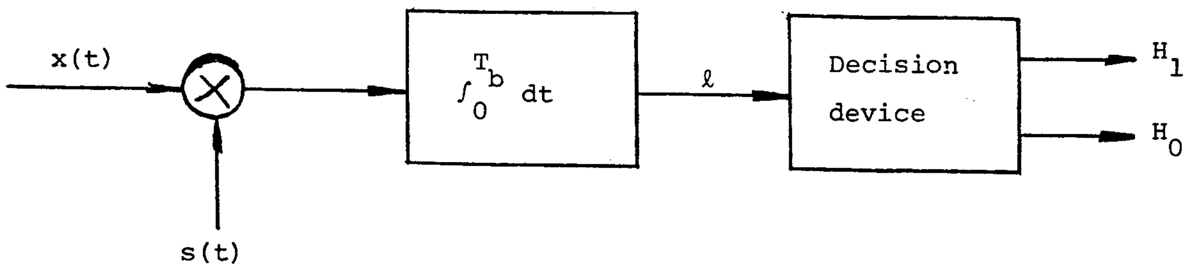
Problem 6.1

(a) ASK with coherent reception



Denoting the presence of symbol 1 or symbol 0 by hypothesis H_1 or H_0, respectively, we may write

$$H_1: \; x(t) = s(t) + w(t)$$

$$H_0: \; x(t) = w(t)$$

where $s(t) = A_c \cos(2\pi f_c t)$, with $A_c = \sqrt{2E_b/T_b}$. Therefore,

$$\ell = \int_0^{T_b} x(t)\, s(t)\, dt$$

If $\ell > E_b/2$, the receiver decides in favor of symbol 1. If $\ell < E_b/2$, it decides in favor of symbol 0.

The conditional probability density functions of the random variable L, whose value is denoted by ℓ, are defined by

$$f_{L|0}(\ell|0) = \frac{1}{\sqrt{\pi N_0 E_b}} \exp\left(-\frac{\ell^2}{N_0 E_b}\right)$$

$$f_{L|1}(\ell|1) = \frac{1}{\sqrt{\pi N_0 E_b}} \exp\left[-\frac{(\ell - E_b)^2}{N_0 E_b}\right]$$

The average probability of error is therefore,

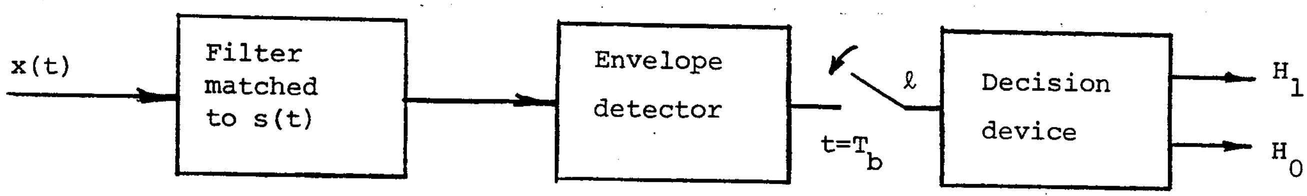
$$P_e = p_0 \int_{E_b/2}^{\infty} f_{L|0}(\ell|0)d\ell + p_1 \int_{-\infty}^{E_b/2} f_{L|1}(\ell|1)d\ell$$

$$= \frac{1}{2}\int_{E_b/2}^{\infty} \frac{1}{\sqrt{\pi N_0 E_b}} \exp\left(-\frac{\ell^2}{N_0 E_b}\right)d\ell + \frac{1}{2}\int_{-\infty}^{E_b/2} \frac{1}{\sqrt{\pi N_0 E_b}} \exp\left[-\frac{(\ell - E_b)^2}{N_0 E_b}\right]d\ell$$

$$= \frac{1}{\sqrt{\pi N_0 E_b}} \int_{E_b/2}^{\infty} \exp\left(-\frac{\ell^2}{N_0 E_b}\right)d\ell$$

$$= \frac{1}{2}\,\mathrm{erfc}\left(\frac{1}{2}\sqrt{E_b/N_0}\right)$$

(b) <u>ASK with noncoherent reception</u>



In this case, the signal s(t) is defined by

$$s(t) = A_c \cos(2\pi f_c t + \theta)$$

where $A_c = \sqrt{2\, E_b/T_b}$, and

$$f_\theta(\theta) = \begin{cases} \frac{1}{2\pi}, & 0 \le \theta \le 2\pi \\ 0, & \text{otherwise} \end{cases}$$

For the case when symbol 0 is transmitted, that is, under hypothesis H_0, we find that the random variable L, at the input of the decision device, is Rayleigh-distributed:

$$f_{L|0}(\ell|0) = \frac{4\ell}{N_0 T_b} \exp\left(-\frac{2\ell^2}{N_0 T_b}\right)$$

For the case when symbol 1 is transmitted, that is, under hypothesis H_1, we find that the random variable L is Rician-distributed:

$$f_{L|1}(\ell|1) = \frac{4\ell}{N_0T_b} \exp\left(-\frac{\ell^2 + A_c^2T_b^2/4}{N_0T_b/2}\right) I_0\left(\frac{2\ell A_c}{N_0}\right)$$

where $I_0(2\ell A_c/N_0)$ is the modified Bessel function of the first kind of zero order.

Before we can obtain a solution for the error performance of the receiver, we have to determine a value for the threshold. Since symbols 1 and 0 occur with equal probability, the minimum probability of error criterion yields:

$$\exp\left(-\frac{A_c^2T_b}{2N_0}\right) I_0\left(\frac{2\ell A_c}{N_0}\right) \underset{H_0}{\overset{H_1}{\gtrless}} 1 \qquad (1)$$

For large values of E_b/N_0, we may approximate $I_0(2\ell A_c/N_0)$ as follows:

$$I_0\left(\frac{2\ell A_c}{N_0}\right) \simeq \frac{\exp(2\ell A_c/N_0)}{\sqrt{4\pi\ell A_c/N_0}}$$

Using this approximation, we may rewrite Eq. (1) as follows:

$$\exp\left[\frac{A_c(4 - A_cT_b)}{2N_0}\right] \underset{H_0}{\overset{H_1}{\gtrless}} \sqrt{\frac{4\pi\ell A_c}{N_0}}$$

Taking the logarithm of both sides of this relation, we get

$$\ell \underset{H_0}{\overset{H_1}{\gtrless}} \frac{A_cT_b}{4} + \frac{1}{2}\sqrt{\frac{\pi\ell N_0}{A_c}}$$

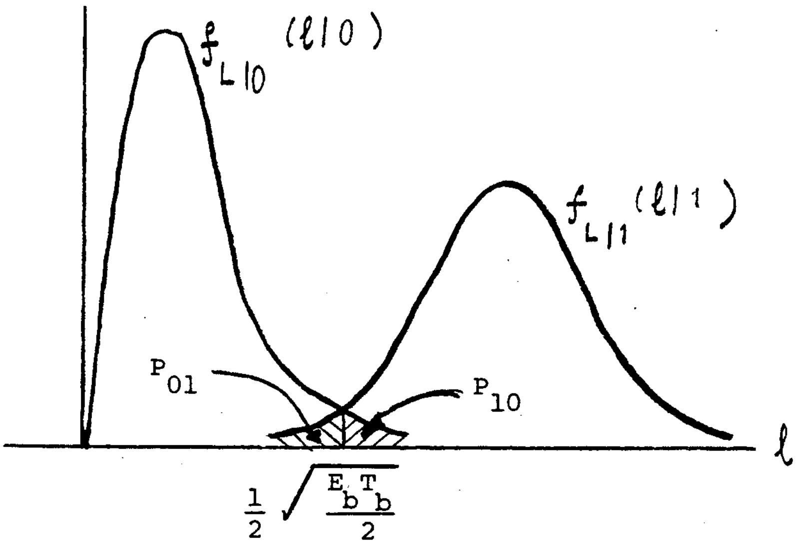
Neglecting the second term on the right hand side of this relation, and using the fact that

$$E_b = \frac{A_c^2T_b}{2}$$

we may write

$$\ell \underset{H_0}{\overset{H_1}{\gtrless}} \frac{1}{2}\sqrt{\frac{E_bT_b}{2}}$$

The threshold $\frac{1}{2}\sqrt{\frac{E_bT_b}{2}}$ is at the point corresponding to the crossover between the two probability density functions, as illustrated below.



The average probability of error is therefore

$$P_e = p_0 P_{10} + p_1 P_{01}$$

where

$$P_{10} = \int_{\sqrt{E_bT_b}/2\sqrt{2}}^{\infty} f_{L|0}(\ell|0)d\ell$$

$$= \int_{\sqrt{E_bT_b}/2\sqrt{2}}^{\infty} \frac{4\ell}{N_0T_b} \exp(-\frac{2\ell^2}{N_0T_b})d\ell$$

$$= \left[-\exp(-\frac{2\ell^2}{N_0T_b})\right]_{\sqrt{E_bT_b}/2\sqrt{2}}^{\infty}$$

$$= \exp(-\frac{E_b}{4N_0})$$

$$P_{01} = \int_0^{\sqrt{E_bT_b}/2\sqrt{2}} f_{L|1}(\ell|1)d\ell$$

$$= \int_0^{\sqrt{E_bT_b}/2\sqrt{2}} \frac{4\ell}{N_0T_b} \exp\left(-\frac{\ell^2 + A_c^2\,T_b^2/4}{N_0T_b/2}\right) I_0\left(\frac{2\ell A_c}{N_0}\right)d\ell$$

$$\simeq \int_0^{\sqrt{E_bT_b}/2\sqrt{2}} \frac{4\ell}{N_0T_b} \exp\left(-\frac{\ell^2 + A_c^2\,T_b^2/4}{N_0T_b/2}\right) \cdot \frac{\exp(2\ell A_c/N_0)}{\sqrt{4\pi\ell A_c/N_0}}\, d\ell$$

$$= \int_0^{\sqrt{E_b T_b}/2\sqrt{2}} \sqrt{\frac{2\ell}{A_c T_b}} \sqrt{\frac{2}{\pi N_0 T_b}} \exp\left[-\frac{(\ell - A_c T_b/2)^2}{N_0 T_b/2}\right] d\ell \qquad (2)$$

The integrand in Eq. (2) is the product of $\sqrt{2\ell/A_c T_b}$ and the probability density function of a Gaussian random variable of mean $A_c T_b/2$ and variance $N_0 T_b/4$. For high values of E_b/N_0, the standard deviation $\sqrt{N_0 T_b/4}$ is much less than the threshold $\sqrt{E_b T_b}/2\sqrt{2}$. Consequently, the area under the portion of the curve from 0 to $\sqrt{E_b T_b}/2\sqrt{2}$ is quite small, that is, $P_{01} \simeq 0$. Then, we may approximate the average probability of error as

$$P_e \simeq p_o P_{10}$$

$$= \frac{1}{2} \exp\left(-\frac{E_b}{4N_0}\right)$$

where it is assumed that symbols 0 and 1 occur with equal probability.

Problem 6.2

The transmitted binary PSK signal is defined by

$$s(t) = \begin{cases} \sqrt{E_b}\phi(t), & 0 \le t \le T_b, \quad \text{symbol } 1 \\ -\sqrt{E_b}\phi(t), & 0 \le t \le T_b, \quad \text{symbol } 0 \end{cases}$$

where the basis function $\phi(t)$ is defined by

$$\phi(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t)$$

The locally generated basis function in the receiver is

$$\phi_{\text{rec}}(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t + \varphi)$$

$$= \sqrt{\frac{2}{T_b}}[\cos(2\pi f_c t)\cos\varphi - \sin(2\pi f_c t)\sin\varphi]$$

where φ is the phase error. The correlator output is given by

$$y = \int_0^{T_b} x(t)\varphi_{\text{rec}}(t)dt$$

where

$$x(t) = s_k(t) + w(t), \qquad k = 1, 2$$

Assuming that f_c is an integer multiple of $1/T_b$, and recognizing that $\sin(2\pi f_c t)$ is orthogonal to $\cos(2\pi f_c t)$ over the interval $0 \le t \le T_b$, we get

$$y = \pm\sqrt{E_b}\cos\varphi + W$$

when the plus sign corresponds to symbol 1 and the minus sign corresponds to symbol 0, and W is a zero-mean Gaussian variable of variance $N_0/2$. Accordingly, the average probability of error of the binary PSK system with phase error φ is given by

$$P_e = \frac{1}{2}\text{erfc}\left(\sqrt{\frac{E_b \cos\varphi}{N_0}}\right)$$

When $\varphi = 0$, this formula reduces to that for the standard PSK system equipped with perfect phase recovery. At the other extreme, when $\varphi = \pm 90°$, P_e attains its worst value of unity.

Problem 6.3

(a) The noiseless PSK signal is given by

$$s(t) = A_c\cos[2\pi f_c t + k_p m(t)]$$

$$= A_c\cos(2\pi f_c t)\cos[k_p m(t)] - A_c\sin(2\pi f_c t)\sin[k_p m(t)]$$

Since $m(t) = \pm 1$, it follows that

$$\cos[k_p m(t)] = \cos(\pm k_p) = \cos(k_p)$$

$$\sin[k_p m(t)] = \sin(\pm k_p) = \pm \sin(k_p) = m(t)\sin(k_p)$$

Therefore,

$$s(t) = A_c\cos(k_p)\cos(2\pi f_c t) - A_c m(t)\sin(k_p)\sin(2\pi f_c t) \tag{1}$$

The VCO output is

$$r(t) = A_v\sin[2\pi f_c t + \theta(t)]$$

The multiplier output is therefore

$$r(t)s(t) = \frac{1}{2} A_c A_v \cos(k_p)\{\sin[\theta(t)] + \sin[4\pi f_c t + \theta(t)]\}$$

$$- \frac{1}{2} A_c A_v m(t)\sin(k_p)\{\cos[\theta(t)] + \cos[4\pi f_c t + \theta(t)]\}$$

The loop filter removes the double-frequency components, producing the output

$$e(t) = \frac{1}{2} A_c A_v \cos(k_p)\sin[\theta(t)] - \frac{1}{2} A_c A_v m(t)\sin(k_p)\cos[\theta(t)]$$

Note that if $k_p = \pi/2$, (i.e., the carrier is fully deviated), there would be no carrier component for the PLL to track.

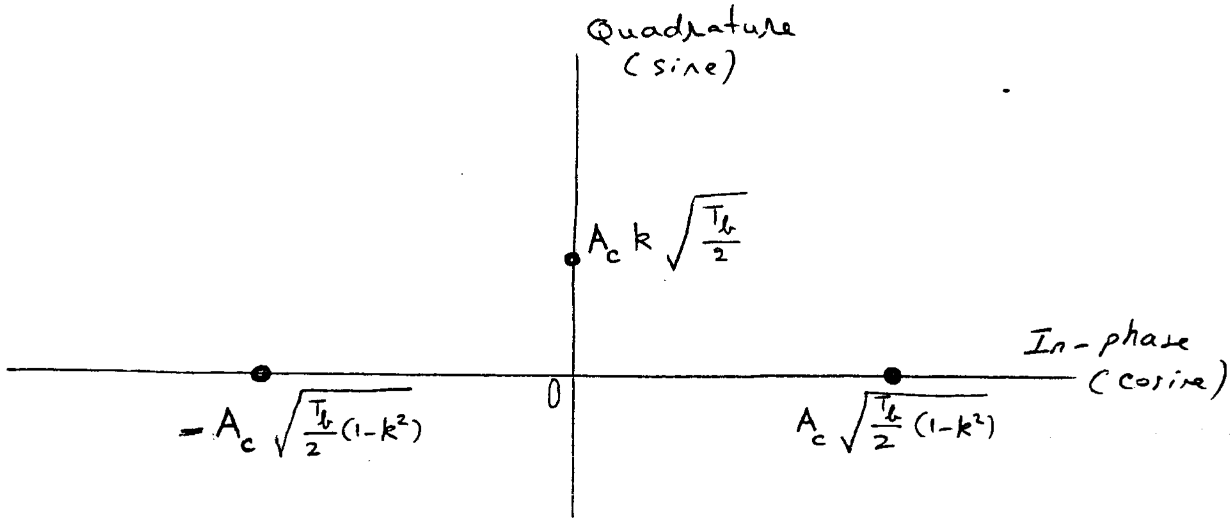
(b) Since the error signal tends to drive the loop into lock (i.e., $\theta(t)$ approaches zero), the loop filter output reduces to

$$e(t) = -\frac{1}{2} A_c A_v \sin(k_p) m(t)$$

which is proportional to the desired data signal m(t). Hence, the phase-locked loop may be used to recover m(t).

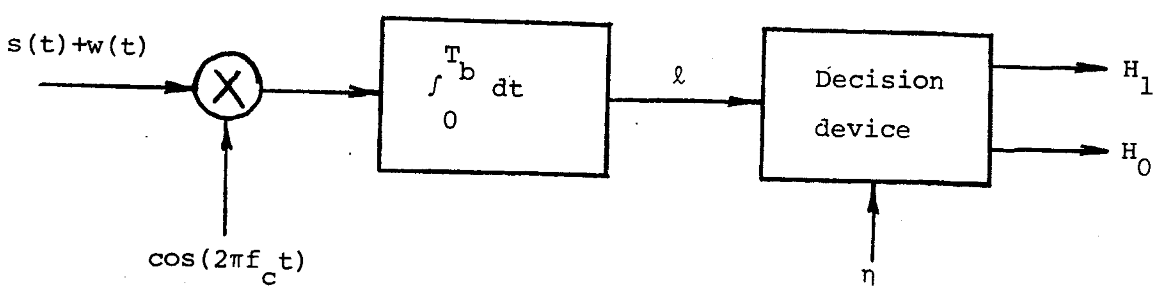
Problem 6.4

(a) The signal-space diagram of the scheme described in this problem is two-dimensional, as shown by



This signal-space diagram differs from that of the conventional PSK signaling scheme in that it is two-dimensional, with a new signal point on the quadrature axis at $A_c k\sqrt{T_b/2}$. If k is reduced to zero, the above diagram reduces to the same form as that shown in Fig. 8.14.

(b)



The signal at the decision device input is

$$\ell = \pm \frac{A_c}{2} \sqrt{1-k^2}\, T_b + \int_0^{T_b} w(t) \cos(2\pi f_c t)dt \tag{1}$$

Therefore, following a procedure similar to that used for evaluating the average probability of error for a conventional PSK system, we find that for the system defined by Eq. (1) the average probability of error is

$$P_e = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{E_b(1-k^2)/N_0}\right)$$

where $E_b = \frac{1}{2} A_c^2 T_b$.

(c) For the case when $P_e = 10^{-4}$ and $k^2 = 0.1$, we get

$$10^{-4} = \frac{1}{2}\,\mathrm{erfc}(u)$$

where $u^2 = \frac{0.9\,E_b}{N_0}$

Using the approximation

$$\mathrm{erfc}(u) \approx \frac{\exp(-u^2)}{\sqrt{\pi}\,u}$$

we obtain

$$\exp(-u^2) - 2\sqrt{\pi} \times 10^{-4}\,u = 0$$

The solution to this equation is u = 2.64. The corresponding value of E_b/N_0 is

$$\frac{E_b}{N_0} = \frac{(2.64)^2}{0.9} = 7.74$$

Expressed in decibels, this value corresponds to 8.9 dB.

(d) For a conventional PSK system, we have

$$P_e = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{E_b/N_0}\right)$$

In this case, we find that

$$\frac{E_b}{N_0} = (2.64)^2 = 6.92$$

Expressed in decibels, this value corresponds to 8.4 dB. Thus, the conventional PSK system requires 0.5 dB less in E_b/N_0 then the modified scheme described herein.

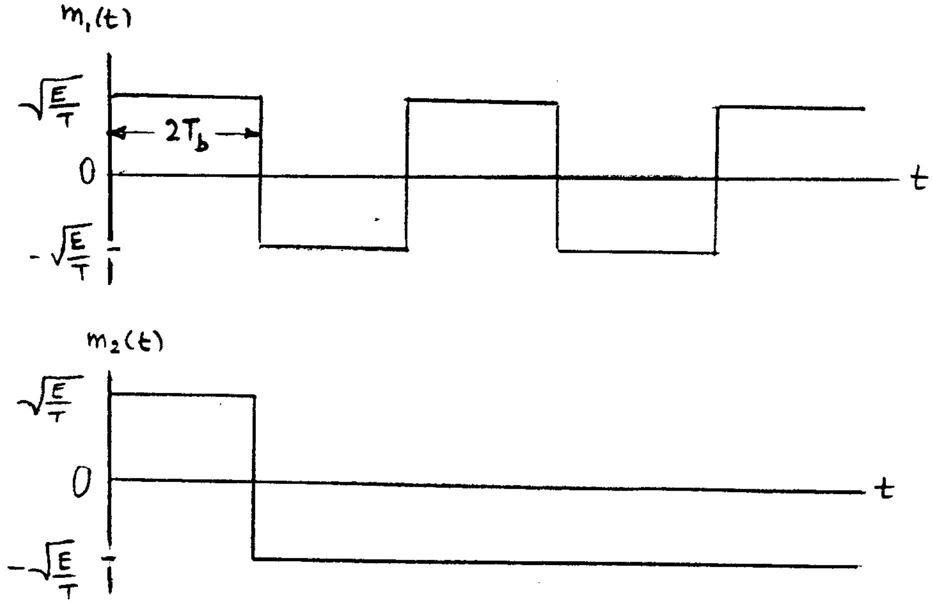
Problem 6.5

(a) The QPSK wave can be expressed as

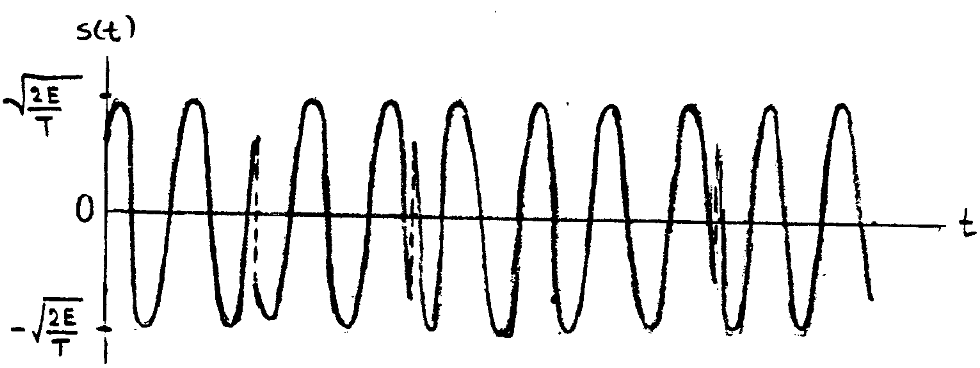
$$s(t) = m_1(t)\cos(2\pi f_c t) + m_2(t)\sin(2\pi f_c t).$$

Dividing the binary wave into dibits and finding $m_1(t)$ and $m_2(t)$ for each dibit:

dibit	11	00	10	00	10
$m_1(t)$	$\sqrt{E/T}$	$-\sqrt{E/T}$	$\sqrt{E/T}$	$-\sqrt{E/T}$	$\sqrt{E/T}$
$m_2(t)$	$\sqrt{E/T}$	$-\sqrt{E/T}$	$-\sqrt{E/T}$	$-\sqrt{E/T}$	$-\sqrt{E/T}$



(b)



Problem 6.6

Let P_{eI} = average probability of symbol error in to the in-phase channel

P_{eQ} = average probability of symbol error in to the quadrature channel

Since the individual outputs of the in-phase and quadrature channels are statistically independent, the overall average probability of correct reception is

$$P_c = (1 - P_{eI})(1 - P_{eQ})$$
$$= 1 - P_{eI} - P_{eQ} + P_{eI} P_{eQ}$$

The overall average probability of error is therefore

$$P_e = 1 - P_c$$
$$= P_{eI} + P_{eQ} - P_{eI} P_{eQ}$$

Problem 6.7

Let **r** denote the received signal vector. Suppose that the signal corresponding to message point $\mathbf{m}_1$ is transmitted. Then, referring to the signal-space diagram of Fig. 1, the conditional probability of error is given

$$
\begin{aligned}
P_{e|\mathbf{m}_1} &= P(\text{r lies in shaded region}) \\
&= P(\text{r lies in } \equiv) + P(\text{R lies in } |||) \\
&\quad - P(\text{r lies in } |||) \\
&= \frac{1}{2}\,\text{erfc}\left(\sqrt{\frac{E}{N_0}}\,\sin\frac{\pi}{M}\right) + \frac{1}{2}\,\text{erfc}\left(\sqrt{\frac{E}{N_0}}\,\sin\frac{\pi}{M}\right) \\
&\quad - P(\text{r lies in } |||)
\end{aligned}
$$

Hence,

$$
P_{e|\mathbf{m}_1} < \text{erfc}\left(\sqrt{\frac{E}{N_0}}\,\sin\frac{\pi}{M}\right)
$$

Assuming that all the message points are equally likely to be transmitted, we have $P_e = P_{e|\mathbf{m}_i}$, and so

$$
P_e < \text{erfc}\left(\sqrt{\frac{E}{N_0}}\,\sin\frac{\pi}{M}\right)
$$

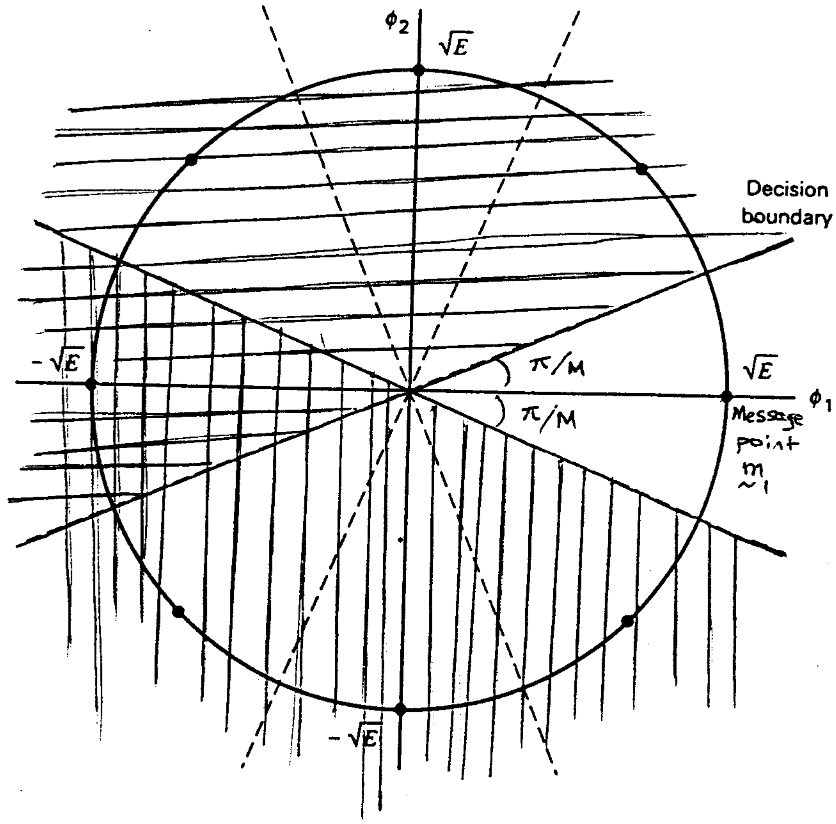

ϕ_2
$\sqrt{E}$
Decision boundary
π/M
π/M
$-\sqrt{E}$
$\sqrt{E}$
ϕ_1
Message point m_1
$-\sqrt{E}$

Figure 1

Problem 6.8

Figures 6.10 and 6.10b of the textbook, reproduced here for convenience of presentation, depict the signal-space diagrams of QPSK and offset QPSK signals, respectively:

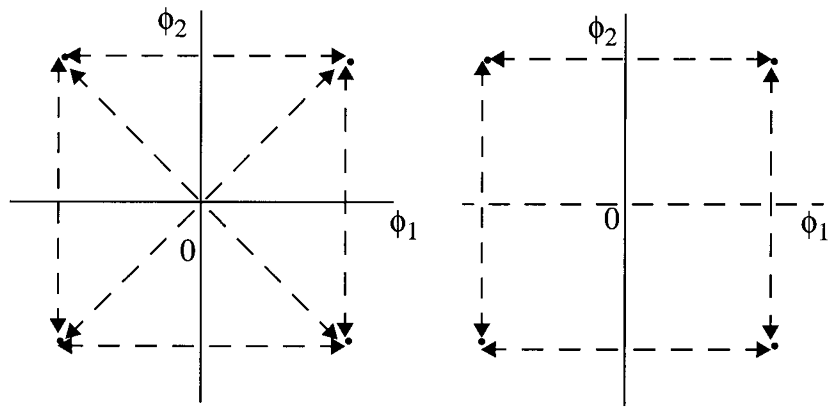


Figure 1

The two parts of this figure clearly show that the signal-space structure of the offset QPSK is basically the same as that of the standard QPSK. They only differ from each other in the way in which transition takes place from one signal point to another. Accordingly, they have the same power spectral density, as shown by

$$S(f) = E_b[\text{sinc}^2(2T_b(f-f_c)) + \text{sinc}^2(2T_b(f+f_c))]$$

where T_b is the bit duration and f_c is the carrier frequency.

Problem 6.9

(a) In vestigial-sideband (VSB) modulation, there are two basis functions:

- The double-bandwidth sinc function, defined by

$$\phi_1(t) = \sqrt{\frac{1}{T}}\text{sinc}\left(\frac{2t}{T}\right)\cos(2\pi f_c t) \tag{1}$$

where T is the symbol period and f_c is the carrier frequency.

- The Hilbert transform of $\phi_1(t)$, defined by

$$\phi_2(t) = \hat{\phi}_1(t)$$

$$= \sqrt{\frac{1}{T}}\operatorname{sinc}\left(\frac{2t}{T}\right)\sin(2\pi f_c t) \tag{2}$$

where it is assumed that $f_c > 2/T$.

(Here we have made use of the Hilbert-transform pair listed as entry 1 in Table A6.4, with the low-pass signal $m(t)$ set equal to $\sqrt{1/T}\operatorname{sinc} 2(t/T)$.)

The basis functions (1) and (2) imply the use of single-sideband modulation, which (as discussed in Chapter 2) is a special form of vestigial sideband modulation. We have chosen these definitions merely to simplify the discussion. The use of VSB substitutes a realizable function for the sinc function that is unrealizable in practice.

Based on the definitions of the basis functions $\phi_1(t)$ and $\phi_2(t)$ given in Eqs. (1) and (2), it may be tempting to choose $2/T$ as the symbol rate for successive transmission of binary symbols using binary VSB. However, such a choice of signaling destroys the orthonormality of $\phi_1(t)$ and $\phi_2(t)$; that is,

$$\int_0^T \phi_i(t)\phi_j\left(t - \frac{T}{2}\right)dt \neq \begin{cases} 1 & \text{for } j = i \\ 0 & \text{for } j \neq i \end{cases}$$

To maintain orthogonality of $\phi_1(t)$ and $\phi_2(t)$, successive translations of these basis functions must be integer multiples of $1/T$, as shown by

$$\int_0^T \phi_i(t)\phi_j(t - kT)dt = \begin{cases} 1 & \text{for } j = i \\ 0 & \text{for } j \neq i \end{cases}$$

for any integer k.

Suppose, however, we restrict k to assume only odd integer values, and choose the carrier frequency f_c to be an odd integer multiple of $1/2T$, that is,

$$f_c = \frac{l}{2T}, \qquad l = \text{odd integer} \tag{3}$$

We then have the following two properties:

(i) $$\int_0^T \phi_1(t)\phi_2\left(t-\frac{kT}{2}\right)dt = 0 \text{ for all odd integer } k \qquad (4)$$

(ii) $$\sin\left(2\pi f_c\left(t-\frac{kT}{2}\right)\right) = \sin(2\pi f_c t)\cos(kl\pi/2) - \cos(2\pi f_c t)\sin(kl\pi/2)$$

$$= \cos(2\pi f_c t) \quad \text{for} \quad \begin{matrix} k = \text{odd integer} \\ l = \text{odd integer} \end{matrix}$$

With such a choice, the implementation of the digital VSB transmission system is equivalent to a time-varying one-dimensional data transmission system, which operates at the rate of $2/T$ dimensions per second.

(b) The optimum receiver for the digital VSB transmission system just described consists of a pair of matched filters, that are matched to the two basis functions $\phi_1(t)$ and $\phi_2(t)$ as defined in Eqs. (1) and (2). However, in order to conform to the design choices imposed on integer k and carrier frequency f_c as described in Eqs. (4) and (3), the instants of time at which the two matched filter outputs are sampled are staggered by $T/2$ with respect to each other. The two sequences of samples so obtained are subsequently interleaved so as to produce a single one-dimensional data stream as the overall receiver output. The delay by $T/2$ is identical to what is actually done in the offset QPSK, thereby establishing the equivalence of the digital VSB system to the offset QPSK.

Problem 6.10

Assuming that modulator initially resides in a phase state of zero, we may construct the following sequence of events in response to the input sequence 01101000.

Step k	Phase θ_{k-1} (radians)	Input dibit	Phase change $\Delta\theta_k$ (radians)	Transmitted phase θ_k (radians)
1	0	01	$3\pi/4$	$3\pi/4$
2	$3\pi/4$	10	$-\pi/4$	$\pi/2$
3	$\pi/2$	10	$-\pi/4$	$\pi/4$
4	$\pi/4$	00	$\pi/4$	$\pi/2$

Problem 6.11

The output of a π/4-shifted QPSK modulator may be expressed in terms of its in-phase and quadrature components as

$$s(t) = \sqrt{\frac{2E}{T}}\cos(i\pi/4)\cos(2\pi f_c t) - \sqrt{\frac{2E}{T}}\sin(i\pi/4)\sin(2\pi f_c t) \qquad i = 0, 1, 2, \ldots, 7$$

The different values of interger i correspond to the eight possible phase states in which the modulator can reside. But, unlike the 8-PSK modulator, the phase states of the π/4-shifted QPSK modulator are divided into two QPSK groups that are shifted by π/4 relative to each other.

Therefore, $s_I(t) = \sqrt{\frac{2E}{T}}\cos(i\pi/4)$

$$s_Q(t) = \sqrt{\frac{2E}{T}}\sin(i\pi/4)$$

The orthonormal-basis functions for π/4-shifted QPSK may be defined as

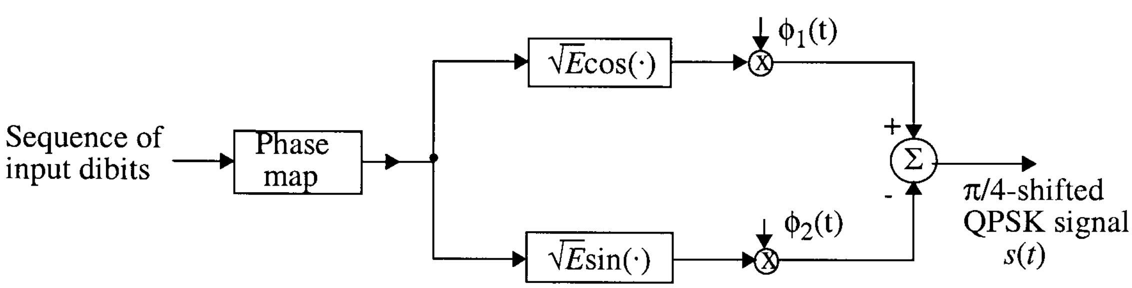
$$\phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t)$$

$$\phi_2(t) = \sqrt{\frac{2}{T}}\sin(2\pi f_c t)$$

Then the π/4-shifted QPSK signal is defined in terms of these two basis functions as

$$s(t) = \sqrt{E}\cos(i\pi/4)\phi_1(t) - \sqrt{E}\sin(i\pi/4)\phi_2(t)$$

On the basis of this representation, we may thus set up the following scheme for generating π/4-shifted QPSK signals:



Problem 6.12

A $\pi/4$-shifted DQPSK signal can be expressed as follows:

$$s(t) = \sqrt{\frac{2E}{T}}\cos(\phi_{k-1} + \Delta\phi_k)\cos(2\pi f_c t) - \sqrt{\frac{2E}{T}}\sin(\phi_{k-1} + \Delta\phi_k)\sin(2\pi f_c t)$$

$$= \sqrt{\frac{2E}{T}}\cos(2\pi f_c t + \phi_{k-1} + \Delta\phi_k)$$

where $\phi_{k-1} + \Delta\phi_k = \phi_k$ and ϕ_{k-1} is the absolute angle of symbol k-1, and $\Delta\phi_k$ is the differentially encoded phase change. In the demodulation process, the change in phase ϕ_k occurring over one symbol interval needs to be determined.

If we demodulate the $\pi/4$-shifted DQPSK signal using a FM discriminator, the output of the FM discriminator is given by

$$v_{\text{out}}(t) = K\frac{d[2\pi f_c t + \phi_k]}{dt}$$

$$= K\left[2\pi f_c + \frac{d\phi_k}{dt}\right]$$

$$= K[2\pi f_c + \Delta\phi_k]$$

where K is a constant. In a balanced FM discriminator, the DC offset $2\pi f_c K$ will not appear at the output. Hence, the output of the FM discriminator is $K\Delta\phi_k$.

Problem 6.13

The output of a $\pi/4$-shifted DQPSK modulator may be expressed as

$$s(t) = \sqrt{\frac{2E}{T}}\cos(\theta_{k-1} + \Delta\theta_k)\cos(2\pi f_c t) - \sqrt{\frac{2E}{T}}\sin(\theta_{k-1} + \Delta\theta_k)\sin(2\pi f_c t)$$

$$= \sqrt{\frac{2E}{T}}[\cos\theta_{k-1}\cos\Delta\theta_k - \sin\theta_{k-1}\sin\Delta\theta_k]\cos(2\pi f_c t)$$

$$-\sqrt{\frac{2E}{T}}[\sin\theta_{k-1}\cos\Delta\theta_k + \cos\theta_{k-1}\sin\Delta\theta_k]\sin(2\pi f_c t)$$

Let $I_k = \cos(\theta_{k-1} + \Delta\theta_k)$ and $Q_k = \sin(\theta_{k-1} + \Delta\theta_k)$. We may then write

$$I_k = \cos\theta_{k-1}\cos\Delta\theta_k - \sin\theta_{k-1}\cos\Delta\theta_k$$

$$= \cos(\theta_{k-2} + \Delta\theta_{k-1})\cos\Delta\theta_k - \sin(\theta_{k-2} + \Delta\theta_{k-1})\cos\Delta\theta_k$$

from which we readily deduce the recursion

$$I_k = I_{k-1}\cos\Delta\theta_k - Q_{k-1}\sin\Delta\theta_k$$

Similarly, we may show that

$$Q_k = \sin\theta_{k-1}\cos\Delta\theta_k + \cos\theta_{k-1}\sin\Delta\theta_k$$

$$= Q_{k-1}\cos\Delta\theta_k + I_{k-1}\sin\Delta\theta_k$$

From the definition of I_k and Q_k, we immediately see that I_k and Q_k may also be expressed as

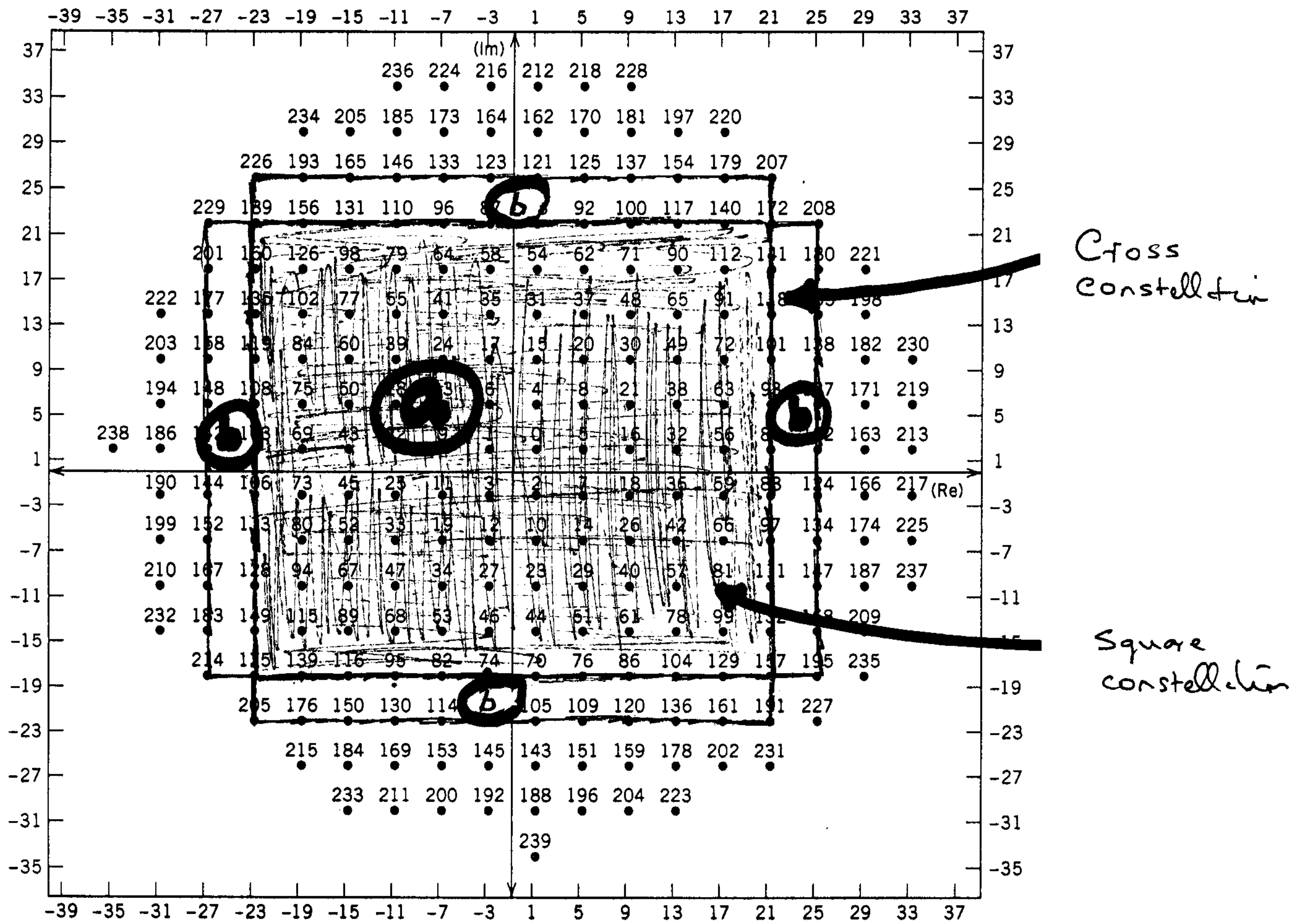
$$I_k = \cos\theta_k$$

and

$$Q_k = \sin\theta_k$$

which are the desired results.

Problem 6.14



Quarter - superconstellation of V.34 modem

Problem 6.15

The transmission bandwidth of 256-QAM signal is

$$B = \frac{2R_b}{\log_2 M}$$

where R_b is the bit rate given by $1/T_b$ and $M = 256$. Thus

$$B_{256} = \frac{2(1/T_b)}{\log_2 256} = \frac{2}{16T_b} = \frac{1}{8T_b}$$

The transmission bandwidth of 64-QAM is

$$B_{64} = \frac{2(1/T_b)}{\log_2 64} = \frac{2}{8T_b} = \frac{1}{4T_b}$$

Hence, the bandwidth advantage of 256-QAM over 64-QAM is

$$\frac{1}{4T_b} - \frac{1}{8T_b} = \frac{1}{8T_b}$$

The average energy of 256-QAM signal is

$$E_{256} = \frac{2(M-1)E_0}{3} = \frac{2(256-1)E_0}{3}$$

$$= 170E_0$$

where E_0 is the energy of the signal with the lowest amplitude. For the 64-QAM signal, we have

$$E_{64} = \frac{2(63)}{3}E_0 = 42E_0$$

Therefore, the increase in average signal energy resulting from the use of 256-QAM over 64-QAM, expressed in dBs, is

$$10\log_{10}\left(\frac{170E_0}{42E_0}\right) \approx 10\log_{10}(4)$$

$$= 6 \text{ dB}$$

Problem 6.16

The probability of symbol error for 16-QAM is given by

$$P_e = 2\left(1 - \frac{1}{\sqrt{M}}\right)\text{erfc}\left(\sqrt{\frac{3E_{\text{av}}}{2(M-1)N_0}}\right)$$

Setting $P_e = 10^{-3}$, we get

$$10^{-3} = 2\left(1 - \frac{1}{4}\right)\text{erfc}\left(\sqrt{\frac{3E_{\text{av}}}{30N_0}}\right)$$

Solving this equation for E_{av}/N_0,

$$\frac{E_{\text{av}}}{N_0} = 58 = 17.6\text{dB}$$

The probability of symbol error for 16-PSK is given by

$$P_e = \text{erfc}\left(\sqrt{\frac{E}{N_0}}\sin(\pi/M)\right)$$

Setting $P_e = 10^{-3}$, we get

$$10^{-3} = \text{erfc}\left(\sqrt{\frac{E}{N_0}}\sin(\pi/16)\right)$$

Solving this equation for E/N_0, we get

$$\frac{E}{N_0} = 142 = 21.5\text{dB}$$

Hence, on the average, the 16-PSK demands 21.5 - 17.6 = 3.9 dB more symbol energy than the 16-QAM for $P_e = 10^{-3}$.

Thus the 16-QAM requires about 4 dB less in signal energy than the 16-PSK for a fixed N_0 and $P_e = 10^{-3}$, However, for this advantage of the 16-QAM over the 16-PSK to be realized, the channel must be linear.

Problem 6.17

(a) An M-ary QAM signal is defined by

$$s_k(t) = \sqrt{\frac{2E}{T}}a_k\cos(2\pi f_c t) - \sqrt{\frac{2E}{T}}a_k\sin(2\pi f_c t) \qquad (1)$$

We can redefine the M-ary QAM signal in terms of a general pulse-shaping function $g(t)$ as

$$s_k(t) = a_k g(t-kT)\cos(2\pi f_c t) - b_k g(t-kT)\sin(2\pi f_c t) \qquad (2)$$

The M-ary QAM signal $s(t)$ for an infinite succession of input symbols can be expressed as

$$s(t) = \sum_{k=-\infty}^{\infty} s_k(t)$$

$$= \sum_{k=-\infty}^{\infty} \{a_k g(t-kT)\cos(2\pi f_c t) - b_k g(t-kT)\sin(2\pi f_c t)\}$$

$$= \text{Re}\left\{\sum_{k=-\infty}^{\infty} (a_k + jb_k)g(t-kT)e^{j2\pi f_c t}\right\}$$

$$= \text{Re}\left\{\sum_{k=-\infty}^{\infty} A_k g(t-kT)e^{j2\pi f_c t}\right\} \qquad (3)$$

where A_k is a complex number defined by

$$A_k = a_k + jb_k$$

By multiplying Eq. (3) by $\exp(-j2\pi f_c kT) \times \exp(j2\pi f_c kT)$, we get

$$s(t) = R_e\left\{\sum_{k=-\infty}^{\infty} A_k g(t-kT)\exp(-j2\pi f_c kT)\exp(j2\pi f_c t)\right\}$$

$$= \text{Re}\left\{\sum_{k=-\infty}^{\infty} \tilde{A}_k \tilde{g}(t-kT)\right\} \qquad (4)$$

where $\tilde{A}_k = A_k \exp(j2\pi f_c kT)$

$$\tilde{g}(t) = g(t)\exp(j2\pi f_c t)$$

The scalar $\tilde{A}_k$ is a rotated version of the complex representation of the kth transmitted signal.

Equation (4), representing a QAM signal, appears to be carrierless, therefore, it is equivalent to a CAP system.

(b) A CAP signal is defined as

$$s(t) = \mathrm{Re}\left\{ \sum_{k=-\infty}^{\infty} \tilde{A}_k \tilde{g}(t-kT) \right\}$$

$$= \mathrm{Re}\left\{ \sum_{k=-\infty}^{\infty} \tilde{A}_k \exp(j2\pi f_c kT) g(t-kT) \exp(j2\pi f_c (t-kT)) \right\}$$

$$= \mathrm{Re}\left\{ A_k \sum_{k=-\infty}^{\infty} g(t-kT) \exp(j2\pi f_c t) \right\}$$

$$= \sum_{k=-\infty}^{\infty} a_k g(t-kT)\cos(2\pi f_c t) - \sum_{k=-\infty}^{\infty} b_k g(t-kT)\sin(2\pi f_c t) \quad (5)$$

Now the pulse shaping functions of CAP signal, $g(t - kT)$, may be replaced by $\sqrt{\frac{2E}{T}}$ for $0 \leq t \leq T$, and the formulaton in Eq. (5) can be rewritten as

$$s(t) = \sum_{k=-\infty}^{\infty} \sqrt{\frac{2E}{T}} a_k \cos(2\pi f_c t) - \sqrt{\frac{2E}{T}} b_k \sin(2\pi f_c t) \quad (6)$$

The kth signal of the signal $s(t)$ defined in Eq. (6) is given by

$$s_k(t) = \sqrt{\frac{2E}{T}} a_k \cos(2\pi f_c t) - \sqrt{\frac{2E}{T}} b_k \sin(2\pi f_c t), \quad 0 \leq t \leq T \quad (7)$$

The signal formulation given in Eq. (7) is recognized as the M-ary QAM signal of Eq. (1).

Problem 6.18

The CAP signal can be expressed as

$$s(t) = \sum_{n=-\infty}^{\infty} a_n p(t-nT) - \sum_{n=-\infty}^{\infty} b_n \hat{p}(t-nT)$$

where $\hat{p}(t)$ is the Hilbert transform of the pulse $p(t)$, and $A_n = a_n + jb_n$. The CAP signal $s(t)$ can be written in the equivalent form:

$$s(t) = \left[\sum_{n=-\infty}^{\infty} a_n \delta(t-nT)\right] \star p(t)$$

$$-\left[\sum_{n=-\infty}^{\infty} b_n \delta(t-nT)\right] \star \hat{p}(t)$$

where $\delta(t)$ is the delta function, and the star denotes convolution in the time domain. Hence, the power spectral density of $s(t)$ is

$$S_s(f) = \frac{\sigma_a^2}{T}|P(f)|^2 + \frac{\sigma_b^2}{T}|\hat{P}(f)|^2$$

where σ_a^2 and σ_b^2 are the variances of symbol a_k and b_k, respectively, where $p(t) \rightleftharpoons P(f)$ and $\hat{p}(t) \rightleftharpoons \hat{P}(f)$. Noting that $|\hat{P}(f)| = P(f)$, we thus have

$$S_s(f) = \frac{\sigma_a^2 + \sigma_b^2}{T}|P(f)|^2$$

Next, noting that

$$\sigma_a^2 + \sigma_b^2 = \sigma_A^2 = \frac{1}{L}\sum_{i=1}^{L}(a_i^2 + b_i^2)$$

we finally get

$$S_s(f) = \sigma_A^2 |P(f)|^2$$

Problem 6.19

From the defining equations (6.74) and (6.75) of the textbook, we have

$$p(t) = g(t)\cos(2\pi f_c t) \tag{1}$$

and

$$\hat{p}(t) = g(t)\sin(2\pi f_c t) \tag{2}$$

Applying the Fourier transform to Eqs. (1) and (2), we get

$$P(f) = \frac{1}{2}[G(f-f_c)+G(f+f_c)] \tag{3}$$

and

$$\hat{P}(f) = \frac{1}{2j}[G(f-f_c)-G(f+f_c)] \tag{4}$$

Accordingly, we may determine $p(t)$ and $\hat{p}(t)$ by proceeding as follows:

- Given $G(f)$, use Eqs. (3) and (4) to evaluate $P(f)$ and $\hat{P}(f)$.
- Using the inverse Fourier transform, compute $p(t) = F^{-1}[P(f)]$ and $\hat{p}(t) = F^{-1}[\hat{P}(f)]$.

Problem 6.20

For binary FSK, the two signal vectors are

$$\mathbf{s}_1 = \begin{bmatrix} \sqrt{E_b} \\ 0 \end{bmatrix}$$

$$\mathbf{s}_2 = \begin{bmatrix} 0 \\ \sqrt{E_b} \end{bmatrix}$$

where E_b is the signal energy per bit. The inner products of these two signal vectors with the observation vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

are as follows, respectively,

$$(\mathbf{x},\mathbf{s}_1) = \sqrt{E_b}\ x_1$$

$$(\mathbf{x},\mathbf{s}_2) = \sqrt{E_b}\ x_2$$

where $(\mathbf{x}, \mathbf{s}_i) = \mathbf{x}^T\mathbf{s}_i$ for i=1,2. The condition

$$\mathbf{x}^T \mathbf{s}_1 > \mathbf{x}^T \mathbf{s}_2$$

is therefore equivalent to

$$\sqrt{E_b}\ x_1 > \sqrt{E_b}\ x_2$$

Cancelling the common factor $\sqrt{E_b}$, we get

$$x_1 > x_2$$

which is the desired condition for making a decision in favor of symbol 1.

Problem 6.21

The bit duration is

$$T_b = \frac{1}{2.5\text{x}10^6\text{Hz}} = 0.4\ \mu s$$

The signal energy per bit is

$$E_b = \frac{1}{2} A_c^2 T_b$$
$$= \frac{1}{2} (10^{-6})^2 \text{ x } 0.4 \text{ x } 10^{-6} = 2 \text{ x } 10^{-19} \text{ joules}$$

(a) Coherent Binary FSK

The average probability of error is

$$P_e = \frac{1}{2} \text{erfc}(\sqrt{E_b/2N_0})$$

$$= \frac{1}{2} \text{erfc}(\sqrt{2\text{x}10^{-19}/4\text{x}10^{-20}})$$

$$= \frac{1}{2} \text{erfc}(\sqrt{5})$$

Using the approximation

$$\text{erfc}(u) \approx \frac{\exp(-u^2)}{\sqrt{\pi}\, u}$$

we obtain the result

$$P_e = \frac{1}{2} \frac{\exp(-5)}{\sqrt{5\pi}} = 0.85 \text{ x } 10^{-3}$$

(b) MSK

$$P_e = \text{erfc}(\sqrt{E_b/N_0})$$

$$= \text{erfc}(\sqrt{10})$$

$$\simeq \frac{\exp(-10)}{\sqrt{10\pi}}$$

$$= 0.81 \times 10^{-5}$$

(c) Noncoherent Binary FSK .

$$P_e = \frac{1}{2}\exp(-\frac{E_b}{2N_0})$$

$$= \frac{1}{2}\exp(-5)$$

$$= 3.37 \times 10^{-3}$$

Problem 6.22

(a) The correlation coefficient of the signals $s_0(t)$ and $s_1(t)$ is

$$\rho = \frac{\int_0^{T_b} s_0(t)s_1(t)dt}{[\int_0^{T_b} s_0^2(t)dt]^{1/2}\,[\int_0^{T_b} s_1^2(t)dt]^{1/2}}$$

$$= \frac{A_c^2 \int_0^{T_b} \cos[2\pi(f_c+\frac{1}{2}\Delta f)t]\cos[2\pi(f_c-\frac{1}{2}\Delta f)t]\,dt}{[\frac{1}{2}A_c^2T_b]^{1/2}\,[\frac{1}{2}A_c^2T_b]^{1/2}}$$

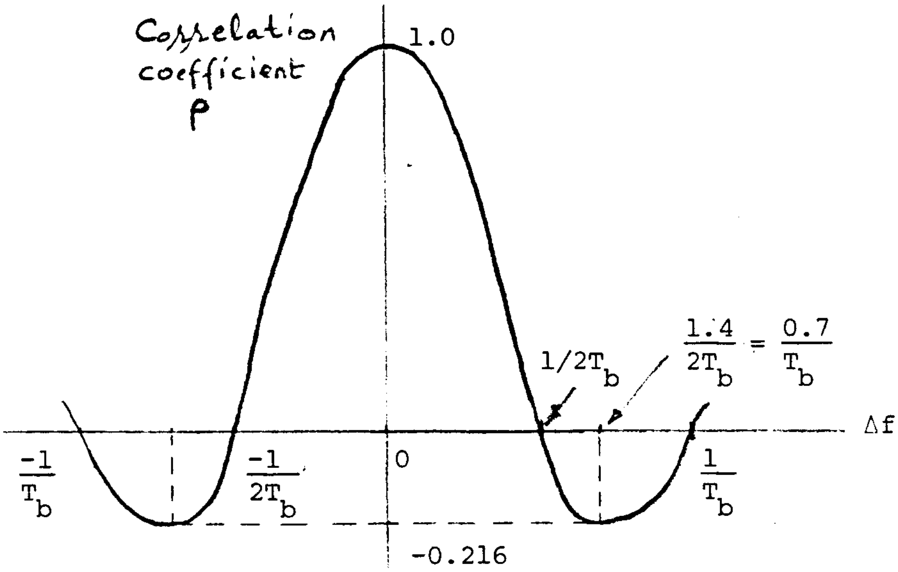
$$= \frac{1}{T_b}\int_0^{T_b} [\cos(2\pi\Delta ft) + \cos(4\pi f_c t)]\,dt$$

$$= \frac{1}{2\pi T_b}\left[\frac{\sin(2\pi\Delta fT_b)}{\Delta f} + \frac{\sin(4\pi f_c T_b)}{2f_c}\right] \qquad (1)$$

Since $f_c \gg \Delta f$, then we may ignore the second term in Eq. (1), obtaining

$$\rho \simeq \frac{\sin(2\pi\Delta fT_b)}{2\pi T_b\,\Delta f} = \text{sinc}(2\Delta fT_b)$$

(b) The dependence of ρ on Δf is as shown in Fig. 1.



Correlation coefficient ρ
1.0
1/2T_b
1.4/2T_b = 0.7/T_b
Δf
-1/T_b
-1/2T_b
0
1/T_b
-0.216

Fig. 1

$s_0(t)$ and $s_1(t)$ are othogonal when $\rho = 0$. Therefore, the minimum value of Δf for which they are orthogonal, is $1/2T_b$.

(c) The average probability of error is given by

$$E_b = \frac{1}{2} \operatorname{erfc}(\sqrt{E_b(1-\rho)/2N_0})$$

The most negative value of ρ is -0.216, occuring at $\Delta f = 0.7/T_b$. The minimum value of P_e is therefore

$$P_{e,min} = \frac{1}{2} \operatorname{erfc}(\sqrt{0.608E_b/N_0})$$

(d) For a coherent binary PSK system, the average probability of error is

$$P_e = \frac{1}{2} \operatorname{erfc}(\sqrt{E_b/N_0})$$

Therefore, the E_b/N_0 of this coherent binary FSK system must be increased by the factor 1/0.608 = 1.645 (or 2.16 dB) so as to realize the same average probability of error as a coherent binary PSK system.

Problem 6.23

(a) Since the two oscillators used to represent symbols 1 and 0 are independent, we may view the resulting binary FSK wave as the sum of two on-off keying (OOK) signals. One OOK signal operates with the oscillator of frequency f_1. The second OOK signal operates with the oscillator of frequency f_2.

The power spectral density of a random binary wave $X_1(t)$, in which symbol 1 is represented by A volts and symbol 0 by zero volts, is given by (see Problem 4.10)

$$S_{X_1}(f) = \frac{A^2}{4} \delta(f) + \frac{A^2 T_b}{4} \operatorname{sinc}^2(fT_b)$$

where T_b is the bit duration. When this binary wave is multiplied by a sinusoidal wave of unit amplitude and frequency $f_c + \Delta f/2$, we get the first OOK signal with

$$A = \sqrt{2E_b/T_b}$$

The power spectral density of this OOK signal equals

$$S_1(f) = \frac{1}{4} [S_{X_1}(f - f_c - \frac{\Delta f}{2}) + S_{X_1}(f + f_c + \frac{\Delta f}{2})]$$

The power spectral density of the random binary wave $X_2(t) = \overline{X_1(t)}$, in which symbol 1 is represented by zero volts and symbol 0 by A volts, is given by

$$S_{X_2}(f) = S_{X_1}(f)$$

When $X_2(t)$ is multiplied by the second sinusoidal wave of unit amplitude and frequency $f_c - \Delta f/2$, we get the second OOK signal whose power spectral density equals

$$S_2(f) = \frac{1}{4}\left[S_{X_2}\left(f - f_c + \frac{\Delta f}{2}\right) + S_{X_2}\left(f + f_c - \frac{\Delta f}{2}\right)\right]$$

The power spectral density of the FSK signal equals:

$$S_{FSK}(f) = S_1(f) + S_2(f)$$

$$= \frac{E_b}{8T_b}\left[\delta\left(f - f_c - \frac{\Delta f}{2}\right) + \delta\left(f + f_c + \frac{\Delta f}{2}\right) + \delta\left(f - f_c + \frac{\Delta f}{2}\right) + \delta\left(f + f_c - \frac{\Delta f}{2}\right)\right]$$

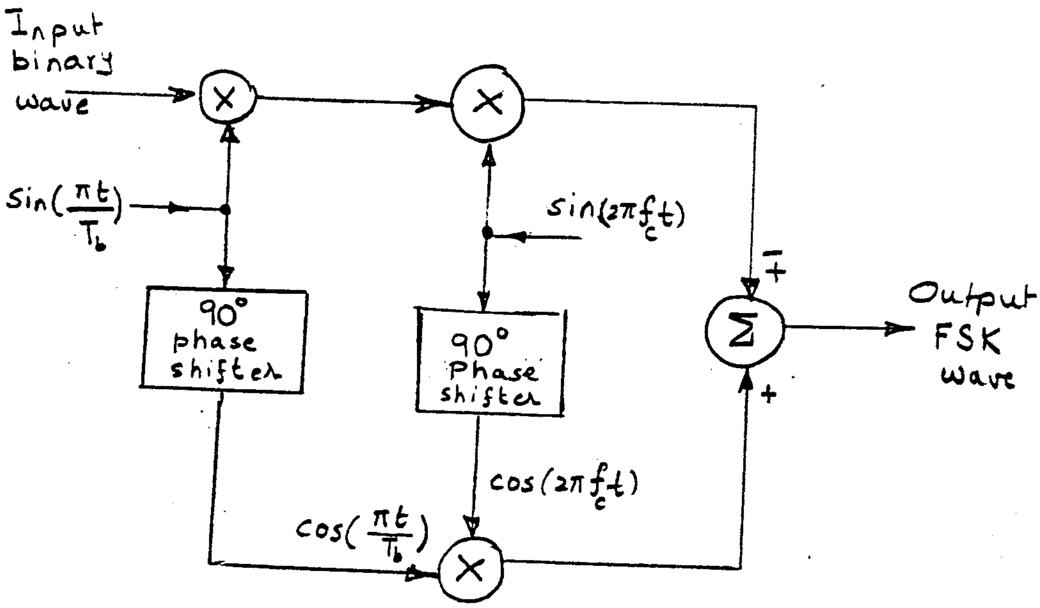
$$+ \frac{E_b}{8}\left\{\mathrm{sinc}^2\left[T_b\left(f - f_c - \frac{\Delta f}{2}\right)\right] + \mathrm{sinc}^2\left[T_b\left(f + f_c + \frac{\Delta f}{2}\right)\right]\right.$$

$$\left. + \mathrm{sinc}^2\left[T_b\left(f - f_c + \frac{\Delta f}{2}\right)\right] + \mathrm{sinc}^2\left[T_b\left(f + f_c - \frac{\Delta f}{2}\right)\right]\right\}$$

This result shows that the power spectrum of this binary FSK wave contains delta functions at $f = f_c \pm \Delta f/2$.

(b) At high values of x, the function sinc(x) falls off as 1/x. Hence, at high frequencies, S_{FSK} falls off as $1/f^2$.

Problem 6.24



Problem 6.25

The similarities between offset QPSK and MSK are that both have a half-symobl delay between the in-phase and quadrature components of each data symbol, and both have the same probability of error.

The differences between the two techniques are: (1) the basis functions for offset QPSK are sinusoids multiplied by a rectangle function, while the basis functions for MSK are sinusoids multiplied by half a cosine pulse, and (2) offset QPSK is a form of phase modulation while MSK is a form of frequency modulation.

Problem 6.26

For coherent MSK, the probability of error is

$$P_e = \text{erfc}(\sqrt{E_b/N_0}) \ ,$$

while for noncoherent MSK, (i.e., noncoherent binary FSK)

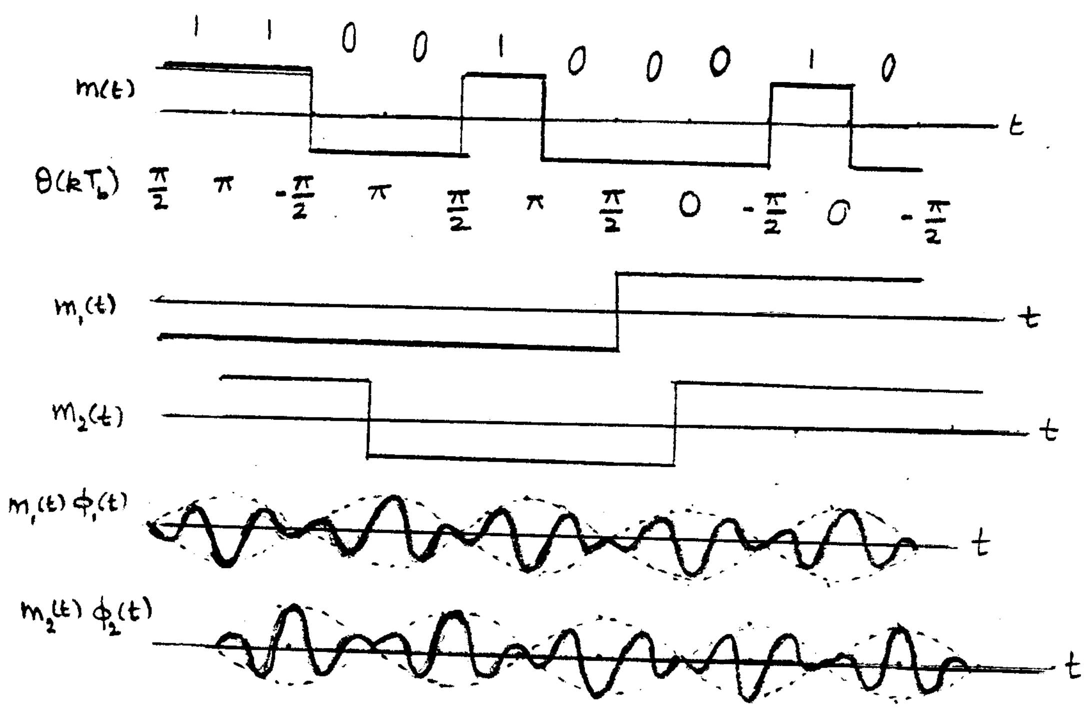
$$P_e = \frac{1}{2}\exp\left(-\frac{E_b}{2N_0}\right) \ .$$

To maintain $P_e = 10^{-5}$ for coherent MSK, $\frac{E_b}{N_0} = 9.8$. To maintain the same probability of symbol error for noncoherent MSK,

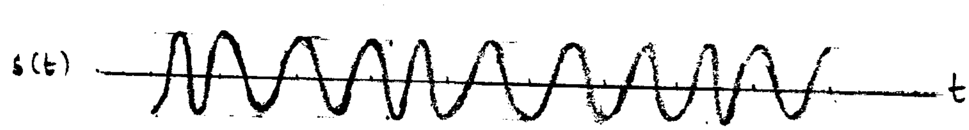
$$\frac{E_b}{N_0} = 21.6,$$ which is an increase of 3.4 dB.

Problem 6.27

(a)



(b)



Problem 6.28

(a) The Fourier transform of $h(t)$ is given by (using entry 5 of the Fourier-transform pairs Table A6.3)

$$H(f) = \frac{\sqrt{\pi}}{\alpha} \cdot \frac{1}{\sqrt{\pi/\alpha}} \exp\left(-\pi \times \frac{f^2}{\pi/\alpha^2}\right)$$

$$= \exp-(f^2\alpha^2) \qquad (1)$$

Substituting $\alpha = (\sqrt{\log 2/2}/W)$ into (1), we get

$$H(f) = \exp\left(-f^2 \frac{\log 2}{2} \times \frac{1}{W^2}\right)$$

$$= \exp\left(-\frac{\log 2}{2}\left(\frac{f}{W}\right)^2\right) \qquad (2)$$

Let f_0 denote the 3-dB cut-off frequency of the GMSK signal. Then, by definition,

$$|H(f_0)| = \frac{1}{\sqrt{2}}|H(0)|$$

$$= \frac{1}{\sqrt{2}}$$

Hence, from Eq. (2) it follows that

$$\exp\left(-\frac{\log 2}{2}\left(\frac{f_0}{W}\right)^2\right) = \frac{1}{\sqrt{2}}$$

or

$$\exp\left(\log 2\left(\frac{f_0}{W}\right)^2\right) = 2$$

Taking the logarithm of both sides, we readily find that

$f_0 = W$

The 3-dB bandwidth (cut-off frequency) of the filter used to shape GMSK signals is therefore W.

(b) The response of the filter to a rectangular pulse of unit amplitude and duration T centered on the origin is given by

$$g(t) = \int_{-T/2}^{T/2} h(t-\tau)d\tau$$

$$= \int_{-T/2}^{T/2} \frac{\sqrt{\pi}}{\alpha}\exp\left[\frac{-\pi^2(t-\tau)^2}{\alpha^2}\right]d\tau \qquad (3)$$

Let $k = \frac{\pi(t-\tau)}{\alpha}$, and

$$dk = -\frac{\pi}{\alpha}d\tau$$

Hence, we may reqrite Eq. (3) as

$$g(t) = -\int_{k_1}^{k_2} \frac{\sqrt{\pi}}{\alpha}\exp(-k^2)\frac{\alpha}{\pi}dk \qquad (4)$$

where $k_1 = \frac{\pi(t+T/2)}{\alpha}$ and

$$k_2 = \frac{\pi(t-T/2)}{\alpha}$$

Equation (4) is finally rewritten as

$$g(t) = -\frac{1}{2}\left\{\frac{2}{\sqrt{\pi}}\int_{0}^{k_2} \exp(-k^2)dk + \frac{2}{\sqrt{\pi}}\int_{k_1}^{0} \exp(-k^2)dk\right\}$$

$$= -\frac{1}{2}\mathrm{erf}(k_2) + \frac{1}{2}\mathrm{erf}(k_1)$$

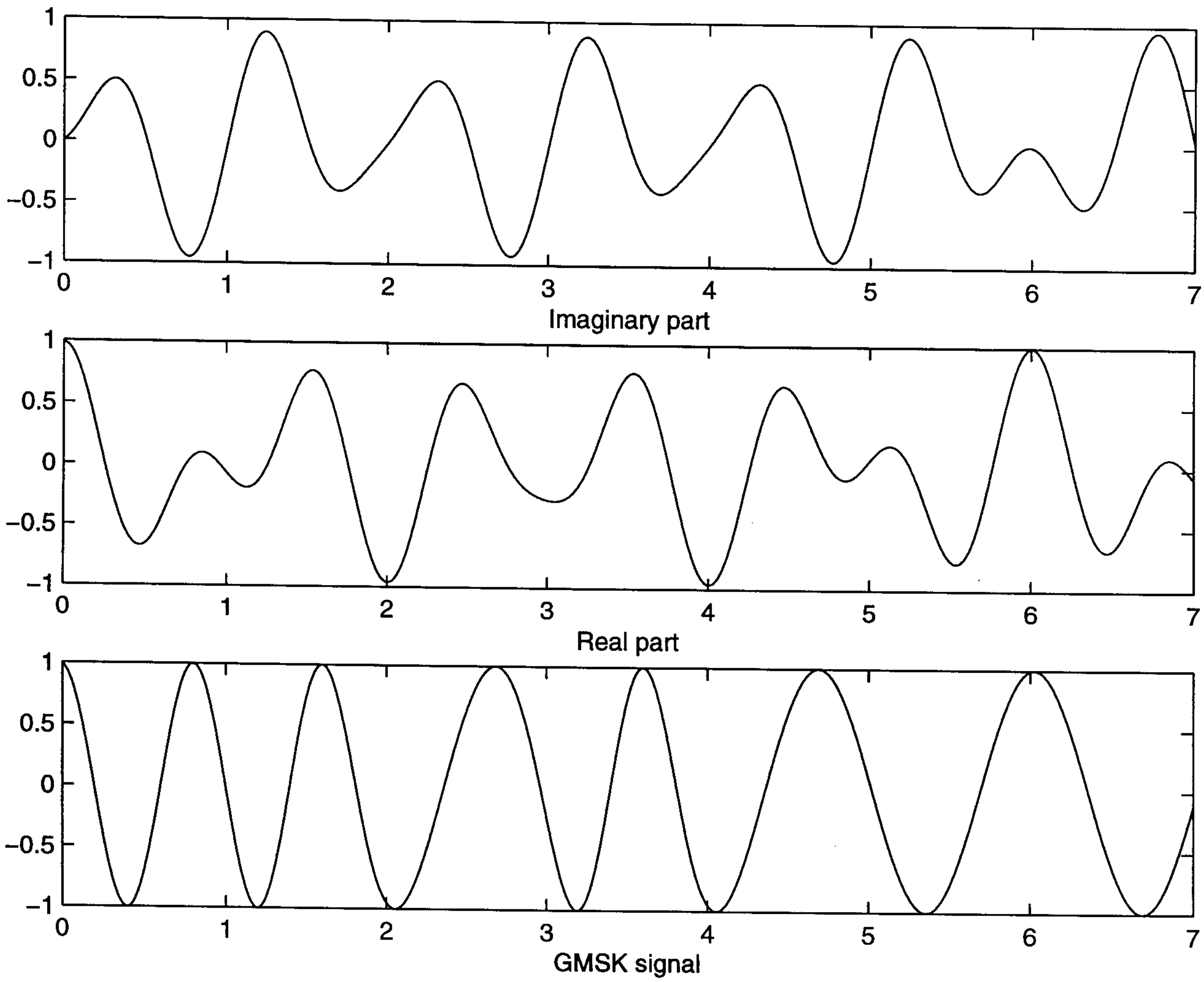
$$= -\frac{1}{2}[1-\mathrm{erfc}(k_2)] + \frac{1}{2}[1-\mathrm{erfc}(k_1)]$$

$$= \frac{1}{2}\mathrm{erfc}(k_2) - \frac{1}{2}\mathrm{erfc}(k_1)$$

$$= \frac{1}{2}\left\{\mathrm{erfc}\left[\frac{\pi(t-T/2)}{\alpha}\right] - \mathrm{erfc}\left[\frac{\pi(t-T/2)}{\alpha}\right]\right\}$$

$$= \frac{1}{2}\left\{\mathrm{erfc}\left[\pi\sqrt{\frac{2}{\log 2}}WT\left(\frac{t}{T}-\frac{1}{2}\right)\right] - \mathrm{erfc}\left[\pi\sqrt{\frac{2}{\log 2}}WT\left(\frac{t}{T}+\frac{1}{2}\right)\right]\right\}$$

Problem 6.29



The GMSK signal, displayed in the bottom waveform, is very similar to that of the MSK signal in Fig. 6.30, both of which are produced by the input sequence 1101000. This objective is indeed the idea behind the GMSK signal.

Problem 6.30

Comparing the standard MSK and Gaussian-filtered GMS signals, we note the following:

(a) Similarities

- For a given input sequence, the waveforms produced by the MSK and GMSK modulators are very similar, as illustrated by comparing the GMSK signal displayed in the solution to Problem 6.29 and the corresponding MSK signal displayed in Fig. 6.30 of the textbook for the input sequence 1101000.
- They both have a constant envelope.

(b) Differences

The use of GMSK results in a slight degradation in performance compared to the standard MSK for a time-bandwidth product $WT_b = 0.3$. However, the GMSK makes up for this loss in performance by providing a more compact power-spectral characteristic.

Problem 6.31

In the binary FSK case, the transmitted signal is defined by

$$s_i(t) = \begin{cases} \sqrt{2\frac{E_b}{T_b}} \cos(2\pi f_i t), & 0 \le r \le T_b \\ 0, & \text{elsewhere} \end{cases} \quad (1)$$

where the carrier frequency f_i equals one of two possible values f_1 and f_2. The transmission of frequency f_1 represents symbol 1, and the transmission of frequency f_2 represents symbol 0. For the noncoherent detection of this frequency-modulated wave, the receiver consists of a pair of matched filters followed by envelope detectors, as in Fig. 1. The filter in the upper channel of the receiver is matched to $\sqrt{2/T_b}\cos(2\pi f_1 t)$ and the filter in the lower channel is matched to $\sqrt{2/T_B}\cos(2\pi f_2 t), 0 \le t \le T_b$. The resulting envelope detector outputs are sampled at $t = T_b$, and their values are compared. Let l_1 and l_2 denote the envelope samples of the upper and lower channels, respectively, Then, if $l_1 > l_2$, the receiver decides in favor of symbol 1, and if $l_1 < l_2$ it decides in favor of symbol 0.

Suppose symbol 1 or frequency f_1 is transmitted. Then a correct decision will be made by the receiver if $l_1 > l_2$. If, however, the noise is such that $l_1 < l_2$, the receiver decides in favor of symbol 0, and an erroneous decision will have been made. To calculate the probability of error, we must have the probability density functions of the random variables L_1 and L_2 whose sample values are denoted by l_1 and l_2, respectively.

When frequency f_1 is transmitted, and there is no synchronism between the receiver and transmitter, the received signal x(t) is of the form

$$\begin{aligned} x(t) &= \sqrt{\frac{2E_b}{T_B}} \cos(2\pi f_1 t + \theta) + w(t) \\ &= \sqrt{\frac{2E_b}{T_b}} \cos\theta \cos(2\pi f_1 t) - \sqrt{\frac{2E_b}{T_b}} \sin\theta \sin(2\pi f_1 t) + w(t), \quad 0 \le t \le T_b \end{aligned} \quad (2)$$

Let

$$x_{ci} = \int_0^{T_b} x(t) \sqrt{\frac{2}{T_b}} \cos(2\pi f_i t) dt, \qquad i = 1,2 \tag{3}$$

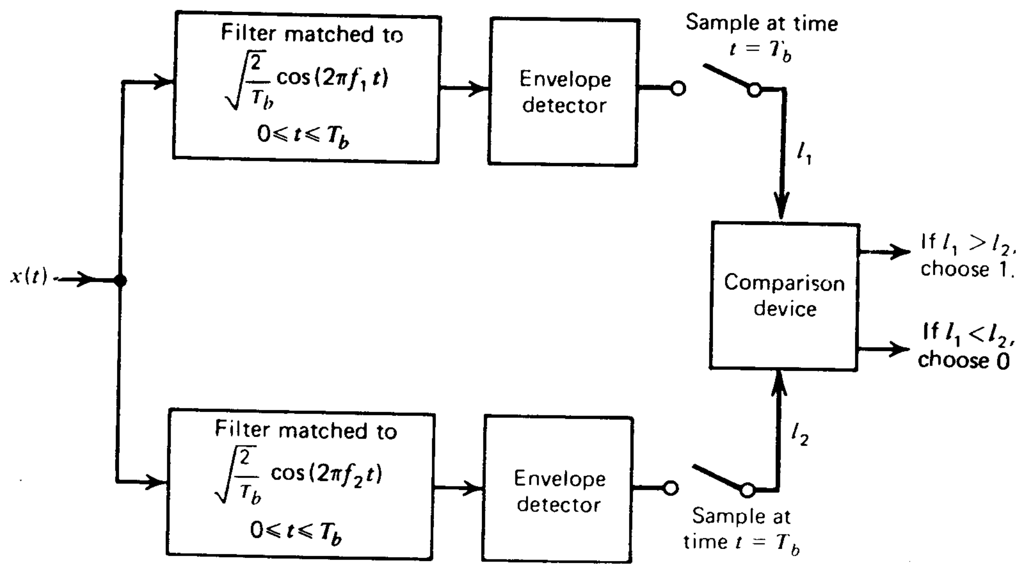


Figure 1

and

$$x_{si} = \int_0^{T_b} x(t) \sqrt{\frac{2}{T_b}} \sin(2\pi f_i t) dt, \qquad i = 1,2, \tag{4}$$

The x_{ci} and x_{si} i=1,2, define the coordinates of the received signal point. Note that, although each transmitted signal $s_i(t)$, i=1,2, is represented by a point in a two-dimensional space, the presence of the unknown phase θ makes it necessary to use four orthonormal basis functions in order to resolve the received signal x(t). With the received signal x(t) having the form shown in Eq. (1), we find that the output of the upper channel in the receiver of Fig. 1 equals

$$l_1 = \sqrt{x_{c1}^2 + x_{s1}^2} \tag{5}$$

where

$$x_{c1} = \sqrt{E_b} \cos\theta + w_{c1} \tag{6}$$

and

$$x_{s1} = -\sqrt{E_b} \sin\theta + w_{s1} \tag{7}$$

On the other hand, the corresponding value of the lower channel output is

$$l_2 = \sqrt{x_{c2}^2 + x_{s2}^2} \tag{8}$$

where

$$x_{c2} = w_{c2} \tag{9}$$

and

$$x_{s2} = w_{s2} \tag{10}$$

The w_{ci} and w_{si}, i=1,2, are related to the noise w(t) as follows:

$$w_{ci} = \int_0^{T_b} w(t) \sqrt{\frac{2}{T_b}} \cos(2\pi f_i t)\,dt, \qquad i = 1,2 \tag{11}$$

and

$$w_{si} = \int_0^{T_b} w(t) \sqrt{\frac{2}{T_b}} \sin(2\pi f_i t)\,dt, \qquad i = 1,2, \tag{12}$$

Accordingly, w_{ci} and w_{si}, i=1,2, are sample values of independent Gaussian random variables of zero mean and variance $N_0/2$.

When symbol 1 or frequency f_1 is transmitted, we see from Eqs. (9) and (10) that x_{c2} and x_{s2} are sample values of two Gaussian and statistically independent random variables, X_{c2} and X_{s2}, with zero mean and variance $N_0/2$. Accordingly, the lower channel output l_2, related to x_{c2} and x_{s2} by Eq. (8), is the sample value of a Rayleigh-distributed random variable L_2. We may thus express the conditional probability density function of L_2, given that symbol 1 was transmitted, as follows:

$$f_{L2|1}(l_2 \mid 1) = \frac{2l_2}{N_0} \exp\left(- \frac{l_2^2}{N_0} \right), \qquad l_2 \geq 0 \tag{13}$$

Again under the condition that symbol 1 or frequency f_1 is transmitted, we see from Eqs. (6) and (7) that x_{c1} and x_{s1} are sample values of two Gaussian and statistically independent random variables, X_{c1} and X_{s1}, with mean values equal to $\sqrt{E_b} \cos \theta$ and $\sqrt{E_b} \sin \theta$, respectively, and variance $N_0/2$. Therefore, the joint probability density function of X_{c1} and X_{s1}, given that symbol 1 was transmitted and that the random phase $\Theta = \theta$, may be expressed as follows

$$f_{X_{c1},X_{s1}|1,\Theta}(x_{c1}, x_{s1} \mid \theta) = \frac{1}{\pi N_0} \exp\left\{ -\frac{1}{N_0} \left[(x_{c1} - \sqrt{E_b} \cos \theta)^2 + (x_{s1} + \sqrt{E_b} \sin \theta)^2 \right] \right\} \tag{14}$$

Define the transformations

$$x_{c1} = l_1 \cos \psi_1 \tag{15}$$

and

$$x_{s1} = l_1 \sin \psi_1 \tag{16}$$

where $\psi_1 = \tan^{-1}(x_{s1}/x_{c1})$, with $0 \leq \psi_1 \leq 2\pi$. Then, applying this transformation and following a procedure similar to that described in Section 5.12, we find that the upper channel output l_1 is the sample value of a Rician-distributed random variable L_1. Hence, the conditional probability density function of L_1, given that symbol 1 was transmitted and that the random phase $\Theta = \theta$, is given by the Rician distribution

$$\begin{aligned} f_{L1|\Theta}(l_1 \mid 1, \theta) &= \int_0^{2\pi} f_{L1,\Psi|1,\Theta}(l_1, \psi \mid 1, \theta) d\psi \\ &= \frac{2l_1}{N_0} \exp\left(-\frac{l_1^2 + E_b}{N_0} \right) I_0\left(\frac{2l_1 \sqrt{E_b}}{N_0} \right), \qquad l_1 \geq 0 \end{aligned} \tag{17}$$

where $I_0(2l_1\sqrt{E_b}/N_0)$ is the modified Bessel function of the first kind of zero order. Since Eq. (17) does not depend on θ, which is to be expected, it follows that the conditional probability density function of L_1, given that symbol 1 was transmitted, is

$$f_{L1|1}(l_1 \mid 1) = \frac{2l_1}{N_0} \exp\left(-\frac{l_1^2 + E_b}{N_0} \right) I_0\left(\frac{2l_1 \sqrt{E_b}}{N_0} \right), \qquad l_1 \geq 0 \tag{18}$$

Note that by putting $E_b = 0$ and recognizing that $I_0(0) = 1$, Eq. (18) reduces to a Rayleigh distribution.

When symbol 1 is transmitted, the receiver makes an error whenever the envelope sample l_2 obtained from the lower channel (due to noise alone) exceeds the envelope sample l_1 obtained from the upper channel (due to signal plus noise), for all possible values of l_1. Consequently, the probability of this error is obtained by integrating $f_{L2|1}(l_2 \mid 1)$ with respect to l_2 from l_1 to infinity, and then averaging over all possible values of l_1. That is to say,

$$\begin{aligned} p_{01} &= P(l_2 > l_1 \mid \text{symbol 1 was sent}) \\ &= \int_0^\infty dl_1 f_{L_1|1} \int_0^\infty dl_2 f_{L_2|1}(l_2 \mid 1) \end{aligned} \tag{19}$$

where the inner integral is the conditional probability of error for a *fixed* value of l_1, given that symbol 1 was transmitted, and the outer integral is the average of this conditional probability for all possible values of l_1. Since the random variable L_2 is Rayleigh-distributed when symbol 1 is transmitted, the inner integral in Eq. (19) is equal to $\exp(-l_1/2N_0)$. Thus, using Eq. (18) in (19), we get

$$p_{01} = \int_0^\infty \frac{2l_1}{N_0} \exp\left(-\frac{2l_1^2 + E_b}{N_0} \right) I_0\left(\frac{2l_1 \sqrt{E_b}}{N_0} \right) dl_1 \tag{20}$$

Define a new variable v related to l_1 by

$$v = \frac{2l_1}{\sqrt{N_0}} \tag{21}$$

Then, changing the variable of integration from l_1 to v, we may rewrite Eq. (20) in the form

$$p_{01} = \frac{1}{2} \exp\left(-\frac{E_b}{2N_0} \right) \int_0^\infty v \exp\left(-\frac{v^2 + a^2}{2} \right) I_0(av)dv \tag{22}$$

where $a = \sqrt{E_b/N_0}$. The integral in Eq. (22) represents the total area under the normalized form of the Rician distribution. Since this integral must be equal to one, we may simplify Eq. (22) as

$$p_{01} = \frac{1}{2}\exp\left(-\frac{E_b}{2N_0}\right) \tag{23}$$

Similarly, when symbol 0 or frequency f_2 is transmitted, we may show that p_{10}, the probability that $l_1 > l_2$ and therefore the probability that the receiver makes an error by deciding in favor of symbol 1, has the same value as in Eq. (23). Thus, averaging p_{10} and p_{01}, we find that the average probability of symbol error for the noncoherent binary FSK equals

$$P_e = \frac{1}{2}\exp\left(-\frac{E_b}{2N_0}\right) \tag{24}$$

which is exactly the same as that in Eq. (6.163) in the textbook.

Comparing the effort involved in the derivation of Eq. (24) presented in this problem with that in deriving Eq. (6.163), we clearly see the elegance of the approach adopted in the textbook.

Problem 6.32

Let

$$x(t) = A_c \cos(2\pi f_c t + \theta)$$

$$= A_c \cos(2\pi f_c t)\cos\theta - A_c \sin(2\pi f_c t)\sin\theta$$

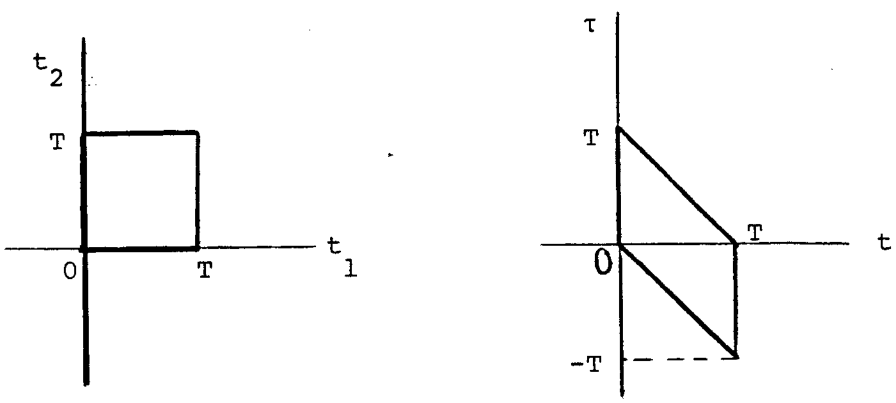
The output of the square-law envelope detector in Fig. P8.2, sampled at time t=T, is given by

$$y(T) = \left[\int_0^T x(t)\cos(2\pi f_c t)\,dt\right]^2 + \left[\int_0^T x(t)\sin(2\pi f_c t)\,dt\right]^2$$

This may be written as

$$y(T) = \int_0^T\int_0^T x(t_1)x(t_2)[\cos(2\pi f_c t_1)\cos(2\pi f_c t_2) + \sin(2\pi f_c t_1)\sin(2\pi f_c t_2)]dt_1 dt_2 \quad (1)$$

Put $t_1 = t$, and $t_2 = t+\tau$. This transformation is illustrated below:



Then, we may rewrite Eq. (1) as follows

$$y(T) = \int_0^T\int_{-t}^{T-t} x(t)x(t+\tau)\,[\cos(2\pi f_c t)\cos(2\pi f_c t + 2\pi f_c \tau)$$

$$+ \sin(2\pi f_c t)\sin(2\pi f_c t + 2\pi f_c \tau)]\,dt\,d\tau \quad (2)$$

However,

$$\cos(2\pi f_c t)\cos(2\pi f_c t + 2\pi f_c \tau) + \sin(2\pi f_c t)\sin(2\pi f_c t + 2\pi f_c \tau) = \cos(2\pi f_c \tau)$$

Therefore, we may simplify Eq. (2) as follows

$$y(T) = \int_0^T\int_{-t}^{T-t} x(t)\,x(t+\tau)\cos(2\pi f_c \tau)\,d\tau\,dt$$

$$= 2\int_0^T\int_0^{T-t} x(t)\,x(t+\tau)\cos(2\pi f_c \tau)\,d\tau\,dt, \quad 0 \le \tau \le T \quad (3)$$

Define

$$R_X(\tau) = \int_0^{T-t} x(t)\, x(t+\tau)\, dt \qquad 0 \leq \tau \leq T$$

Then, we may rewrite Eq. (3) in terms of $R_X(\tau)$ as follows

$$y(T) = 2 \int_0^T R_X(\tau) \cos(2\pi f_c \tau)\, d\tau$$

$$= 2\, S_X(f_c) \qquad (3)$$

where

$$S_X(f) = \int_0^T R_X(\tau) \cos(2\pi f_c \tau)\, d\tau$$

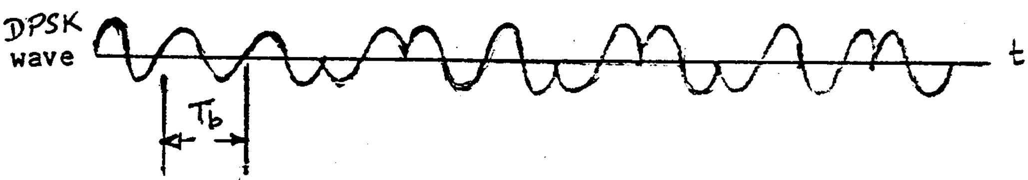
Equation (3) is the desired result.

Problem 6.33

(a)

b_k		1	1	0	0	1	0	0	0	1	0
d_{k-1}		1	1	1	0	1	1	0	1	0	0
d_k	1	1	1	0	1	1	0	1	0	0	1
Transmitted phase	0	0	0	π	0	0	π	0	π	π	0

The waveform of the DPSK signal is thus as follows:



(b) Let x_I = output of the integrator in the in-phase channel

x_Q = output of the integrator in the quadrature channel

x_I' = one-bit delayed version of x_I

x_Q' = one-bit delayed version of x_Q

l_I = in-phase channel output

$= x_I x_I'$

l_Q = quadrature channel output

$= x_Q x_Q'$

$y = l_I + l_Q$

Transmitted phase (radians)	0	0	0	π	0	0	π	0	π	π	0
Polarity of x_I	+	+	+	-	+	+	-	+	-	-	+
Polarity of x_I'		+	+	+	-	+	+	-	+	-	-
Polarity of l_I		+	+	-	-	+	-	-	-	+	-
Polarity of x_Q	-	-	-	+	-	-	+	-	+	+	-
Polarity of x_Q'		-	-	-	+	-	-	+	-	+	+
Polarity of l_Q		+	+	-	-	+	-	-	-	+	-
Polarity of y		+	+	-	-	+	-	-	-	+	-
Reconstructed data stream		1	1	0	0	1	0	0	0	1	0

Problem 6.34

Coherent M-ary PSK requires exact knowledge of the carrier frequency and phase for the receiver to be accurately synchronized to the transmitter. When carrier recovery at the receiver is impractical, we may use differential encoding based on the phase difference between successive symbols at the cost of some degradation in performance. If the incoming data are encoded by a phase-shift rather than by absolute phase, the receiver performs detection by comparing the phase of one symbol with that of the previous symbol, and the need for a coherent reference is thereby eliminated. This procedure is the same as that described for binary DPSK. The exact calculation of probability of symbol error for the differential detection of differential M-ary PSK (commonly referred to as M-ary DPSK) is much too complicated for $M > 2$. However, for large values of E/N_0 and $M \geq 4$, the probability of symbol error is approximately given by

$$P_e \approx \operatorname{erfc}\left(\sqrt{\frac{2E}{N_0}}\sin\left(\frac{\pi}{2M}\right)\right), \qquad M \geq 4 \tag{1}$$

For coherent M-ary PSK, the corresponding formula for the average probability of symbol error is approximately given by

$$P_e \approx \operatorname{erfc}\left(\sqrt{\frac{E}{N_0}}\sin\left(\frac{\pi}{M}\right)\right) \tag{2}$$

(a) Comparing the approximate formulas of Eqs. (1) and (2), we see that for $M \geq 4$ an M-ary DPSK system attains the same probability of symbol error as the corresponding coherent M-ary PSK system provided that the transmitted energy per symbol is increased by the following factor:

$$k(M) = \frac{\sin^2\left(\frac{\pi}{M}\right)}{2\sin^2\left(\frac{\pi}{2M}\right)}, \qquad M \geq 4$$

(b) For example, $k(4) = 1.7$. That is, differential QPSK (which is noncoherent) is approximately 2.3 dB poorer in performance than coherent QPSK.

Problem 6.35

(a) For coherent binary PSK,

$$P_e = \frac{1}{2}\,\mathrm{erfc}\left(\frac{E_b}{N_0}\right) .$$

For P_e to equal 10^{-4}, $\sqrt{E_b/N_0} = 2.64$. This yields $E_b/N_0 = 7.0$. Hence $E_b = 3.5 \times 10^{-10}$. The required average carrier power is 0.35 mW.

(b) For DPSK,

$$P_e = \frac{1}{2}\exp\left(-\frac{E_b}{N_0}\right) .$$

For P_e to equal 10^{-4}, we have $\frac{E_b}{N_0} = 8.5$. Hence $E_b = 4.3 \times 10^{-10}$. The required average power is 0.43 mW.

Problem 6.36

(a) For a coherent PSK system, the average probability of error is

$$P_e = \frac{1}{2}\,\mathrm{erfc}\left[\sqrt{(E_b/N_0)_1}\right]$$

$$\simeq \frac{1}{2}\,\frac{\exp[-(E_b/N_0)_1]}{\sqrt{\pi}\,\sqrt{(E_b/N_0)_1}} \qquad (1)$$

For a DPSK system, we have

$$P_e = \frac{1}{2}\exp[-\,(E_b/N_0)_2] \qquad (2)$$

Let

$$\left(\frac{E_b}{N_0}\right)_2 = \left(\frac{E_b}{N_0}\right)_1 + \delta$$

Then, we may use Eqs. (1) and (2) to obtain

$$\sqrt{\pi}\,\sqrt{(E_b/N_0)_1} = \exp\,\delta$$

We are given that

$$\left(\frac{E_b}{N_0}\right)_1 = 7.2$$

Hence,

$$\begin{aligned}\delta &= \ln[\sqrt{7.2\pi}] \\ &= 1.56\end{aligned}$$

Therefore,

$$10\,\log_{10}\left(\frac{E_b}{N_0}\right)_1 = 10\,\log_{10}\,7.2 = 8.57\text{ dB}$$

$$\begin{aligned}10\,\log_{10}\left(\frac{E_b}{N_0}\right)_2 &= 10\,\log_{10}(7.2 + 1.56) \\ &= 9.42\text{ dB}\end{aligned}$$

The separation between the two (E_b/N_0) ratios is therefore $9.42 - 8.57 = 0.85$ dB.

(b) For a coherent PSK system, we have

$$\begin{aligned}P_e &= \frac{1}{2}\,\mathrm{erfc}[\sqrt{(E_b/N_0)_1}] \\ &\simeq \frac{1}{2}\,\frac{\exp[-(E_b/N_0)_1]}{\sqrt{\pi}\,\sqrt{(E_b/N_0)_1}}\end{aligned} \tag{3}$$

For a QPSK system, we have

$$\begin{aligned}P_e &= \mathrm{erfc}[\sqrt{(E_b/N_0)_2}] \\ &\simeq \frac{\exp[-(E_b/N_0)_2]}{\sqrt{\pi}\,\sqrt{(E_b/N_0)_2}}\end{aligned} \tag{4}$$

Here again, let

$$\left(\frac{E_b}{N_0}\right)_2 = \left(\frac{E_b}{N_0}\right)_1 + \delta$$

Then we may use Eqs. (3) and (4) to obtain

$$\frac{1}{2} = \frac{\exp(-\delta)}{\sqrt{1 + \delta/(E_b/N_0)_1}} \tag{5}$$

Taking logarithms of both sides:

$$-\ln 2 = -\delta - 0.5 \ln[1 + \delta/(E_b/N_0)_1]$$

$$\simeq -\delta - 0.5 \frac{\delta}{(E_b/N_0)_1}$$

Solving for δ:

$$\delta \simeq \frac{\ln 2}{1 + 0.5/(E_b/N_0)_1}$$

$$= 0.65$$

Therefore,

$$10 \log_{10}\left(\frac{E_b}{N_0}\right)_1 = 10 \log_{10}(7.2) = 8.57 \text{ dB}$$

$$10 \log_{10}\left(\frac{E_b}{N_0}\right)_2 = 10 \log_{10}(7.2 + .65)$$

$$= 8.95 \text{ dB.}$$

The separation between the two (E_b/N_0) ratios is 8.95 - 8.57 = 0.38 dB.

(c) For a coherent binary FSK system, we have

$$P_e = \frac{1}{2} \operatorname{erfc}[\sqrt{(E_b/2N_0)_1}]$$

$$= \frac{1}{2} \frac{\exp\left(-\frac{1}{2}\left(\frac{E_b}{N_0}\right)_1\right)}{\sqrt{\pi}\sqrt{(E_b/2N_0)_1}} \tag{6}$$

For a noncoherent binary FSK system, we have

$$P_e = \frac{1}{2} \exp\left(-\frac{1}{2}\left(\frac{E_b}{N_0}\right)_2\right) \tag{7}$$

Hence,

$$\sqrt{\frac{\pi}{2}\left(\frac{E_b}{N_0}\right)_1} = \exp\left(\frac{\delta}{2}\right) \tag{8}$$

We are given that $(E_b/N_0)_1 = 13.5$. Therefore,

$$\delta = \ln\left(\frac{13.5\ \pi}{2}\right)$$

$$= 3.055$$

We thus find that

$$10 \log_{10}\left(\frac{E_b}{N_0}\right)_1 = 10 \log_{10}(13.5)$$

$$= 11.3 \text{ dB}$$

$$10 \log_{10}\left(\frac{E_b}{N_0}\right)_2 = 10 \log_{10}(13.5 + 3.055)$$

$$= 12.2 \text{ dB}$$

Hence, the separation between the two (E_b/N_0) ratios is 12.2 - 11.3 = 0.9 dB.

(d) For a coherent binary FSK system, we have

$$P_e = \frac{1}{2} \operatorname{erfc}[\sqrt{(E_b/2N_0)_1}]$$

$$= \frac{1}{2} \frac{\exp\left(-\frac{1}{2}\left(\frac{E_b}{N_0}\right)_1\right)}{\sqrt{\pi}\ \sqrt{(E_b/2N_0)_1}} \tag{9}$$

For a MSK system, we have

$$P_e = \frac{1}{2} \operatorname{erfc}[\sqrt{(E_b/2N_0)_2}] \tag{10}$$

$$\simeq \frac{\exp[-\frac{1}{2}\left(\frac{E_b}{N_0}\right)_2]}{\sqrt{\pi}\ \sqrt{(E_b/2N_0)_2}} \tag{10}$$

Hence, using Eqs. (9) and (10), we

$$\ln 2 - \frac{1}{2} \ln[1 + \frac{\delta}{(E_b/N_0)_1}] = \frac{1}{2}\,\delta \tag{11}$$

Noting that

$$\frac{\delta}{(E_b/N_0)_1} << 1$$

we may approximate Eq. (11) to obtain

$$\ln 2 - \frac{1}{2}\,[\frac{\delta}{(E_b/N_0)_1}] \simeq \frac{1}{2}\,\delta \tag{11}$$

Solving for δ, we obtain

$$\delta = \frac{2 \ln 2}{1 + \frac{1}{(E_b/N_0)_1}}$$

$$= \frac{2 \times 0.693}{1 + \frac{1}{13.5}}$$

$$= 1.29$$

We thus find that

$$10 \log_{10}\left(\frac{E_b}{N_0}\right)_1 = 10 \log_{10}(13.5) = 10 \times 1.13 = 11.3 \text{ dB}$$

$$10 \log_{10}\left(\frac{E_b}{N_0}\right)_2 = 10 \log_{10}(13.5 + 1.29) = 11.7 \text{ dB}$$

Therefore, the separation between the two (E_b/N_0) ratios is 11.7 - 11.3 = 0.4 dB.

Problem 6.37

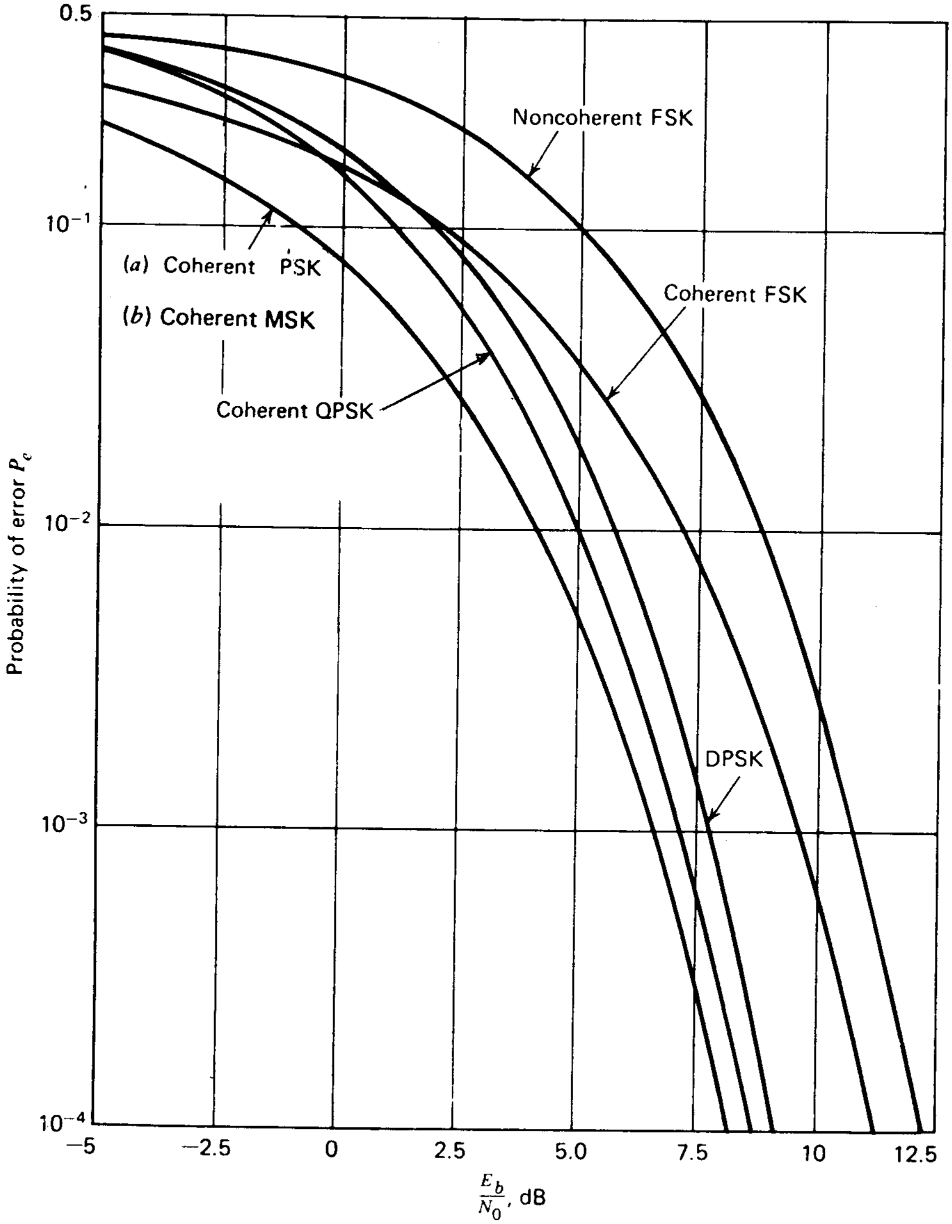


Figure 1 Comparison of the noise performances of different PSK and FSK systems.

The important point to note here, in comparison to the results plotted in Fig. 1 is that the error performance of the coherent QPSK is slightly degraded with respect to that of coherent PSK and coherent MSK. Otherwise, the observations made in Section 8.18 still hold here.

Problem 6.38

The average power for any modulation scheme is

$$P = \frac{E_b}{T_b} .$$

This can be demonstrated for the three types given by integrating their power spectral densities from $-\infty$ to ∞,

$$P = \int_{-\infty}^{\infty} S(f)\, df$$

$$= \frac{1}{4} \int_{-\infty}^{\infty} [S_B(f - f_c) + S_B(f + f_c)]df$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} S_B(f)df .$$

The baseband power spectral densities for each of the modulation techniques are:

	PSK	QPSK	MSK
$S_B(f)$	$2E_b \ \text{sinc}^2(fT_b)$	$4E_b \ \text{sinc}^2(2fT_b)$	$\frac{32E_b}{\pi^2} \left[\frac{\cos(2\pi fT_b)}{16f^2T_b^2 - 1}\right]^2$

Since $\int_{-\infty}^{\infty} a \ \text{sinc}^2(ax)dx = 1$, $P = \frac{E_b}{T_b}$ is easily derived for PSK and QPSK. For MSK we have

$$P = \frac{16E_b}{\pi^2} \int_{-\infty}^{\infty} \left[\frac{\cos(2\pi f T_b)}{16f^2T_b^2 - 1}\right]^2 df$$

$$= \frac{16E_b}{\pi^2 T_b} \int_{-\infty}^{\infty} \frac{\cos^2(2\pi x)}{(16x^2 - 1)^2} dx$$

$$= \frac{8E_b}{\pi^2 T_b} \int_{-\infty}^{\infty} \frac{1 + \cos(4\pi x)}{16x^2(x^2 - \frac{1}{16})} dx$$

$$= \frac{E_b}{16\pi^2 T_b} \int_{-\infty}^{\infty} \frac{\cos 0 + \cos(4\pi x)}{(x^2 - \frac{1}{16})^2} dx$$

From integral tables, (see Appendix AII-6)

$$\int_0^x \frac{\cos(ax)dx}{(b^2 - x^2)^2} = \frac{\pi}{4b^3}[\sin(ab) - ab\cos(ab)]$$

For $a = 0$, the integral is 0.

For $a = 4\pi$, $b = \frac{1}{4}$, we have

$$P = \frac{E_b}{16\pi^2 T_b} \int_{-\infty}^{\infty} \frac{\cos(ax)}{(b^2 - x^2)^2} dx = \frac{E_b}{T_b}$$

For the three schemes, the values of $S(f_c)$ are as follows:

	PSK	QPSK	MSK
$S(f_c)$	$\frac{E_b}{2}$	E_b	$\frac{8E_b}{\pi^2}$

Hence, the noise equivalent bandwidth for each technique is as follows:

	PSK	QPSK	MSK
Bandwidth	$\frac{1}{T_b}$	$\frac{1}{2T_b}$	$\frac{0.62}{T_b}$

Problem 6.39

(a) Table 1, presented below, describes the differential quadrant coding for the V.32 modem of Fig. 6.48a in the textbook, which may operate with nonredundant coding at 9,600 b/s. The entries in the table correspond to the following:

Present inputs: $Q_{1,n}Q_{2,n}$
Previous outputs: $I_{1,n\text{-}1}I_{2,n\text{-}1}$
Present outputs: $I_{1,n}\ I_{2,n}$

Table 1

Input dibit		Previous output dibit		Present output dibit	
$Q_{1,n}$	$Q_{2,n}$	$I_{1,n\text{-}1}$	$I_{2,n\text{-}1}$	$I_{1,n}$	$I_{2,n}$
0	1	0	0	0	0
0	1	0	1	0	1
0	1	1	0	1	0
0	1	1	1	1	1
0	0	0	0	0	1
0	0	0	1	1	1
0	0	1	0	0	0
0	0	1	1	1	0
1	0	0	0	1	1
1	0	0	1	1	0
1	0	1	0	0	1
1	0	1	1	0	0
1	1	0	0	1	0
1	1	0	1	0	0
1	1	1	0	1	1
1	1	1	1	0	1

(b) Table 2, presented below, describes the mapping from the four bits $I_{1,n-1}I_{2,n-1}$, $Q_{3,n}Q_{4,n}$ to the output coordinates of the V.32 modem.

Table 2

Present output dibit		Present input dibit		Output coordinates		
$I_{1,n}$	$I_{2,n}$	$Q_{3,n}$	$Q_{4,n}$	ϕ_1	ϕ_2	
0	1	0	0	1	-1	4th quadrant
0	1	0	1	1	-3	
0	1	1	0	3	-1	
0	1	1	1	3	-3	
0	0	0	0	-1	-1	3rd quadrant
0	0	0	1	-3	-1	
0	0	1	0	-1	-3	
0	0	1	1	-3	-3	
1	0	0	0	-1	1	2nd quadrant
1	0	0	1	-1	3	
1	0	1	0	-3	1	
1	0	1	1	-3	3	
1	1	0	0	1	1	1st quadrant
1	1	0	1	3	1	
1	1	1	0	1	3	
1	1	1	1	3	3	

(b) We are given the current input quadbit:

$$Q_{1,n}Q_{2,n}Q_{3,n}Q_{4,n} = 0001$$

and the previous output dibit:

$$I_{1,n-1}I_{2,n-1} = 01$$

From Table 1, we find that the resulting present output dibit is

$$I_{1,n}I_{2,n} = 11$$

Hence, using this result, together with the given input dibit $Q_{3,n}Q_{4,n} = 01$ in Table 2, we find that the coordinates of the modem output are as follows:

$\phi_1 = 3$, and $\phi_2 = 1$

We may check this result by consulting Table 6.10 and Fig. 6.49 of the textbook. With $Q_{1,n}Q_{2,n} = 00$ we find from Table 6.10 that the modem experiences a phase change of 90°. With $I_{1,n-1}\, I_{2,n-1} = 01$, we find from Fig. 6.49 that the modem was previously residing in the fourth quadrant. Hence, with a rotation of 90° in the counterclockwise direction, the modem moves into the first quadrant. With $Q_{3,n}Q_{4,n} = 01$, we readily find from Fig. 6.49 that

$\phi_1 = 3$, and $\phi_2 = 1$

which is exactly the same as the result deduced from Tables 1 and 2 of the solutions manual.

For another example, suppose we are given

$$Q_{1,n}Q_{2,n}Q_{3,n}Q_{4,n} = 1011$$

and

$$I_{1,n-1}I_{2,n-1} = 11$$

Then, from Table 1, we find that

$$I_{1,n}I_{2,n} = 00$$

Next, from Table 2, we find that the output coordinates are $\phi_1 = -3$ and $\phi_2 = -3$. Confirmation that these results are in perfect accord with the calculations based on Table 6.10 and Figure 6.49 is left as an exercise for the reader.

<u>Problem 6.40</u>

(a) The average signal-to-noise ratio is defined by

$$(\text{SNR})_{\text{av}} = \frac{P_{\text{av}}}{\sigma^2} \tag{1}$$

where P_{av} is the average transmitted power, and σ^2 is the channel noise variance. The transmitted signal is defined by

$$s_k(t) = a_k\cos(2\pi f_c t) - b_k\sin(2\pi f_c t), \qquad 0 \le t \le T$$

where (a_k, b_k) is the kth symbol of the QAM signal, and T is the symbol duration. The power spectrum of $s_k(t)$ has the following graphical form:

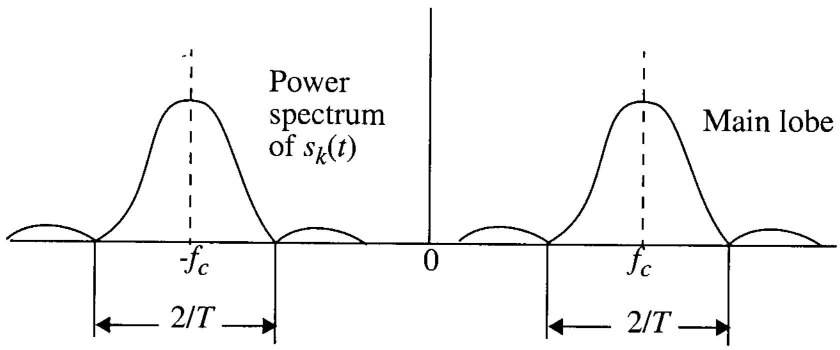


Fig. 1

On the basis of this diagram, we may use the null-to-null bandwidth of the power spectrum in Fig. 1 as the channel bandwidth:

$$B = \frac{2}{T} \text{ or } T = \frac{2}{B}$$

The average transmitted power is

$$P_{av} = \frac{1}{T}E_{av} = \frac{BE_{av}}{2} \quad (2)$$

where E_{av} is the average signal energy per symbol.

To calculate the noise variance σ^2, ref[illegible] the following figure:

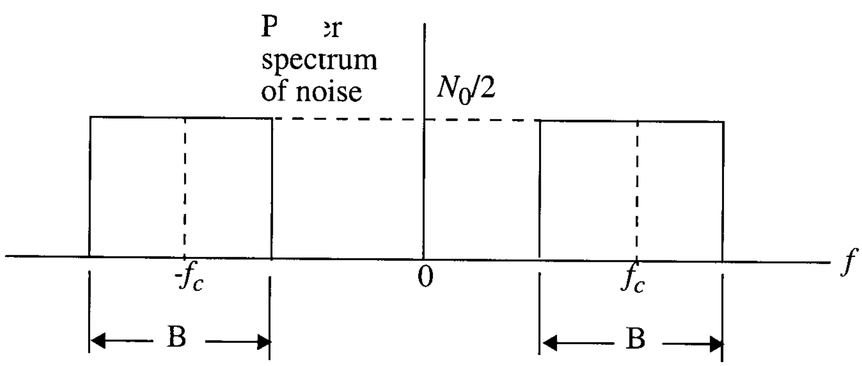


Fig. 2

The noise variance is therefore

$$\sigma^2 = N_0 B \qquad (3)$$

Hence, substituting Eqs. (2) and (3) into (1):

$$(\text{SNR})_{\text{av}} = \frac{BE_{\text{av}}/2}{N_0 B}$$

$$= \frac{1}{2}\left(\frac{E_{\text{av}}}{N_0}\right)$$

Expressing the SNR in decibels, we may thus write

$$10\log_{10}(\text{SNR})_{\text{av}} = -3 + 10\log_{10}\left(\frac{E_{\text{av}}}{N_0}\right), \text{ dB}$$

Given the value $10\log_{10}$, $(E_{\text{av}}/N_0) = 20$ dB or $E_{\text{av}}/N_0 = 100$, we thus have

$$10\log_{10}(\text{SNR})_{\text{av}} = 17 \text{ dB}$$

(b) With M = 16, the average probability of symbol error is

$$P_e = 2\left(1 - \frac{1}{\sqrt{M}}\right)\text{erfc}\left(\sqrt{\frac{3E_{\text{av}}}{2(M-1)N_0}}\right)$$

$$= 2\left(1 - \frac{1}{4}\right)\text{erfc}\left(\sqrt{\frac{100}{10}}\right)$$

$$= 1.16 \times 10^{-5}$$

Problem 6.41

We are given the following set of passband basis functions:

$$\{\phi(t)\cos(2\pi f_n t), \phi(t)\sin(2\pi f_n t)\}_{n=1}^{N}$$

where $f_n = \frac{n}{T}$, $\quad n = 1, 2, \cdots, N$

and $\phi(t) = \sqrt{\frac{2}{T}}\operatorname{sinc}\left(\frac{t}{T}\right)$, $\quad -\infty < t < \infty$

<u>Property 1</u>

$$\int_{-\infty}^{\infty} (\phi(t)\cos(2\pi f_n t))(\phi(t)\sin(2\pi f_n t))dt = 0 \quad \text{for all } n \tag{1}$$

To prove this property, we use the following relation from Fourier transform theory:

$$\int_{-\infty}^{\infty} (\phi(t)\cos(2\pi f_n t))(\phi(t)\sin(2\pi f_n t))dt = 0 \quad \text{for all } n \tag{2}$$

where $g_i(t) \rightleftharpoons G_i(f)$ for $i = 1,2$, and the asterisk denotes complex conjugation. For the problem at hand, we have

$$g_1(t) = \sqrt{\frac{2}{T}}\operatorname{sinc}\left(\left(\frac{t}{T}\right)\cos(2\pi f_n t)\right)$$

$$g_2(t) = \sqrt{\frac{2}{T}}\operatorname{sinc}\left(\left(\frac{t}{T}\right)\sin(2\pi f_n t)\right)$$

The Fourier transform of the sinc function is

$$F\left[\operatorname{sinc}\left(\frac{t}{T}\right)\right] = T\operatorname{rect}(fT)$$

where

$$\operatorname{rect}(fT) = \left\{\begin{array}{ll} 1 & \text{for } -\frac{1}{2T} \le f \le \frac{1}{2T} \\ 0 & \text{otherwise} \end{array}\right\}$$

Hence,

$$G_1(f) = \sqrt{\frac{T}{2}}[\operatorname{rect}((f - f_n)T) + \operatorname{rect}((f + f_n)T)]$$

$$G_2(f) = \frac{1}{j}\sqrt{\frac{T}{2}}[\mathrm{rect}((f-f_n)T) - \mathrm{rect}((f+f_n)T)]$$

Let I_1 denote the integral on the left-hand side of Eq. (1). We may then use Eq. (2) to write

$$I_1 = j\left(\frac{T}{2}\right)\int_{-\infty}^{\infty}[\mathrm{rect}^2((f-f_n)T) - \mathrm{rect}^2((f+f_n)T)]df \qquad (3)$$

where the integrand is depicted as follows:

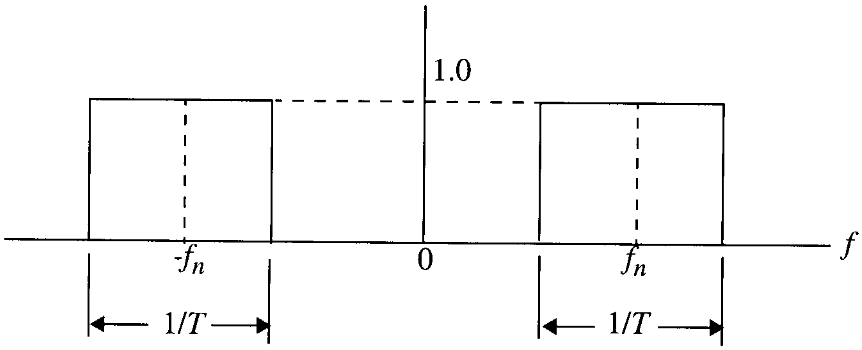


Fig. 1

From Fig. 1 we immediately see that the areas under the two rectangular functions are exactly equal. Hence, Eq. (3) is zero, thereby proving Property 1 for any n.

Property 2

$$\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2}}\phi(t)e^{j2\pi f_n t}\right)\left(\frac{1}{\sqrt{2}}\phi(t)e^{j2\pi f_n t}\right)^* dt = \begin{cases} 1 \text{ for } k = n \\ 2 \text{ for } k \neq n \end{cases} \qquad (4)$$

Let I_2 denote the integral in Eq. (4). When $k = n$, we have

$$I_2 = \left(\frac{T}{2}\right)\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2}}\phi(t)e^{j2\pi f_n t}\right)\left(\frac{1}{\sqrt{2}}\phi(t)e^{(-j)2\pi f_n t}\right)dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\phi^2(t)dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\left(\sqrt{\frac{2}{T}}\mathrm{sinc}\left(\frac{t}{T}\right)\right)^2 dt$$

$$= \frac{1}{T}\int_{-\infty}^{\infty} \text{sinc}^2\left(\frac{t}{T}\right) dt$$

Using Rayleigh's energy theorem, namely,

$$\int_{-\infty}^{\infty} g^2(t)dt = \int_{-\infty}^{\infty} |G(f)|^2 df$$

we may write

$$\frac{1}{T}\int_{-\infty}^{\infty} \text{sinc}^2\left(\frac{t}{T}\right) dt = \text{sinc}^2(\lambda)d\lambda, \qquad \lambda = t/T$$

$$= \int_{-\infty}^{\infty} \text{rect}^2(f) df$$

$$= 1$$

which proves Property 2 for $k = n$.

To prove Property 2 for $k \neq n$, let

$$g_1(t) = \phi(t) e^{j2\pi f_n t}$$

$$= \sqrt{\frac{2}{T}} \text{sinc}\left(\frac{t}{T}\right) e^{j2\pi f_n t}$$

$$g_2(t) = \phi(t) e^{j2\pi f_k t}$$

$$= \sqrt{\frac{2}{T}} \text{sinc}\left(\frac{t}{T}\right) e^{j2\pi f_k t}, \qquad f_k \neq f_n$$

Then applying the following relation from Fourier transform theory,

$$\int_{-\infty}^{\infty} g_1(t) g_2^*(t) \; dt = \int_{-\infty}^{\infty} G_1(f) G_2^*(f) \; df$$

we may rewrite the integral I_2 of Eq. (4) as

$$I_2 = \frac{2}{T}\int_{-\infty}^{\infty} F\left[\operatorname{sinc}\left(\frac{t}{T}\right)e^{j2\pi f_n t}\right]F\left[\operatorname{sinc}\left(\frac{t}{T}\right)e^{j2\pi f_n t}\right]df$$

Since $f_n = n/T$ by definition, we may depict the two Fourier transforms constituting the integrand of I_2 as shown in Fig. 2 for the worst possible case of $k = n+1$:

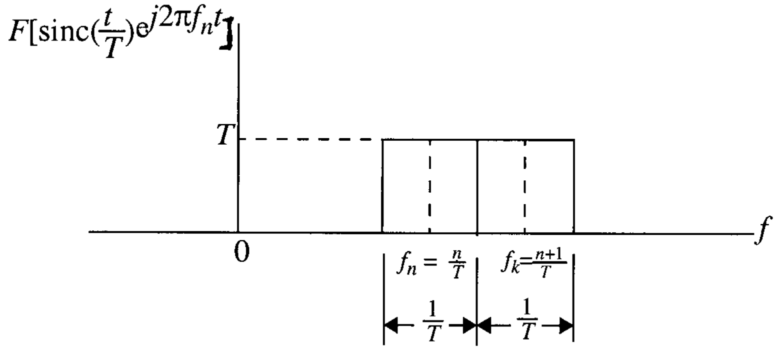


Fig. 2

From Fig. 2 we immediately see that the Fourier transforms of $\phi(t)e^{j2\pi f_n t}$ and $\phi(t)e^{j2\pi f_k t}$ will never overlap for $k \neq n$. Hence, the integral I_2 is zero, proving the rest of Property 2.

Property 3

$$\int_{-\infty}^{\infty} (\phi(t) \ast h(t))e^{j2\pi f_n t}(\phi(t) \ast h(t)e^{j2\pi f_k t})^* \, dt = 0 \quad \text{for} \quad k \neq n$$

where the star denotes convolution. From the convolution theorem, we have

$$F[\phi(t) \ast h(t)] = \Phi(f)H(f)$$

where $\Phi(f) = F[\phi(t)]$ and $\Phi(f) = F[h(t)]$. For $k \neq n$, the picture portrayed in Fig. 2 remains equally valid except for the fact that the basic rectangular spectrum is now replaced by that rectangular spectrum multiplied by the frequency response $H(f)$. This multiplication does not affect the nonoverlapping nature of the two spectra representing $(\phi(t) \ast h(t))e^{j2\pi f_n t}$ and $(\phi(t) \ast h(t)e^{j2\pi f_k t})$ for $k \neq n$, hence, proving Property 3.

Problem 6.42

Step 1 - Set $k = 0$ and the initial noise-to-signal ratio $NSR(k) = 0$. Sort the subchannels used in ascending order (i.e., from the smallest to largest ones).

Step 2 - Update the number of subchannels used by setting $k = k+1$.

Step 3 - Compute $NSR(k+1) = NSR(k) + \frac{\sigma_k^2}{g_k^2}$

Step 4 - Set $\lambda(k) = \frac{1}{k}[P_k + \Gamma NSR(k)]$

Step 5 - If $P_k = \lambda(k) - \Gamma\left(\frac{\sigma_k^2}{g_k^2}\right) < 0$

then compute $P_l = \lambda(k-1) - P\left(\frac{\sigma_l^2}{P_l^2}\right)$

and

$$B_l = \log_2\left(1 + \frac{P_l g_l^2}{\Gamma \sigma_l^2}\right)$$

for $l = 1, 2, \cdots, k-1$

Otherwise, go to step 2.

For notations, refer to Section 6.12.

(The algorithm presented here is adapted from T. Starr, J.M. Cioffi, and P.J. Silverman (1999);' see the bibliography.)

<u>Problem 6.43</u>

(a) $P_1 + P_2 + P_3 = P$ (1)

$$P_1 - K = -\Gamma\frac{\sigma^2}{g_1^2} = -\Gamma\sigma^2 \quad (2)$$

$$P_2 - K = -\Gamma\frac{\sigma^2}{g_2^2} = -\Gamma\frac{\sigma^2}{l_1} \quad (3)$$

$$P_3 - K = -\Gamma\frac{\sigma^2}{g_3^2} = -\Gamma\frac{\sigma^2}{l_2} \tag{4}$$

Adding Eqs. (2), (3) and (4), and then using Eq. (1):

$$3K = P + \Gamma\sigma^2\left(1 + \frac{1}{l_1} + \frac{1}{l_2}\right)$$

Solving for K, we thus have

$$K = \frac{P}{3} + \frac{\Gamma\sigma^2}{3}\left(1 + \frac{1}{l_1} + \frac{1}{l_2}\right)$$

With this value of K at hand, we next solve for P_1, P_2, and P_3, obtaining

$$P_1 = \frac{P}{3} + \frac{\Gamma\sigma^2}{3}\left(\frac{1}{l_1} + \frac{1}{l_2} - 2\right)$$

$$P_2 = \frac{P}{3} + \frac{\Gamma\sigma^2}{3}\left(1 + \frac{1}{l_2} - \frac{1}{l_1}\right)$$

$$P_3 = \frac{P}{3} + \frac{\Gamma\sigma^2}{3}\left(1 + \frac{1}{l_1} - \frac{2}{l_2}\right)$$

(b) $$P_1 = \frac{10}{3} + \frac{1}{3}\left(\frac{3}{2} + 3 - 2\right)$$

$$= \frac{1}{3}(10 + 2.5)$$

$$= \frac{12.5}{3}$$

$$P_2 = \frac{10}{3} + \frac{1}{3}(1 + 3 - 3)$$
$$= \frac{11}{3}$$

$$P_3 = \frac{10}{3} + \frac{1}{3}\left(1 + \frac{3}{2} - 6\right)$$

$$= \frac{10}{3} + \frac{1}{3}(-3.5)$$

$$= \frac{6.5}{3}$$

Problem 6.44

(a) Using matrix notation, the channel output vector is defined by

$$\mathbf{x} = \mathbf{Ha} + \mathbf{w}$$

where $\mathbf{H}$ is the channel matrix, $\mathbf{a}$ is the transmitted signal vector, and $\mathbf{w}$ is the channel noise vector. Applying the singular value decomposition to $\mathbf{H}$, we may write

$$\mathbf{H} = \mathbf{U}\left[\Lambda \vdots \mathbf{0}\right]\mathbf{V}^{\dagger}$$

where † denotes Hermitian transposition. The vector-coding receiver uses a bank of discrete matched filters defined as the rows of orthonormal matrix $\mathbf{U}^{\dagger}$. Thus passing $\mathbf{x}$ through this bank of matched filters, we get

$$\mathbf{X} = \mathbf{U}^{\dagger}\mathbf{x}$$

$$= \mathbf{U}^{\dagger}(\mathbf{Ha} + \mathbf{w})$$

$$= \mathbf{U}^{\dagger}\left(\mathbf{U}\left[\Lambda \vdots \mathbf{0}\right]\mathbf{V}^{\dagger}\mathbf{a} + \mathbf{w}\right) \qquad (1)$$

Defining

$$\mathbf{A} = \mathbf{V}^{\dagger}\mathbf{a}$$

and recognizing that $\mathbf{U}^{\dagger}\mathbf{U} = \mathbf{I}$ (identity matrix), Eq. (1) reduces to

$$\mathbf{X} = \Lambda\mathbf{A} + \mathbf{W} \qquad (2)$$

where

$$\mathbf{W} = \mathbf{U}^{\dagger}\mathbf{w}$$

Each element of the vector **X** in Eq. (2) represents an independent channel, and so we write

$$X_n = \lambda_n A_n + W_n, \qquad n = 1, 2, \cdots, N$$

(b) In the multichannel transmission model, the channel capacity of the entire system in bits per transmission is given by

$$R = \frac{1}{N}\sum_{n=1}^{N+\nu} R_n$$

$$= \frac{1}{2N}\sum_{n=1}^{N+\nu} \log_2\left(1 + \frac{P_n}{\Gamma\sigma^2}\right)$$

$$= \frac{1}{2}\log_2\left[\prod_{n=1}^{N}\left(1 + \frac{P_n}{\Gamma\sigma_n^2}\right)\right]^{1/(n+\nu)} \tag{3}$$

where ν is the length of the channel impulse response. We may also express the R as follows:

$$R = \frac{1}{2}\log_2\left(1 + \frac{1}{\Gamma}(\text{SNR})_{\text{vector coding}}\right) \tag{4}$$

Hence, combining (3) and (4):

$$\Gamma + (\text{SNR})_{\text{vector coding}} = \Gamma\left(\prod_{n=1}^{N}\left(1 + \frac{P_n}{\Gamma\sigma_n^2}\right)\right)^{\frac{1}{(N+\nu)}}$$

$$(\text{SNR})_{\text{vector coding}} = \Gamma\left(\prod_{n=1}^{N}\left(1 + \frac{P_n}{\Gamma\sigma_n^2}\right)\right)^{\frac{1}{(N+\nu)}} - \Gamma$$

(c) As the block length goes to infinity, we may ignore ν, in which case the channel matrix **H** becomes nearly an N x N matrix. Therefore, **H** may be decomposed as

$$\mathbf{H} = \mathbf{Q}^H\Lambda\mathbf{Q}$$

where $\mathbf{Q}$ is an orthonormal matrix, and Λ is a diagonal matrix of eigenvalues (i.e., singular values). Correspondingly, the singular values approach the magnitude of the Fourier transform.

Even though a vector coding receiver and discrete multitone receiver converge to the same performance, they are not the same:

- The subchannel gains are the same in both cases, but in vector modulation all subchannels have zero phase, while in DMT the subchannels have arbitrary phase angles.
- Unlike DMT, a vector-coding system does not require the use of a cyclic prefix.
- The computational complexity of the vector-coding multichannel system is much greater than that of its DMT counterpart.

Problem 6.45†

	Impulse noise (produced in a subscriber loop plant)	Narrowband interference (picked up from a nearby AM radio station)
Discrete multitone (DMT) system	The DMT receiver spreads the energy of an impulse over many subchannels, thereby reducing its degrading effect	Due to the sinc(x) spectral characteristic of each subchannel, the DMT receiver tends to pass the interference into many subchannels. Since the signal power in each subchannel is only 1/N of the total signal power, susceptibility of the receiver to narrowband interference can be acute.
Carrierless amplitude/phase (CAP) modulation system	The CAP system relies on the use of a single carrier. It is therefore susceptible to impulse noise. (This problem may be mitigated by the use of a powerful code such as the Reed-Solomon code.)	The use of an adaptive equalizer is based on the mean-square error criterion. The effect of narrowband interference can therefore be reduced by creating a notch in the receiver performance at the frequency of the interferer.

†The comparative points made in the table presented herein are based on Saltzberg (1998); see the bibliography

Problem 6.46

In M-ary FSK, the transmitted signal is defined by

$$s_i(t) = \sqrt{\frac{2E}{T}}\cos\left(\frac{\pi}{T}(n_c + i)t\right), \qquad 0 \le t \le T \text{ and } i = 1, 2, \cdots, M \tag{1}$$

where E is the symbol energy, T is the symbol period, and the carrier frequency $f_c = n_c/2T$ for some fixed integer n_c. The signals $s_i(t)$ for $i = 1, 2, \cdots, M$ constitute an orthogonal set over the interval $0 \le t \le T$, as shown by

$$\int_0^T s_i(t)s_j(t)dt = 0 \text{ for } i \ne j \tag{2}$$

Each frequency in Eq. (1) (i.e., specified value of integer i) is modulated with binary data. The net result is a set of parallel carriers, each of which contains a certain portion of the incoming user's data. What we have just described is a form of orthogonal frequency-division multiplexing (OFDM).

Problem 6.47

(a) The M-ary PSK signal is given by

$$y(t) = \sqrt{\frac{2E}{T}}\cos\left(2\pi f_c t + \frac{2\pi}{M}(i-1)\right), \qquad i = 1, 2, \cdots, M \tag{1}$$

The output of the Mth power-law device is the Mth power of the input signal $y(t)$:

$$z(t) = \left(\frac{2E}{T}\right)^{\frac{M}{2}} \cos^M\left(2\pi f_c t + \frac{2\pi}{M}(i-1)\right) \tag{2}$$

The signal $z(t)$ generates a frequency component at Mf_c, which can be used to drive a phase-locked loop tuned to Mf_c. Specifically, expanding Eq. (2), we get

$$z(t) = \left(\frac{2E}{T}\right)^{\frac{M}{2}} \left\{ \binom{M}{\frac{M}{2}}\frac{1}{2^M} + \frac{1}{2^{M-1}} \sum_{k=1}^{(M/2)} \binom{M}{\frac{M}{2}-k} \cos\left[2\pi(2k)f_c t + (2k)\frac{2\pi}{M}(i-1)\right] \right\}$$

$$= \left(\frac{2E}{T}\right)^{\frac{M}{2}} \left\{ \frac{1}{2^{M-1}}\binom{M}{0}\cos[2\pi M f_c t + 2\pi(i-1)] \right\}$$

$$+\frac{1}{2^{M-1}}\binom{M}{1}\cos\left[2\pi(M-1)f_c t+\frac{(M-1)2\pi(i-1)}{M}\right]$$

$$+\ldots+\frac{1}{2^{M-1}}\binom{M}{(M/2)-1}\cos\left(4\pi f_c t+\frac{4\pi}{M}(i-1)\right)$$

$$\left.+\binom{M}{M/2}\frac{1}{2^M}\right\}$$

Therefore according to the first term of this series expansion, the output of the Mth power law device contains a tone of frequency Mf_c, where f_c is the original carrier frequency.

(b) The phase-locked loop is set to a frequency equal to Mf_c. The phase-locked loop acts as a narrow-band filter, thereby passing the sinusoidal component of frequency Mf_c and rejecting the other components.

(c) Consider, for example, the simple case of binary PSK. Since a squaring loop contains a squaring device at its input end, it is clear that changing the sign of the input signal leaves the sign of the recovered carrier unaltered. In other words, the squaring loop with M=2 exhibits a 180° phase ambiguity. Generalizing this result, we may say that Mth power loop for M-ary PSK exhibits M phase ambiguities in the interval $[0,2\pi]$.

One method of resolving the phase ambiguity problem is to exploit differential encoding. Specifically, the incoming data sequence is first differentially encoded before modulation, resulting in a small degradation in noise performance. This method is called the coherent detection of differentially encoded M-ary PSK. As such, this method of modulation is different from the M-ary DPSK considered in Problem 6.34. For the special case of coherent detection of differentially encoded binary PSK, the average probability of symbol error is given by

$$P_e = \operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)-\frac{1}{2}\operatorname{erfc}^2\left(\sqrt{\frac{E_b}{N_0}}\right) \qquad (3)$$

In the region where $(E_b/N_0) >> 1$, the second term on the right-hand side of Eq. (3) has a negligible effect; hence, this modulation scheme has an average probability of symbol error practically the same as that for coherent QPSK or MSK. For the coherent detection of differentially encoded QPSK, the average probability of symbol error is given by

$$P_e = 2\operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)-2\operatorname{erfc}^2\left(\sqrt{\frac{E_b}{N_0}}\right)+\operatorname{erfc}^3\left(\sqrt{\frac{E_b}{N_0}}\right)-\frac{1}{4}\operatorname{erfc}^4\left(\sqrt{\frac{E_b}{N_0}}\right)$$

For large E_b/N_0, this average probability of symbol error is approximately twice that of coherent QPSK.

Problem 6.48

(a) Assuming that the input data sequence $x[n]$ is measured in volts, and recognizing that the symbol a_n is dimensionless, then from Eq. (6.271) we find that the error signal $e[n]$ is also measured in volts. Then, with the phase estimate $\hat{\theta}[n]$ measured in radians, it follows that the step-size parameter γ for carrier recovery in Eq. (6.272) is measured in radians/volts.

(b) From Eq. (6.282) defining the error signal in terms of the input data sequence, we see that the error signal is measured in volts squared. Hence, with $c[n]$ responsible for timing recovery, measured in seconds, it follows that the step-size parameter γ for timing recovery in Eq. (6.286) is measured in seconds/volts^2.

Problem 6.49

(a) The complex envelope of the received waveform is given by

$$\tilde{r}(t) = \tilde{s}(t) + \tilde{w}(t) \tag{1}$$

where $\tilde{s}(t) = e^{j(2\pi\nu t + \theta)} \sum_{k=0}^{L_0-1} a_k g(t - kT - \tau)$

and $w(t)$ is the channel noise. The parameter ν represents the frequency offset, θ is the carrier phase we want to estimate, τ is the timing error, $\{a_k\}$ is the sequence of information symbols, T is the symbol period, and $g(t)$ is the signaling pulse shape.

The likelihood function $L(r|\tilde{\theta})$ is given by

$$L(r|\tilde{\theta}) = \exp\left(\frac{1}{N_0}\int_0^{T_0} Re\{\tilde{r}(t)\tilde{s}(t)\}dt -- \frac{1}{2N_0}\int_0^{T_0} |\tilde{s}(t)|^2 dt\right) \tag{2}$$

where $\tilde{s}(t) = e^{j(2\pi\nu t + \theta)} \sum_{k=0}^{L_0\text{-}1} a_k g(t - kT - \tau)$

Since $|\tilde{s}(t)|$ is independent of the carrier phase θ, the log-likelihood function of θ is given by

$$l(\theta)=\log(L(r|\tilde{\theta})) = Re\left\{\int_0^{T_0} \tilde{r}(t)\tilde{s}^*(t)dt\right\} \tag{3}$$

where $\int_0^{T_0} \tilde{r}(t)\tilde{s}^*(t)dt = e^{-j\tilde{\theta}}\sum_{k=0}^{L_0-1} a_k^* x(k)$

where $x(k)$ represents the sample taken at time $t = kT + \tau$ in the formula for convolution:

$$x(t)=[r(t)e^{-j2\pi\nu t}] \ast g(-t)$$

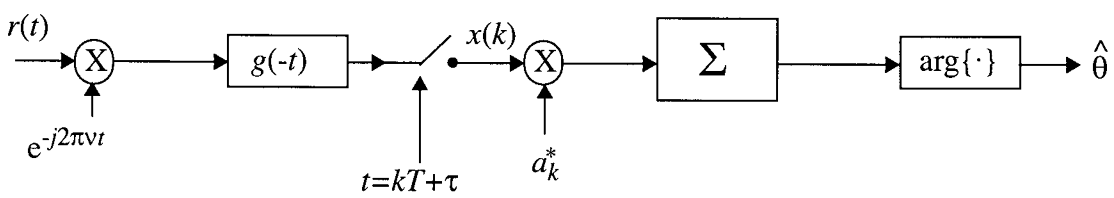
Therefore,

$$l(\theta) = Re\left\{e^{-j\tilde{\theta}}\sum_{k=0}^{L_0-1} a_k^* x(k)\right\}$$

The maximum of $l(\theta)$, i.e., maximum likelihood estimation of θ, is achieved for

$$\hat{\theta} = \arg\left\{\sum_{k=0}^{L_0-1} a_k^* x(k)\right\} \tag{4}$$

(b) Hence, from Eq. (4) we deduce the following system for estimating the phase θ :



Problem 6.51

Matlab codes

```
% Problem 6.51 a. CS: Haykin
%effect of a dispersive channel on BPSK signals
% M. Sellathurai


% number of bits and number of samples per bit
no_of_syms =10;
no_of_bits=no_of_syms*1;
samples_per_bit=16;


% generating bits
Bits=[1 1 0 1 1 0 1 0 0 1]';


% generating QPSK signals
[syms]=BPSK_mod(no_of_bits, Bits);

ts=1e-3/16;
l=length(syms);

% baseband signal
s=zeros(samples_per_bit*(l),1);
for k=1:l-1
for kk=0:(samples_per_bit-1)
s((k-1)*samples_per_bit+kk+1,1)=syms(1,k);
end
end

t=0:ts:(length(s)-1)*ts;

% channel bandwidths 2B=12, filter order 2N=10
B=6; N=5;
H12=butter_channel(2*B,N);
TT12=conv(H12, s);

% channel bandwidths  2B=16 filter order 2N=10
B=8;
H16=butter_channel(2*B,N);
TT16=conv(H16, s);
```

```
% channel bandwidth 2*B=20 filter order 2N=10
B=10;
H20=butter_channel(2*B,N);
TT20=conv(H20, s);


%channel bandwidth  2*B=24 filter order 2N=10
B=12;
H24=butter_channel(2*B,N);
TT24=conv(H24, s);

%  channel bandwidth 2*B=30 filter order 2N=10
B=15;
H30=butter_channel(2*B,N);
TT30=conv(H30, s);

% prints

subplot(2,3,1)
hold on
for k=1:10
plot(k, [syms(k)]','o')
 line([k, k], [0 syms(k)])
end
xlabel('Bit period');
title('Transmitted bits');
hold off


subplot(2,3,2)
[ m start]=max(real(H12));
hold on
plot(t,s,'--');
plot(t,real(TT12(start:160+start-1)));
xlabel('time (s)');
title('Baseband BPSK, BW=12kHz');
hold off

subplot(2,3,3)
[m start]=max(real(H16));
hold on
plot(t,s,'--');
plot(t,real(TT16(start:160+start-1)));
xlabel('time (s)');
title(' Baseband BPSK, BW=16kHz');
hold off
```

```
subplot(2,3,4)
[m start]=max(real(H20));
hold on
plot(t,s,'--');
plot(t,real(TT20(start:160+start-1)));
xlabel('time (s)');
title('Baseband BPSK, BW=20kHz');
hold off

subplot(2,3,5)
[m start]=max(real(H24));
hold on
plot(t,s,'--');
plot(t,real(TT24(start:160+start-1)));
xlabel('time (s)');
title('Baseband BPSK, BW=24kHz');
hold off

subplot(2,3,6)
[m start]=max(real(H30));
hold on
plot(t,s,'--');
plot(t,real(TT30(start:160+start-1)));
xlabel('time (s)');
title('Baseband BPSK, BW=30kHz');
hold off
```

```
% Problem 6.51 b. CS: Haykin
%effect of a dispersive channel on QPSK signals
% M. Sellathurai


% number of bits and number of samples per bit
no_of_syms =5;
no_of_bits=no_of_syms*2;
samples_per_bit=16;

% generating bits
%Bits=round(rand(no_of_bits,1));
Bits=[1 1 0 1 1 0 1 0 0 1]';

% generating QPSK signals
[syms]=QPSK_mod(no_of_bits, Bits);

l=length(syms);

% baseband signal
s=zeros(samples_per_bit*(l-1),1);
for k=1:l-1
for kk=0:(samples_per_bit-1)
s((k-1)*samples_per_bit+kk+1,1)=syms(1,k);
end
end

t=0:ts:(length(s)-1)*ts;

% channel bandwidths 2B=12, filter order 2N=10
B=6; N=5;
H12=butter_channel(2*B,N);
TT12=conv(H12, s);

% channel bandwidths  2B=16 filter order 2N=10
B=8;
H16=butter_channel(2*B,N);
TT16=conv(H16, s);


% channel bandwidth 2*B=20 filter order 2N=10
B=10;
H20=butter_channel(2*B,N);
TT20=conv(H20, s);
```

```
%channel bandwidth  2*B=24 filter order 2N=10
B=12;
H24=butter_channel(2*B,N);
TT24=conv(H24, s);

%  channel bandwidth 2*B=30 filter order 2N=10
B=15;
H30=butter_channel(2*B,N);
TT30=conv(H30, s);


% prints
subplot(2,3,1)
hold on
for k=1:10
plot(k, [2*Bits(k)-1]','o')
line([k, k], [0 (2*Bits(k)-1)])
end
xlabel('Bit period');
title('Transmitted bits');
hold off


subplot(2,3,2)
[ m start]=max(real(H12));
hold on
plot(t,s,'--');
plot(t,real(TT12(start:64+start-1)));
xlabel('time (s)');
title('Baseband QPSK, BW=12kHz');
hold off

subplot(2,3,3)
[m start]=max(real(H16));
hold on
plot(t,s,'--');
plot(t,real(TT16(start:64+start-1)));
xlabel('time (s)');
title(' Baseband QPSK, BW=16kHz');
hold off

subplot(2,3,4)
[m start]=max(real(H20));
hold on
plot(t,s,'--');
plot(t,real(TT20(start:64+start-1)));
```

```
xlabel('time (s)');
title('Baseband QPSK, BW=20kHz');
hold off

subplot(2,3,5)
[m start]=max(real(H24));
hold on
plot(t,s,'--');
plot(t,real(TT24(start:64+start-1)));
xlabel('time (s)');
title('Baseband QPSK, BW=24kHz');
hold off

subplot(2,3,6)
[m start]=max(real(H30));
hold on
plot(t,s,'--');
plot(t,real(TT30(start:64+start-1)));
xlabel('time (s)');
title('Baseband QPSK, BW=30kHz');
hold off
```

```
% Problem 6.51 c. CS: Haykin
%effect of a dispersive channel on MSK signals
% M. Sellathurai


% number of bits and number of samples per bit
no_of_syms =5;
no_of_bits=no_of_syms*2;
samples_per_bit=16;

% generating bits
Bits=[1 1 0 1 1 0 1 0 0 0]';

% generating QPSK signals
[s,phase]=MSK_mod(no_of_bits,samples_per_bit,Bits);

% channel bandwidths 2B=12, filter order 2N=10
B=6; N=5;
H12=butter_channel(2*B,N);
TT12=conv(H12, s);

% channel bandwidths  2B=16 filter order 2N=10
B=8;
H16=butter_channel(2*B,N);
TT16=conv(H16, s);


% channel bandwidth 2*B=20 filter order 2N=10
B=10;
H20=butter_channel(2*B,N);
TT20=conv(H20, s);


%channel bandwidth  2*B=24 filter order 2N=10
B=12;
H24=butter_channel(2*B,N);
TT24=conv(H24, s);

%  channel bandwidth 2*B=30 filter order 2N=10
B=15;
H30=butter_channel(2*B,N);
TT30=conv(H30, s);
ts=1e-3/16;
t=0:ts:(length(s)-1)*ts

% prints
```

```
subplot(2,3,1)
hold on
for k=1:10
plot(k, [2*Bits(k)-1]','o')
line([k, k], [0 (2*Bits(k)-1)])
end
xlabel('Bit period');
title('Transmitted bits');
hold off


subplot(2,3,2)
[ m start]=max(real(H12));
hold on
plot(t,abs(s),'--');
plot(t,abs(TT12(start+5:165+start-1)));
xlabel('time (s)');
title('MSK (envelope), BW=12kHz');
hold off
axis([0, 0.01,0.9,1.1 ])

subplot(2,3,3)
[m start]=max(real(H16));
hold on
plot(t,abs(s),'--');
plot(t,abs(TT16(start+5:165+start-1)));
xlabel('time (s)');
title(' MSK  (envelope), BW=16kHz');
hold off
axis([0, 0.01,0.9,1.1 ])

subplot(2,3,4)
[m start]=max(real(H20));
hold on
plot(t,abs(s),'--');
plot(t,abs(TT20(start+5:165+start-1)));
xlabel('time (s)');
title('MSK  (envelope), BW=20kHz');
hold off
axis([0, 0.01,0.9,1.1 ])

subplot(2,3,5)
[m start]=max(real(H24));
hold on
plot(t,abs(s),'--');
plot(t,abs(TT24(start+5:165+start-1)));
```

```
xlabel('time (s)');
title('MSK  (envelope), BW=24kHz');
hold off
axis([0, 0.01,0.9,1.1 ])

subplot(2,3,6)
[m start]=max(real(H30));
hold on
plot(t,abs(s),'--');
plot(t(1:155),abs(TT30(start+5:160+start-1)));
xlabel('time (s)');
title('MSK  (envelope), BW=30kHz');
hold off
axis([0, 0.01,0.9,1.1 ])
```

```
% Problem 6.51 d. CS: Haykin
%effect of a dispersive channel on GMSK signals
% M. Sellathurai


% number of bits and number of samples per bit
no_of_syms =5;
no_of_bits= no_of_syms*2;
samples_per_bit=16;

% generating bits
Bits=[1 1 0 1 1 0 1 0 0 0]';

% generating GMSK signals, WTb=0.3
[s, phase]=GMSK_mod(no_of_bits,samples_per_bit,Bits);

% channel bandwidths 2B=12, filter order 2N=10
B=6; N=5;
H12=butter_channel(2*B,N);
TT12=conv(H12, s);

% channel bandwidths  2B=16 filter order 2N=10
B=8;
H16=butter_channel(2*B,N);
TT16=conv(H16, s);


% channel bandwidth 2*B=20 filter order 2N=10
B=10;
H20=butter_channel(2*B,N);
TT20=conv(H20, s);


%channel bandwidth  2*B=24 filter order 2N=10
B=12;
H24=butter_channel(2*B,N);
TT24=conv(H24, s);

%  channel bandwidth 2*B=30 filter order 2N=10
B=15;
H30=butter_channel(2*B,N);
TT30=conv(H30, s);

ts=1e-3/16;
t=0:ts:(length(s)-1)*ts
```

```
% prints
subplot(2,3,1)
hold on
for k=1:10
plot(k, [2*Bits(k)-1]','o')
line([k, k], [0 (2*Bits(k)-1)])
end
xlabel('Bit period');
title('Transmitted bits');
hold off
subplot(2,3,2)

[ m start]=max(real(H12));
hold on
plot(t,abs(s),'--');
plot(t,abs(TT12(start:160+start-1)));
xlabel('time (s)');
title('GMSK (envelope), BW=12kHz');
hold off
axis([0, 0.01,0.9,1.1 ])

subplot(2,3,3)
[m start]=max(real(H16));
hold on
plot(t,abs(s),'--');
plot(t,abs(TT16(start:160+start-1)));
xlabel('time (s)');
title(' GMSK  (envelope), BW=16kHz');
hold off
axis([0, 0.01,0.9,1.1 ])

subplot(2,3,4)
[m start]=max(real(H20));
hold on
plot(t,abs(s),'--');
plot(t,abs(TT20(start:160+start-1)));
xlabel('time (s)');
title('GMSK  (envelope), BW=20kHz');
hold off
axis([0, 0.01,0.9,1.1 ])

subplot(2,3,5)
[m start]=max(real(H24));
hold on
plot(t,abs(s),'--');
plot(t,abs(TT24(start:160+start-1)));
```

```
xlabel('time (s)');
title('GMSK  (envelope), BW=24kHz');
hold off
axis([0, 0.01,0.9,1.1 ])

subplot(2,3,6)
[m start]=max(real(H30));
hold on
plot(t,abs(s),'--');
plot(t,abs(TT30(start:160+start-1)));
xlabel('time (s)');
title('GMSK  (envelope), BW=30kHz');
hold off
axis([0, 0.01,0.9,1.1 ])
```

```
function [amp]=BPSK_mod(no_of_bits, b)
% used in problem 6.51(a), CS: Haykin
% BPSK modulation
% Mathini Sellathurai
amp=[];

l=1;
m=size(b,1);

for k=1:1:m
if (b(k)==0 )
amp(l)= (-1);
elseif (b(k)==1 )
amp(l)= 1;
end
l=l+1;
end
```

```
function [amp]=QPSK_mod(no_of_bits, b)
% used in problem 6.51(b), CS: Haykin
% QPSK modulation
% Mathini Sellathurai
amp=[];

l=1;
m=size(b,1);
for k=1:2:m

if (b(k)==0 & b(k+1) == 0)
amp(l)= (-1+i*-1)/sqrt(2);
elseif (b(k)==1 & b(k+1) == 0)
amp(l)= (1-i*1)/sqrt(2);
elseif (b(k)==1 & b(k+1) == 1)
amp(l)= (1+i*1)/sqrt(2);
else (b(k)==0& b(k+1) == 1)
amp(l)= (-1+i*1)/sqrt(2);

end
l=l+1
end
```

```
function [amp,phase]=MSK_mod(no_of_bits, samples_per_bit, b)
% used in problem 6.51(c), CS: Haykin
% MSK signal generator
% Mathini Sellathurai

amp=[];

ini_phase=0;


for k=1:no_of_bits

ee=b(k);

for kk=0:samples_per_bit-1

% NRZ signal generator
if ee==0
      ee=-1;
elseif ee==1
      ee=1;
end

      phase((k-1)*samples_per_bit+kk+1)=ini_phase +ee*(pi/(2*samples_per_bit));
      ini_phase=phase((k-1)*samples_per_bit+kk+1);

end
end

phase=rem(phase,2*pi);
in=cos(phase);
quad=sin(phase);
amp=in+i*quad;
```

```
function [amp, phase1]=GMSK_mod(no_of_bits, samples_per_bit, b)
% used in problem 6.51(d), CS: Haykin
% GMSK signal generator
% Mathini Sellathurai

amp=[];

for k =1:no_of_bits

 %Generating NRZ sequence
  if  b(k,1)==0
im_bits(k,1)= -1;
  else
im_bits(k,1) =1;
  end
end

impulse_bits=im_bits;
Bits_to_transmit=max(size(impulse_bits));
BT=0.6;
inphase=0;
data(1,4)=0;
t=0;
for i=0:3

for k=0:(samples_per_bit -1)
       co =GMSK_co(i-2,k+8,samples_per_bit,BT);
       qmskcoef(1,i*samples_per_bit+k+1)=co;
end
end

for bitcount=1:Bits_to_transmit
       ini_phase=inphase;
       ini_phase=rem(ini_phase+data(1,4)*pi/2,2*pi);

data(1,1)=impulse_bits(bitcount,1);
for i =4:-1:2
data(1,i)=data(1,i-1);
end

inphase=ini_phase;

for pha_loop=1:samples_per_bit
      phase=inphase;
for i=0:3
phase=phase+pi/2*data(1,i+1)*qmskcoef(1,samples_per_bit*i+pha_loop);
```

```
end
samples_store(1,t+1)=t;
t=t+1;
phase=rem(phase,2*pi);
phase1(1, pha_loop+samples_per_bit*(bitcount-1))=phase;
rephase(1,pha_loop+samples_per_bit*(bitcount-1))=cos(phase);
quphase(1,pha_loop+samples_per_bit*(bitcount-1))=sin(phase);
end
end

amp=rephase+j*quphase;
```

```
function [co] = GMSK_co(a, b, samples_per_bit,bt)
% used in GMSK signal generation Problem 651d, CS: Haykin
% Mathini Sellathurai

alpha=bt*5.336446225;
T=a+b/samples_per_bit;
co=T*erf(T*alpha)+exp(-alpha*alpha*T*T)/(alpha *1.772453855);
co=co-(T-1)*erf((T-1)*alpha)-exp(-alpha*alpha*(T-1)*(T-1))/(alpha*sqrt(pi));
co=0.5+0.5*co;
```

```
function  hb=butter_channel(f,N)
% Used in Problem 6.51
% Butterworth filter of order 2N=10;
% M. Sellathurai

[B, A]=butter(N, f/64);
[H,w]=freqz(B,A,128,'whole');
hb=ifft(H);
```

Answer to Problem 6.51

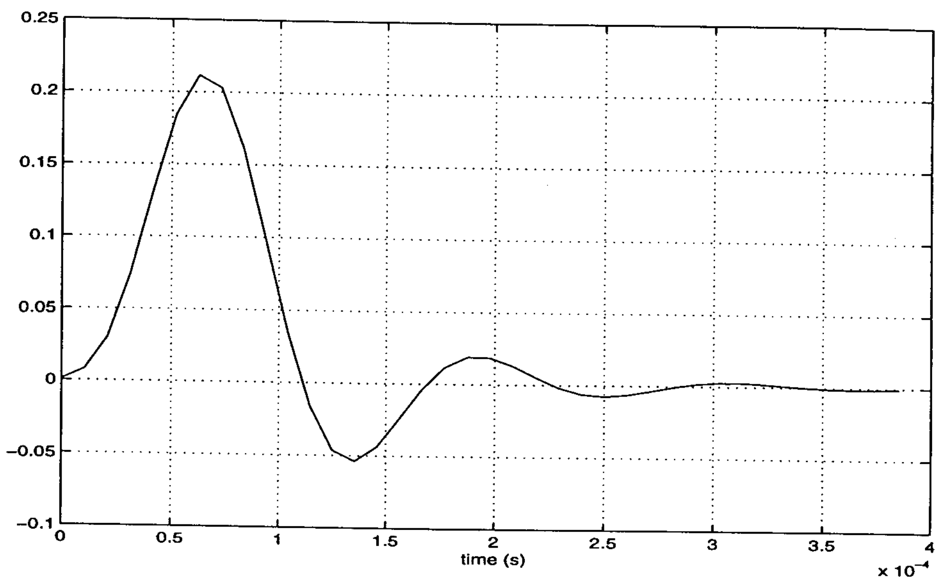
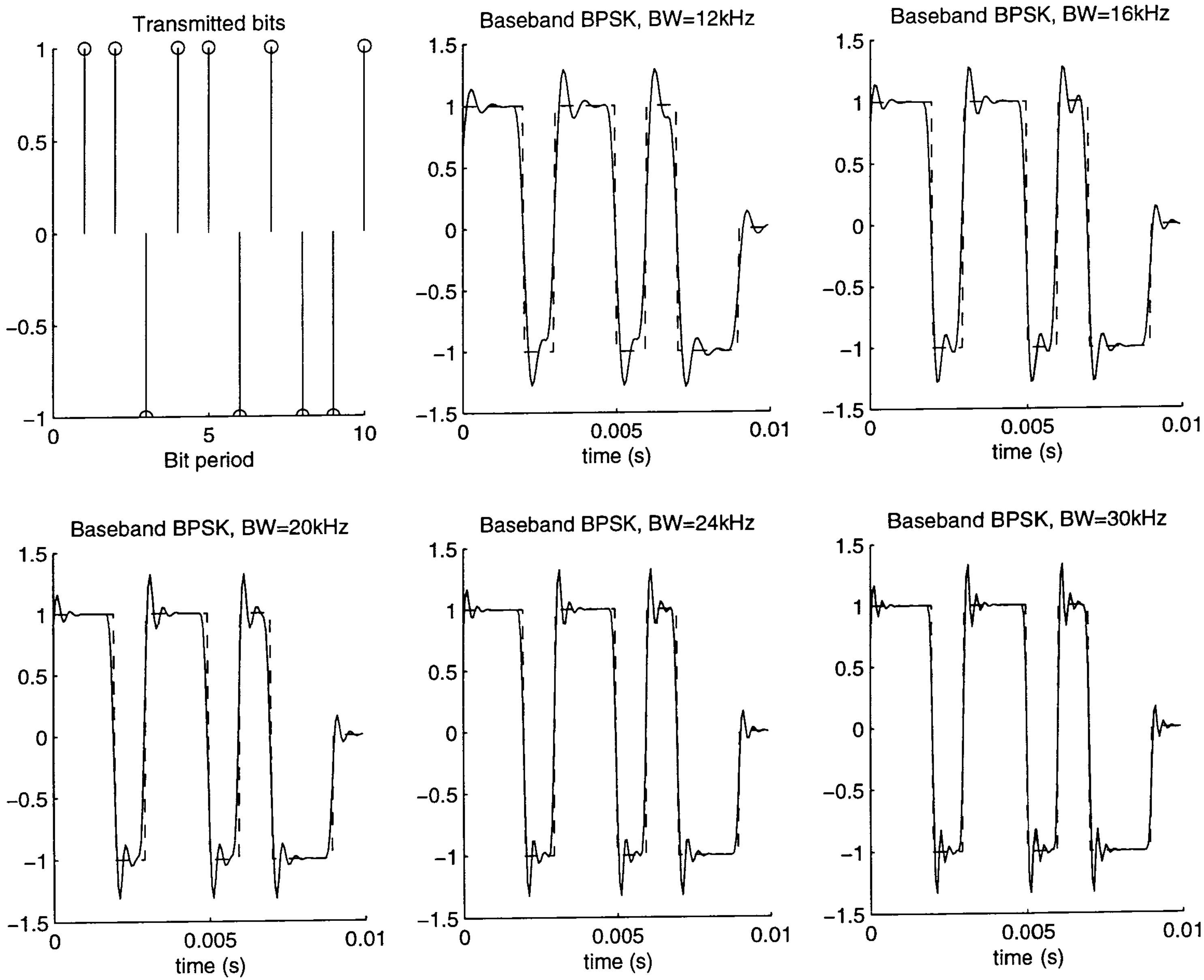
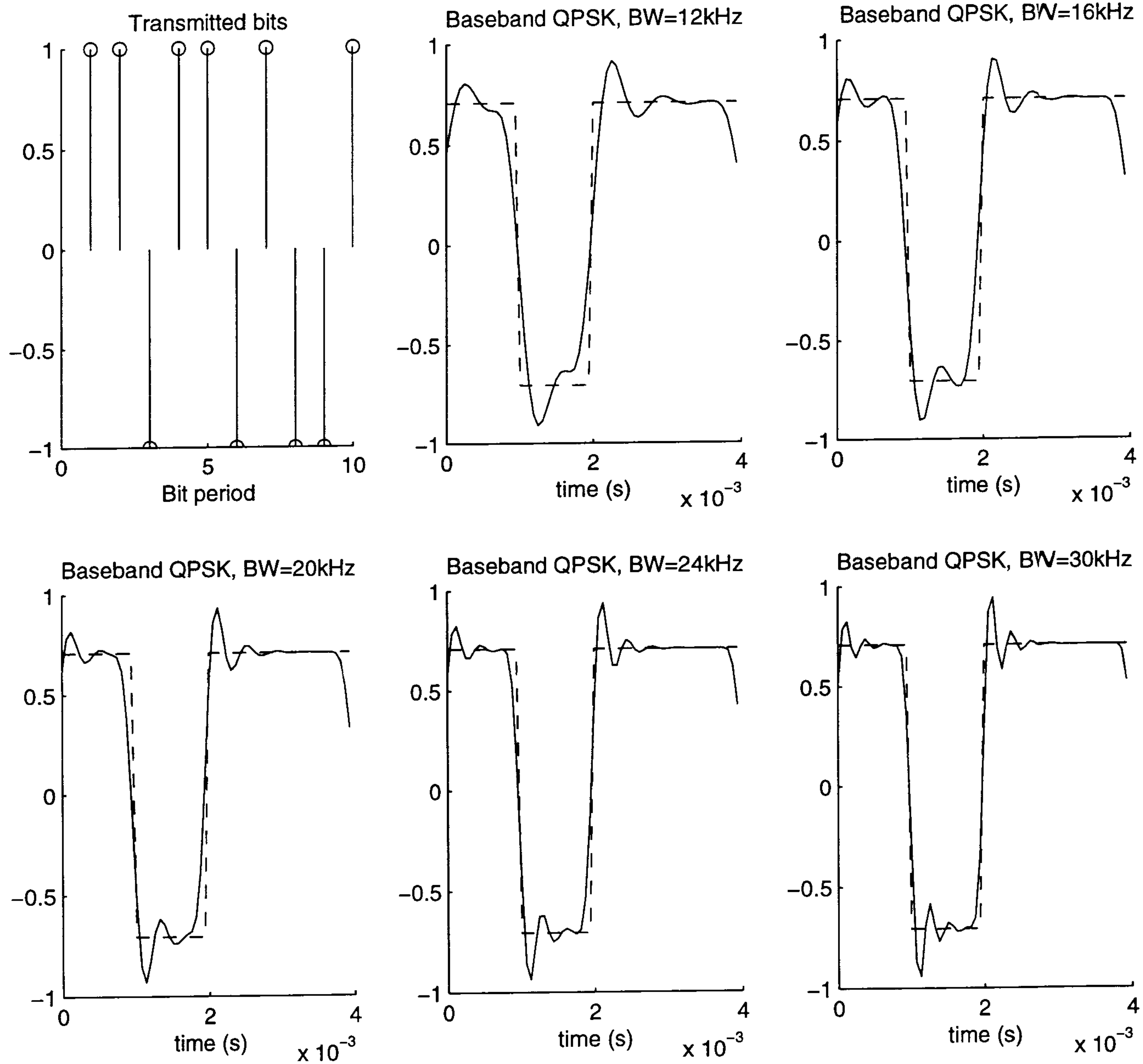
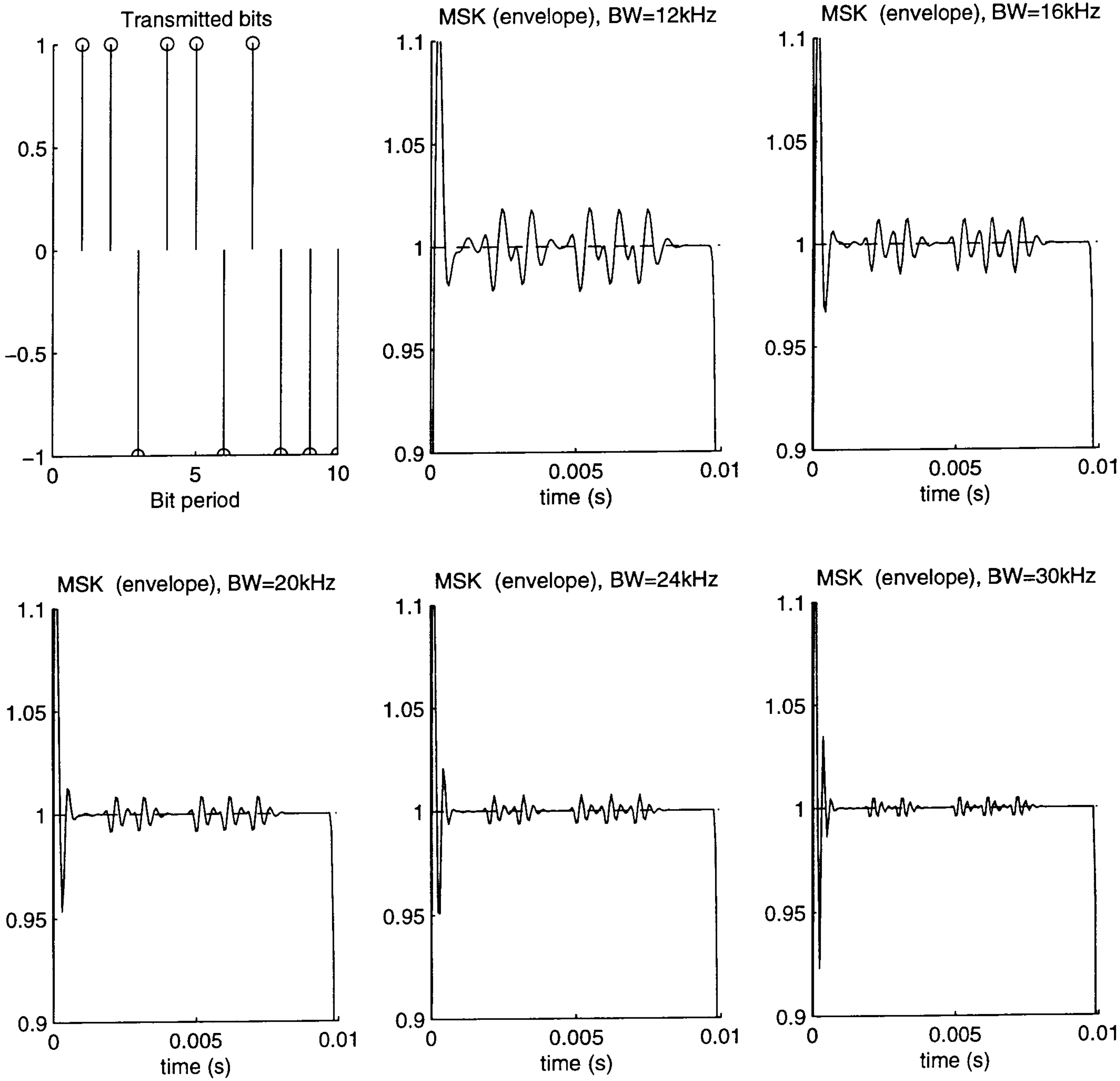


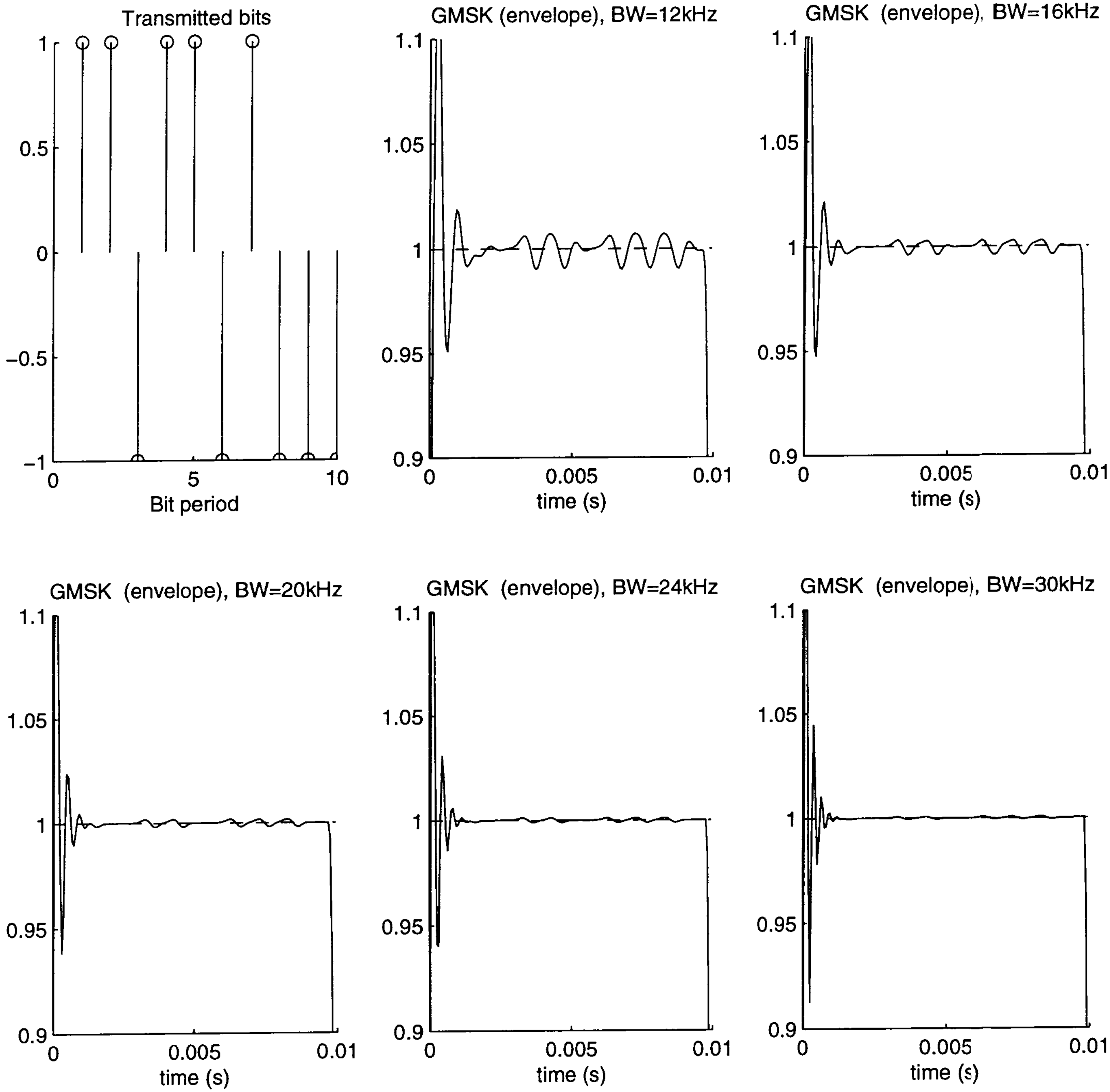
Figure 1: Butterworth baseband filter of order N=5


Transmitted bits
Bit period
Baseband BPSK, BW=12kHz
Baseband BPSK, BW=16kHz
Baseband BPSK, BW=20kHz
Baseband BPSK, BW=24kHz
Baseband BPSK, BW=30kHz
time (s)



Transmitted bits
Bit period
Baseband QPSK, BW=12kHz
Baseband QPSK, BW=16kHz
Baseband QPSK, BW=20kHz
Baseband QPSK, BW=24kHz
Baseband QPSK, BW=30kHz
time (s)
x 10^-3
1
0.5
0
−0.5
−1
0
2
4
5
10


Transmitted bits
Bit period
MSK (envelope), BW=12kHz
MSK (envelope), BW=16kHz
MSK (envelope), BW=20kHz
MSK (envelope), BW=24kHz
MSK (envelope), BW=30kHz
time (s)


Transmitted bits
Bit period
GMSK (envelope), BW=12kHz
GMSK (envelope), BW=16kHz
GMSK (envelope), BW=20kHz
GMSK (envelope), BW=24kHz
GMSK (envelope), BW=30kHz
time (s)
1.1
1.05
1
0.95
0.9
0
0.005
0.01
0.5
-0.5
-1
5
10

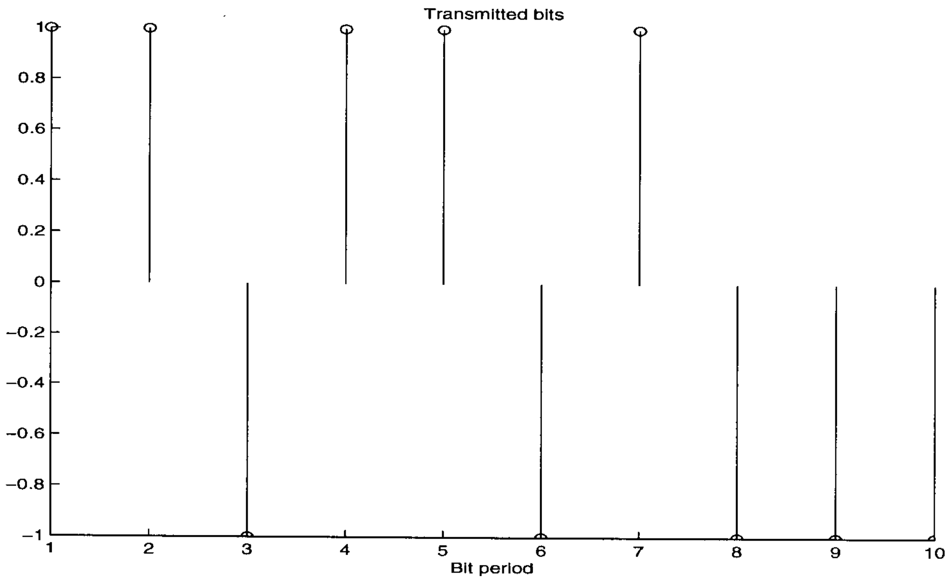


Figure 2 Transmitted bits

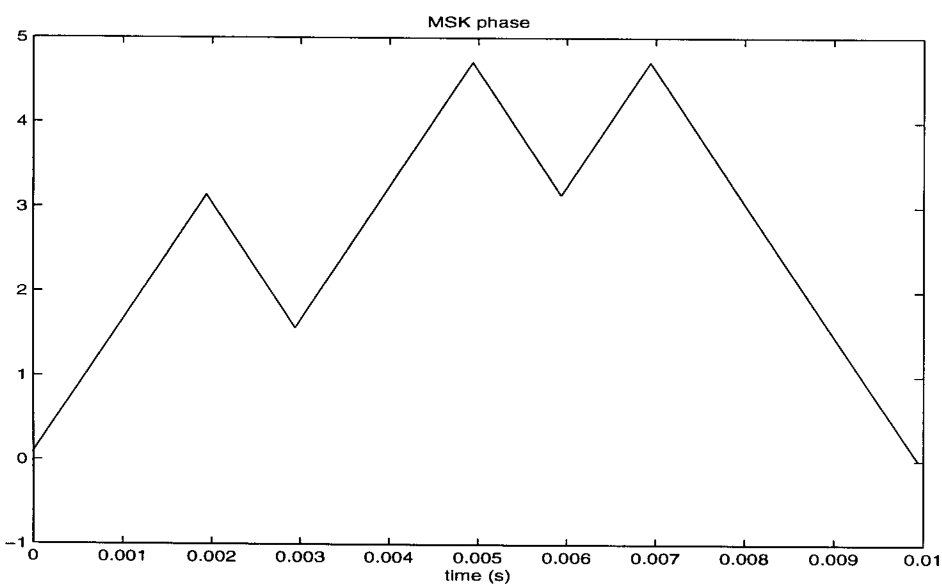


Figure 3 Phase of baseband MSK signal

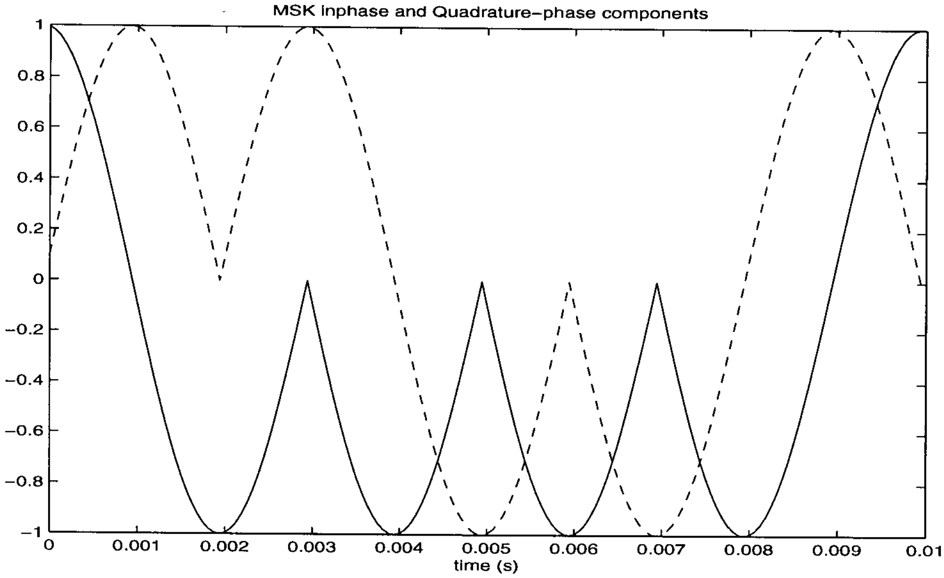


Figure 4 I and Q components of baseband MSK signal

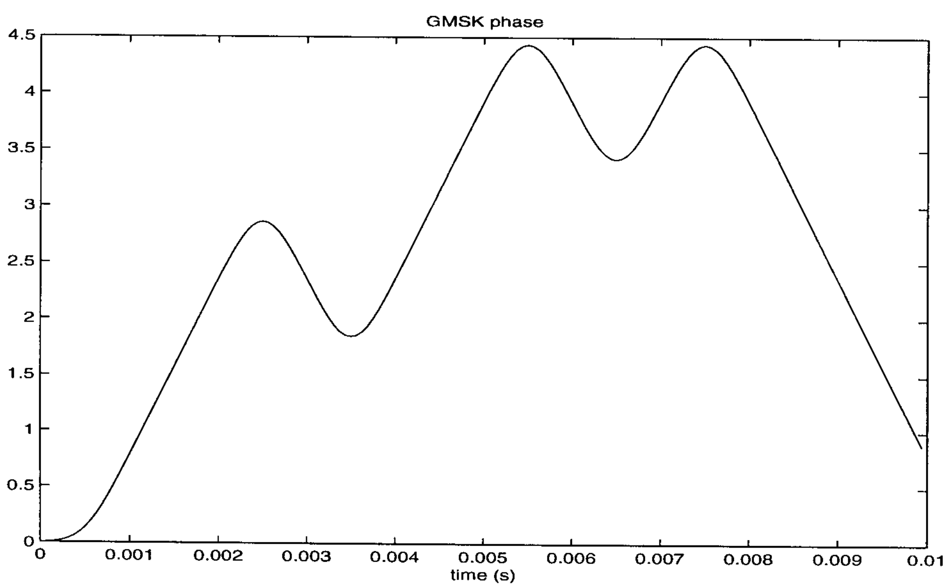


Figure 5 Phase of baseband GMSK signal

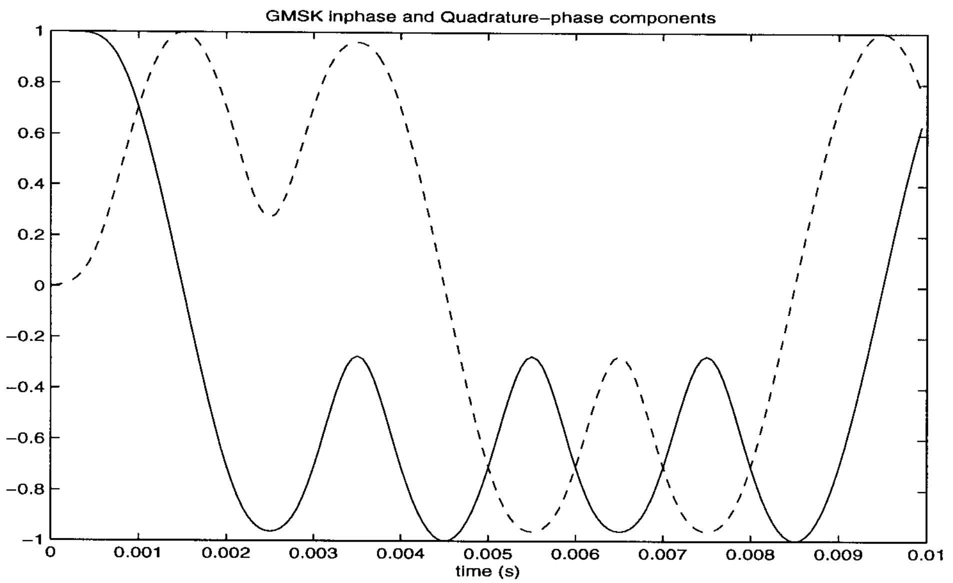


Figure 6: I and Q components of baseband GMSK signal