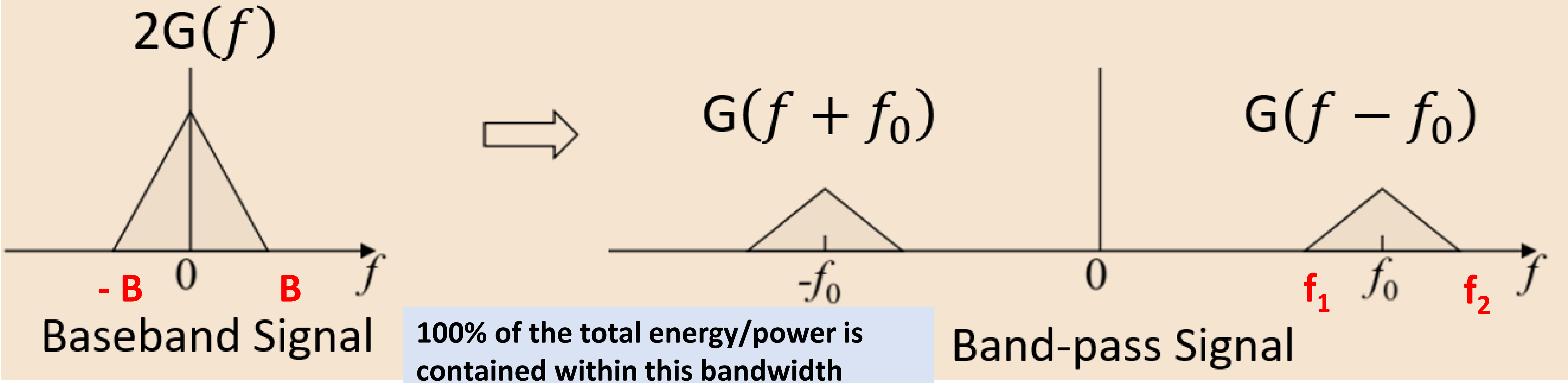
$$\text{G} = 10\log\left(\frac{\frac{{V_o}^2}{R}}{\frac{{V_i}^2}{R}}\right) = 20log_{10}\left(\frac{V_o}{V_i}\right) \text{ dB}$$

- For a transfer function H(f), G becomes

$$\text{G} = 20log_{10}(H(f)) \text{ dB}$$

Definitions of Bandwidth: Absolute Bandwidth

- Here, the Fourier transform of a signal is non-zero only within a certain frequency band.
- Low-pass Signals: If $G(f) = 0$ for $|f| > B$, then $g(t)$ is absolutely band-limited to B Hz and **B.W = B**
- Bandpass Signals: When $G(f) \neq 0$ for $f_1 < |f| < f_2$, then the absolute bandwidth is **$B.W = f_2 - f_1$.**

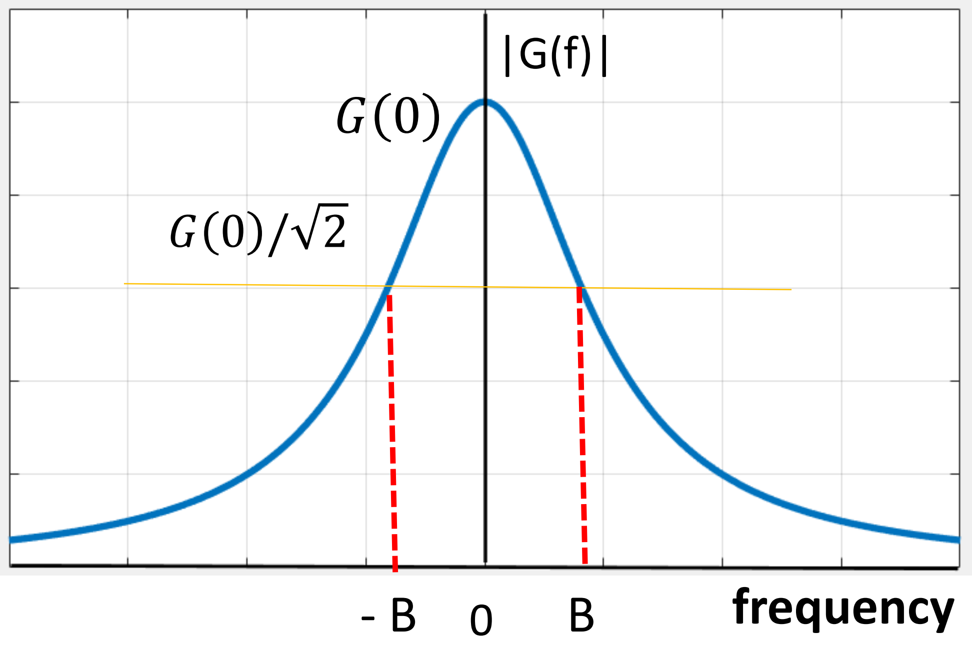


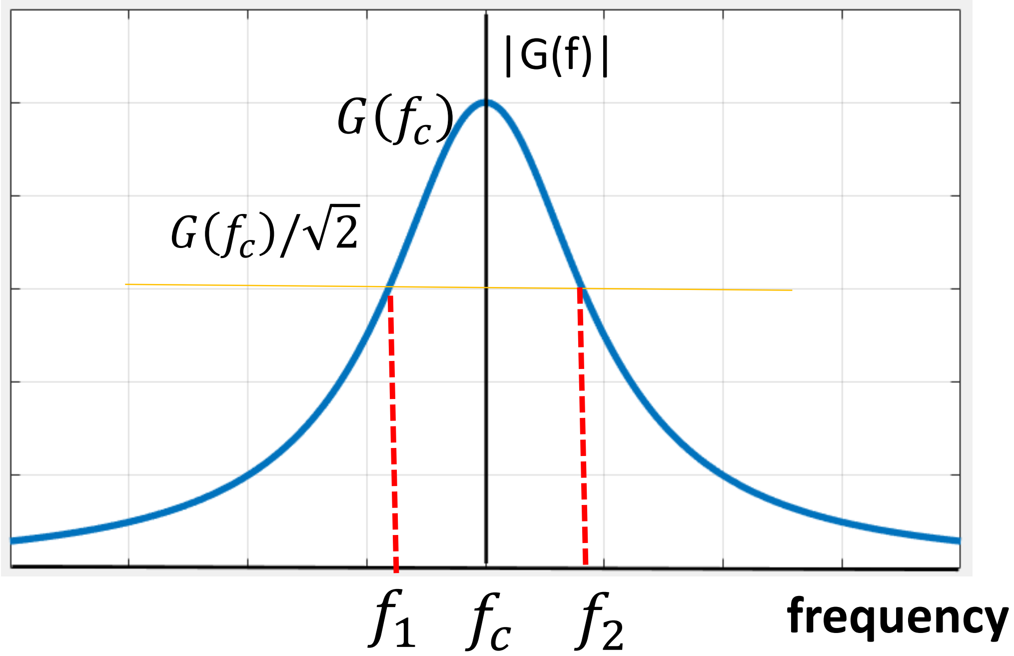
100% of the total energy/power is contained within this bandwidth

Definitions of Bandwidth: 3-dB (half power points) Bandwidth

- The range of frequencies from 0 to some frequency B at which |G(f)| drops to $\frac{1}{\sqrt{2}}$ of its maximum value (for a low pass signal).
- As for a band pass signal, the B.W = $f_2 - f_1$.

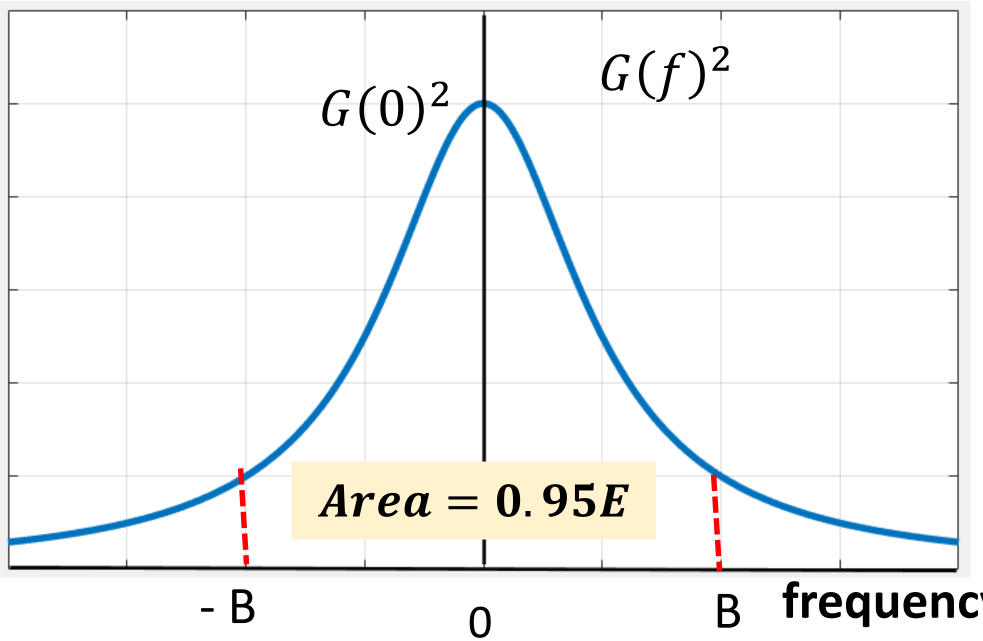
$$G = 20\,log_{10}\left(\frac{G(B)}{G(0)}\right) = 20\,log_{10}\left(\frac{G(0)/\sqrt{2}}{G(0)}\right) = -20log_{10}(\sqrt{2}) = -3\,dB$$





Definitions of Bandwidth: The 95 % (energy or power) Bandwidth

- Here, the B.W is defined as the band of frequencies where the area under the energy spectral density (or power spectral density) is at least 95% (or 99%) of the total area.
- Total Signal Energy $E = \int_{-\infty}^{\infty} |G(f)|^2\, df = 2\int_{0}^{\infty} |G(f)|^2 df = \int_{-\infty}^{\infty} |g(t)|^2\, dt$
- The 95% energy bandwidth B should satisfy the relationship
- $\int_{-B}^{B} |G(f)|^2 df = 0.95 \int_{-\infty}^{\infty} |G(f)|^2\, df = 0.95\, E$
- $\boldsymbol{\int_{-B}^{B} |G(f)|^2 df = 0.95\, E}$



Definitions of Bandwidth: Equivalent Rectangular Bandwidth

- It is the width of a fictitious rectangular spectrum such that the power in that rectangular band is equal to the energy associated with the actual spectrum. Let B_{eq} be the equivalent rectangular bandwidth. To find B_{eq} we set

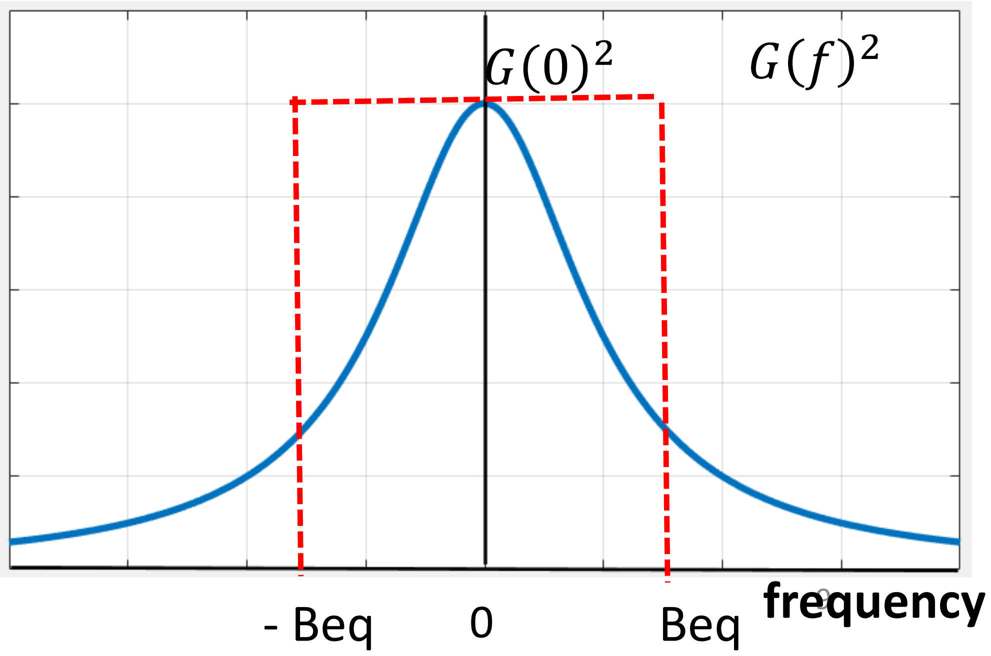
Area under fictitious rectangle = Total Signal Energy E

Area of red rectangle = Area under the blue curve

$$|G(0)|^2 * 2B_{eq} = \int_{-\infty}^{\infty} |G(f)|^2 df = E$$

$$|G(0)|^2 * 2B_{eq} = 2\int_{0}^{\infty} |G(f)|^2 df$$

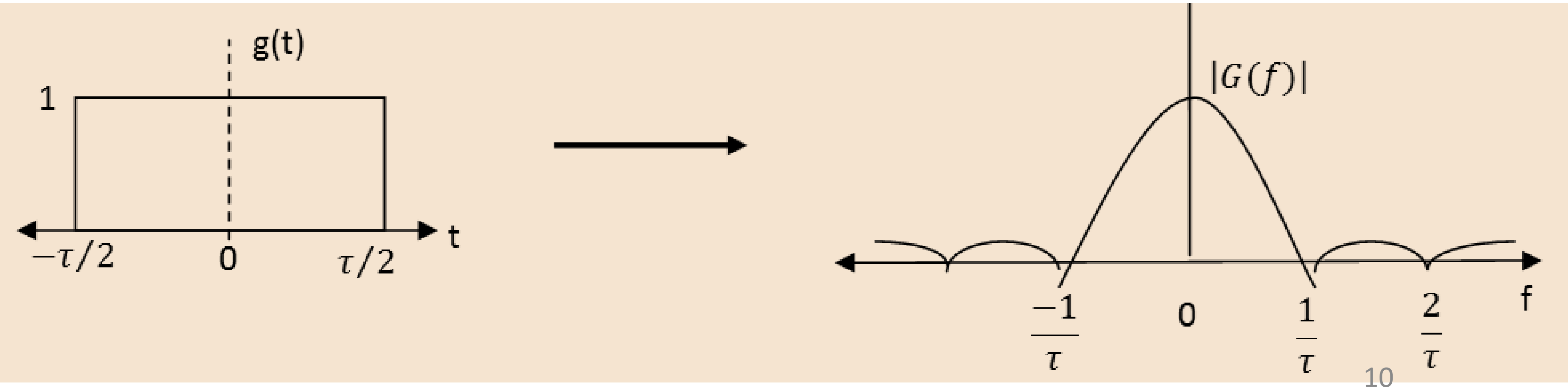
$$\boldsymbol{B_{eq} = \frac{1}{|G(0)|^2} \int_0^{\infty} |G(f)|^2 df}$$



Definitions of Bandwidth: Null – to – Null Bandwidth

- For baseband signals, the null bandwidth is taken to be the band from zero to the first null in the envelope of the magnitude spectrum.
- For example, consider the rectangular pulse g(t), for which the Fourier transform is G(f). Note that
- $rect\left(\frac{t}{\tau}\right) \rightarrow \tau\ \mathrm{sinc} f\tau = \tau \frac{sin\pi f\tau}{\pi f\tau}.$
- The zero crossings occur when sin(πfτ) = 0
- $\pi f\tau = n\pi \rightarrow f = \frac{n}{\tau}$; n = 1,2, … The smallest value of $n = 1$, gives
- $\boldsymbol{Null\ Bandwidth = \frac{1}{\tau}.}$
- For a band pass signal,

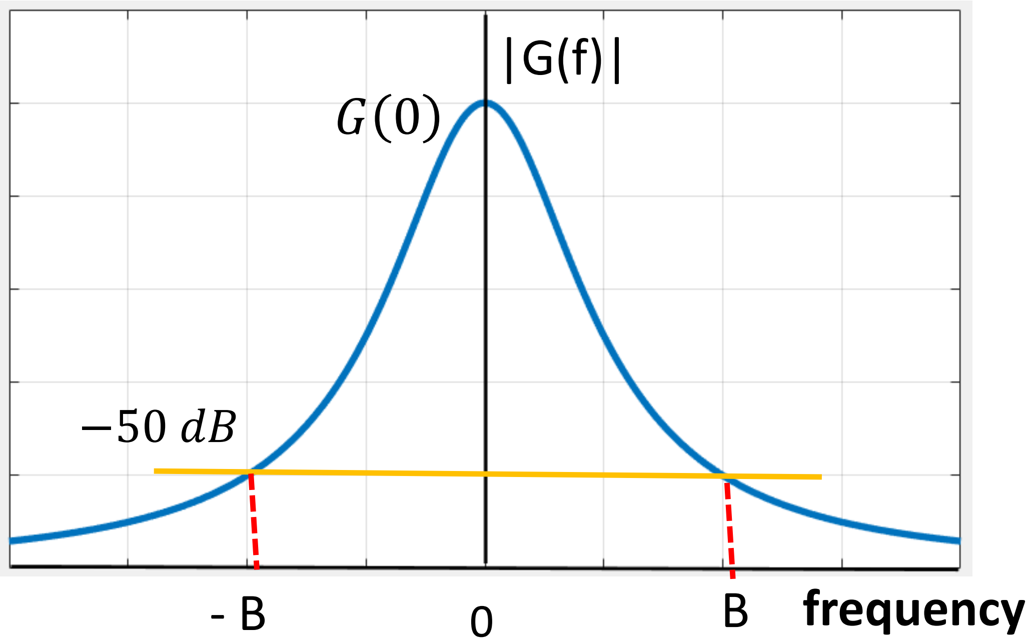
  B.W= $f_2 - f_1$



Definitions of Bandwidth: Bounded Spectrum Bandwidth

- The range of frequencies from 0 to some frequency B at which |G(f)| drops to , say, $-50dB$ relative to its maximum value (for a low pass signal).

$$-50\ dB = 20\ log_{10}\left(\frac{G(B)}{G(0)}\right)$$



Definitions of Bandwidth: RMS bandwidth

- The RMS bandwidth of a signal g(t) is defined as
- $$B_{rms} = \sqrt{\left(\frac{\int_{-\infty}^{\infty} f^2|G(f)|^2 \mathrm{df}}{\int_{-\infty}^{\infty} |G(f)|^2 \mathrm{df}}\right)} = \sqrt{\left(\frac{\int_{-\infty}^{\infty} f^2|G(f)|^2 \mathrm{df}}{E_g}\right)}$$
- In an analogous way, the corresponding RMS duration of g(t) is
- $$T_{rms} = \sqrt{\left(\frac{\int_{-\infty}^{\infty} t^2|g(t)|^2 \mathrm{dt}}{\int_{-\infty}^{\infty} |g(t)|^2 \mathrm{dt}}\right)} = \sqrt{\left(\frac{\int_{-\infty}^{\infty} t^2|g(t)|^2 \mathrm{dt}}{E_g}\right)}$$
- (here g(t) is assumed to be centered around the origin).
- **Remark:** The time bandwidth product is $\boldsymbol{(T_{rms})(B_{rms}) \geq \frac{1}{4\pi}}$ (the proof is beyond the scope of this presentation).

Example: 95% Energy Bandwidth of the Exponential Pulse

- Find the 95% energy bandwidth for the exponential pulse $g(t) = Ae^{-\alpha t}\ u(t)$.
- **Solution**: The Fourier transform of g(t) is

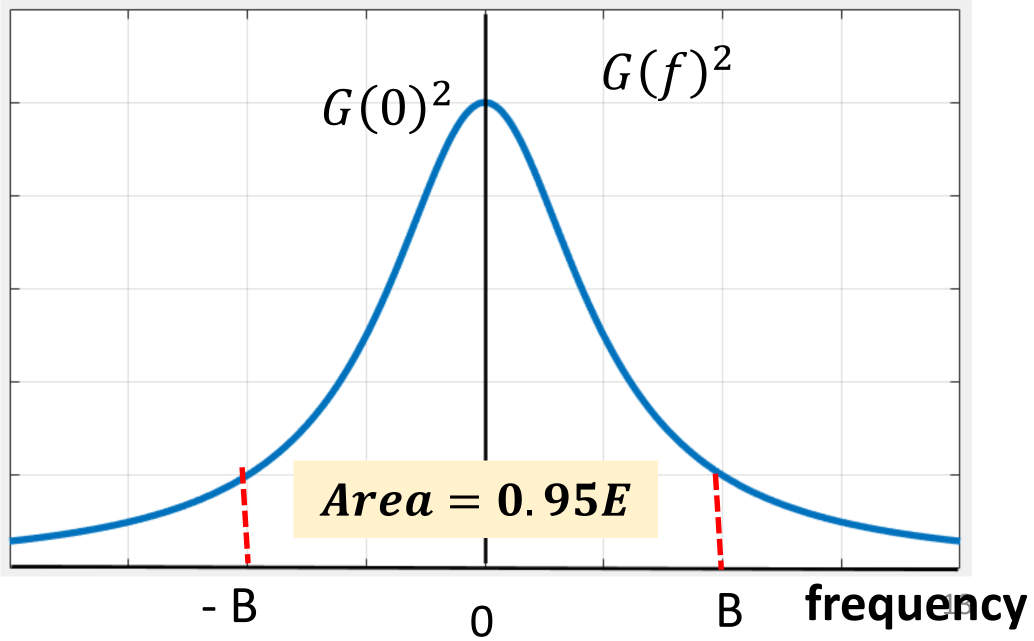
$$G(f) = \frac{A}{\alpha + j2\pi f}$$

- The total energy in g(t) (calculated in the time domain) is
- $E_g = \int_0^\infty |g(t)|^2\, dt = \int_0^\infty A^2 e^{-2\alpha t}\, dt = \frac{A^2}{2\alpha}$
- Let B be the 95% energy bandwidth, then the energy contained within B is
- $E_B = \int_{-B}^{B} |G(f)|^2\, df = \int_{-B}^{B} \frac{A^2}{(\alpha^2 + (2\pi f)^2)}\, df$
- $E_B = \frac{2A^2}{2\pi\alpha} tan^{-1} \frac{2\pi B}{\alpha}$
- B should be chosen such that it satisfies the condition
- $E_B = 0.95 E_g$

$$\mathbf{\frac{2A^2}{2\pi\alpha} tan^{-1} \frac{2\pi B}{\alpha} = 0.95 \left(\frac{A^2}{2\alpha}\right)}$$

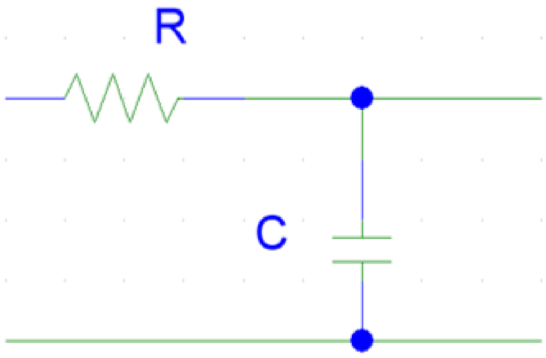
- The 95% energy bandwidth is, therefore

$$\mathbf{B_{95\%} = 2\alpha}$$

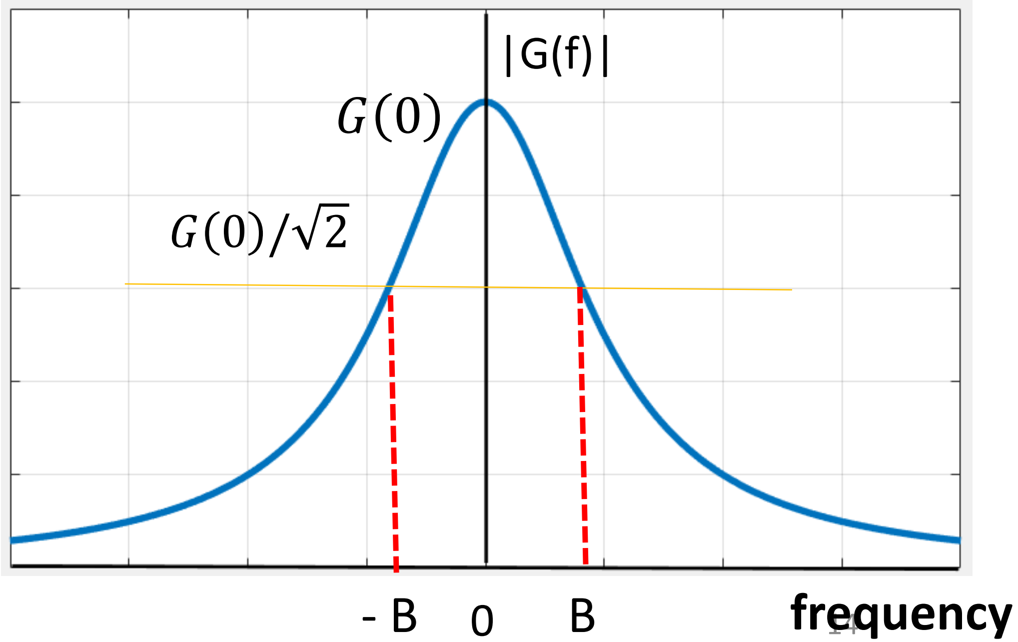


Example: 3-dB Bandwidth of the First Order RC Circuit

- **Example**: Find the 3-dB bandwidth of a first order RC low pass filter
- **Solution**: The transfer function of the circuit is
- $H(f) = \frac{\frac{1}{j2\pi fC}}{R+\frac{1}{j2\pi fC}} = \frac{1}{1+j2\pi fRC}$



- The magnitude of $H(f)$ is
- $|H(f)| = \frac{1}{\sqrt{1+(2\pi fRC)^2}}$
- The 3-dB bandwidth is some frequency $f = B$ at which $|H(f)|$ drops to $1/\sqrt{2}$ of its maximum value. Note that the maximum value of $|H(f)|$ is 1 and occurs at $f = 0$. Therefore, B should satisfy
- $|H(B)| = \frac{1}{\sqrt{1+(2\pi BRC)^2}} = \frac{1}{\sqrt{2}}$
- From this relationship, we notice that at the 3-dB point, $2\pi BRC = 1$
- Therefore, $\boldsymbol{B = \frac{1}{2\pi RC}}$



Example: Bandwidth of a Periodic Rectangular Signal

- **Example**: Find the 93% power bandwidth for the periodic square function define over one period as $g(t) = \begin{cases} 2A, & \frac{-T_0}{4} \le t \le \frac{T_0}{4} \\ -A, & o.w \end{cases}$
- **Solution**: The average power, computed using the time average, is
- $P_{av} = \frac{1}{T_0}\int_0^{T_0} |g(t)|^2 \ dt = \frac{1}{T_0}\left[4A^2\frac{T_0}{2} + A^2\frac{T_0}{2}\right] = \frac{5A^2T_0}{2T_0} = \frac{5A^2}{2} \Rightarrow \boldsymbol{P_{av} = 2.5A^2}$
- Also, by using the Parseval's theorem, the average power can be computed as:
- $P_{av} = |C_0|^2 + 2\sum_{n=1}^{\infty}|C_n|^2$
- We recall that the Fourier coefficients for this signal were found in the lecture on Fourier series. Using these values, we get
- $P_{av} = \left(\frac{A}{2}\right)^2 + 2\sum_{n=1}^{\infty}\frac{(3A)^2}{(n\pi)^2} \Rightarrow \boldsymbol{P_{av} = \frac{A^2}{4} + 2A^2\sum_{n=1}^{\infty}\frac{(3)^2}{(n\pi)^2}}$

Example: Bandwidth of a Periodic Rectangular Signal

- $P_{av} = \left(\frac{A}{2}\right)^2 + 2\sum_{n=1}^{\infty}\frac{(3A)^2}{(n\pi)^2} \Rightarrow \boldsymbol{P_{av} = \frac{A^2}{4} + 2A^2\sum_{n=1}^{\infty}\frac{(3)^2}{(n\pi)^2} = 2.5A^2}$
- Let us take n = 1, then the power in the DC and the fundamental frequency is
- $P_1 = A^2\left\{0.25 + 2\left(\frac{9}{\pi^2}\right)\right\} = 2.073A^2 \Rightarrow \frac{P_1}{P_{av}} = \frac{2.073A^2}{2.5A^2} = 82.95\%$
- The fraction of power in these two terms relative to the total average power is only 82.95%. The 93% power limit is not yet reached. So, let us add one more term.
- When n = 3, the power in the DC, the fundamental term, and the third harmonic is
- $P_3 = A^2\left\{0.25 + 2\left(\frac{3^2}{\pi^2} + \frac{3^2}{3^2\pi^2}\right)\right\} = 2.276A^2 \Rightarrow \frac{P_3}{P_{av}} = \frac{2.276A^2}{2.5A^2} = 91.05\%.$
- The fraction of power in these three terms relative to the total average power is now 91.05%. Still, the 93% power limit is not reached yet. So, let us add one more term.
- For n = 5, the power in the DC, the fundamental term, the third harmonic, and the fifth harmonic is
- $P_5 = A^2\left\{0.25 + 2\left(\left(\frac{3}{\pi}\right)^2 + \left(\frac{3}{3\pi}\right)^2 + \left(\frac{3}{5\pi}\right)^2\right)\right\} = 2.349A^2 \quad \Rightarrow \frac{P_5}{P_{av}} = \frac{2.349A^2}{2.5A^2} = 93.97\%.$
- With n=5, the 93% power limit has been reached. Therefore, the 93% power B.W is $\boldsymbol{B_{93\%} = 5f_0}$.

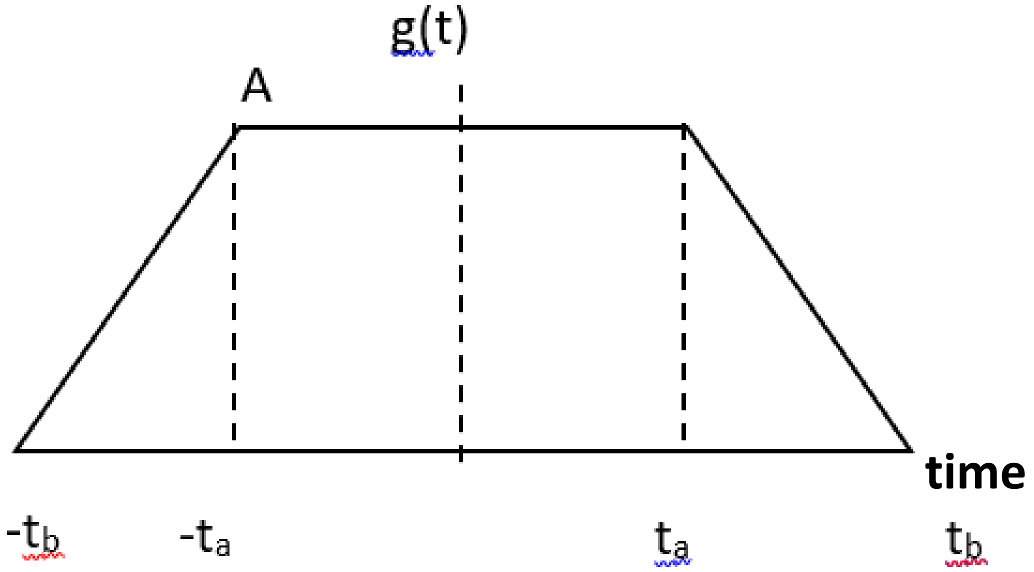
Time-Bandwidth Product

- One more time, to illustrate the time – bandwidth product ($(Bandwidth)(Time\ Duration) \geq Constant$), consider the equivalent rectangular bandwidth defined earlier as
- $B_{eq} = \frac{\int_{-\infty}^{\infty} |G(f)|^2 \mathrm{df}}{2|G(0)|^2}$
- Analogous to this definition, we define an equivalent rectangular time duration as
- $T_{eq} = \frac{\left(\int_{-\infty}^{\infty} |g(t)| dt\right)^2}{\int_{-\infty}^{\infty} |g(t)|^2 \mathrm{dt}}$
- The time bandwidth product is
- $B_{eq} T_{eq} = \left(\frac{\int_{-\infty}^{\infty} |G(f)|^2 \mathrm{df}}{2|G(0)|^2}\right) \left(\frac{\left(\int_{-\infty}^{\infty} |g(t)| dt\right)^2}{\int_{-\infty}^{\infty} |g(t)|^2 \mathrm{dt}}\right)$
- Note that $\int_{-\infty}^{\infty} |g(t)|^2$ dt = $\int_{-\infty}^{\infty} |G(f)|^2$ df;
- Rayleigh energy theorem.
- Note also that $G(0) = \int_{-\infty}^{\infty} g(t) dt.$
- Using these two relations, we get
- $\boldsymbol{B_{eq} T_{eq} = \frac{1}{2} \frac{\left(\int_{-\infty}^{\infty} |g(t)| dt\right)^2}{|\int_{-\infty}^{\infty} g(t) \mathrm{dt}|^2}}$
- Case 1: When g(t) is positive for all time t, then |g(t)| = g(t) and $B_{eq} T_{eq}$ becomes
- $B_{eq} T_{eq} = \frac{1}{2}$
- Case 2 : For a general g(t) that can take on positive as well as negative values, $B_{eq} T_{eq}$ satisfies the inequality
- $B_{eq} T_{eq} \geq \frac{1}{2}$
- Note : For B_{rms} and T_{rms} , the time – bandwidth product satisfies the inequality $B_{rms} T_{rms} \geq \frac{1}{4\pi}$

Example: Bandwidth of a Trapezoidal Signal

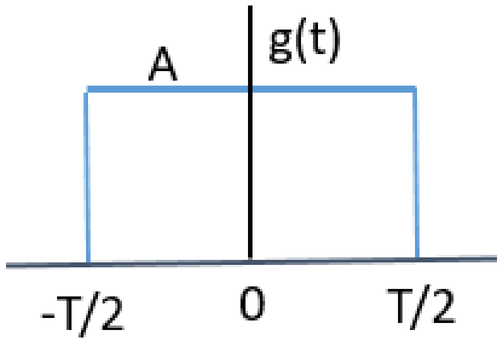
- **Example**: Find the equivalent rectangular bandwidth, B_{eq}, for the trapezoidal pulse shown.
- **Solution:**
- $T_{eq} = \frac{\left(\int_{-\infty}^{\infty} |g(t)| dt\right)^2}{\int_{-\infty}^{\infty} |g(t)|^2 \mathrm{dt}}$
- $\int_{-\infty}^{\infty} |g(t)| dt \ = A\,(\,t_a + t_b)$
- $\int_{-\infty}^{\infty} |g(t)|^2 dt \ = \frac{2A^2}{3}\,(\,2t_a + t_b)$
- $T_{eq} = \frac{3}{2}\frac{(t_a+t_b)^2}{(2t_a+t_b)}$
- $B_{eq} = \frac{0.5}{T_{eq}} = \frac{2t_a+t_b}{3(t_a+t_b)^2}.$

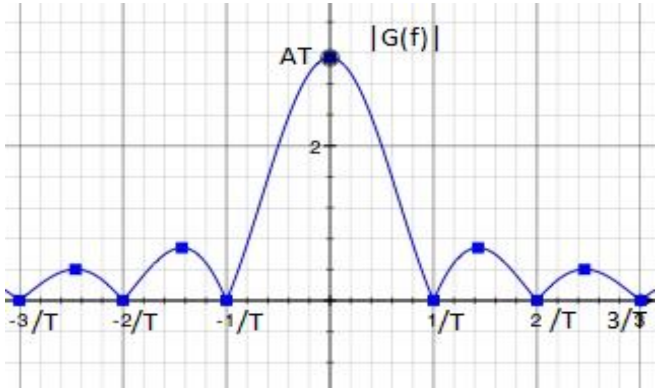
- **Remark**: Note that using this method we were able to determine the signal bandwidth without the need to go through the Fourier transform.

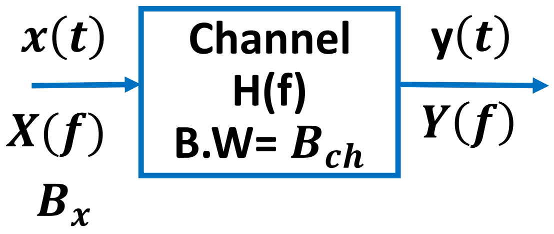


Pulse Response and Rise-time

- In this lecture, we will investigate the relationship that should exist between the pulse bandwidth and the channel bandwidth. As we know, the rectangular pulse contains significant high frequency components. When that pulse is passed through a band-limited channel, the channel will alter the shape of the input resulting in linear distortion (amplitude and phase)
- This subject is of particular importance, especially, when we study the transmission of data over band-limited channels. In the simplest form, a binary digit 1 may be represented by a pulse , $0 \leq t \leq T_b$, while binary digit 0 may be represented by the negative pulse $-A$, $0 \leq t \leq T_b$. Therefore, in order to retrieve the transmitted data, the channel bandwidth must be wide enough to accommodate the transmitted data.
- To convey this idea in a simple form, we first consider the response of a first order low pass filter to a unit step function and then to a pulse.

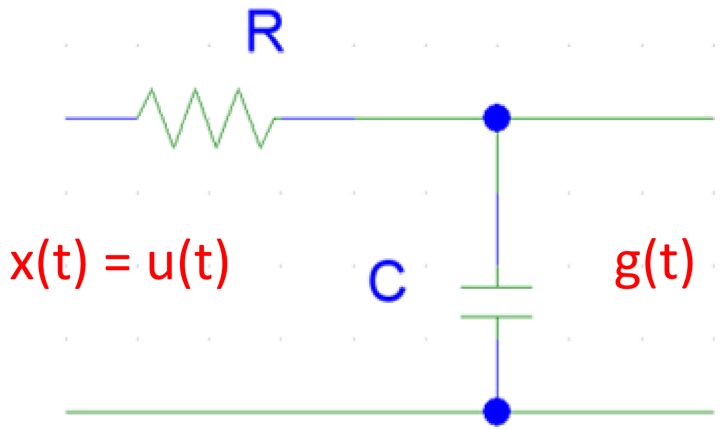






Step Response of a First Order System

- Let $x(t) = u(t)$ be applied to a first order RC circuit. This first order filter is a fair representation of a low-pass communication channel
- The system differential equation is
- $x(t) = Ri(t) + g(t) = RC\frac{dg(t)}{dt} + g(t)$
- where g(t) is the channel output.

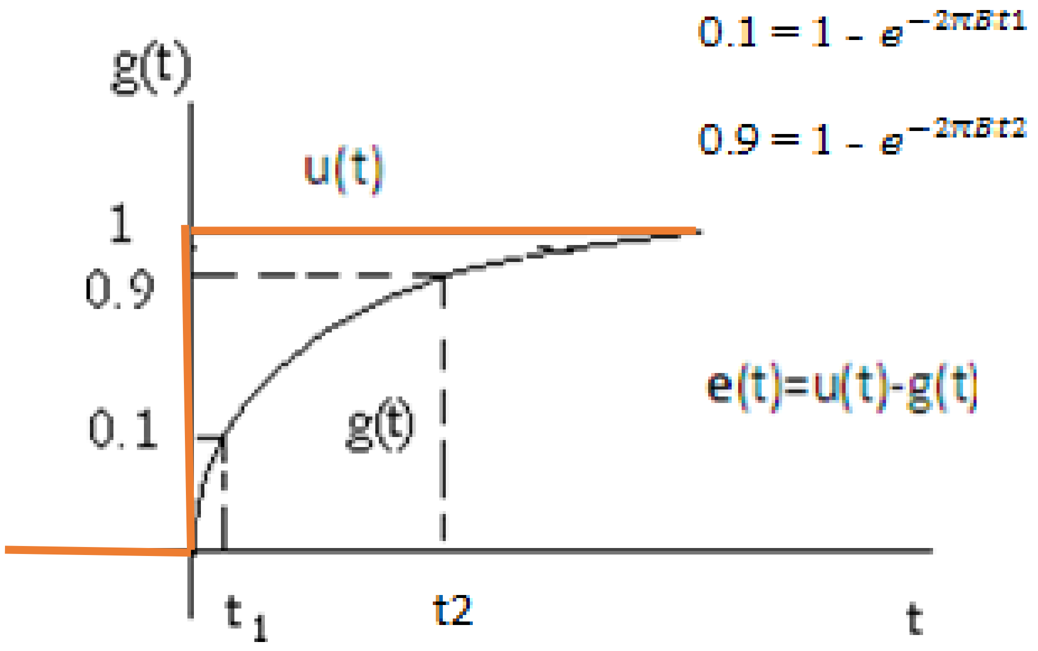


- Now let $x(t) = u(t)$. The system D.E. becomes $x(t)$
- $RC\frac{dg(t)}{dt} + g(t) = u(t)$
- The solution to this first order system is
- $g(t) = (1 - e^{-t/RC})u(t)$
- The 3- dB bandwidth of the channel (was derived in a previous example in this chapter) is $\boldsymbol{B_{ch} = \frac{1}{2\pi RC}}$
- The output $g(t)$, expressed in terms of B_{ch} becomes

$$\boldsymbol{g(t) = (1 - e^{-2\pi B_{ch} t})u(t)}$$

Step Response and Rise-time of a First Order System

- Define the difference between the input and the output as
- $e(t) = u(t) - g(t) = e^{-2\pi B_{ch} t}$
- Note that $e(t)$ decreases as B_{ch} increases. This means that as the channel bandwidth increases, the output becomes closer and closer to the input.
- In the ideal case, when the channel bandwidth becomes infinity, the output becomes a step function.
- In essence, to reproduce a step function (or a rectangular pulse), **a channel with infinite bandwidth is needed**.



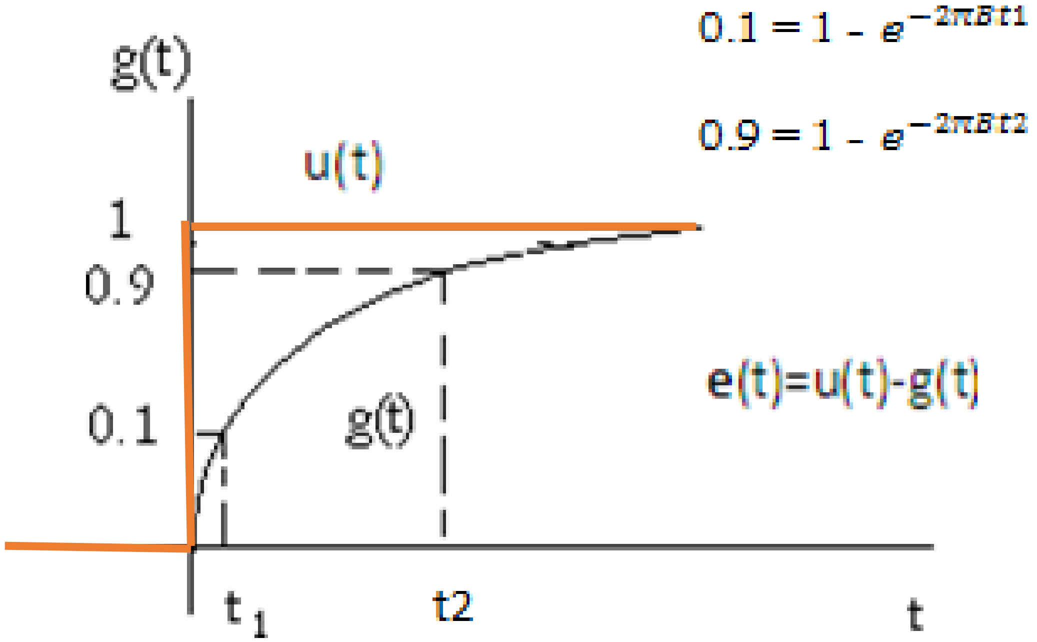
Step Response and Rise-time of a First Order System

- The rise-time is a measure of the speed of rise of the output of a system due to step function applied at its input.
- One common measure is the 10-90 % rise-time, defined as the time it takes for the output to rise between 10% to 90% of the final steady state value when a unit step function is applied to the system input.
- The 10% - 90% rise-time for the first order RC circuit considered above is

$$\boldsymbol{T_r = t_2 - t_1 = \frac{0.35}{B_{ch}}}$$

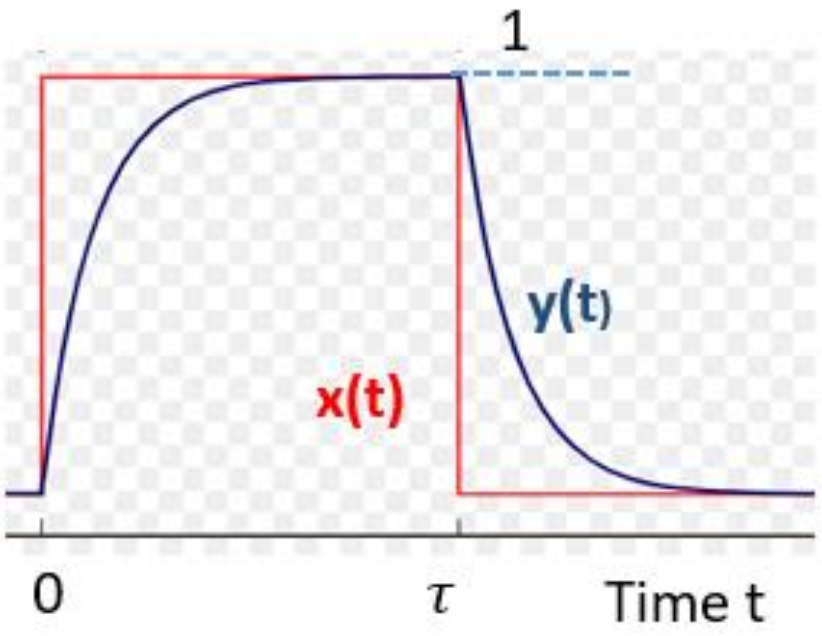
- **Exercise:** For the system above, verify that the rise-time is given as $T_r = \frac{0.35}{B_{ch}}$.

- From this result, we conclude that **increasing the bandwidth of the channel will decrease the rise-time**, implying a faster response.



Pulse Response of a First Order System

- It is the response of the circuit to a pulse of duration τ. For the same RC circuit, considered above, let us apply the pulse
- $x(t) = u(t) - u(t - \tau)$
- Using the linearity and time invariance properties, the output due to the pulse can be obtained from the step response g(t) as



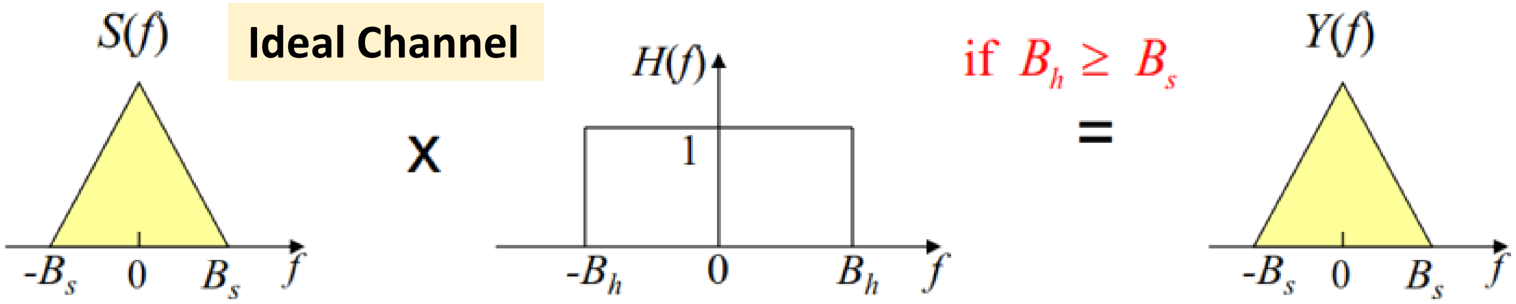
- $y(t) = g(t) - g(t - \tau)$
- $y(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-2\pi B_{ch} t} & 0 < t < \tau \\ (1 - e^{-2\pi B_{ch}\tau})e^{-2\pi B_{ch}(t-\tau)} & t > \tau \end{cases}$
- This response is sketched in the figure below.
- From the equation above, we observe that the output $y(t)$ approximates the input $x(t)$ provided that ($y(\tau) > 0.99$)

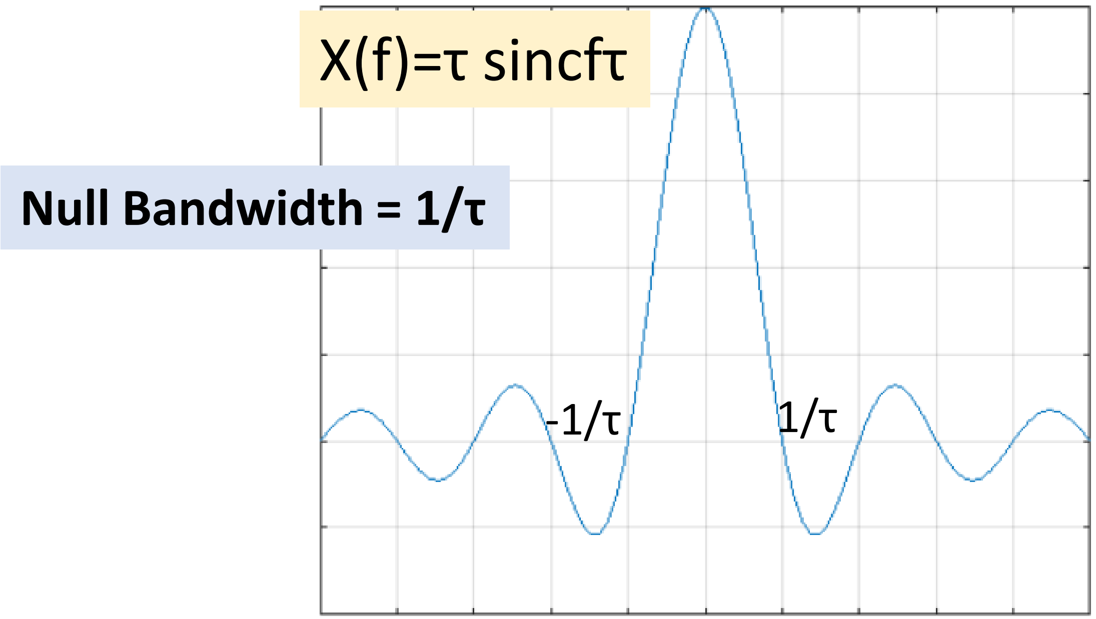
$$B_{ch}\tau \geq 1 \quad \text{or} \quad B_{ch} \geq \frac{1}{\tau}$$

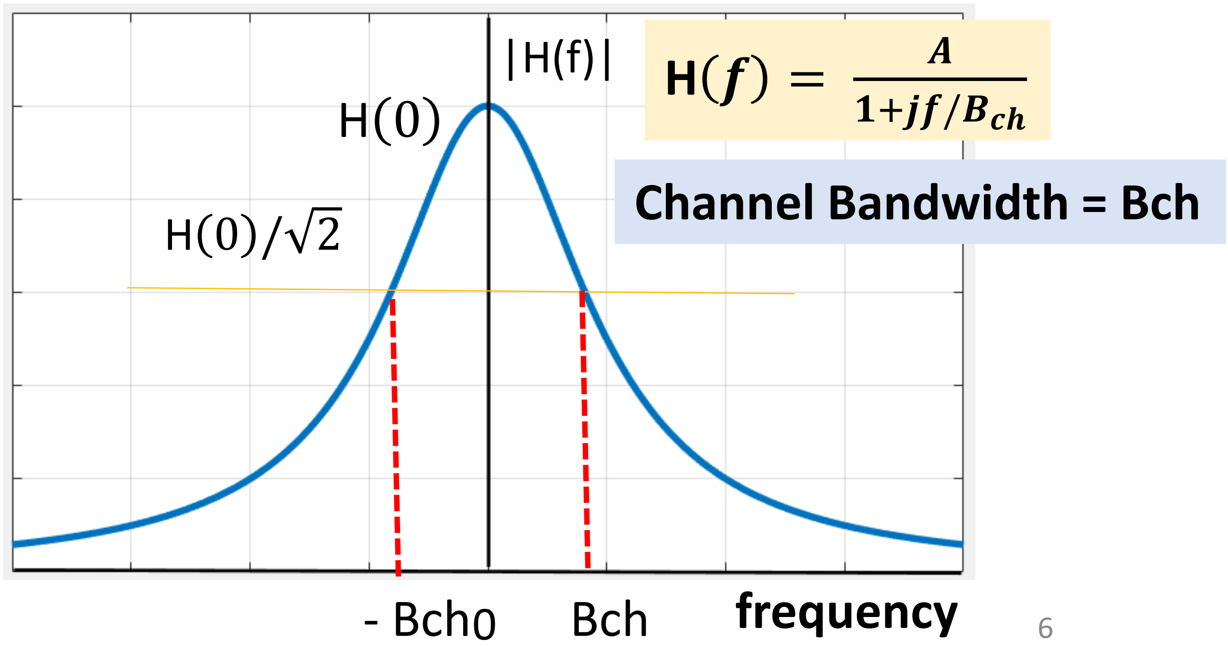
Pulse Response of a First Order System

- The figure below shows the Fourier transform of the input and the channel.
- To reproduce the input, the channel bandwidth should be wider than the message bandwidth
- Y (f) = X(f) H(f)
- $Y(f) \simeq X(f)$

- If the channel bandwidth is much wider than the message bandwidth, then







Relationship to data transmission

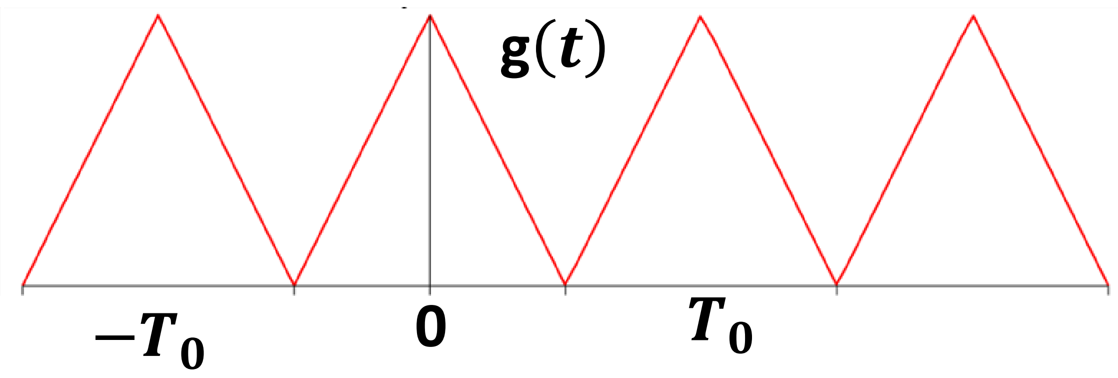
- In digital communication systems, data are transmitted at a rate of R_b bits/sec. The time allocated for each bit is $\tau = \frac{1}{R_b}$. To enable the receiver to recognize the transmitted bit within its allocated slot and to prevent cross talk between neighboring time slots, we require that

$$\boldsymbol{B_{ch} \geq \frac{1}{\tau} = R_b}$$

- **Result**: the channel bandwidth in binary digital communication systems should be larger than the rate of the data sent over the channel.

Autocorrelation and Spectral Density

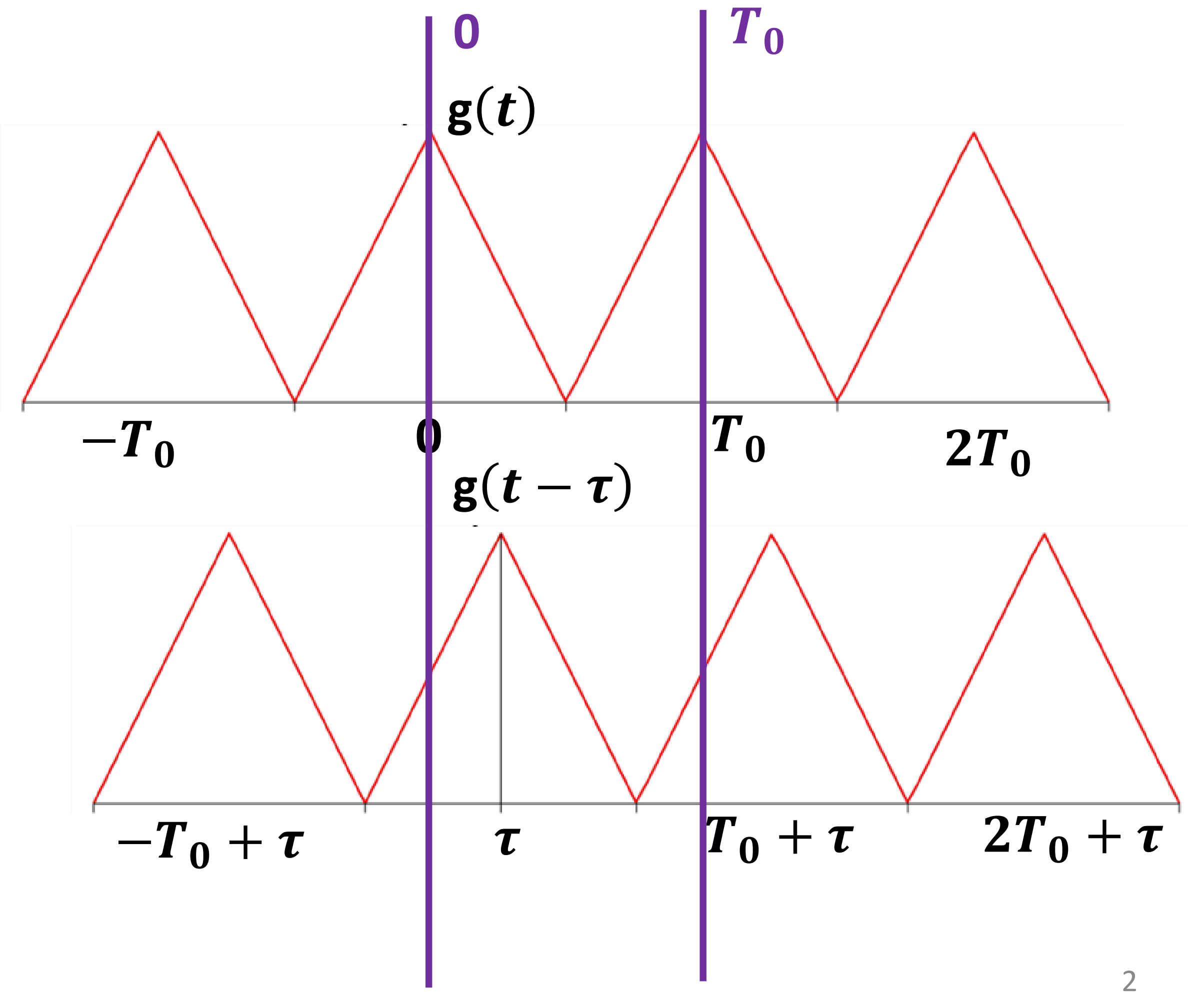
- In this lecture, we define the autocorrelation function of a signal. Also, we present the relationship between the autocorrelation function and the power/energy spectral density.
- In this discussion, we restrict our attention to real signals. First, we consider power signals and then energy signals.
- **Definition**: The autocorrelation function of a signal $g(t)$ is a measure of similarity between $g(t)$ and a delayed version of $g(t)$.



Correlation and Spectral Density

- **Autocorrelation function of a periodic power signal**
- The autocorrelation function of a periodic power signal $g(t)$ with period T_0 is

$$R_g(\tau) = \frac{1}{T_0}\int_0^{T_0} g(t)g(t-\tau)dt$$



Properties of $R_g(\tau)$

- $R_g(\tau = 0) = \frac{1}{T_0}\int_0^{T_0} g(t)^2 dt$; is the total average signal power.
- $R_g(\tau)$ is an even function of τ, i.e., $R_g(\tau) = R_g(-\tau)$.

Correlation and Spectral Density

Properties of $R_g(\tau)$: $R_g(\tau) = \frac{1}{T_0}\int_0^{T_0} g(t)g(t-\tau)dt$

- $R_g(\tau)$ has a maximum (positive) magnitude at τ = 0 , i.e. $|R_g(\tau)| \leq R_g(0)$.

Proof of this property:

Consider the quadratic quantity

$$[\, g(t) \pm g(t+\tau)]^2 \geq 0$$

Taking the time average ($< \mathrm{y(t)} > \frac{1}{T_0}\int_0^{T_0} y(t)dt$) of both sides, and expanding, we get

$$< \{[g(t) \pm g(t+\tau)]^2\} > \; \geq 0$$

$$< \{\, g(t)^2\} > \; + < \{\, g(t+\tau)^2\} > \pm\, 2 < \{\, g(t)g(t+\tau)\} > \geq 0$$

But, $< \{\, g(t)^2\} > \; = R_g(0)$ and $R_g(0) = < \{\, g(t+\tau)^2\} >$ as well.

Combining these results, we get: $-R_g(0) < R_g(\tau) < R_g(0).$

Correlation and Spectral Density

- **Theorem**: The autocorrelation function $R_g(\tau)$ of a periodic signal g(t) is also periodic with the same period T_0.
- **Proof:** $R_g(\tau) = \frac{1}{T_0}\int_0^{T_0} g(t)g(t-\tau)dt$
  - Expand $g(t)$ in a complex Fourier series $g(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$.
  - Form the delayed signal $g(t-\tau) = \sum_{m=-\infty}^{\infty} C_m e^{jm\omega_0 (t-\tau)}$
  - Perform the integration over a complete period T_0 , making use of orthogonality. The result is:
- $\boldsymbol{R_g(\tau) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0\tau} = \sum_{n=-\infty}^{\infty} |C_n|^2 e^{jn\omega_0\tau}}$; Fourier series expansion of $R_g(\tau)$.
- $D_n = |C_n|^2$ Fourier coefficients of $R_g(\tau)$; C_n Fourier coefficients of $g(t)$.
- Note that the real Fourier coefficients D_n of $R_g(\tau)$ are related to the complex Fourier coefficients C_n of $g(t)$ by the relation $D_n = |C_n|^2$.
- The Fourier transform of the autocorrelation function is
- $\boldsymbol{S_g(f) = \mathfrak{I}\{R_g(\tau)\} = \sum_{n=-\infty}^{\infty} |C_n|^2 \delta(f - nf_0)}$; Discrete spectrum
- This is, of course, the power spectral density of $g(t)$, which we considered earlier.

Autocorrelation of a periodic sinusoidal signal

- **Example**: Find the auto-correlation function and power spectral density of the sine signal $g(t) = A\cos(2\pi f_0 t + \theta)$, where A and θ are constants.
- **Solution**: As we know, $g(t)$ is a periodic signal. Therefore, we find $R_g(\tau)$ using the definition
- $$R_g(\tau) = \frac{1}{T_0}\int_0^{T_0} g(t)g(t-\tau)dt$$
- $$R_g(\tau) = \frac{1}{T_0}\int_0^{T_0} A\cos(2\pi f_0 t + \theta)A\cos(2\pi f_0 t - 2\pi f_0 \tau + \theta)dt$$
- $$R_g(\tau) = \frac{A^2}{2T_0}\int_0^{T_0}[\cos(4\pi f_0 t - 2\pi f_0 \tau + 2\theta) + \cos(2\pi f_0 \tau)]dt$$
- $$R_g(\tau) = \frac{A^2}{2T_0}[0 + \cos(2\pi f_0 \tau)T_0]$$
- $R_g(\tau) = \frac{A^2}{2}\cos(2\pi f_0 \tau)$; Periodic with period T_0.
- $S_g(f) = \frac{A^2}{4}\{\delta(f - f_0) + \delta(f + f_0)\}$; power spectral density

Correlation and Spectral Density

Autocorrelation function of energy signals

- When g(t) is an energy signal, $R_g(\tau)$ is defined as

$$\boldsymbol{R_g(\tau) = \int_{-\infty}^{\infty} g(t)g(t-\tau)dt}$$

Properties of R(τ)

- $R_g(\tau = 0) = \int_{-\infty}^{\infty} g(t)^2 dt;$ is the total signal energy.
- $R_g(\tau)$ is an even function of τ, i.e., $R_g(\tau) = R_g(-\tau)$.
- $R_g(\tau)$ has a maximum (positive) magnitude at τ = 0 , i.e. $|R_g(\tau)| \leq R_g(0)$.

Correlation and Spectral Density

Theorem: The autocorrelation function of an energy signal and its energy spectral density (a continuous function of frequency) are **Fourier transform pairs**, i.e.,

- $S_g(f) = \Im\{R_g(\tau)\} = \int_{-\infty}^{\infty} R_g(\tau)e^{-j2\pi f\tau}d\tau;$
- $R_g(\tau) = \int_{-\infty}^{\infty} S_g(f)e^{j2\pi f\tau}df.$

Proof: The autocorrelation function is defined as:

- $R_g(\tau) = \int_{-\infty}^{\infty} g(\lambda)g(\lambda-\tau)d\,\lambda$
- In this integral we have replaced t by λ (both are dummy variables of integration). With this substitution, we can rewrite the integral as
- $R_g(\tau) = \int_{-\infty}^{\infty} g(\lambda)g(-(\tau-\lambda))d\,\lambda$
- One can realize that $R_g(\tau)$ is nothing but the convolution of $g(\tau)$ and $g(-\tau)$. That is,
- $R_g(\tau) = g(\tau) * g(-\tau)$
- Taking the Fourier transform of both sides, we get
- $F\{R_g(\tau)\} = G(f)G^*(f)$, Therefore $\boldsymbol{S_g(f) = \Im\{R_g(\tau)\} = |G(f)|^2}$.

Example: Autocorrelation of a non-periodic signal

- **Example**: Determine the autocorrelation function of the sinc pulse

$$g(t) = Asinc2Wt.$$

- **Solution:** Using the duality property of the Fourier transform, we deduce that
- $$G(f) = \frac{A}{2W} rect(\frac{f}{2W})$$
- The energy spectral density of $g(t)$ is
- $$S_g(f) = |G(f)|^2 = (\frac{A}{2W})^2 rect(\frac{f}{2W})$$
- Taking the inverse Fourier transform, we get the autocorrelation function
- $$R_g(\tau) = \frac{A^2}{2W} sinc2W\tau$$

$$rect\left(\frac{t}{T}\right) \leftrightarrow TsinfT$$

$$sinc2Wt \leftrightarrow \frac{1}{2W} rect(\frac{f}{2W})$$

Autocorrelation function of the rectangular pulse

- **Example:** Find the autocorrelation function of the pulse $g(t) = rect\left(\frac{t-0.5T}{T}\right),\ T = 1.$
- **Solution**: As we saw earlier, this pulse is an energy signal and therefore, we can find its $R_g(\tau)$ as: $R_g(\tau) = \int_{\tau}^{1}(A)(A)dt = A^2(1-\tau)$; $0 < \tau < 1$
- Using the even symmetry property of the autocorrelation function, we can find $R_g(\tau)$ for -ve values of τ as:
- $R_g(\tau) = A^2(1+\tau)$; $-1 < \tau < 0$
- This function is sketched below. Note that that the maximum value occurs at τ = 0 and that g(t) and g(t-τ) become uncorrelated for τ = 1 sec, which is the duration of the pulse.
- The energy spectral density is $S_g(f) = \Im\{R_g(\tau)\} = A^2((sincf)^2)$