CHAPTER 5

Problem 5.1

(a) Unipolar NRZ code.

The pair of signals $s_1(t)$ and $s_2(t)$ used to represent binary symbols 1 and 0, respectively are defined by

$$s_1(t) = \sqrt{\frac{E_b}{T_b}}, \qquad 0 \le t \le T_b$$

$$s_2(t) = 0, \qquad 0 \le t \le T_b$$

where E_b is the transmitted signal energy per bit and T_b is the bit duration. From the definitions of $s_1(t)$ and $s_2(t)$, it is clear that, in the case of unipolar NRZ signals, there is only one basis function of unit energy. The basis function is given by

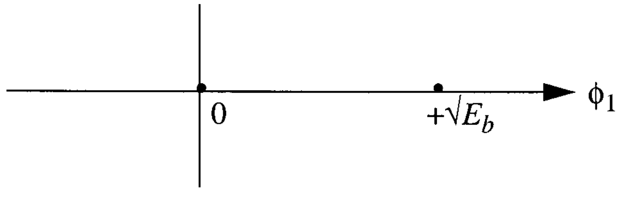
$$\phi_1(t) = \sqrt{\frac{1}{T_b}}, \qquad 0 \le t \le T_b$$

Then, we may expand the transmitted signals $s_1(t)$ and $s_2(t)$ in terms of $\phi_1(t)$ as follows:

$$s_1(t) = \sqrt{E_b}\phi_1(t), \qquad 0 \le t \le T_b$$

$$s_2(t) = 0, \qquad 0 \le t \le T_b$$

Hence, the signal-space diagram for unipolar NRZ code is $(+\sqrt{E_b}, 0)$, as shown



(b) Polar NRZ code.

In this code, binary symbols 1 and 0 are defined by

$$s_1(t) = +\sqrt{\frac{E_b}{T_b}}, \qquad 0 \le t \le T_b$$

$$s_2(t) = -\sqrt{\frac{E_b}{T_b}}, \qquad 0 \le t \le T_b$$

The basis function is given by

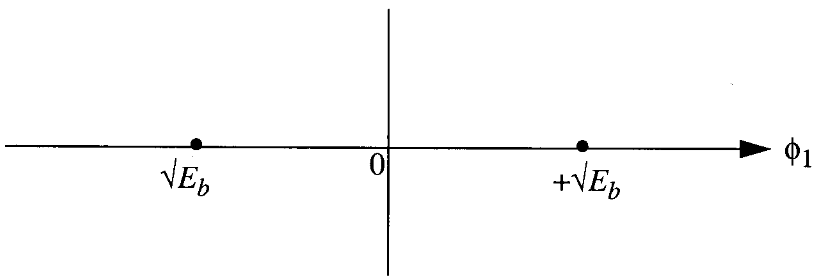
$$\phi_1(t) = \sqrt{\frac{1}{T_b}}, \qquad 0 \le t \le T_b$$

Then, the transmitted signals in terms of $\phi_1(t)$ are as follows:

$$s_1(t) = \sqrt{E_b}\phi_1(t) \qquad 0 \le t \le T_b$$

$$s_2(t) = -\sqrt{E_b}\phi_1(t) \qquad 0 \le t \le T_b$$

Hence, the signal-space diagram for the polar NRZ code is $(+\sqrt{E_b}, -\sqrt{E_b})$ as shown below:



(c) Unipolar return-to-zero code.

In this third code, binary symbols 1 and 0 are defined by

$$s_1(t) = +\sqrt{\frac{E_b}{T_b}}, \qquad 0 \le t \le T_b/2$$

$$= 0 \qquad T_b/2 \le t \le T_b$$

$$s_2(t) = 0 \qquad 0 \le t \le T_b$$

The energy of signal $s_1(t)$ is

$$E_1 = \int_0^{T_b/2} \left(\sqrt{\frac{E_b}{T_b}}\right)^2 dt + \int_{T_b/2}^{T_b} 0\,dt$$

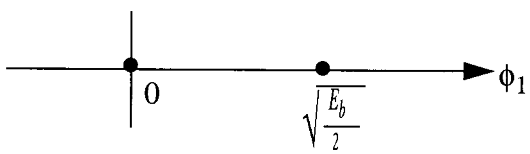
$$= \frac{E_b}{2}$$

The energy of signal $s_2(t)$ is zero.

The basis function is given by

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{\frac{E_b}{2}}}$$

The signal-space diagram for the RZ code is as follows:



(d) Manchester code

Binary symbols 1 and 0 are defined by

$$s_1(t) = \begin{cases} \sqrt{\frac{E_b}{T_b}}, & 0 \le t \le T_b/2 \\ -\sqrt{\frac{E_b}{T_b}}, & T_b/2 \le t \le T_b \end{cases}$$

$$s_2(t) = \begin{cases} -\sqrt{\frac{E_b}{T_b}}, & 0 \le t \le T_b/2 \\ +\sqrt{\frac{E_b}{T_b}}, & T_b/2 \le t \le T_b \end{cases}$$

The energy of signal $s_1(t)$ is

$$E_1 = \int_0^{T_b/2} \left(\sqrt{\frac{E_b}{T_b}}\right)^2 dt + \int_{T_b/2}^{T_b} \left(\sqrt{\frac{E_b}{T_b}}\right)^2 dt$$

$$= E_b$$

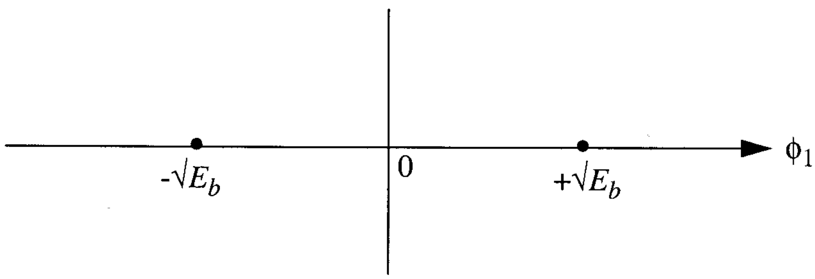
Similarly, the energy of symbol $s_2(t)$ is

$$E_2 = E_b$$

The basis function is given by

$$\phi_1(t) = = \frac{s_1(t)}{\sqrt{E_b}}$$

The signal-space diagram of the Manchester code is thus as follows:



Thus all the four line codes in this problem are one-dimensional.

Problem 5.2

The given 8-level PAM signal is defined by

$$s_i(t) = A_i \text{rect}\left(\frac{t}{T} - \frac{T}{2}\right)$$

The energy of signal $s_i(t)$ is given by

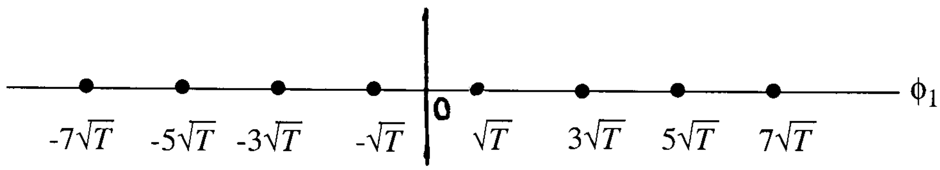
$$E_i = \int_0^T (A_i)^2 dt$$

$$= A_i^2 T, \qquad A_i = \pm 1, \pm 3, \pm 5, \pm 7$$

The basis function is given by

$$\phi_1(t) = \frac{s_i(t)}{\sqrt{E_i}} = \frac{s_i(t)}{A_i\sqrt{T}}$$

The signal-space diagram of the 8-level PAM signal is as follows:



Problem 5.3

Consider the signals $s_1(t)$, $s_2(t)$, $s_3(t)$, and $s_4(t)$ shown in Fig. 1a. We wish to use the Gram-Schmidt orthogonalization procedure to find an orthonormal basis for this set of signals.

Step 1 We note that the energy of signal $s_1(t)$ is

$$E_1 = \int_0^T s_1^2(t)dt$$

$$= \int_0^{T/3} (1)^2 dt$$

$$= \frac{T}{3}$$

The first basis function $\phi_1(t)$ is therefore

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}}$$

$$= \left\{ \begin{array}{ll} \sqrt{3/T}, & 0 \le t \le T/3 \\ 0, & \text{otherwise} \end{array} \right\}$$

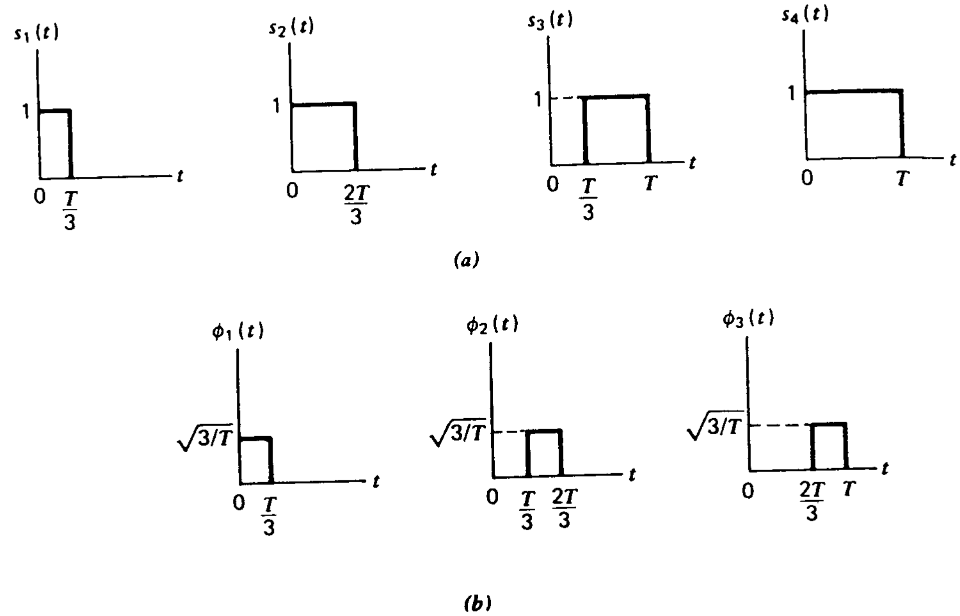


Figure 1

Step 2 Evaluating the projection of $s_2(t)$ onto $\phi_1(t)$, we find that

$$s_{21} = \int_0^T s_2(t)\phi_1(t)dt$$

$$= \int_0^{T/3} (1)\left(\sqrt{\frac{3}{T}}\right)dt$$

$$= \sqrt{\frac{3}{T}}$$

The energy of signal $s_2(t)$ is

$$E_2 = \int_0^T s_2^2(t)$$

$$= \int_0^{2T/3} (1)^2 dt$$

$$= \frac{2T}{3}$$

The second basis function $\phi_2(t)$ is therefore

$$\phi_2(t) = \frac{s_2(t) - s_{21}\phi_1(t)}{\sqrt{E_2 - s_{21}^2}}$$

$$= \begin{cases} \sqrt{3/T}, & T/3 \le 2T/3 \\ 0 & \text{otherwise} \end{cases}$$

Step 3 Evaluating the projection of $s_3(t)$ onto $\phi_1(t)$,

$$s_{31} = \int_0^T s_3(t)\phi_1(t)dt$$

$$= 0$$

and the coefficient s_{32} equals

$$s_{32} = \int_0^T s_3(t)\phi_1(t)dt$$

$$= \int_{T/3}^{2T/3} (1)\left(\sqrt{\frac{3}{T}}\right)dt$$

$$= \sqrt{\frac{3}{T}}$$

The corresponding value of the intermediate function $g_i(t)$, with $i = 3$, is therefore

$$g_3(t) = s_3(t) - s_{31}\phi_1(t) - s_{32}\phi_2(t)$$

$$= \begin{cases} 1, & 2T/3 \le t \le T \\ 0, & \text{elsewhere} \end{cases}$$

Hence, the third basis function $\phi_3(t)$ is

$$\phi_3(t) = \frac{g_3(t)}{\sqrt{\int_0^T g_3^2(t)dt}}$$

$$= \begin{cases} \sqrt{3/T}, & 2T/3 \le t \le T \\ 0, & \text{elsewhere} \end{cases}$$

The orthogonalization process is now complete.

The three basis functions $\phi_1(t)$, $\phi_2(t)$, and $\phi_3(t)$ form an orthonormal set, as shown in Fig. 1b. In this example, we thus have $M = 4$ and $N = 3$, which means that the four signals $s_1(t)$, $s_2(t)$, $s_3(t)$, and $s_4(t)$ described in Fig. 1a do not form a linearly independent set. This is readily confirmed by noting that $s_4(t) = s_1(t) + s_3(t)$. Moreover, we note that any of these four signals can be expressed as a linear combination of the three basis functions, which is the essence of the Gram-Schmidt orthogonalization procedure.

Problem 5.4

(a) We first observe that $s_1(t)$, $s_2(t)$ and $s_3(t)$ are linearly independent.

The energy of $s_1(t)$ is

$$E_1 = \int_0^1 (2)^2 dt = 4$$

The first basis function is therefore

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}}$$

$$= \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{otherwsie} \end{cases}$$

Define

$$s_{21} = \int_0^T s_2(t)\ \phi_1(t)dt$$

$$= \int_0^1 (-4)(1)dt = -4$$

$$g_2(t) = s_2(t) - s_{21}\phi_1(t)$$

$$= \begin{cases} -4, & 1 \le t \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Hence, the second basis function is

$$\phi_2(t) = \frac{g_2(t)}{\sqrt{\int_0^T g_2^2(t)dt}}$$

$$= \begin{cases} -1, & 1 \leq t \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

Define

$$s_{31} = \int_0^T s_3(t)\ \phi_1(t)dt$$

$$= \int_0^1 (3)(1)dt = 3$$

$$s_{32} = \int_T^{2T} s_3(t)\ \phi_2(t)dt$$

$$= \int_1^2 (3)(-1)dt = -3$$

$$g_3(t) = s_3(t) - s_{31}\ \phi_1(t) - s_{32}\ \phi_2(t)$$

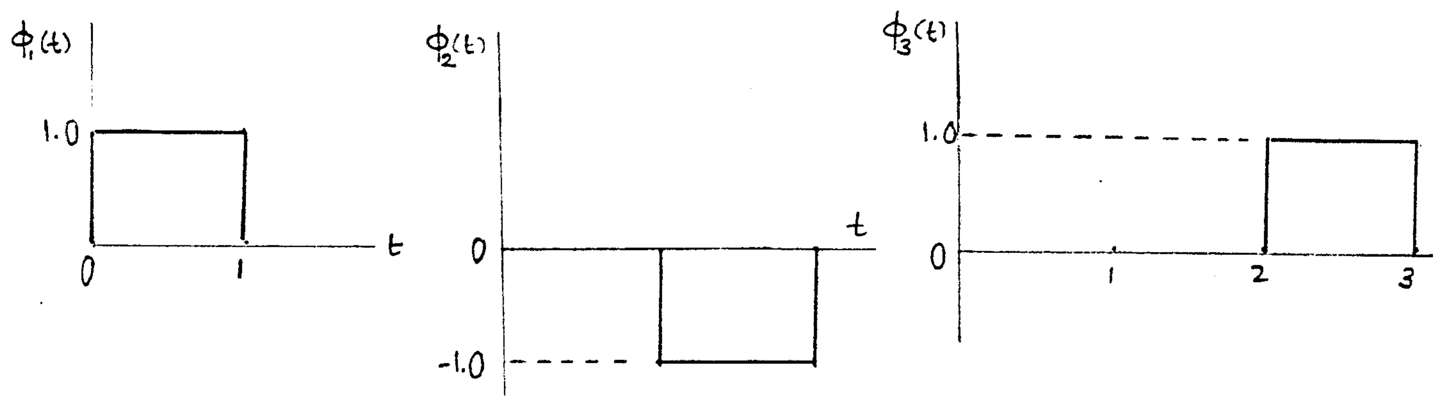
$$= \begin{cases} 3, & 2 \leq t \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

Hence, the thrid basis function is

$$\phi_3(t) = \frac{g_3(t)}{\sqrt{\int_0^T g_3^2(t)dt}}$$

$$= \begin{cases} 1, & 2 \leq t \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

The three basis functions are as follows (graphically)



(b) $s_1(t) = 2\phi_1(t)$

$s_2(t) = -4\phi_1(t) + 4\phi_2(t)$

$s_3(t) = 3\phi_1(t) - 3\phi_2(t) + 3\phi_3(t)$

Problem 5.5

Signals $s_1(t)$ and $s_2(t)$ are orthogonal to each other. The energy of $s_1(t)$ is

$$E_1 = \int_0^{T/2} 1^2 dt + \int_{T/2}^{T} (-1)^2 dt = T$$

The energy of $s_2(t)$ is

$$E_2 = \int_0^{T} 1^2 dt = T$$

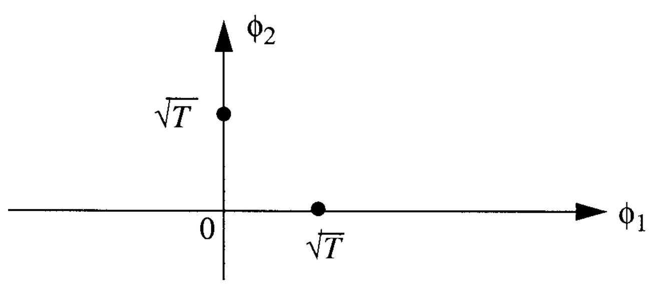
To represent the orthogonal signals $s_1(t)$ and $s_2(t)$, we need two basis functions. The first basis function is given by

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \frac{s_1(t)}{\sqrt{T}}$$

The second basis function is given by

$$\phi_2(t) = \frac{s_2(t)}{\sqrt{E_2}} = \frac{s_2(t)}{\sqrt{T}}$$

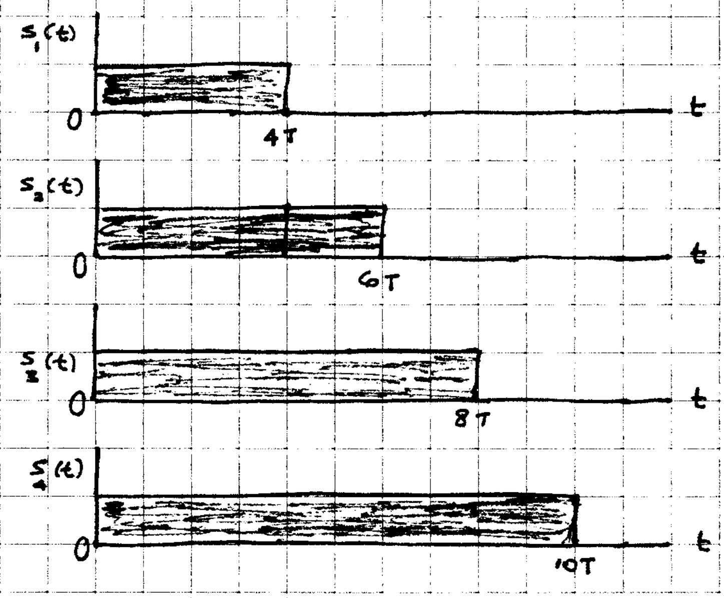
The signal-space diagram for $s_1(t)$ and $s_2(t)$ is as shown below:



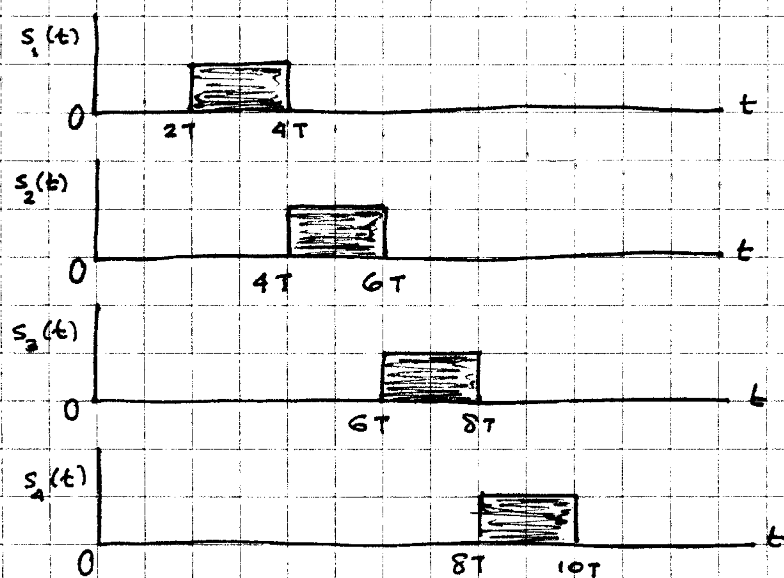
Problem 5.6

The common properties of PDM and PPM are as follows: In both cases a time parameter of the pulse is modulated and the pulses have a constant amplitude. In PDM, the samples of the message signals are used to vary the duration of the individual pulses, as illustrated in Fig. 1a for $M = 4$ on the next page. In PPM, the position of the pulse is varied in accordance with the message, while keeping the duration of the pulse constant, as illustrated in Fig. 1b for $M = 4$.

From these two illustrative figures, it is perfectly clear that the set of PDM signals is nonorthogonal, whereas the PDM signals form an orthogonal set.



(a) Pulse-duration modulation



(b) Pulse-position modulation

Figure 1

Problem 5.7

(a) The biorthogonal signals are defined as the negatives of orthogonal signals. Consider for example the two orthogonal signals $s_1(t)$ and $s_2(t)$ defined as follows:

$$s_1(t) = \sqrt{E}\phi_1(t)$$

$$s_2(t) = \sqrt{E}\phi_2(t)$$

where $\phi_1(t)$ and $\phi_2(t)$ are orthonormal basis functions. The biorthogonal signals are given by $-s_1(t)$ and $-s_2(t)$, which are respectively expressed in terms of the basis functions as $\sqrt{E}\phi_1(t)$ and $-\sqrt{E}\phi_2(t)$. Hence, the inclusion of these two biorthogonal signals leaves the dimensionality of the signal-space diagram unchanged. This result holds for the general case of M orthogonal signals.

(b) The signal-space diagram for the biorthogonal signals corresponding to those shown in Fig. P5.5 is as shown in Fig. 1a. Incorporating this diagram with that of the solution to Problem 5.5, we get the 4-signal constellation shown in Fig. 1b.

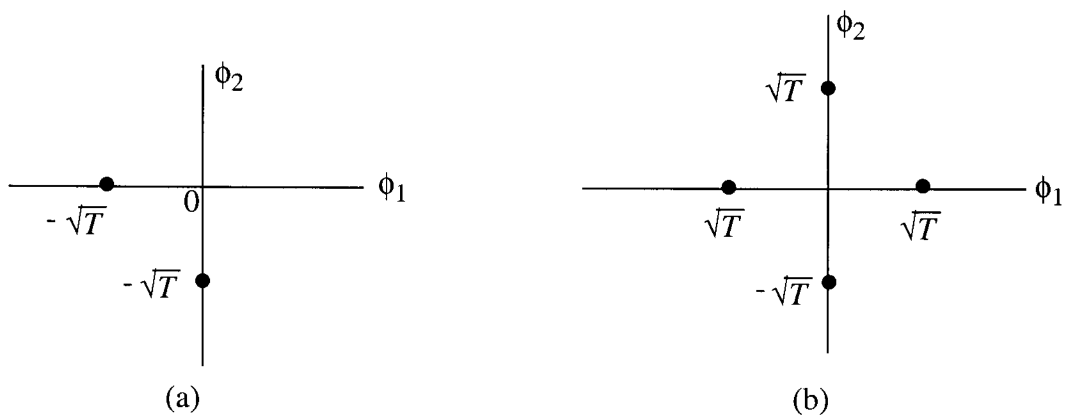


Figure 1

Problem 5.8

(a) A pair of signals $s_i(t)$ and $s_k(t)$, belonging to an N-dimensional signal space, can be represented as linear combinations of N orthonormal basis functions. We thus write

$$s_i(t) = \sum_{j=1}^{N} s_{ij}\phi_j(t), \quad \begin{array}{l} 0 \le t \le T \\ i = 1, 2 \end{array} \qquad (1)$$

where the coefficients of the expansion are defined by

$$s_{ij} = \int_0^T s_i(t)\phi_j(t)dt, \qquad \begin{array}{l} i = 1, 2 \\ j = 1, 2 \end{array} \tag{2}$$

The real-valued basis functions $\phi_1(t)$ and $\phi_2(t)$ are orthonormal. Hence,

$$\int_0^T \phi_i(t)\phi_j(t) \quad = \delta_{ij} = \begin{cases} 1, & \text{if } i=j \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

The set of coefficients $\{s_{ij}\}_{j=1}^{N}$ may be viewed as an N-dimensional vector defined by

$$\mathbf{s}_i = \begin{bmatrix} s_{i1} \\ s_{i2} \\ \vdots \\ s_{iN} \end{bmatrix}, \quad i = 1, 2, \ldots, M \tag{4}$$

where M is the number of signals in the setl, with $M \geq N$. The inner product of the pair of signal $s_i(t)$ and $s_k(t)$ is given by

$$\int_0^T s_i(t)s_k(t)dt \tag{5}$$

By substituting (1) in (5), we get the following result for the inner product:

$$\int_0^T \left[\sum_{j=1}^{N} s_{ij}\phi_j(t)\right] \left[\sum_{l=1}^{N} s_{kl}\phi_l(t)\right] dt$$

$$= \sum_{j=1}^{N} \sum_{l=1}^{N} s_{ij}s_{kl}\int_0^T \phi_j(t)\phi_l(t)dt \tag{6}$$

Since the $\phi_j(t)$ form an orthonormal set, then, in accordance with the two conditions of Eq. (3) and (4), the inner product of $s_i(t)$ and $s_k(t)$ reduces to

$$\int_0^T s_i(t)s_k(t)dt = \sum_{j=1}^{N} s_{ij}s_{kj}$$

$$= \mathbf{s}_i^T \mathbf{s}_k$$

(b) Consider next the squared Euclidean distance between $\mathbf{s}_i$ and $\mathbf{s}_k$, which can be expressed as follows:

$$\|\mathbf{s}_i - \mathbf{s}_k\|^2 = (\mathbf{s}_i - \mathbf{s}_k)^T (\mathbf{s}_i - \mathbf{s}_k)$$

$$= \mathbf{s}_i^T \mathbf{s}_i + \mathbf{s}_k^T \mathbf{s}_k - 2\mathbf{s}_i^T \mathbf{s}_k$$

$$= \int_0^T s_i^2(t)dt + \int_0^T s_k^2(t)dt - 2\int_0^T s_i(t)s_k dt$$

$$= \int_0^T (s_i(t) - s_k(t))^2 dt$$

Problem 5.9

Consider the pair of complex-valued signals $s_1(t)$ and $s_2(t)$, which are defined by

$$s_1(t) = a_{11}\phi_1(t) + a_{12}\phi_2(t) \quad (1)$$

$$s_2(t) = a_{21}\phi_1(t) + a_{22}\phi_2(t) \quad (2)$$

The basis functions $\phi_1(t)$ and $\phi_2(t)$ are real-valued and the coefficients a_{11}, a_{12}, a_{21} and a_{22} are complex-valued. We may denote the complex-valued coefficients as follows:

$a_{11} = \alpha_{11} + j\beta_{11}$
$a_{12} = \alpha_{12} + j\beta_{12}$
$a_{21} = \alpha_{21} + j\beta_{21}$
$a_{22} = \alpha_{22} + j\beta_{22}$

On this basis, we may represent the signals $s_1(t)$ and $s_2(t)$ by the following respective pair of vectors:

$$\mathbf{g}_1 = \begin{bmatrix} \alpha_{11} \\ \beta_{11} \\ \alpha_{12} \\ \beta_{12} \end{bmatrix}, \quad \mathbf{g}_2 = \begin{bmatrix} \alpha_{21} \\ \beta_{21} \\ \alpha_{22} \\ \beta_{22} \end{bmatrix}$$

The angle subtended between the vectors $\mathbf{g}_1$ and $\mathbf{g}_2$ is

$$\cos\theta = \frac{\mathbf{g}_1^T \mathbf{g}_2}{\|\mathbf{g}_1\| \cdot \|\mathbf{g}_2\|}$$

$$= \frac{\alpha_{11}\alpha_{21} + \beta_{11}\beta_{21} + \alpha_{12}\alpha_{22} + \beta_{11}\beta_{22}}{\sqrt{\alpha_{11}^2\beta_{11}^2 + \alpha_{12}^2 + \beta_{12}^2} \cdot \sqrt{\alpha_{21}^2\beta_{21}^2 + \alpha_{22}^2 + \beta_{22}^2}}$$

$$= \frac{\mathbf{s}_1^H s_2}{\|\mathbf{s}_1\| \cdot \|\mathbf{s}_2\|}$$

where $\mathbf{s}_1 = \begin{bmatrix} a_{11} \\ a_{12} \end{bmatrix}$ and $\mathbf{s}_2 = \begin{bmatrix} a_{21} \\ a_{22} \end{bmatrix}$ are complex vectors. Recognizing that

$$\cos\theta = \frac{\mathbf{s}_2^H s_1}{\|\mathbf{s}_1\| \cdot \|\mathbf{s}_2\|} \le 1$$

we may go on to write

$$\frac{\int_{-\infty}^{\infty} s_1(t) s_2^*(t)\,dt}{\left(\int_{-\infty}^{\infty} |s_1(t)|^2 dt\right)^{1/2} \left(\int_{-\infty}^{\infty} |s_2(t)|^2 dt\right)^{1/2}} \le 1$$

The complex form of the Schwarz unequality becomes

$$\left|\int_{-\infty}^{\infty} s_1(t) s_2^*(t)\,dt\right|^2 \le \int_{-\infty}^{\infty} |s_1(t)|^2 dt \int_{-\infty}^{\infty} |s_2(t)|^2 dt$$

The equality holds when $s_1(t)$ and $s_2(t)$ are co-linear, that is, $s_1(t) = ks_2(t)$ where k is any real-valued constant.

Problem 5.10

$$E[X_j W'(t_k)] = E[(s_{ij} + W_j)W'(t_k)]$$

$$E[s_{ij}\, W'(t_k)] = s_{ij}\, E[W'(t_k)] = 0$$

We also note that

$$W'(t_k) = W(t_k) - \sum_{i=1}^{N} W_i\, \phi_i(t_k)$$

We therefore have

$$E[X_j W'(t_k)] = E[W_j\, W'(t_k)]$$

$$= E[W_j W(t_k) - \sum_{i=1}^{N} \phi_i(t_k)\, E[W_j W_i]$$

But $E[W_j\, W(t_k)] = E[W(t_k) \int_0^T W(t)\, \phi_j(t)dt] = \int_0^T \phi_j(t)\, E[W(t_k)W(t)]dt$

$$= \int_0^T \phi_j(t) \cdot \frac{N_0}{2}\, \delta(t-t_k)dt = \frac{N_0}{2}\, \phi_j(t_k)$$

$$E[W_j W_i] = \begin{cases} \frac{N_0}{2}, & i=j \\ 0 & i \neq j \end{cases}$$

Hence, we get the final result

$$E[X_j\, W'(t_k)] = \frac{N_0}{2}\, \phi_j(t_k) - \frac{N_0}{2}\, \phi_j(t_k)$$

$$= 0.$$

Problem 5.11

For the noiseless case, the received signal $r(t) = s(t)$, $0 \le t \le T$.

(a) The correlator output is

$$y(T) = \int_0^T r(\tau)s(\tau)d\tau$$

$$y(T) = \int_0^T s^2(\tau)d\tau$$

$$= \int_0^T \sin^2\left(\frac{8\pi\tau}{T}\right)d\tau$$

$$= \int_0^T \frac{1}{2}\left[1 - \cos\left(\frac{16\pi\tau}{T}\right)\right]d\tau$$

$$= T/2$$

(b) The matched filter is defined by the impulse response

$$h(t) = s(T - t)$$

The matched filter output is therefore

$$y(t) = \int_{-\infty}^{\infty} r(\lambda)h(t-\lambda)d\lambda$$

$$= \int_{-\infty}^{\infty} s(\lambda)s(T-t+\lambda)d\lambda$$

$$= \int_{-\infty}^{\infty} \sin\left(\frac{8\pi\lambda}{T}\right)\sin\left(\frac{8\pi(T-t+\lambda)}{T}\right)d\lambda$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \cos\left[\frac{8\pi(T-t)}{T}\right]d\lambda - \frac{1}{2}\int_{-\infty}^{\infty} \cos\left[\frac{8\pi(T-t+\lambda)}{T}\right]d\lambda$$

Since $-T < \lambda \leq 0$, we have

$$y(t) = \frac{1}{2}\cos\left(\frac{8\pi(t-T)}{T}\right)\lambda\Bigg]_{\lambda=-T}^{\lambda=0}$$

$$- \frac{1}{2}\cdot\frac{T}{8\pi}\sin\left(\frac{8\pi(T-t+2\lambda)}{T}\right)\Bigg]_{\lambda=-T}^{\lambda=0}$$

$$= \frac{T}{2}\cos\left(\frac{8\pi(t-T)}{T}\right) - \frac{T}{16\pi}\sin\left(\frac{8\pi(t-T)}{T}\right) - \frac{T}{16\pi}\sin\left(\frac{8\pi(t+T)}{T}\right)$$

(c) When the matched filter output is sampled at $t = T$, we get

$$y(T) = T/2$$

which is exactly the same as the correlator output determined in part (a).

Problem 5.12

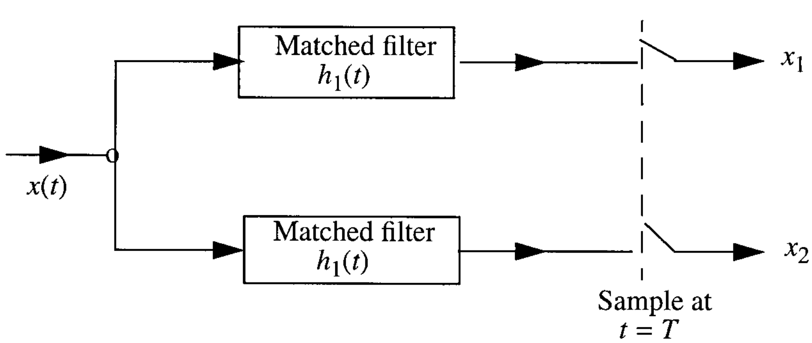
(a) The matched filter for signal $s_1(t)$ is defined by the impulse response

$$h_1(t) = s_1(T-t)$$

The matched filter for signal $s_2(t)$ is defined by the impulse response

$$h_2(t) = s_2(T-t)$$

The matched filter receiver is as follows



The receiver decides in favor of $s_2(t)$ if, for the noisy received signal,

$$x(t) = s_k(t) + w(t), \qquad \begin{array}{l} 0 \le t \le T \\ k = 1, 2 \end{array}$$

we find that $x_1 > x_2$. On the other hand, if $x_2 > x_1$, it decides in favor of $s_2(t)$. If $x_1 = x_2$, the decision is made by tossing a fair coin.

(b) Energy of signal $s_1(t)$ is given by

$$E_1 = \int_0^T (1)^2 dt + \int_T^{2T} (-1)^2 dt + \int_{2T}^{3T} (1)^2 dt$$

$$= 3T = E$$

Energy of signal $s_2(t)$ is

$$E_2 = \int_0^{T/2} (-1)^2 dt + \int_{T/2}^{3T/2} (1)^2 dt + \int_{3T/2}^{5T/2} (-1)^2 dt + \int_{5T/2}^{3T} (1)^2 dt$$

$$= 3T = E$$

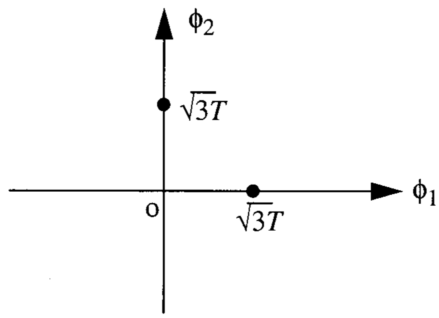
The orthornomal basis functions for the signal-space diagram of these two orthogonal signals are given by

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{3T}}$$

and

$$\phi_2(t) = \frac{s_2(t)}{\sqrt{3T}}$$

The signal-space diagram of signals s_1 and s_2 is as follows:



The distance between the two signal points $s_1(t)$ and $s_2(t)$ is

$$d = \sqrt{2E} = \sqrt{6T}$$

The average probability of error is therefore

$$P_e = \frac{1}{2}\text{erfc}\left(\frac{1}{2}\frac{d}{\sqrt{N_0}}\right)$$

$$= \frac{1}{2}\text{erfc}\left(\frac{1}{2}\sqrt{\frac{2E}{N_0}}\right)$$

For E/N_0, we therefore have

$$P_e = \frac{1}{2}\text{erfc}\left(\frac{1}{2}\sqrt{2 \times 4}\right)$$

$$= \frac{1}{2}\text{erfc}(\sqrt{2})$$

$$= 4 \times 10^{-2}$$

<u>Problem 5.13</u>

Energy of binary symbol 1 represented by signal $s_1(t)$ is

$$E_1 = \int_0^{T/2} (+1)^2 dt + \int_{T/2}^{T} (-1)^2 dt = T$$

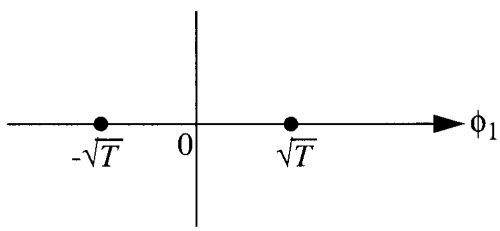
Energy of binary symbol 0 represented by signal $s_2(t)$ is the same as shown by

$$E_2 = \int_0^{T/2} (-1)^2 dt + \int_{T/2}^{T} (\ +1)^2 dt = T$$

The only basis function of the signal-space diagram is

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \frac{s_1(t)}{\sqrt{T}}$$

The signal-space diagram of the Manchester code using the doublet pulse is as follows:



Hence, the distance between the two signal points is $d = 2\sqrt{T}$. The average probability of error over an AWGN channel is given by

$$P_e = \frac{1}{2}\text{erfc}\left(\frac{d}{2\sqrt{N_0}}\right) = \frac{1}{2}\text{erfc}\left(\sqrt{\frac{T}{N_0}}\right)$$

Problem 5.14

(a) Let Z denote the total observation space, which is divided into two parts Z_0 and Z_1. Whenever an observation falls in Z_0, we say H_0, and whenever an observation falls in Z_1, we say H_1. Thus, expressing the risk R in terms of the conditional probability density functions and the decision regions, we may write

$$R = C_{00}\, p_0 \int_{Z_0} f_{\underline{X}|H_0}(\underline{x}|H_0)d\underline{x}$$

$$+ C_{10}\, p_0 \int_{Z_1} f_{\underline{X}|H_0}(\underline{x}|H_0)d\underline{x}$$

$$+ C_{11}\, p_1 \int_{Z_1} f_{\underline{X}|H_1}(\underline{x}|H_1)d\underline{x}$$

$$+ C_{01}\, p_1 \int_{Z_0} f_{\underline{X}|H_1}(\underline{x}|H_1)d\underline{x} \qquad (1)$$

For an N-dimensional observation space, the integrals in Eq. (1) are N-fold integrals.

To find the Bayes test, we must choose the decision regions Z_0 and Z_1 in such a manner that the risk R will be minimized. Because we require that a decision be made, this means that we must assign each point $\underline{x}$ in the observation space Z to Z_0 or Z_1; thus

$$Z = Z_0 + Z_1$$

Hence, we may rewrite Eq. (1) as

$$R = p_0C_{00} \int_{Z_0} f_{\underline{X}|H_0}(\underline{x}|H_0)d\underline{x} + p_0C_{10} \int_{Z-Z_0} f_{\underline{X}|H_0}(\underline{x}|H_0)d\underline{x}$$

$$+ p_1C_{11} \int_{Z-Z_0} f_{\underline{X}|H_1}(\underline{x}|H_1)d\underline{x} + p_1C_{01} \int_{Z_0} f_{\underline{X}|H_1}(\underline{x}|H_1)d\underline{x} \qquad (2)$$

We observe that

$$\int_Z f_{\underline{X}|H_0}(\underline{x}|H_0)d\underline{x} = \int_Z f_{\underline{X}|H_1}(\underline{x}|H_1)d\underline{x} = 1$$

Hence, Eq. (2) reduces to

$$R = p_0C_{10} + p_1C_{11}$$

$$+ \int_{Z_0} \{-[p_0(C_{10}-C_{00})f_{\underline{X}|H_0}(\underline{x}|H_0)] + [p_1(C_{01}-C_{11})f_{\underline{X}|H_1}(\underline{x}|H_1)]\}d\underline{x} \qquad (3)$$

The first two terms in Eq. (3) represent the fixed cost. The integral represents the cost controlled by those points $\underline{x}$ that we assign to Z_0. Since $C_{10} > C_{00}$ and $C_{01} > C_{11}$, we find that the two terms inside the square brackets are positive. Therefore, all values of $\underline{x}$ where the first term is larger than the second should be included in Z_0 because they contribute a negative amount to the integral. Similarly, all values of $\underline{x}$ where the second term is larger than the first should be excluded from Z_0 (i.e., assigned to Z_1) because they would contribute a positive amount to the integral. Values of $\underline{x}$ where the two terms are equal have no effect on the cost and may be assigned arbitrarily. Thus the decision regions are defined by the following statement: If

$$p_1(C_{01} - C_{11}) f_{\underline{X}|H_1}(\underline{x}|H_1) > p_0(C_{10} - C_{00}) f_{\underline{X}|H_0}(\underline{x}|H_0),$$

assign $\underline{x}$ to Z_1 and consequently say that H_1 is true. If the reverse is true, assign $\underline{x}$ to Z_0 and say H_0 is true.

Alternatively, we may write

$$\frac{f_{\underline{X}|H_1}(\underline{x}|H_1)}{f_{\underline{X}|H_0}(\underline{x}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{p_0(C_{10} - C_{00})}{p_1(C_{01} - C_{11})}$$

The quantity on the left is the likelihood ratio:

$$\Lambda(\underline{x}) = \frac{f_{\underline{X}|H_1}(\underline{x}|H_1)}{f_{\underline{X}|H_0}(\underline{x}|H_0)}$$

Let

$$\lambda = \frac{p_0(C_{10} - C_{00})}{p_1(C_{01} - C_{11})}$$

Thus, Bayes criterion yields a likelihood ratio test described by

$$\Lambda(\underline{x}) \underset{H_0}{\overset{H_1}{\gtrless}} \lambda$$

(b) For the minimum probability of error criterion, the likelihood ratio test is described by

$$\Lambda(\underline{x}) \underset{H_0}{\overset{H_1}{\gtrless}} \frac{p_0}{p_1}$$

Thus, we may view the minimum probability of error criterion as a special case of the Bayes criterion with the cost values defined as

$$C_{00} = C_{11} = 0$$

$$C_{10} = C_{01}$$

That is, the cost of a correct decision is zero, and the cost of an error of one kind is the same as the cost of an error of the other kind.

Problem 5.15

From the signal-space diagrams derived in the solution to Problem 5.1, we immediately observe the following:

1. Unipolar NRZ and unipolar RZ codes are non-minimum energy signals.
2. Polar NRZ and Manchester codes are minimum energy signals.

Problem 5.16

The orthonormal matrix that transforms the signal constellation shown in Fig. 5.11(a) of the textbook into the one shown in Fig. 5.11(b) is

$$\mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

To prove this statement, we note that the constellations of Fig. 5.11(a) is defined by the four points $\{(\alpha, \alpha), (-\alpha, \alpha), (-\alpha, -\alpha), (\alpha, -\alpha)\}$. The new constellation is defined by

$\mathbf{s}_{i,\text{ rotate}} = \mathbf{Q}\mathbf{s}_i$, which for $i = 1$ yields

$$\mathbf{s}_{1,\text{ rotate}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix} \qquad \text{for } \alpha = 1.$$

Similarly, $\mathbf{s}_{2,\text{ rotate}} = \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix}$

$$\mathbf{s}_{3,\text{ rotate}} = \begin{pmatrix} -\sqrt{2} \\ 0 \end{pmatrix}$$

$$\mathbf{s}_{4,\text{ rotate}} = \begin{pmatrix} 0 \\ -\sqrt{2} \end{pmatrix}$$

Hence, the transformation from Fig. 5.11(a) to Fig. 5.11(b) is given by $\mathbf{Q}$, except for a scaling factor.

Problem 5.17

(a) The minimum distance between any two adjacent signal points in the constellation of Fig. P5.17a of the textbook is

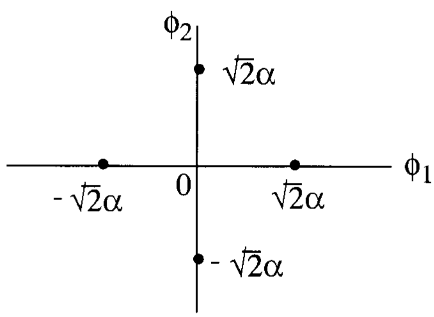
$$d_{\min}^{(a)} = 2\alpha$$

The minimum distance between any two adjacent signal points in the constellation of Fig. P5.17b of the textbook is

$$d_{\min}^{(b)} = \sqrt{(\sqrt{2}\alpha)^2 + (\sqrt{2}\alpha)^2} = 2\alpha$$

which is the same as $d_{\min}^{(a)}$. Hence, the average probability of symbol error using the constellation of Fig. P5.17a is the same as that of Fig. P5.17b.

(b) The constellation of Fig. P5.17a has minimum energy, whereas that of Fig. P5.17b is of non-minimum energy. Applying the minimum energy translate to the constellation of Fig. P5.17b, which involves translating it bodily to the left along the ϕ_1-axis by the amount $\sqrt{2}\alpha$, we get the corresponding minimum energy configuration:



Problem 5.18

Consider a set of three orthogonal signals denoted by $\{s_i(t)\}_{i=0}^{2}$, each with energy E_s. The average of these three signals is

$$a(t) = \frac{1}{3}\sum_{i=0}^{2} s_i(t)$$

Applying the minimum energy translate to the signal set $\{s_i(t)\}_{i=0}^{2}$, we get a new signal set defined by

$$s_i'(t) = s_i(t) - a(t), \qquad i = 0, 1, 2 \tag{1}$$

The signal energy of the new set is

$$E = \int_{-\infty}^{\infty} (s_i'(t))^2 dt$$

$$= \int_{-\infty}^{\infty} s_i^2(t)dt - 2\int_{-\infty}^{\infty} s_i(t)a(t)dt + \int_{-\infty}^{\infty} a(t)dt$$

$$= E_s - \frac{2}{3}E_s + \frac{1}{9}(3E_s)$$

$$= \frac{2}{3}E_s$$

The correlation coefficient ρ_{ij} between the signals $s_i'(t)$ and $s_j'(t)$ is given by

$$\rho_{ij} = \frac{E[s_i'(t)(s_j'(t))]}{E}$$

$$= \frac{3}{2E_s}\int_{-\infty}^{\infty} (s_i(t) - a(t))(s_j(t) - a(t))dt$$

$$= \frac{3}{2E_s}\left(\int_{-\infty}^{\infty} s_i(t)s_j(t)dt - \int_{-\infty}^{\infty} a(t)(s_i(t) + s_j(t))dt + \int_{-\infty}^{\infty} a^2(t)dt\right) \tag{2}$$

Since $s_i(t)$ and $s_j(t)$ are orthogonal by choice, Eq. (2) reduces to

$$\rho_{ij} = \frac{3}{2E_s}\left(0 - \frac{1}{3}E_s - \frac{1}{3}E_s + \frac{1}{9}(3E_s)\right)$$

$$= -\frac{1}{2} \quad \text{for } i \neq j$$

which is the maximum negative correlation that characterizes a simplex signal with $M = 3$. thus, the signal set $\{s_i'(t)\}_{i=0}^{2}$ defined in Eq. (1) is indeed a simplex signal.

To represent the signal set $\{s_i'(t)\}_{i=0}^{2}$ in geometric terms, we use the Gram-Schmidt orthogonalization procedure. Specifically, we first set

$$\phi_0(t) = \frac{s_0'(t)}{\sqrt{E}} \tag{3}$$

or equivalently

$$s_0'(t) = \sqrt{E}\phi_0(t)$$

The projection of $s_1'(t)$ unto $\phi_0(t)$ is

$$\begin{aligned} s_{10} &= \int_{-\infty}^{\infty} s_1'(t)\phi_0(t)dt \\ &= \frac{1}{\sqrt{E}}\int_{-\infty}^{\infty} s_1'(t)s_0'(t)dt \\ &= \sqrt{E}\left(\frac{1}{E}\int_{-\infty}^{\infty} s_1'(t)s_0'(t)dt\right) \\ &= \sqrt{E}\left(-\frac{1}{2}\right) \end{aligned}$$

The second basis function is therefore

$$\begin{aligned} \phi_1(t) &= \frac{s_1'(t) - s_{10}\phi_0(t)}{\sqrt{E - s_{10}^2}} \\ &= \frac{s_1'(t) + (\sqrt{E}/2)(\phi_0(t))}{\sqrt{E-(E/4)}} \\ &= \frac{2}{\sqrt{3E}}\left(s_1'(t) + \frac{\sqrt{E}}{2}\phi_0(t)\right) \end{aligned}$$

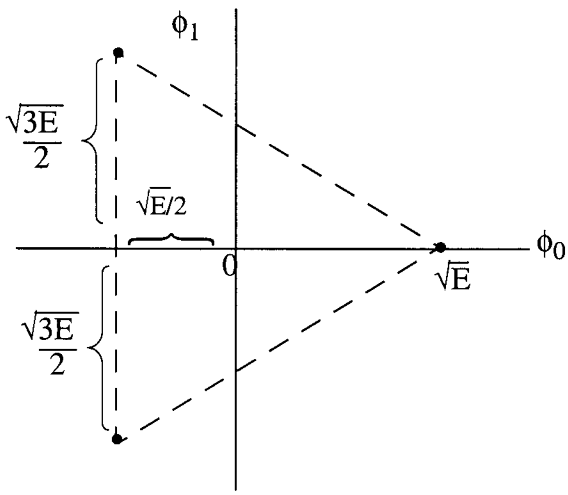
Accordingly, we may express $s_1'(t)$ in terms of the basis functions $\phi_0(t)$ and $\phi_1(t)$ as

$$s_1'(t) = -\frac{\sqrt{E}}{2}\phi_0(t) + \frac{\sqrt{3E}}{2}\phi_1(t) \tag{4}$$

The remaining signal $s_2'(t)$ may be expressed in terms of $\phi_0(t)$ and $\phi_1(t)$ as

$$s_2'(t) = -\frac{\sqrt{E}}{2}\phi_0(t) - \frac{\sqrt{3E}}{2}\phi_1(t) \tag{5}$$

Thus, using Eqs. (3) to (5), we may represent the simplex code by the following signal-space diagram:



Problem 5.19

(a) An upper bound on the complementary error function is given by

$$\text{erfc}(u) < \frac{\exp(-u^2)}{\sqrt{\pi}\ u}$$

Hence, we may bound the given P_e as follows:

$$P_e = \frac{1}{2}\text{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right) < \frac{\exp\left(-\frac{E_b}{N_0}\right)}{2\sqrt{\pi E_b N_0}} = \frac{1}{2}\exp\left(-\frac{E_b}{N_0}\right) \times \sqrt{\frac{N_0}{\pi E_b}} \tag{1}$$

For large positive u, we may further simplify the upper bound on the complementary error function as shown here:

$$\text{erfc}(u) < \frac{\exp(-u^2)}{\sqrt{\pi}}$$

Correspondingly, we may bound P_e as follows:

$$P_e < \frac{\exp(-E_b/N_0)}{2\sqrt{\pi}} \qquad (2)$$

(b) For $E_b/N_0 = 9$, we get the following results:

(i) The exact calculation of P_e yields

$$P_e = \frac{1}{2}\text{erfc}(3)$$

$$= 1.0 \times 10^{-5}$$

(ii) Using the bound in (1), we have the approximate value:

$$P_e \approx \frac{\exp(-9)}{6\sqrt{\pi}}$$

$$= 1.16 \times 10^{-5}$$

(iii) Using the looser bound of (2), we have

$$P_e \approx \frac{\exp(-9)}{2\sqrt{\pi}}$$

$$= 3.48 \times 10^{-5}$$

As expected, the first bound is more accurate than the second bound for calculating P_e.

Problem 5.20

According to Eq. (5.91) of the textbook, the probability of error is over-bounded as follows:

$$P_e(m_i) \le \frac{1}{2} \sum_{\substack{k=1 \\ k \ne i}}^{M} \operatorname{erfc}\left(\frac{d_{ik}}{2\sqrt{N_0}}\right), \quad i = 1, 2, \cdots, M \tag{1}$$

where d_{ik} is the distance between message points s_i and s_k. With the M transmitted messages assumed equally likely, the average probability of symbol error is overbounded as follows:

$$\begin{aligned} P_e &= \frac{1}{M} \sum_{i=1}^{M} P_e(m_i) \\ &\le \frac{1}{2M} \sum_{i=1}^{M} \sum_{\substack{k=1 \\ k \ne i}}^{M} \operatorname{erfc}\left(\frac{d_{ik}}{2\sqrt{N_0}}\right) \end{aligned} \tag{2}$$

The second line of Eq. (2) defines the union bound on the average probability of symbol error for any set of M equally likely signals in an AWGN channel. Equation (2) is particularly useful for the special case of a signal set that has a *symmetric geometry,* which is of common occurrence in practice. In such a case, the conditional error probability $P_e(m_i)$ is the same for all i, and so we may simplify Eq. (2) as

$$\begin{aligned} P_e &= P_e(m_i) \\ &\le \frac{1}{2} \sum_{\substack{k=1 \\ k \ne i}}^{M} \operatorname{erfc}\left(\frac{d_{ik}}{2\sqrt{N_0}}\right), \quad \text{for all } i \end{aligned} \tag{3}$$

The complementary error function may be upper-bounded as follows:

$$\operatorname{erfc}\left(\frac{d_{ik}}{2\sqrt{N_0}}\right) \le \frac{1}{\sqrt{\pi}} \exp\left(-\frac{d_{ik}^2}{2N_0}\right)$$

Hence, we may rewrite Eq. (3) as

$$P_e \le \frac{1}{2\sqrt{\pi}} \sum_{\substack{k=1 \\ k \ne i}}^{M} \exp\left(-\frac{d_{ik}^2}{2N_0}\right) \quad \text{for all } i \tag{4}$$

Provided that the transmitted signal energy is high enough compared to the noise spectral density N_0, the exponential term with the smallest distance d_{ik} will dominate the summation in Eq. (4). Accordingly, we may approximate the bound on P_e as

$$P_e \le \frac{M_{\min}}{2\sqrt{\pi}} \exp\left[-\min_{\substack{i,k \\ i \ne k}}\left(\frac{d_{ik}^2}{2N_0}\right)\right] \tag{5}$$

where $M_{\min}$ is the number of transmitted signals that attain the minimum Euclidean distance for each m_i. Equation (5) describes a simplified form of the union bound for a symmetric signal set, which is easy to calculate.