Solutions Manual

for

Communication Systems

4th Edition

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Preface

This Manual is written to accompany the fourth edition of my book on Communication Systems. It consists of the following:

- Detailed solutions to all the problems in Chapters 1 to 10 of the book
- MATLAB codes and representative results for the computer experiments in Chapters 1, 2, 3, 4, 6, 7, 9 and 10

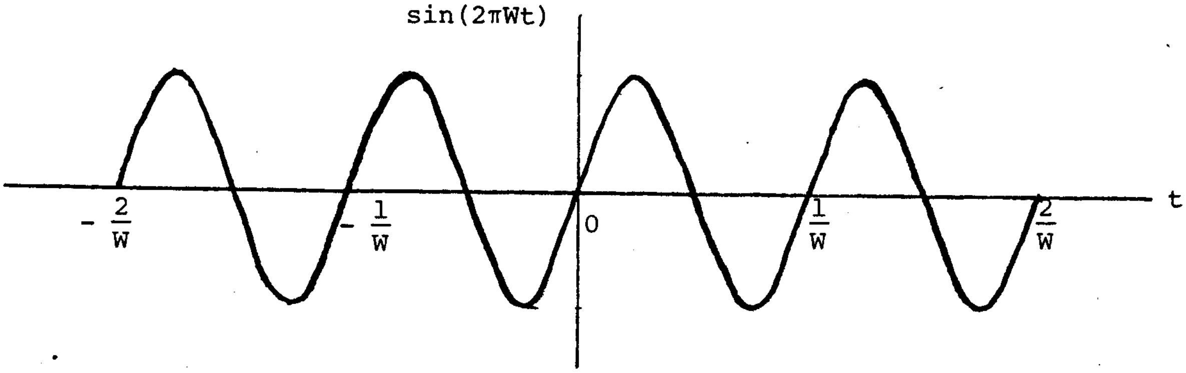
I would like to express my thanks to my graduate student, Mathini Sellathurai, for her help in solving some of the problems and writing the above-mentioned MATLAB codes. I am also grateful to my Technical coordinator, Lola Brooks for typing the solutions to new problems and preparing the manuscript for the Manual.

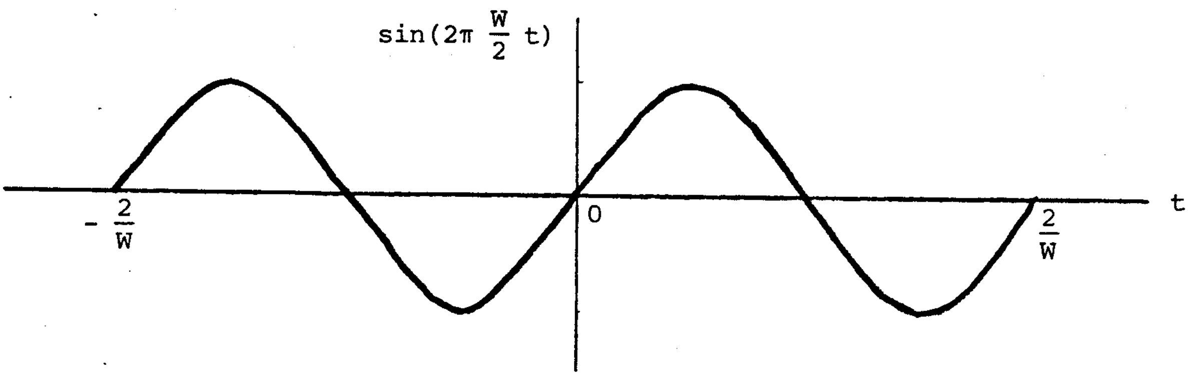
Simon Haykin
Ancaster
April 29, 2000

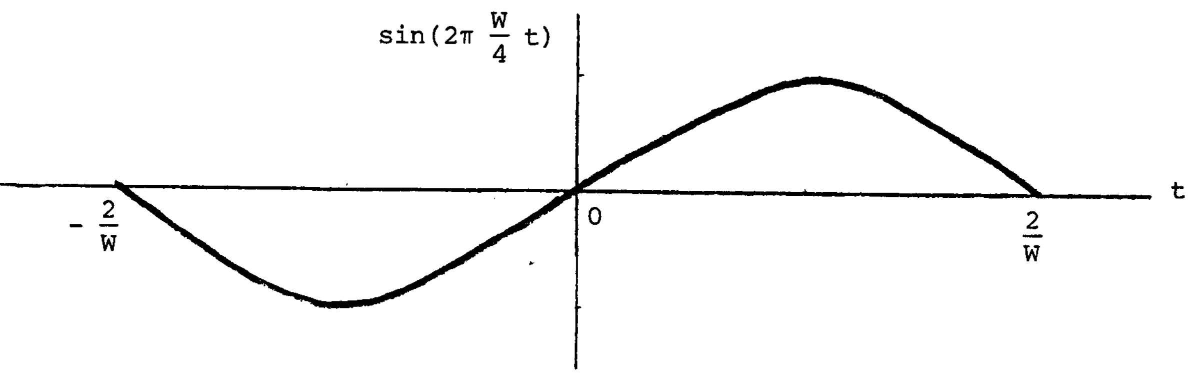
CHAPTER 1

Problem 1.1

As an illustration, three particular sample functions of the random process X(t), corresponding to F = W/4, W/2, and W, are plotted below:







To show that X(t) is nonstationary, we need only observe that every waveform illustrated above is zero at $t = 0$, positive for $0 < t < 1/2W$, and negative for $-1/2W < t < 0$. Thus, the probability density function of the random variable $X(t_1)$ obtained by sampling X(t) at $t_1 = 1/4W$ is identically zero for negative argument, whereas the probability density function of the random variable $X(t_2)$ obtained by sampling X(t) at $t = -1/4W$ is nonzero only for negative arguments. Clearly, therefore,

$f_{X(t_1)}(x_1) \neq f_{X(t_2)}(x_2)$, and the random process X(t) is nonstationary.

Problem 1.2

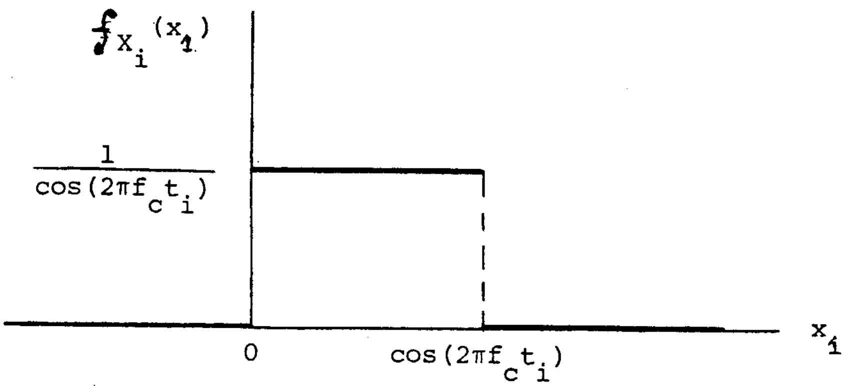
$$X(t) = A\cos(2\pi f_c t)$$

Therefore,

$$X_i = A\cos(2\pi f_c t_i)$$

Since the amplitude A is uniformly distributed, we may write

$$f_{X_i}(x_1) = \begin{cases} \dfrac{1}{\cos(2\pi f_c t_i)}, & 0 \le x_1 \le \cos(2\pi f_c t_i) \\ 0, & \text{otherwise} \end{cases}$$



Similarly, we may write

$$X_{i+\tau} = A\cos[2\pi f_c(t_i+\tau)]$$

and

$$f_{X_{i+\tau}}(x_2) = \begin{cases} \dfrac{1}{\cos[2\pi f_c(t_i+\tau)]}, & 0 \le x_2 \le \cos[2\pi f_c(t_i+\tau)] \\ 0, & \text{otherwise} \end{cases}$$

We thus see that $f_{X_i}(x_1) \ne f_{X_{i+\tau}}(x_2)$, and so the process X(t) is nonstationary.

Problem 1.3

(a) The integrator output at time t is

$$Y(t) = \int_0^t X(\tau)\, d\tau$$

$$= A\int_0^t \cos(2\pi f_c \tau)\, d\tau$$

$$= \frac{A}{2\pi f_c} \sin(2\pi f_c t)$$

Therefore,

$$E[Y(t)] = \frac{\sin(2\pi f_c t)}{2\pi f_c} E[A] = 0$$

$$\text{Var}[Y(t)] = \frac{\sin^2(2\pi f_c t)}{(2\pi f_c)^2} \text{Var}[A]$$

$$= \frac{\sin^2(2\pi f_c t)}{(2\pi f_c)^2} \sigma_A^2 \qquad (1)$$

$Y(t)$ is Gaussian-distributed, and so we may express its probability density function as

$$f_{Y(t)}(y) = \frac{\sqrt{2\pi}\, f_c}{\sigma_A \sin(2\pi f_c t)} \exp\left[- \frac{2\pi^2 f_c^2}{\sin^2(2\pi f_c t)\, \sigma_A^2} y^2\right]$$

(b) From Eq. (1) we note that the variance of $Y(t)$ depends on time t, and so $Y(t)$ is nonstationary.

(c) For a random process to be ergodic it has to be stationary. Since $Y(t)$ is nonstationary, it follows that it is not ergodic.

Problem 1.4

(a) The expected value of $Z(t_1)$ is

$$E[Z(t_1)] = \cos(2\pi t_1)\, E[X] + \sin(2\pi t_1)\, E[Y]$$

Since $E[X] = E[Y] = 0$, we deduce that

$$E[Z(t_1)] = 0$$

Similarly, we find that

$$E[Z(t_2)] = 0$$

Next, we note that

$$\text{Cov}[Z(t_1)Z(t_2)] = E[Z(t_1)Z(t_2)]$$

$$= E\{[X \cos(2\pi t_1) + Y \sin(2\pi t_1)][X \cos(2\pi t_2) + Y \sin(2\pi t_2)]\}$$

$$= \cos(2\pi t_1) \cos(2\pi t_2)\, E[X^2]$$

$$+ [\cos(2\pi t_1)\sin(2\pi t_2) + \sin(2\pi t_1)\cos(2\pi t_2)]E[XY]$$

$$+ \sin(2\pi t_1)\sin(2\pi t_2)E[Y^2]$$

Noting that

$$E[X^2] = \sigma_X^2 + \{E[X]\}^2 = 1$$

$$E[Y^2] = \sigma_Y^2 + \{E[Y]\}^2 = 1$$

$$E[XY] = 0$$

we obtain

$$\mathrm{Cov}[Z(t_1)Z(t_2)] = \cos(2\pi t_1)\cos(2\pi t_2)+\sin(2\pi t_1)\sin(2\pi t_2)$$

$$= \cos[2\pi(t_1-t_2)] \quad (1)$$

Since every weighted sum of the samples of the process Z(t) is Gaussian, it follows that Z(t) is a Gaussian process. Furthermore, we note that

$$\sigma_{Z(t_1)}^2 = E[Z^2(t_1)] = 1$$

This result is obtained by putting $t_1=t_2$ in Eq. (1). Similarly,

$$\sigma_{Z(t_2)}^2 = E[Z^2(t_2)] = 1$$

Therefore, the correlation coefficient of $Z(t_1)$ and $Z(t_2)$ is

$$\rho = \frac{\mathrm{Cov}[Z(t_1)Z(t_2)]}{\sigma_{Z(t_1)}\sigma_{Z(t_2)}}$$

$$= \cos[2\pi(t_1-t_2)]$$

Hence, the joint probability density function of $Z(t_1)$ and $Z(t_2)$

$$f_{Z(t_1),Z(t_2)}(z_1, z_2) = C \exp[-Q(z_1,z_2)]$$

where

$$C = \frac{1}{2\pi\sqrt{1-\cos^2[2\pi(t_1-t_2)]}}$$

$$= \frac{1}{2\pi \sin[2\pi(t_1-t_2)]}$$

$$Q(z_1,z_2) = \frac{1}{2 \sin^2[2\pi(t_1-t_2)]} \{z_1^2 - 2 \cos[2\pi(t_1-t_2)]z_1z_2 + z_2^2\}$$

(b) We note that the covariance of $Z(t_1)$ and $Z(t_2)$ depends only on the time difference t_1-t_2. The process $Z(t)$ is therefore wide-sense stationary. Since it is Gaussian it is also strictly stationary.

Problem 1.5

(a) Let

$$X(t) = A + Y(t)$$

where A is a constant and Y(t) is a zero-mean random process. The autocorrelation function of X(t) is

$$\begin{aligned} R_X(\tau) &= E[X(t+\tau)\,X(t)] \\ &= E\{[A + Y(t+\tau)]\,[A + Y(t)]\} \\ &= E[A^2 + A\,Y(t+\tau) + A\,Y(t) + Y(t+\tau)\,Y(t)] \\ &= A^2 + R_Y(\tau) \end{aligned}$$

which shows that $R_X(\tau)$ contains a constant component equal to A^2.

(b) Let

$$X(t) = A_c \cos(2\pi f_c t + \theta) + Z(t)$$

where $A_c \cos(2\pi f_c t+\theta)$ represents the sinusoidal component of X(t) and θ is a random phase variable. The autocorrelation function of X(t) is

$$\begin{aligned} R_X(\tau) &= E[X(t+\tau)\,X(t)] \\ &= E\{[A_c \cos(2\pi f_c t + 2\pi f_c \tau + \theta) + Z(t+\tau)]\,[A_c \cos(2\pi f_c t + \theta) + Z(t)]\} \\ &= E[A_c^2 \cos(2\pi f_c t + 2\pi f_c \tau + \theta)\cos(2\pi f_c t + \theta)] \\ &\quad + E[Z(t+\tau)\,A_c \cos(2\pi f_c t + \theta)] \\ &\quad + E[A_c \cos(2\pi f_c t + 2\pi f_c \tau + \theta)\,Z(t)] \\ &\quad + E[Z(t+\tau)\,Z(t)] \\ &= (A_c^2/2)\cos(2\pi f_c \tau) + R_Z(\tau) \end{aligned}$$

which shows that $R_X(\tau)$ contains a sinusoidal component of the same frequency as X(t).

Problem 1.6

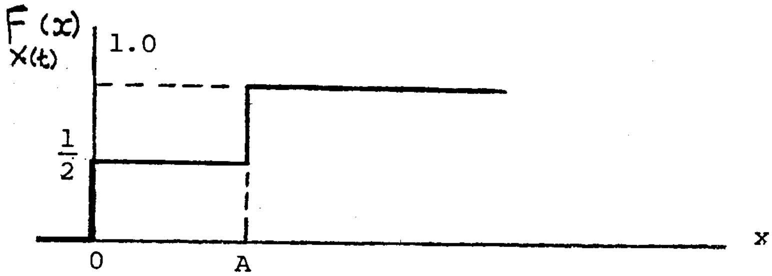
(a) We note that the distribution function of X(t) is

$$F_{X(t)}(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2}, & 0 \le x \le A \\ 1, & A < x \end{cases}$$

and the corresponding probability density function is

$$f_{X(t)}(x) = \frac{1}{2}\,\delta(x) + \frac{1}{2}\,\delta(x - A)$$

which are illustrated below:



(b) By ensemble-averaging, we have

$$E[X(t)] = \int_{-\infty}^{\infty} x\, f_{X(t)}(x)\, dx$$

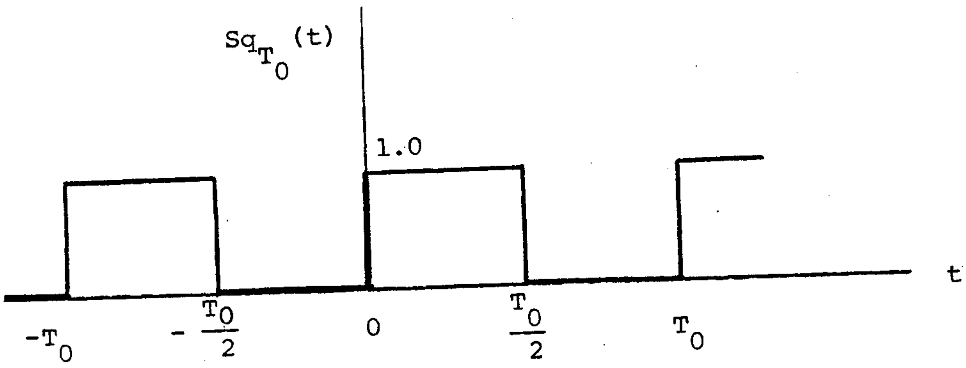
$$= \int_{-\infty}^{\infty} x\, [\tfrac{1}{2}\,\delta(x) + \tfrac{1}{2}\,\delta(x - A)]\, dx$$

$$= \frac{A}{2}$$

The autocorrelation function of X(t) is

$$R_X(\tau) = E[X(t+\tau)\, X(t)]$$

Define the square function $Sq_{T_0}(t)$ as the square-wave shown below:



Then, we may write

$$R_X(\tau) = E[A\ Sq_{T_0}(t - t_d + \tau) \cdot A\ Sq_{T_0}(t - t_d)]$$

$$= A^2 \int_{-\infty}^{\infty} Sq_{T_0}(t - t_d + \tau)\ Sq_{T_0}(t - t_d)\ f_{T_d}(t_d)\ dt_d$$

$$= A^2 \int_{-T_0/2}^{T_0/2} Sq_{T_0}(t - t_d + \tau)\ Sq_{T_0}(t - t_d) \cdot \frac{1}{T_0}\ dt_d$$

$$= \frac{A^2}{2}\left(1 - 2\frac{|\tau|}{T_0}\right),\quad |\tau| \leq \frac{T_0}{2}.$$

Since the wave is periodic with period T_0, $R_X(\tau)$ must also be periodic with period T_0.

(c) On a time-averaging basis, we note by inspection of Fig. P1.6 that the mean is

$$\langle x(t)\rangle = \frac{A}{2}$$

Next, the autocorrelation function

$$\langle x(t+\tau)x(t)\rangle = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t+\tau)\ x(t)\ dt$$

has its maximum value of $A^2/2$ at $\tau = 0$, and decreases linearly to zero at $\tau = T_0/2$. Therefore,

$$\langle x(t+\tau)\ x(t)\rangle = \frac{A^2}{2}\left(1 - 2\frac{|\tau|}{T_0}\right),\ |\tau| \leq \frac{T_0}{2}.$$

Again, the autocorrelation must be periodic with period T_0.

(d) We note that the ensemble-averaging and time-averaging procedures yield the same set of results for the mean and autocorrelation functions. Therefore, X(t) is ergodic in both the mean and the autocorrelation function. Since ergodicity implies wide-sense stationarity, it follows that X(t) must be wide-sense stationary.

Problem 1.7

(a) For $|\tau| > T$, the random variables X(t) and X(t+τ) occur in different pulse intervals and are therefore independent. Thus,

$$E[X(t)\,X(t+\tau)] = E[X(t)]\,E[X(t+\tau)], \qquad |\tau| > T.$$

Since both amplitudes are equally likely, we have $E[X(t)] = E[x(t+\tau)] = A/2$. Therefore, for $|\tau| > T$,

$$\therefore R_X(\tau) = \frac{A^2}{4}.$$

For $|\tau| \leq T$, the random variables occur in the same pulse interval if $t_d < T - |\tau|$. If they do occur in the same pulse interval,

$$E[X(t)\,X(t+\tau)] = \frac{1}{2}A^2 + \frac{1}{2}0^2 = \frac{A^2}{2}.$$

We thus have a conditional expectation:

$$E[X(t)\,X(t+\tau)] = A^2/2, \qquad t_d < T - |\tau|$$

$$= A^2/4, \text{ otherwise.}$$

Averaging over t_d, we get

$$R_X(\tau) = \int_0^{T-|\tau|} \frac{A^2}{2T}\,dt_d + \int_{T-|\tau|}^{T} \frac{A^2}{4T}\,dt_d$$

$$= \frac{A^2}{4}\left(1 - \frac{|\tau|}{T}\right) + \frac{A^2}{4}, \qquad |\tau| \leq T$$

(b) The power spectral density is the Fourier transform of the autocorrelation function. The Fourier transform of

$$g(\tau) = 1 - \frac{|\tau|}{T}, \quad |\tau| \leq T$$

$$= 0, \text{ otherwise,}$$

is given by

$$G(f) = T\,\text{sinc}^2(fT).$$

Therefore,

$$S_X(f) = \frac{A^2}{4}\,\delta(f) + \frac{A^2T}{4}\,\text{sinc}^2(fT) .$$

We next note that

$$\frac{A^2}{4}\int_{-\infty}^{\infty} \delta(f)\, df = \frac{A^2}{4} ,$$

$$\frac{A^2}{4}\int_{-\infty}^{\infty} T\text{sinc}^2(fT)\, df = \frac{A^2}{4} ,$$

$$\int_{-\infty}^{\infty} S_X(f)\, df = R_X(0) = \frac{A^2}{2} .$$

It follows therefore that half the power is in the dc component.

Problem 1.8

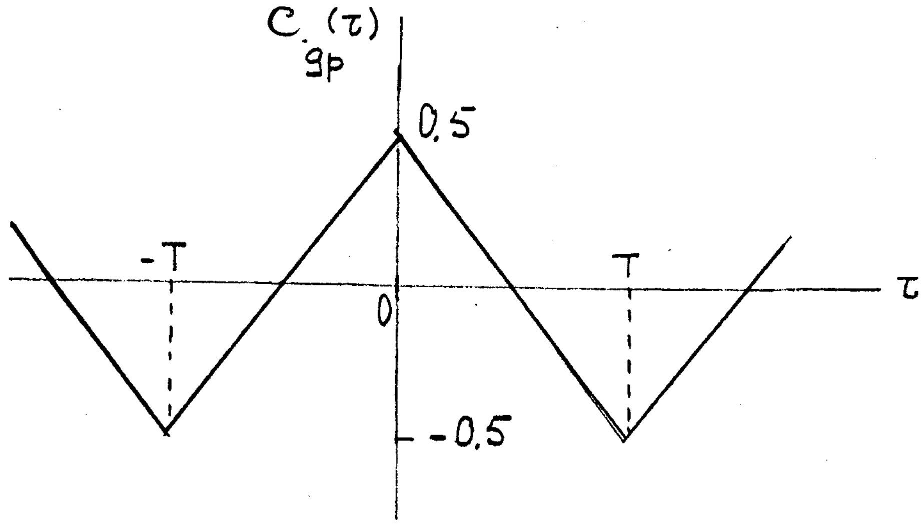
Since

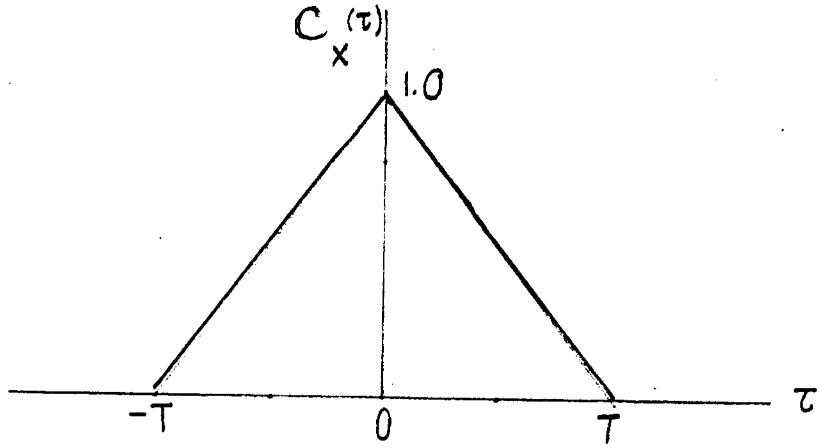
$$Y(t) = g_p(t) + X(t) + \sqrt{3/2}$$

and $g_p(t)$ and $X(t)$ are uncorrelated, then

$$C_Y(\tau) = C_{g_p}(\tau) + C_X(\tau)$$

where $C_{gp}(\tau)$ is the autocovariance of the periodic component and $C_X(\tau)$ is the autocovariance of the random component. $C_Y(\tau)$ is the plot in figure P1.8 shifted down by 3/2, removing the dc component. $C_{gp}(\tau)$ and $C_X(\tau)$ are plotted below:





Both $g_p(t)$ and $X(t)$ have zero mean.

(a) The average power of the periodic component $g_p(t)$ is therefore,

$$\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g_p^2(t)\,dt = C_{g_p}(0) = \frac{1}{2}$$

(b) The average power of the random component $X(t)$ is

$$E[X^2(t)] = C_X(0) = 1$$

Problem 1.9

(a) $R_{XY}(\tau) = E[X(t+\tau)\,Y(t)]$

Replacing τ with $-\tau$:

$$R_{XY}(-\tau) = E[X(t-\tau)\,Y(t)]$$

Next, replacing t-τ with t, we get

$$R_{XY}(-\tau) = E[Y(t+\tau)\ X(t)]$$
$$= R_{YX}(\tau)$$

(b) Form the non-negative quantity

$$E[\{X(t+\tau) \pm Y(t)\}^2] = E[X^2(t+\tau) \pm 2X(t+\tau)\ Y(t) + Y^2(t)]$$
$$= E[X^2(t+\tau)] \pm 2E[X(t+\tau)\ Y(t)] + E[Y^2(t)]$$
$$= R_X(0) \pm 2R_{XY}(\tau) + R_Y(0)$$

Hence,

$$R_X(0) \pm 2R_{XY}(\tau) + R_Y(0) \geq 0$$

or

$$|R_{XY}(\tau)| \leq \frac{1}{2}[R_X(0) + R_Y(0)]$$

Problem 1.10

(a) The cascade connection of the two filters is equivalent to a filter with impulse response

$$h(t) = \int_{-\infty}^{\infty} h_1(u)\ h_2(t-u)\ du$$

The autocorrelation function of Y(t) is given by

$$R_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)\ h(\tau_2)\ R_X(\tau - \tau_1 + \tau_2)\ d\tau_1\ d\tau_2$$

(b) The cross-correlation function of V(t) and Y(t) is

$$R_{VY}(\tau) = E[V(t+\tau)\ Y(t)]$$

The Y(t) and V(t+τ) are related by

$$Y(t) = \int_{-\infty}^{\infty} V(\lambda)\ h_2(t-\lambda)\ d\lambda$$

Therefore,

$$R_{VY}(\tau) = E[V(t+\tau)\int_{-\infty}^{\infty} V(\lambda)\ h_2(t-\lambda)\ d\lambda]$$

$$= \int_{-\infty}^{\infty} h_2(t-\lambda)\, E[V(t+\tau)\, V(\lambda)]\, d\lambda$$

$$= \int_{-\infty}^{\infty} h_2(t-\lambda)\, R_V(t+\tau-\lambda)\, d\lambda$$

Substituting λ for $t-\lambda$:

$$R_{VY}(\tau) = \int_{-\infty}^{\infty} h_2(\lambda)\, R_V(\tau+\lambda)\, d\lambda$$

The autocorrelation function $R_V(\tau)$ is related to the given $R_X(\tau)$ by

$$R_V(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_1(\tau_1)\, h_1(\tau_2)\, R_X(\tau-\tau_1+\tau_2)\, d\tau_1 d\tau_2$$

Problem 1.11

(a) The cross-correlation function $R_{YX}(\tau)$ is

$$R_{YX}(\tau) = E[Y(t+\tau)\, X(t)]$$

The Y(t) and X(t) are related by

$$Y(t) = \int_{-\infty}^{\infty} X(u)\, h(t-u)\, du$$

Therefore,

$$R_{YX}(\tau) = E[\int_{-\infty}^{\infty} X(u)X(t)\, h(t+\tau-u)\, du]$$

$$= \int_{-\infty}^{\infty} h(t+\tau-u)\, E[X(u)X(t)]\, du$$

$$= \int_{-\infty}^{\infty} h(t+\tau-u)\, R_X(u-t)\, du$$

Replacing $t+\tau-u$ by u:

$$R_{YX}(\tau) = \int_{-\infty}^{\infty} h(u)\, R_X(\tau-u)\, du$$

(b) Since $R_{XY}(\tau) = R_{YX}(-\tau)$, we have

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} h(u)\, R_X(-\tau-u)\, du$$

Since $R_X(\tau)$ is an even function of τ:

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} h(u)\, R_X(\tau+u)\, du$$

Replacing u by -u:

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} h(-u)\, R_X(\tau-u)\, du$$

(c) If X(t) is a white noise process with zero mean and power spectral density $N_0/2$, we may write

$$R_X(\tau) = \frac{N_0}{2}\delta(\tau)$$

Therefore,

$$R_{YX}(\tau) = \frac{N_0}{2}\int_{-\infty}^{\infty} h(u)\, \delta(\tau-u)\, du$$

Using the sifting property of the delta function:

$$R_{YX}(\tau) = \frac{N_0}{2} h(\tau)$$

That is,

$$h(\tau) = \frac{2}{N_0} R_{YX}(\tau)$$

This means that we may measure the impulse response of the filter by applying a white noise of power spectral density $N_0/2$ to the filter input, cross-correlating the filter output with the input, and then multiplying the result by $2/N_0$.

Problem 1.12

(a) The power spectral density consists of two components:

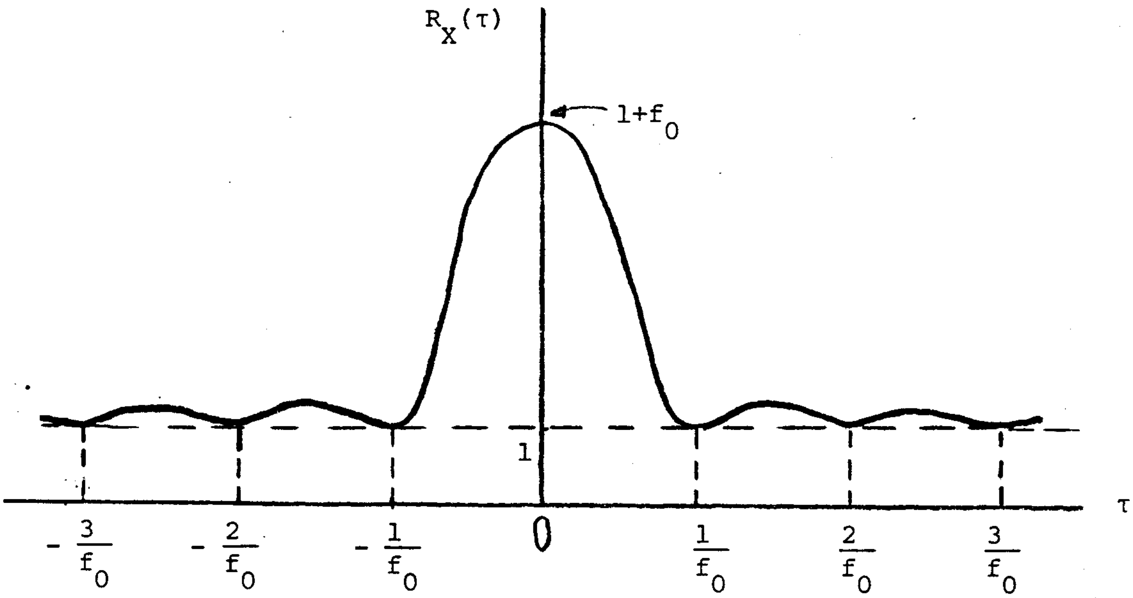
(1) A delta function $\delta(t)$ at the origin, whose inverse Fourier transform is one.

(2) A triangular component of unit amplitude and width $2f_0$, centered at the origin; the inverse Fourier transform of this component is $f_0 \operatorname{sinc}^2(f_0\tau)$.

Therefore, the autocorrelation function of X(t) is

$$R_X(\tau) = 1 + f_0 \operatorname{sinc}^2(f_0\tau)$$

which is sketched below:



(b) Since $R_X(\tau)$ contains a constant component of amplitude 1, it follows that the dc power contained in X(t) is 1.

(c) The mean-square value of X(t) is given by

$$E[X^2(t)] = R_X(0)$$
$$= 1 + f_0$$

The ac power contained in X(f) is therefore equal to f_0.

(d) If the sampling rate is f_0/n, where n is an integer, the samples are uncorrelated. They are not, however, statistically independent. They would be statistically independent if X(t) were a Gaussian process.

Problem 1.13

The autocorrelation function of $n_2(t)$ is

$$R_{N_2}(t_1,t_2) = E[n_2(t_1)\, n_2(t_2)]$$

$$= E\{[n_1(t_1)\cos(2\pi f_c t_1 + \Theta) - n_1(t_1)\sin(2\pi f_c t_1+\Theta)]$$
$$\cdot [n_1(t_2)\cos(2\pi f_c t_2 + \Theta) - n_1(t_2)\sin(2\pi f_c t_2 + \Theta)]\}$$
$$= E[n_1(t_1)\, n_1(t_2)\cos(2\pi f_c t_1 + \Theta)\cos(2\pi f_c t_2 + \Theta)$$
$$- n_1(t_1)\, n_1(t_2)\cos(2\pi f_c t_1 + \Theta)\sin(2\pi f_c t_2 + \Theta)$$
$$- n_1(t_1)\, n_1(t_2)\sin(2\pi f_c t_1 + \Theta)\cos(2\pi f_c t_2 + \Theta)$$

$$+ n_1(t_1)\, n_1(t_2) \sin(2\pi f_c t_1 + \Theta) \sin(2\pi f_c t_2 + \Theta)]$$

$$= E\{n_1(t_1)\, n_1(t_2) \cos[2\pi f_c(t_1 - t_2)]$$

$$- n_1(t_1)\, n_1(t_2) \sin[2\pi f_c(t_1 + t_2) + 2\Theta]\}$$

$$= E[n_1(t_1)\, n_1(t_2)] \cos[2\pi f_c(t_1 - t_2)]$$

$$- E[n_1(t_1)\, n_1(t_2)] \cdot E\{\sin[2\pi f_c(t_1 + t_2) + 2\Theta]\}$$

Since Θ is a uniformly distributed random variable, the second term is zero, giving

$$R_{N_2}(t_1, t_2) = R_{N_1}(t_1, t_2) \cos[2\pi f_c(t_1 - t_2)]$$

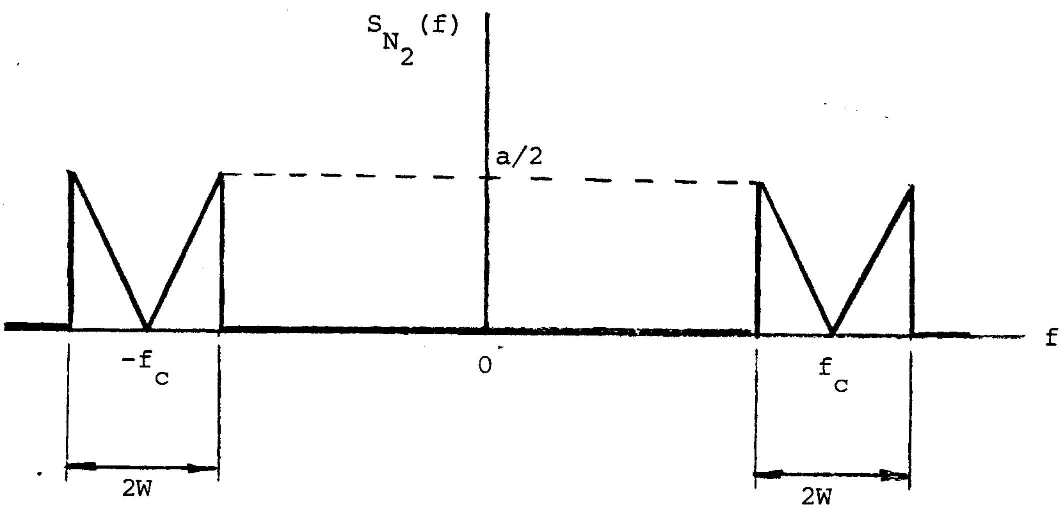
Since $n_1(t)$ is stationary, we find that in terms of $\tau = t_1 - t_2$:

$$R_{N_2}(\tau) = R_{N_1}(\tau) \cos(2\pi f_c \tau)$$

Taking the Fourier transforms of both sides of this relation:

$$S_{N_2}(f) = \frac{1}{2} [S_{N_1}(f + f_c) + S_{N_1}(f - f_c)]$$

With $S_{N_1}(f)$ as defined in Fig. P1.3, we find that $S_{N_2}(f)$ is as shown below:



Problem 1.14

The power spectral density of the random telegraph wave is

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)\, \exp(-j2\pi f\tau)\, d\tau$$

$$= \int_{-\infty}^{0} \exp(2\nu\tau)\, \exp(-j2\pi f\tau)\, d\tau$$

$$+ \int_{0}^{\infty} \exp(-2\nu\tau)\, \exp(-j2\pi f\tau)\, d\tau$$

$$= \frac{1}{2(\nu - j\pi f)} \left[\exp(2\nu\tau - j2\pi f\tau)\right]_{-\infty}^{0}$$

$$- \frac{1}{2(\nu + j\pi f)} \left[\exp(-2\nu\tau - j2\pi f\tau)\right]_{0}^{\infty}$$

$$= \frac{1}{2(\nu - j\pi f)} + \frac{1}{2(\nu + j\pi f)}$$

$$= \frac{\nu}{\nu^2 + \pi^2 f^2}$$

The transfer function of the filter is

$$H(f) = \frac{1}{1 + j2\pi f RC}$$

Therefore, the power spectral density of the filter output is

$$S_Y(f) = |H(f)|^2\, S_X(f)$$

$$= \frac{\nu}{[1 + (2\pi f RC)^2](\nu^2 + \pi^2 f^2)}$$

To determine the autocorrelation function of the filter output, we first expand $S_Y(f)$ in partial fractions as follows

$$S_Y(f) = \frac{\nu}{1 - 4R^2C^2\nu^2} \left[- \frac{1}{(1/2RC)^2 + \pi^2 f^2} + \frac{1}{\nu^2 + \pi^2 f^2}\right]$$

Recognizing that

$$\exp(-2\nu|t|) \rightleftharpoons \frac{\nu}{\nu^2 + \pi^2 f^2}$$

$$\exp(-|t|/RC) \rightleftharpoons \frac{1/2RC}{(1/2RC)^2 + \pi^2 f^2}$$

we obtain the desired result:

$$R_Y(\tau) = \frac{\nu}{1-4R^2C^2\nu^2}\left[\frac{1}{\nu}\exp(-2\nu|\tau|) - 2RC\exp\left(-\frac{|\tau|}{RC}\right)\right]$$

Problem 1.15

We are given

$$y(t) = \int_{t-T}^{t} x(\tau)d\tau$$

For $x(t) = \delta(t)$, the impulse response of this running integrator is, by definition,

$$h(t) = \int_{t-T}^{t} \delta(\tau)d\tau$$

$$= 1 \text{ for } t - T \le 0 \le t \text{ or, equivalently, } 0 \le t \le T$$

Correspondingly, the frequency response of the running integrator is

$$H(f) = \int_{-\infty}^{\infty} h(t)\exp(-j2\pi ft)dt$$

$$= \int_{0}^{T} \exp(-j2\pi ft)dt$$

$$= \frac{1}{j2\pi ft}[1 - \exp(-j2\pi fT)]$$

$$= T\,\mathrm{sinc}\,(fT)\exp(-j\pi fT)$$

Hence the power spectral density $S_Y(f)$ is defined in terms of the power spectral density $S_X(f)$ as follows

$$S_Y(f) = |H(f)|^2 S_X(f)$$

$$= T^2 \mathrm{sinc}^2(fT) S_X(f)$$

Problem 1.16

We are given a filter with the impulse response

$$h(t) = \begin{cases} a\exp(-at), & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases}$$

The frequency response of the filter is therefore

$$H(f) = \int_{-\infty}^{\infty} h(t)\exp(-j2\pi ft)dt$$

$$= \int_0^T a\exp(-at)\exp(-j2\pi ft)dt$$

$$= a\int_0^T \exp(-(a+j2\pi f)t)dt$$

$$= \frac{a}{a+j2\pi f}[-\exp(-(a+j2\pi f)t]_0^T$$

$$= \frac{a}{a+j2\pi f}[1 - \exp(-(a+j2\pi f)T]$$

$$= \frac{a}{a+j2\pi f}[1 - e^{-at}(\cos(2\pi fT) - j\sin(2\pi fT))$$

The squared magnitude response is

$$|H(f)|^2 = \left[\frac{a^2}{a^2+4\pi^2 f^2}(1 - e^{-aT}\cos(2\pi fT))^2 + (e^{-aT}\sin(2\pi fT))^2\right]$$

$$= \frac{a^2}{a^2+4\pi^2 f^2}[1 - 2e^{-aT}\cos(2\pi fT) + e^{-2aT}(\cos^2(2\pi fT) + \sin^2(2\pi fT))]$$

$$= \frac{a^2}{a^2+4\pi^2 f^2}[1 - 2e^{-aT}\cos(2\pi fT) + e^{-2aT}]$$

Correspondingly, we may write

$$S_Y(f) = \frac{a^2}{a^2+4\pi^2 f^2}[1 - 2e^{-aT}\cos(2\pi fT) + e^{-2aT}]S_X(f)$$

Problem 1.17

The autocorrelation function of X(t) is

$$R_X(\tau) = E[X(t+\tau)\, X(t)]$$

$$= A^2\, E[\cos(2\pi Ft + 2\pi F\tau - \Theta)\cos(2\pi Ft - \Theta)]$$

$$= \frac{A^2}{2} E[\cos(4\pi Ft + 2\pi F\tau - 2\Theta) + \cos(2\pi F\tau)]$$

Averaging over Θ, and noting that Θ is uniformly distributed over 2π radians, we get

$$R_X(\tau) = \frac{A^2}{2} E[\cos(2\pi F\tau)]$$

$$= \frac{A^2}{2} \int_{-\infty}^{\infty} f_F(f)\cos(2\pi f\tau)\, df \qquad (1)$$

Next, we note that $R_X(\tau)$ is related to the power spectral density by

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f)\cos(2\pi f\tau)\, df \qquad (2)$$

Therefore, comparing Eqs. (1) and (2), we deduce that the power spectral density of X(t) is

$$S_X(f) = \frac{A^2}{2} f_F(f)$$

When the frequency assumes a constant value, f_c (say), we have

$$f_F(f) = \frac{1}{2}\delta(f-f_c) + \frac{1}{2}\delta(f+f_c)$$

E12

and, correspondingly,

$$S_X(f) = \frac{A^2}{4}\,\delta(f-f_c) + \frac{A^2}{4}\,\delta(f+f_c)$$

Problem 1.18

Let σ_X^2 denote the variance of the random variable X_k obtained by observing the random process X(t) at time t_k. The variance σ_X^2 is related to the mean-square value of X_k as follows

$$\sigma_X^2 = E[X_k^2] - \mu_X^2$$

where $\mu_X = E[X_k]$. Since the process X(t) has zero mean, it follows that

$$\sigma_X^2 = E[X_k^2]$$

Next we note that

$$E[X_k^2] = \int_{-\infty}^{\infty} S_X(f)df$$

We may therefore define the variance σ_X^2 as the total area under the power spectral density $S_X(f)$ as

$$\sigma_X^2 = \int_{-\infty}^{\infty} S_X(f)df \tag{1}$$

Thus with the mean $\mu_X = 0$ and the variance σ_X^2 defined by Eq. (1), we may express the probability density function of X_k as follows

$$f_{X_k}(x) = \frac{1}{\sqrt{2\pi}\,\sigma_X}\exp\left(-\frac{x^2}{2\sigma_X^2}\right)$$

Problem 1.19

The input-output relation of a full-wave rectifier is defined by

$$Y(t_k) = |X(t_k)| = \begin{cases} X(t_k), & X(t_k) \geq 0 \\ -X(t_k), & X(t_k) \leq 0 \end{cases}.$$

The probability density function of the random variable $X(t_k)$, obtained by observing the input random process at time t_k, is defined by

$$f_{X(t_k)}(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp(-\frac{x^2}{2\sigma^2})$$

To find the probability density function of the random variable $Y(t_k)$, obtained by observing the output random process, we need an expression for the inverse relation defining $X(t_k)$ in terms of $Y(t_k)$. We note that a given value of $Y(t_k)$ corresponds to 2 values of $X(t_k)$, of equal magnitude and opposite sign. We may therefore write

$$X(t_k) = -Y(t_k), \qquad X(t_k) < 0$$

$$X(t_k) = Y(t_k), \qquad X(t_k) > 0 .$$

In both cases, we have

$$\left| \frac{dX(t_k)}{dY(t_k)} \right| = 1 .$$

The probability density function of $Y(t_k)$ is therefore given by

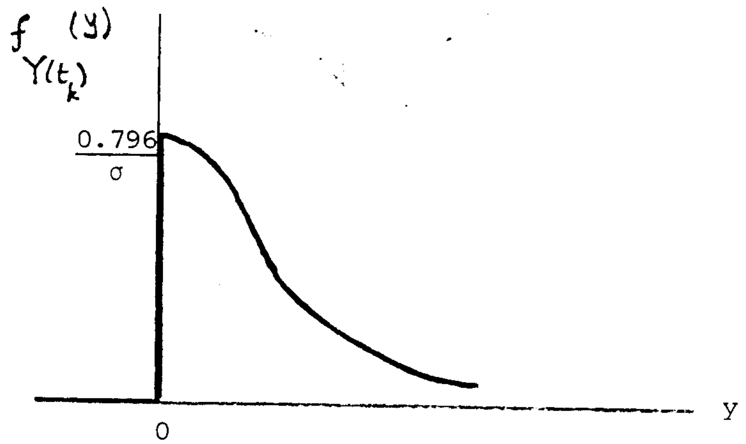
$$f_{Y(t_k)}(y) = f_{X(t_k)}(x = -y) \cdot \left| \frac{dX(t_k)}{dY(t_k)} \right| + f_{X(t_k)}(x = y) \cdot \left| \frac{dX(t_k)}{dY(t_k)} \right|$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{\sigma} \exp(-\frac{y^2}{2\sigma^2})$$

We may therefore write

$$f_{Y(t_k)}(y) = \begin{cases} \sqrt{\frac{2}{\pi}} \frac{1}{\sigma} \exp(-\frac{y^2}{2\sigma^2}), & y \geq 0 \\ 0, & y < 0 . \end{cases}$$

which is illustrated below:



Problem 1.20

(a) The probability density function of the random variable $Y(t_k)$, obtained by observing the rectifier output $Y(t)$ at time t_k, is

$$f_{Y(t_k)}(y) = \begin{cases} \frac{1}{\sqrt{2\pi y}\ \sigma_X} \exp\left(-\frac{y^2}{2\sigma_X}\right), & y \geq 0 \\ 0, & y < 0 \end{cases}$$

where $\sigma_X^2 = E[X^2(t_k)] - \{E[X(t_k)]\}^2$

$= E[X^2(t_k)]$

$= R_X(0)$

The mean value of $Y(t_k)$ is therefore

$$E[Y(t_k)] = \int_{-\infty}^{\infty} y\ f_{Y(t_k)}(y)\ dy$$

$$= \frac{1}{\sqrt{2\pi}\ \sigma_X} \int_0^{\infty} \sqrt{y}\ \exp\left(-\frac{y^2}{2\sigma_X^2}\right) dy \qquad (1)$$

Put

$$\frac{y}{\sigma_X^2} = u^2$$

Then, we may rewrite Eq. (1) as

$$E[Y(t_k)] = \sqrt{\frac{2}{\pi}}\, \sigma_X^2 \int_0^\infty u^2 \exp(-\frac{u^2}{2})\, du$$

$$= \sigma_X^2$$

$$= R_X(0)$$

(b) The autocorrelation function of Y(t) is

$$R_Y(\tau) = E[Y(t+\tau)\, Y(t)]$$

Since $Y(t) = X^2(t)$, we have

$$R_Y(\tau) = E[X^2(t+\tau)\, X^2(t)]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1^2\, x_2^2\, f_{X(t_k+\tau),X(t_k)}(x_1,x_2)\, dx_1\, dx_2 \qquad (2)$$

The $X(t_k+\tau)$ and $X(t_k)$ are jointly Gaussian with a joint probability density function defined by

$$f_{X(t_k+\tau),X(t_k)}(x_1, x_2) = \frac{1}{2\pi\, \sigma_X^2 \sqrt{1-\rho_X^2(\tau)}} \exp[-\frac{x_1^2 - 2\rho_X(\tau)\, x_1\, x_2 + x_2^2}{2\sigma_X^2\, (1-\rho_X^2(\tau))}]$$

where $\sigma_X^2 = R_X(0)$,

$$\rho_X(\tau) = \frac{\mathrm{Cov}[X(t_k+\tau)X(t_k)]}{\sigma_X^2},$$

$$= \frac{R_X(\tau)}{R_X(0)}$$

Rewrite Eq. (2) in the form:

$$R_Y(\tau) = \frac{1}{2\pi\, \sigma_X^2 \sqrt{1-\rho_X^2(\tau)}} \int_{-\infty}^{\infty} x_2^2 \exp(-\frac{x_2^2}{2\sigma_X^2})\, g(x_2)\, dx_2 \qquad (3)$$

where

$$g(x_2) = \int_{-\infty}^{\infty} x_1^2 \exp\{-\frac{[x_1 - \rho_X(\tau)\, x_2]^2}{2\sigma_X^2[1 - \rho_X^2(\tau)]}\}\, dx_1$$

Let

$$u = \frac{x_1 - \rho_X(\tau)\, x_2}{\sigma_X \sqrt{1-\rho_X^2(\tau)}}$$

Then, we may express $g(x_2)$ in the form

$$g(x_2) = \sigma_X \sqrt{1-\rho_X^2(\tau)} \int_{-\infty}^{\infty} \exp(-\frac{u^2}{2}) \{\rho_X^2(\tau)\, x_2^2 + \sigma_X^2[1 - \rho_X^2(\tau)]u^2 + 2\sigma_X\, \rho_X \sqrt{1-\rho_X^2(\tau)}\, ux_2\}\, du$$

However, we note that

$$\int_{-\infty}^{\infty} \exp(-\frac{u^2}{2})\, du = \sqrt{2\pi}$$

$$\int_{-\infty}^{\infty} u \exp(-\frac{u^2}{2})\, du = 0$$

$$\int_{-\infty}^{\infty} u^2 \exp(-\frac{u^2}{2})\, du = \sqrt{2\pi}$$

Hence,

$$g(x_2) = \sigma_X \sqrt{2\pi[1-\rho_X^2(\tau)]}\, \{\rho_X^2(\tau)\, x_2^2 + \sigma_X^2\, [1 - \rho_X^2(\tau)]\}$$

Thus, from Eq. (3):

$$R_Y(\tau) = \frac{1}{\sqrt{2\pi}\, \sigma_X} \int_{-\infty}^{\infty} x_2^2 \exp(-\frac{x_2^2}{2\sigma_X^2}) \{\rho_X^2(\tau)\, x_2^2 + \sigma_X^2\, [1-\rho_X^2(\tau)]\}\, dx_2$$

Using the results:

$$\int_{-\infty}^{\infty} x_2^2 \exp(-\frac{x_2^2}{2\sigma_X^2})\, dx_2 = \sqrt{2\pi}\, \sigma_X^3$$

$$\int_{-\infty}^{\infty} x_2^4 \exp(-\frac{x_2^2}{2\sigma_X^2})\, dx_2 = 3\sqrt{2\pi}\, \sigma_X^5$$

we obtain,

$$R_Y(\tau) = 3\sigma_X^4\, \rho_X^2(\tau) + \sigma_X^4\,[1 - \rho_X^2(\tau)]$$

$$= \sigma_X^4\,[1 + 2\rho_X^2(\tau)]$$

Since $\sigma_X^2 = R_X(0)$

$$\rho_X(\tau) = \frac{R_X(\tau)}{R_X(0)}$$

we obtain

$$R_Y(\tau) = R_X^2(0)\,[1 + 2\,\frac{R_X^2(\tau)}{R_X^2(0)}]$$

$$= R_X^2(0) + 2R_X^2(\tau)$$

The autocovariance function of Y(t) is therefore

$$C_Y(\tau) = R_Y(\tau) - \{E[Y(t_k)]\}^2$$
$$= R_X^2(0) + 2R_X^2(\tau) - R_X^2(0)$$
$$= 2R_X^2(\tau)$$

Problem 1.21

(a) The random variable $Y(t_1)$ obtained by observing the filter output of impulse response $h_1(t)$, at time t_1, is given by

$$Y(t_1) = \int_{-\infty}^{\infty} X(t_1-\tau)\, h_1(\tau)\, d\tau$$

The expected value of $Y(t_1)$ is

$$m_{Y_1} = E[Y(t_1)]$$

$$= H_1(0)\, m_X$$

where

$$H_1(0) = \int_{-\infty}^{\infty} h_1(\tau)\, d\tau$$

The random variable $Z(t_2)$ obtained by observing the filter output of impulse response $h_2(t)$, at time t_2, is given by

$$Z(t_2) = \int_{-\infty}^{\infty} X(t_2-u)\, h_2(u)\, du$$

The expected value of $Z(t_2)$ is

$$m_{Z_2} = E[z(t_2)]$$

$$= H_2(0)\, m_X$$

where

$$H_2(0) = \int_{-\infty}^{\infty} h_2(u)\, du$$

The covariance of $Y(t_1)$ and $Z(t_2)$ is

$$\mathrm{Cov}[Y(t_1)Z(t_2)] = E[(Y(t_1) - \mu_{Y_1})(Z(t_2) - \mu_{Z_2})]$$

$$= E[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (X(t_1-\tau) - \mu_X)\, (X(t_2-u) - \mu_X)\, h_1(\tau)\, h_2(u)\, d\tau\, du]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E[(X(t_1-\tau) - \mu_X)\, (X(t_2-u) - \mu_X)]\, h_1(\tau)\, h_2(u)\, d\tau\, du$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C_X(t_1-t_2-\tau+u)\, h_1(\tau)\, h_2(u)\, d\tau\, du$$

where $C_X(\tau)$ is the autocovariance function of $X(t)$. Next, we note that the variance of $Y(t_1)$ is

$$\sigma^2_{Y_1} = E[(Y(t_1) - \mu_{Y_1})^2]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C_X(\tau-u)\, h_1(\tau)\, h_1(u)\, d\tau\, du$$

and the variance of $Z(t_2)$ is

$$\sigma^2_{Z_2} = E[(Z(t_2) - \mu_{Z_2})^2]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C_X(\tau-u)\, h_2(\tau)\, h_2(u)\, d\tau\, du$$

The correlation coefficient of $Y(t_1)$ and $Z(t_2)$ is

$$\rho = \frac{\mathrm{cov}[Y(t_1)Z(t_2)]}{\sigma_{Y_1}\ \sigma_{Z_2}}$$

Since $X(t)$ is a Gaussian process, it follows that $Y(t_1)$ and $Z(t_2)$ are jointly Gaussian with a probability density function given by

$$f_{Y(t_1),Z(t_2)}(y_1,z_2) = K \exp[-Q(y_1,z_2)]$$

where

$$K = \frac{1}{2\pi\sigma_{Y_1}\sigma_{z_2}\sqrt{1-\rho^2}}$$

$$Q(y_1,z_2) = \frac{1}{2(1-\rho^2)}\left[\left(\frac{y_1-\mu_{Y1}}{\sigma_{Y_1}}\right)^2 - 2\rho\left(\frac{y_1-\mu_{Y1}}{\sigma_{Y_1}}\right)\left(\frac{z_2-\mu_{Z2}}{\sigma_{Z_2}}\right) + \left(\frac{z_2-\mu_{Z2}}{\sigma_{Z_2}}\right)^2\right]$$

(b) The random variables $Y(t_1)$ and $Z(t_2)$ are uncorrelated if and only if their covariance is zero. Since $Y(t)$ and $Z(t)$ are jointly Gaussian processes, it follows that $Y(t_1)$ and $Z(t_2)$ are statistically independent if $\mathrm{Cov}[Y(t_1)Z(t_2)]$ is zero. Therefore, the necessary and sufficient condition for $Y(t_1)$ and $Z(t_2)$ to be statistically independent is that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_X(t_1-t_2-\tau+u)\ h_1(\tau)\ h_2(u)\ d\tau\ du = 0$$

for choices of t_1 and t_2.

Problem 1.22

(a) The filter output is

$$Y(t) = \int_{-\infty}^{\infty} h(\tau)\, X(t-\tau)\, d\tau$$

$$= \frac{1}{T} \int_{0}^{T} X(T-\tau)\, d\tau$$

Put $T-\tau=u$. Then, the sample value of $Y(t)$ at $t=T$ equals

$$Y = \frac{1}{T} \int_{0}^{T} X(u)\, du$$

The mean of Y is therefore

$$E[Y] = E[\frac{1}{T} \int_{0}^{T} X(u)\, du]$$

$$= \frac{1}{T} \int_{0}^{T} E[X(u)]\, du$$

$$= 0$$

The variance of Y is

$$\sigma_Y^2 = E[Y^2] - \{E[Y]\}^2$$

$$= R_Y(0)$$

$$= \int_{-\infty}^{\infty} S_Y(f)\, df$$

$$= \int_{-\infty}^{\infty} S_X(f)\, |H(f)|^2\, df$$

But

$$H(f) = \int_{-\infty}^{\infty} h(t) \exp(-j2\pi ft)\, dt$$

$$= \frac{1}{T} \int_{0}^{T} \exp(-j2\pi ft)\, dt$$

$$= \frac{1}{T} \left[\frac{\exp(-j2\pi ft)}{-j2\pi f}\right]_{0}^{T}$$

$$= \frac{1}{j2\pi fT} [1 - \exp(-j2\pi fT)]$$

$$= \text{sinc}(fT) \exp(-j\pi fT)$$

Therefore,

$$\sigma_Y^2 = \int_{-\infty}^{\infty} S_X(f)\, \text{sinc}^2(fT)\, df$$

(b) Since the filter input is Gaussian, it follows that Y is also Gaussian. Hence, the probability density function of Y is

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma_Y} \exp\left(-\frac{y^2}{2\sigma_Y^2}\right)$$

where σ_Y^2 is defined above.

Problem 1.23

(a) The power spectral density of the noise at the filter output is given by

$$S_N(f) = \frac{N_0}{2} \left| \frac{j2\pi fL}{R+j2\pi fL} \right|^2$$

$$S_N(f) = \frac{N_0}{2} \frac{(2\pi f L/R)^2}{1+(2\pi f L/R)^2}$$

$$= \frac{N_0}{2} [1 - \frac{1}{1+(2\pi f L/R)^2}]$$

The autocorrelation function of the filter output is therefore

$$R_N(\tau) = \frac{N_0}{2} [\delta(\tau) - \frac{R}{2L} \exp(- \frac{R}{L} |\tau|)]$$

(b) The mean of the filter output is equal to H(0) times the mean of the filter input. The process at the filter input has zero mean. The value H(0) of the filter's transfer function H(f) is zero. It follows therefore that the filter output also has a zero mean.

The mean-square value of the filter output is equal to $R_N(0)$. With zero mean, it follows therefore that the variance of the filter output is

$$\sigma_N^2 = R_N(0)$$

Since $R_N(\tau)$ contains a delta function $\delta(\tau)$ centered on $\tau = 0$, we find that, in theory, σ_N^2 is infinitely large.

Problem 1.24

(a) The noise equivalent bandwidth is

$$W_N = \frac{1/2}{|H(0)|^2} \int_{-\infty}^{\infty} |H(f)|^2 \, df$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \frac{df}{1 + (f/f_0)^{2n}}$$

$$= \int_{0}^{\infty} \frac{df}{1 + (f/f_0)^{2n}}$$

$$= \frac{\pi f_0}{2n \sin(\pi/2n)}$$

$$= \frac{f_0}{\mathrm{sinc}(1/2n)}$$

(b) When the filter order n approaches infinity, we have

$$W_N = f_0 \lim_{n \to \infty} \frac{1}{\mathrm{sinc}(1/2n)}$$

$$= f_0$$

Problem 1.25

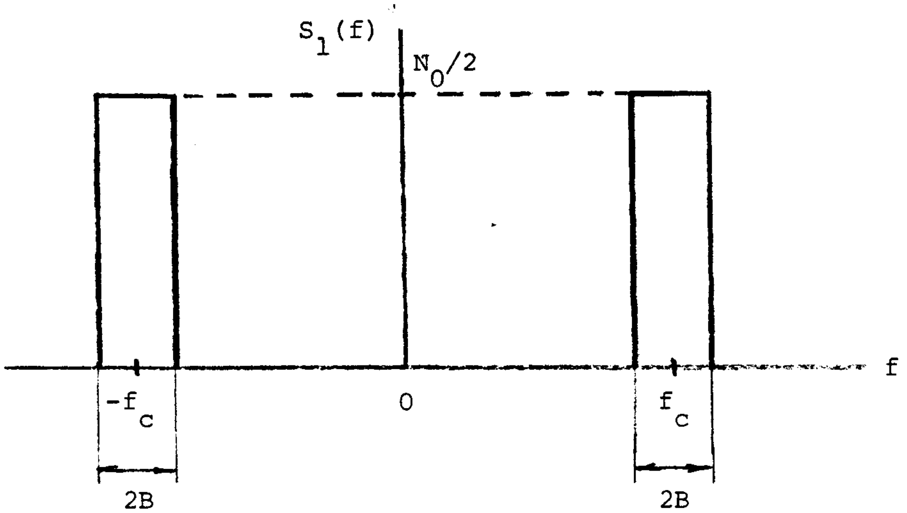
The process X(t) defined by

$$X(t) = \sum_{k=-\infty}^{\infty} h(t - \tau_k),$$

where $h(t - \tau_k)$ is a current pulse at time τ_k, is stationary for the following simple reason. There is no distinguishing origin of time.

Problem 1.26

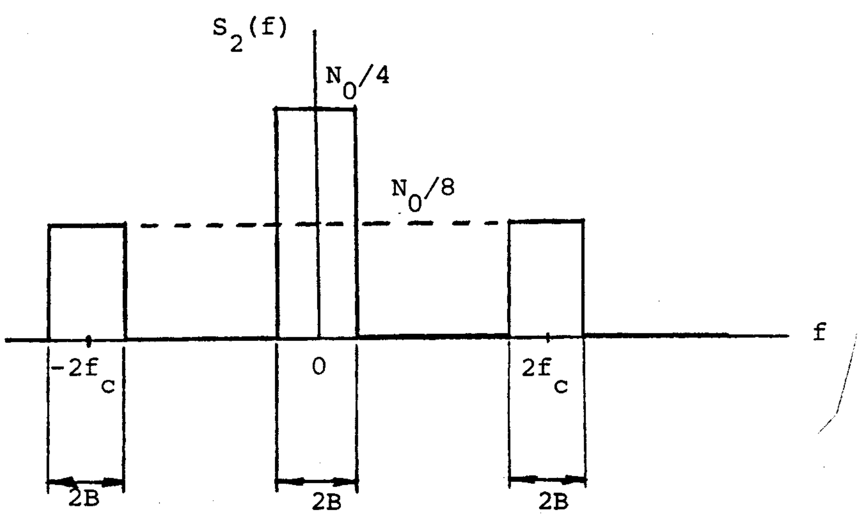
(a) Let $S_1(f)$ denote the power spectral density of the noise at the first filter output. The dependence of $S_1(f)$ on frequency is illustrated below:



Let $S_2(f)$ denote the power spectral density of the noise at the mixer output. Then, we may write

$$S_2(f) = \frac{1}{4}[S_1(f+f_c) + S_1(f-f_c)]$$

which is illustrated below:



The power spectral density of the noise n(t) at the second filter output is therefore defined by

$$S_0(f) = \begin{cases} \dfrac{N_0}{4}, & -B<f<B \\ 0, & \text{otherwise} \end{cases}$$

The autocorrelation function of the noise n(t) is

$$R_o(\tau) = \frac{N_0 B}{2}\,\mathrm{sinc}(2B\tau)$$

(b) The mean value of the noise at the system output is zero. Hence, the variance and mean-square value of this noise are the same. Now, the total area under $S_0(f)$ is equal to $(N_0/4)(2B) = N_0B/2$. The variance of the noise at the system output is therefore $N_0B/2$.

(c) The maximum rate at which n(t) can be sampled for the resulting samples to be uncorrelated is 2B samples per second.

Problem 1.27

(a) The autocorrelation function of the filter output is

$$R_X(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1)\, h(\tau_2)\, R_W(\tau-\tau_1+\tau_2)\, d\tau_1\, d\tau_2$$

Since $R_W(\tau) = (N_0/2)\,\delta(\tau)$, we find that the impulse response h(t) of the filter must satisfy the condition:

$$R_X(\tau) = \frac{N_0}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1)\, h(\tau_2)\, \delta(\tau-\tau_1+\tau_2)\, d\tau_1\, d\tau_2$$

$$= \frac{N_0}{2} \int_{-\infty}^{\infty} h(\tau+\tau_2)\, h(\tau_2)\, d\tau_2$$

(b) For the filter output to have a power spectral density equal to $S_X(f)$, we have to choose the transfer function H(f) of the filter such that

$$S_X(f) = \frac{N_0}{2} |H(f)|^2$$

or

$$|H(f)| = \sqrt{\frac{2S_X(f)}{N_0}}$$

Problem 1.28

(a) Consider the part of the analyzer in Fig. 1.19 defining the in-phase component $n_I(t)$, reproduced here as Fig. 1:

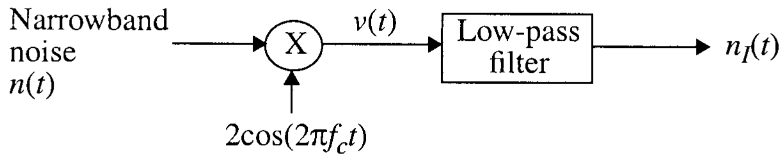


Figure 1

For the multiplier output, we have

$$v(t) = 2n(t)\cos(2\pi f_c t)$$

Applying Eq. (1.55) in the textbook, we therefore get

$$S_V(f) = [S_N(f-f_c) + S_N(f+f_c)]$$

Passing $v(t)$ through an ideal low-pass filter of bandwidth B, defined as one-half the bandwidth of the narrowband noise $n(t)$, we obtain

$$S_{N_I}(f) = \begin{cases} S_V(f) & \text{for } -B \le f \le B \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} S_N(f-f_c) + S_N(f+f_c) & \text{for } -B \le f \le B \\ 0 & \text{otherwise} \end{cases} \qquad (1)$$

For the quadrature component, we have the system shown in Fig. 2:

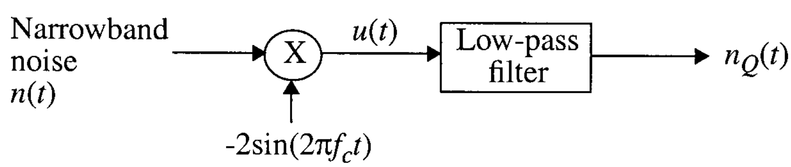


Fig. 2

The multiplier output $u(t)$ is given by

$$u(t) = -2n(t)\sin(2\pi f_c t)$$

Hence,

$$S_U(f) = [S_N(f-f_c) + S_N(f+f_c)]$$

and

$$S_{N_Q}(f) = \begin{cases} S_U(f) & \text{for } -B \le f \le B \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} S_N(f-f_c) + S_N(f+f_c) & \text{for } -B \le f \le B \\ 0 & \text{otherwise} \end{cases} \qquad (2)$$

Accordingly, from Eqs. (1) and (2) we have

$$S_{N_I}(f) = S_{N_Q}(f)$$

(b) Applying Eq. (1.78) of the textbook to Figs. 1 and 2, we obtain

$$S_{N_I N_Q}(f) = |H(f)|^2 S_{VU}(f) \qquad (3)$$

where

$$|H(f)| = \begin{cases} 1 & \text{for } -B \le f \le B \\ 0 & \text{otherwise} \end{cases}$$

Applying Eq. (1.23) of the textbook to the problem at hand:

$$R_{VU}(\tau) = 2R_N(\tau)\sin(2\pi f_c \tau) = \frac{1}{j}R_N(\tau)(e^{j2\pi f_c \tau} - e^{-j2\pi f_c \tau})$$

Applying the Fourier transform to both sides of this relation:

$$S_{VU}(t) = \frac{1}{j}(S_N(f-f_c) - S_N(f+f_c)) \qquad (4)$$

Substituting Eq. (4) into (3):

$$S_{N_I N_Q}(f) = \begin{cases} j[S_N(f+f_c) - S_N(f-f_c)] & \text{for } -B \le f \le B \\ 0 & \text{otherwise} \end{cases}$$

which is the desired result.

Problem 1.29

If the power spectral density $S_N(f)$ of narrowband noise $n(t)$ is symmetric about the midband frequency f_c we then have

$$S_N(f-f_c) = S_N(f+f_c) \quad \text{for} \quad -B \le f \le B$$

From part (b) of Problem 1.28, the cross-spectral densities between the in-phase noise component $n_I(t)$ and quadrature noise component $n_Q(t)$ are zero for all frequencies:

$$S_{N_I N_Q}(f) = 0 \quad \text{for all} \quad f$$

This, in turn, means that the cross-correlation functions $R_{N_I N_Q}(\tau)$ and $R_{N_Q N_I}(\tau)$ are both zero, that is,

$$E[N_I(t_k+\tau)N_Q(t_k)] = 0$$

which states that the random variables $N_I(t_k + \tau)$ and $N_Q(t_k)$, obtained by observing $n_I(t)$ at time $t_k + \tau$ and observing $n_Q(t)$ at time t_k, are orthogonal for all t.

If the narrow-band noise $n(t)$ is Gaussian, with zero mean (by virtue of the narrowband nature of $n(t)$), then it follows that both $N_I(t_k + \tau)$ and $N_Q(t_k)$ are also Gaussian with zero mean. We thus conclude the following:

- $N_I(t_k + \tau)$ and $N_Q(t_k)$ are both uncorrelated
- Being Gaussian and uncorrelated, $N_I(t_k + \tau)$ and $N_Q(t_k)$ are therefore statistically independent.

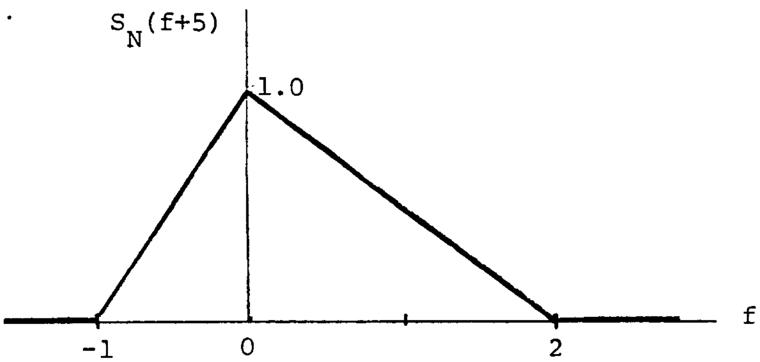
That is, the in-phase noise component $n_I(t)$ and quadrature noise component $n_Q(t)$ are statistically independent.

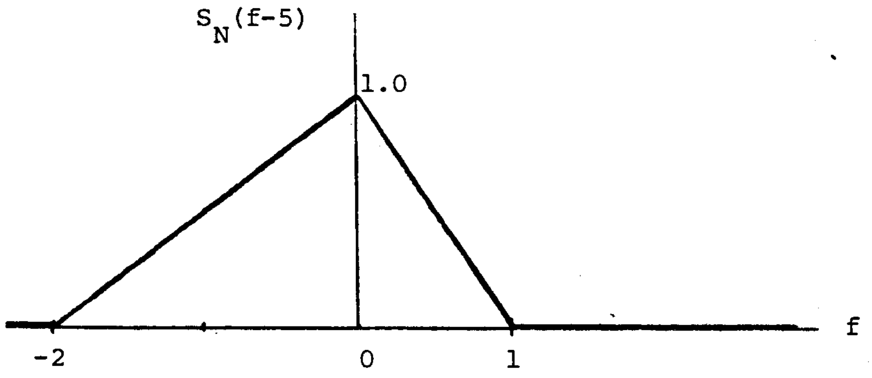
Problem 1.30

(a) The power spectral density of the in-phase component or quadrature component is defined by

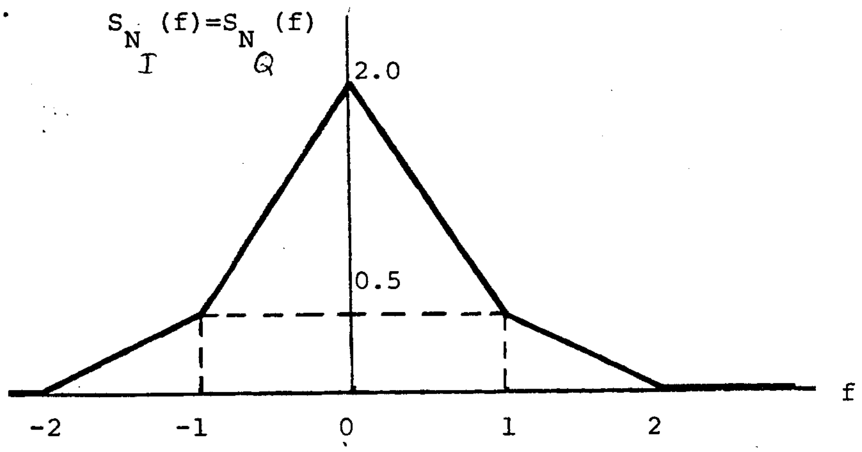
$$S_{N_I}(f) = S_{N_Q}(f) = \begin{cases} S_N(f+f_c) + S_N(f-f_c), & -B \le f \le B \\ 0 & \text{otherwise} \end{cases}$$

We note that, for $-2 \le f \le 2$, the $S_N(f+5)$ and $S_N(f-5)$ are as shown below:





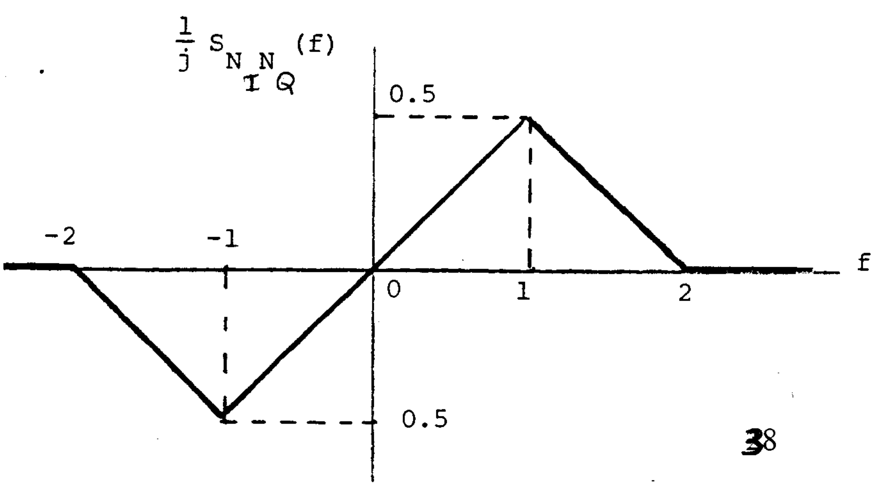
We thus find that $S_{N_I}(f)$ or $S_{N_Q}(f)$ is as shown below:



(b) The cross-spectral density $S_{N_I N_Q}(f)$ is defined by

$$S_{N_I N_Q}(f) = \begin{cases} j[S_N(f+f_c) - S_N(f-f_c)] , & -B \leq f \leq B \\ 0 , & \text{otherwise} \end{cases}$$

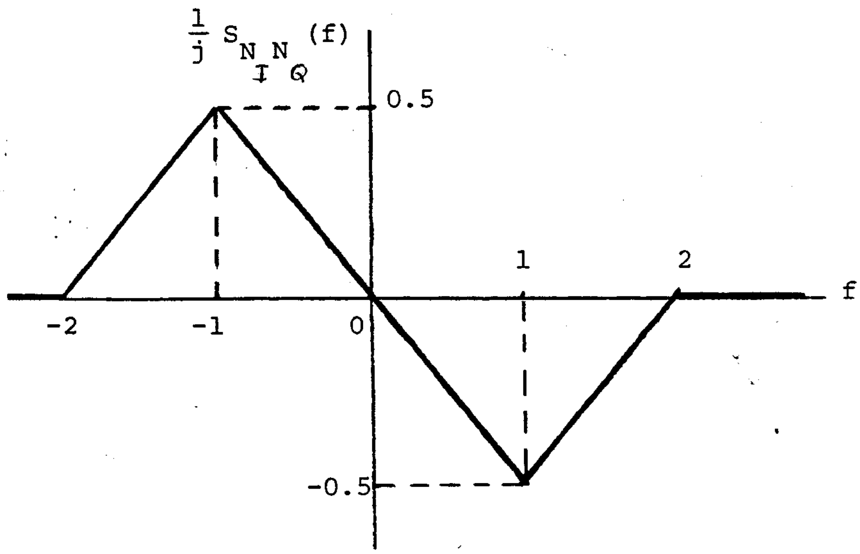
We therefore find that $S_{N_I N_Q}(f)/j$ is as shown below:



Next, we note that

$$S_{N_I N_Q}(f) = S^*_{N_I N_Q}(f)$$

We thus find that $S_{N_I N_Q}(f)$ is as shown below:



Problem 1.31

(a) Express the noise n(t) in terms of its in-phase and quadrature components as follows:

$$n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)$$

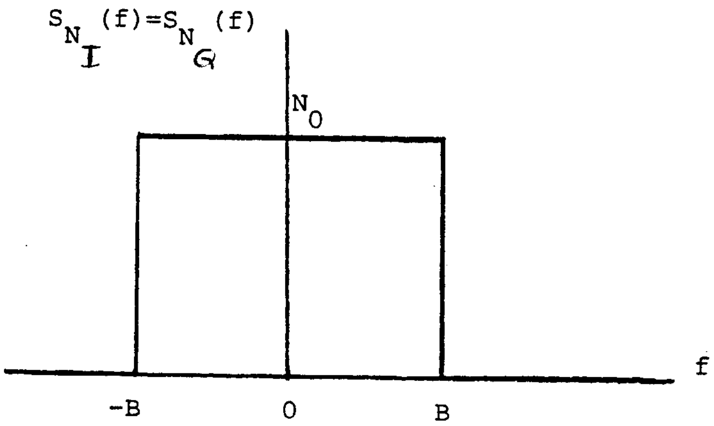
The envelope of n(t) is

$$r(t) = \sqrt{n_I^2(t) + n_Q^2(t)}$$

which is Rayleigh-distributed. That is

$$f_R(r) = \begin{cases} \frac{r}{\sigma^2}\exp(-\frac{r^2}{2\sigma^2}), & r \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

To evaluate the variance σ^2, we note that the power spectral density of $n_I(t)$ or $n_Q(t)$ is as follows



Since the mean of n(t) is zero, we find that

$$\sigma^2 = 2\, N_0 B$$

Therefore,

$$f_R(r) = \begin{cases} \dfrac{r}{2N_0B} \exp\left(-\dfrac{r^2}{4N_0B}\right), & r \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

(b) The mean value of the envelope is equal to $\sqrt{\pi N_0 B}$, and its variance is equal to 0.858 N_0B.

Problem 1.32

Autocorrelation of a Sinusoidal Wave Plus White Gaussian Noise

In this computer experiment, we study the statistical characterization of a random process $X(t)$ consisting of a sinusoidal wave component $A\cos(2\pi f_c t + \Theta)$ and a white Gaussian noise process $W(t)$ of zero mean and power spectral density $N_0/2$. That is, we have

$$X(t) = A\cos 2\pi f_c t + \Theta + W(t) \tag{1}$$

where Θ is a uniformly distributed random variable over the interval$(-\pi,\pi)$. Clearly, the two components of the process $X(t)$ are independent. The autocorrelation function of $X(t)$ is therefore the sum of the individual autocorrelation functions of the signal (sinusoidal wave) component and the noise component, as shown by

$$R_X(\tau) = \frac{A^2}{2}\cos(2\pi f_c \tau) + \frac{N_0}{2}\delta(\tau) \tag{2}$$

This equation shows that for $|\tau| > 0$, the autocorrelation function $R_X(\tau)$ has the same sinusoidal waveform as the signal component. We may generalize this result by stating that the presence of a periodic signal component corrupted by additive white noise can be detected by computing the autocorrelation function of the composite process $X(t)$.

The purpose of the experiment described here is to perform this computation using two different methods: (a) ensemble averaging, and (b) time averaging. The signal of interest consists of a sinusoidal signal of frequency $f_c = 0.002$ and phase $\theta = -\pi/2$, truncated to a finite duration T = 1000; the amplitude A of the sinusoidal signal is set to $\sqrt{2}$ to give unit average power. A particular realization $x(t)$ of the random process $X(t)$ consists of this sinusoidal signal and additive white Gaussian noise; the power spectral density of the noise for this realization is $(N_0/2) = 1000$. The original sinusoidal is barely recognizable in $x(t)$.

(a) For *ensemble-average computation* of the autocorrelation function, we may proceed as follows:

- Compute the product $x(t + \tau)x(t)$ for some fixed time t and specified time shift τ, where $x(t)$ is a particular realization of the random process $X(t)$.
- Repeat the computation of the product $x(t + \tau)x(t)$ for M independent realizations (i.e., sample functions) of the random process X(t).
- Compute the average of these computations over M.
- Repeat this sequence of computations for different values of τ.

The results of this computation are plotted in Fig. 1 for $M = 50$ realizations. The picture portrayed here is in perfect agreement with theory defined by Eq. (2). The important point to note here is that the ensemble-averaging process yields a clean estimate of the true autocorrelation function $R_X(\tau)$ of the random process $X(t)$. Moreover, the presence of the sinusoidal signal is clearly visible in the plot of $R_X(\tau)$ versus t.

(b) For the time-average estimation of the autocorrelation function of the process $X(t)$, we invoke ergodicity and use the formula

$$R_X(\tau) = \lim_{T \to \infty} R_x(\tau, T) \tag{3}$$

where $R_x(\tau,T)$ is the time-averaged autocorrelation function:

$$R_x(\tau, T) = \frac{1}{2T}\int_{-T}^{T} x(t+\tau)x(t)dt \tag{4}$$

The $x(t)$ in Eq. (4) is a particular realization of the process $X(t)$, and $2T$ is the total observation interval. Define the *time-windowed function*

$$x_T(t) = \begin{cases} x(t), & -T \le t \le T \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

We may then rewrite Eq. (4) as

$$R_x(\tau, T) = \frac{1}{2T}\int_{-\infty}^{\infty} x_T(t+\tau)x_T(t)dt \tag{6}$$

For a specified time shift τ, we may compute $R_x(\tau,T)$ directly using Eq. (6). However, from a computational viewpoint, it is more efficient to use an *indirect* method based on Fourier transformation. First, we note From Eq. (6) that the time-averaged autocorrelation function $R_x(\tau,T)$ may be viewed as a scaled form of convolution in the τ-domain as follows:

$$R_x(\tau, T) = \frac{1}{2T}\int_{-\infty}^{\infty} x_T(\tau) \star \; x_T(-\tau) \tag{7}$$

where the star denotes convolution and $x_T(\tau)$ is simply the time-windowed function $x_T(t)$ with t replaced by τ. Let $X_T(f)$ denote the Fourier transform $x_T(\tau)$; note that $X_T(f)$ is the same as the Fourier transform $X(f,T)$. Since convolution in the τ-domain is transformed into multiplication in the frequency domain, we have the Fourier-transform pair:

$$R_x(\tau, T) = \frac{1}{2T}\int_{-\infty}^{\infty} |X_T(f)|^2 \tag{8}$$

The parameter $|X_T(f)|^2/2T$ is recognized as the periodogram of the process $X(t)$. Equation (8) is a mathematical description of the *correlation theorem*, which may be formally stated as follows: *The time-averaged autocorrelation function of a sample function pertaining to a random process and its periodogram, based on that sample function, constitute a Fourier-transform pair.*

We are now ready to describe the indirect method for computing the time-averaged autocorrelation function $R_x(\tau,T)$:

- Compute the Fourier transform $X_T(f)$ of time-windowed function $x_T(\tau)$.
- Compute the periodogram $|X_T(f)|^2/2T$.
- Compute the inverse Fourier transform of $|X_T(f)|^2/2T$.

To perform these calculations on a digital computer, the customary procedure is to use the fast Fourier transform (FFT) algorithm. With $x_T(\tau)$ uniformly sampled, the computational procedure described herein yields the desired values of $R_x(\tau,T)$ for $\tau = 0, \Delta, 2\Delta, \cdots, (N-1)\Delta$ where Δ is the sampling period and N is the total number of samples used in the computation. Figure 2 presents the results obtained in the time-averaging approach of "estimating" the autocorrelation function $R_X(\tau)$ using the indirect method for the same set of parameters as those used for the ensemble-averaged results of Fig. 1. The symbol $\hat{R}_X(\tau)$ is used to emphasize the fact that the computation described here results in an "estimate" of the autocorrelation function $R_X(\tau)$. The results presented in Fig. 2 are for a signal-to-noise ratio of + 10dB, which is defined by

$$\text{SNR} = \frac{A^2/2}{N_0/(2T)} = \frac{A^2 T}{N_0} \qquad (9)$$

On the basis of the results presented in Figures 1 and 2 we may make the following observations:

- The ensemble-averaging and time-averaging approaches yield similar results for the autocorrelation function $R_X(\tau)$, signifying the fact that the random process $X(t)$ described herein is indeed ergodic.
- The indirect time-averaging approach, based on the FFT algorithm, provides an efficient method for the estimation of $R_X(\tau)$ using a digital computer.
- As the SNR is increased, the numerical accuracy of the estimation is improved, which is intuitively satisfying.

1 Problem 1.32

Matlab codes

```
% Problem 1.32a CS: Haykin
% Ensemble average autocorrelation
% M. Sellathurai


clear all
A=sqrt(2);
N=1000; M=1; SNRdb=0;
e_corrf_f=zeros(1,1000);
f_c=2/N;
t=0:1:N-1;

for trial=1:M

% signal
s=cos(2*pi*f_c*t);

%noise
snr = 10^(SNRdb/10);
wn = (randn(1,length(s)))/sqrt(snr)/sqrt(2);

%signal plus noise
s=s+wn;

% autocorrelation
[e_corrf]=en_corr(s,s, N);

%Ensemble-averaged autocorrelation
e_corrf_f=e_corrf_f+e_corrf;
end

%prints
plot(-500:500-1,e_corrf_f/M);
xlabel('(\tau)')
ylabel('R_X(\tau)')
```

```
% Problem 1.32b CS: Haykin
% time-averaged estimation of autocorrelation
% M. Sellathurai


clear all
A=sqrt(2);
N=1000; SNRdb=0;
f_c=2/N;
t=0:1:N-1;

% signal
s=cos(2*pi*f_c*t);%noise

%noise
snr = 10^(SNRdb/10);
wn = (randn(1,length(s)))/sqrt(snr)/sqrt(2);

%signal plus noise
s=s+wn;

% time -averaged autocorrelation
[e_corrf]=time_corr(s,N);

%prints
plot(-500:500-1,e_corrf);
xlabel('(\tau)')
ylabel('R_X(\tau)')
```

```
function [corrf]=en_corr(u, v, N)% funtion to compute the autocorreation/ cross-correlati
% ensemble average
% used in problem 1.32, CS: Haykin
% M. Sellathurai, 10 june 1999.

max_cross_corr=0;
tt=length(u);

for m=0:tt
shifted_u=[u(m+1:tt) u(1:m)];
corr(m+1)=(sum(v.*shifted_u))/(N/2);
if (abs(corr)>max_cross_corr)
max_cross_corr=abs(corr);
end
end

corr1=flipud(corr);
corrf=[corr1(501:tt) corr(1:500)];
```

```
function [corrf]=time_corr(s,N)
% funtion to compute the autocorreation/ cross-correlation
% time average
% used in problem 1.32, CS: Haykin
% M. Sellathurai, 10 june 1999.

X=fft(s);
X1=fftshift((abs(X).^2)/(N/2));
corrf=(fftshift(abs(ifft(X1))));
```

Answer to Problem 1.32

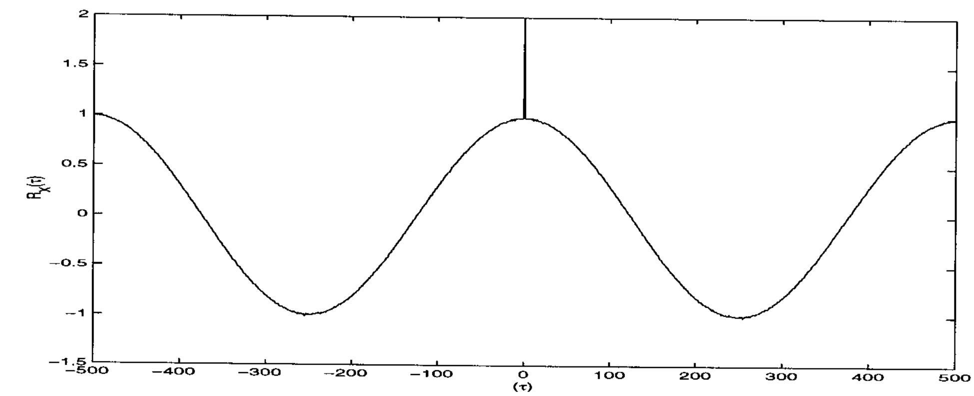


Figure 1: Ensemble averaging

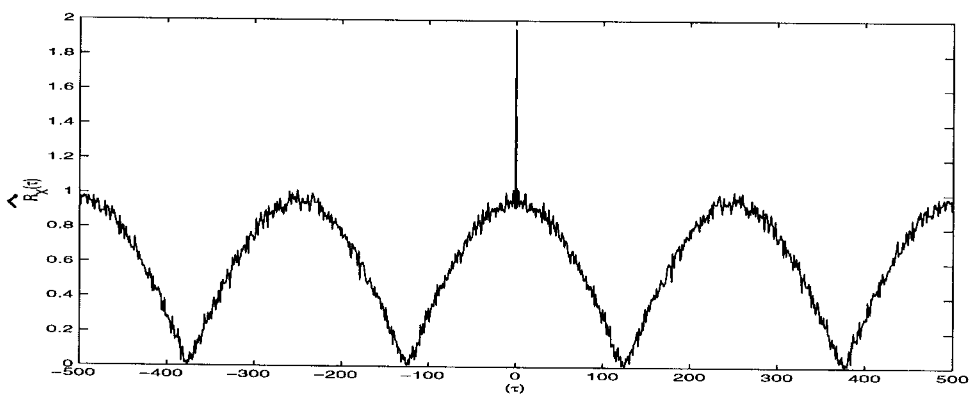


Figure 2: Time averaging

Problem 1.33

Matlab codes

```
% Problem 1.33 CS: Haykin
% multipath channel
% M. Sellathurai


clear all
Nf=0;Xf=0; % initializing counters

N=10000; % number of samples
M=2;  P=10;

a=1; % line of sight component component

for i=1:P

A=sqrt(randn(N,M).^2 + randn(N,M).^2);

xi=A.*cos(cos(rand(N,M)*2*pi) + rand(N,M)*2*pi); % inphase cpmponent
xq=A.*sin(cos(rand(N,M)*2*pi) + rand(N,M)*2*pi); % quadrature phase component

xi=(sum(xi'));
xq=(sum(xq'));

ra=sqrt((xi+a).^2+ xq.^2) ; % rayleigh, rician fading

[h X]=hist(ra,50);

Nf=Nf+h;
Xf=Xf+X;

end

Nf=Nf/(P);
Xf=Xf/(P);

% print
plot(Xf,Nf/(sum(Xf.*Nf)/20))
xlabel('v')
ylabel('f_v(v)')
```

Answer to Problem 1.33

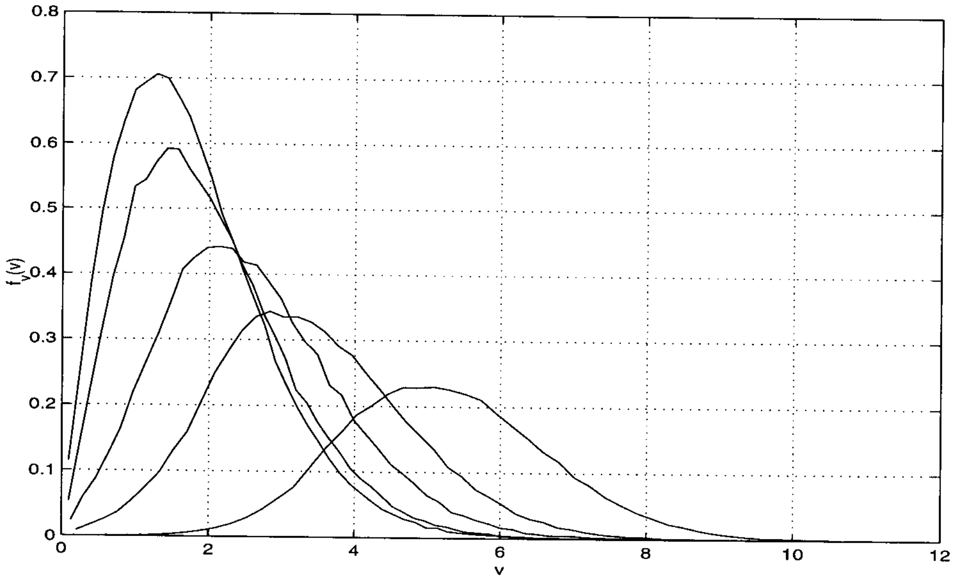


Figure 1 Rician distribution