

Chapter 9

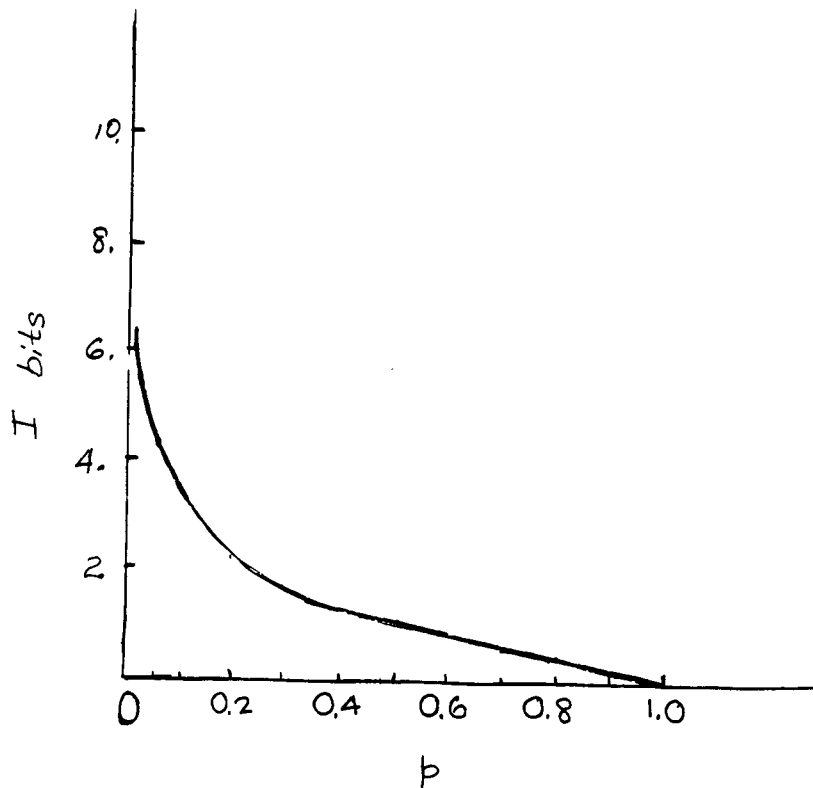
Fundamental limits in Information Theory

Problem 9.1

Amount of information gained by the occurrence of an event of probability p is

$$I = \log_2\left(\frac{1}{p}\right) \text{ bits}$$

I varies with p as shown below:



Problem 9.2

Let the event $S=s_k$ denote the emission of symbol s_k by the source, Hence,

$$I(s_k) = \log_2\left(\frac{1}{p}\right) \text{ bits}$$

s_k	s_0	s_1	s_2	s_3
p_k	0.4	0.3	0.2	0.1
$I(s_k)$ bits	1.322	1.737	2.322	3.322

Problem 9.3

Entropy of the source is

$$\begin{aligned} H(S) &= p_0 \log_2\left(\frac{1}{p_0}\right) + p_1 \log_2\left(\frac{1}{p_1}\right) + p_2 \log_2\left(\frac{1}{p_2}\right) + p_3 \log_2\left(\frac{1}{p_3}\right) \\ &= \frac{1}{3} \log_2(3) + \frac{1}{6} \log_2(6) + \frac{1}{4} \log_2(4) + \frac{1}{4} \log_2(4) \\ &= 0.528 + 0.431 + 0.5 + 0.5 \\ &= 1.959 \text{ bits} \end{aligned}$$

Problem 9.4

Let X denote the number showing on a single roll of a die. With a die having six faces, we note that p_X is $1/6$. Hence, the entropy of X is

$$\begin{aligned} H(X) &= p_X \log_2\left(\frac{1}{p_X}\right) \\ &= \frac{1}{6} \log_2(6) = 0.431 \text{ bits} \end{aligned}$$

Problem 9.5

The entropy of the quantizer output is

$$H = - \sum_{k=1}^4 P(X_k) \log_2 P(X_k)$$

where X_k denotes a representation level of the quantizer. Since the quantizer input is Gaussian with zero mean, and a Gaussian density is symmetric about its mean, we find that

$$P(X_1) = P(X_4)$$

$$P(X_2) = P(X_3)$$

The representation level $X_1 = 1.5$ corresponds to a quantizer input $+1 \leq Y < \infty$. Hence,

$$\begin{aligned} P(X_1) &= \int_1^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy \\ &= \frac{1}{2} - \frac{1}{2} \operatorname{erf}\left(\frac{4}{\sqrt{2}}\right) \\ &= 0.1611 \end{aligned}$$

The representation level $X_2 = 0.5$ corresponds to the quantizer input $0 \leq Y < 1$. Hence,

$$\begin{aligned} P(X_2) &= \int_0^1 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy \\ &= \frac{1}{2} \operatorname{erf}\left(\frac{4}{\sqrt{2}}\right) \\ &= 0.3389 \end{aligned}$$

Accordingly, the entropy of the quantizer output is

$$\begin{aligned} H &= - 2 \left[0.1611 \log_2 \left(\frac{1}{0.1611} \right) + 0.3389 \log_2(0.3389) \right] \\ &= 1.91 \text{ bits} \end{aligned}$$

Problem 9.6

(a) For a discrete memoryless source:

$$P(\sigma_i) = P(s_{i_1}) P(s_{i_2}) \dots P(s_{i_n})$$

Noting that $M = K^n$, we may therefore write

$$\begin{aligned} \sum_{i=0}^{M-1} P(\sigma_i) &= \sum_{i=0}^{M-1} P(s_{i_1}) P(s_{i_2}) \dots P(s_{i_n}) \\ &= \sum_{i_1=0}^{K-1} \sum_{i_2=0}^{K-1} \dots \sum_{i_n=0}^{K-1} P(s_{i_1}) P(s_{i_2}) \dots P(s_{i_n}) \\ &= \sum_{i_1=0}^{K-1} P(s_{i_1}) \sum_{i_2=0}^{K-1} P(s_{i_2}) \dots \sum_{i_n=0}^{K-1} P(s_{i_n}) \\ &= 1 \end{aligned}$$

(b) For $k = 1, 2, \dots, n$, we have

$$\sum_{i=0}^{M-1} P(\sigma_i) \log_2 \left(\frac{1}{p_{i_k}} \right) = \sum_{i=0}^{M-1} P(s_{i_1}) P(s_{i_2}) \dots P(s_{i_n}) \log_2 \left(\frac{1}{p_{i_k}} \right)$$

For $k=1$, say, we may thus write

$$\begin{aligned} \sum_{i=0}^{M-1} P(\sigma_i) \log_2 \left(\frac{1}{p_{i_1}} \right) &= \sum_{i=0}^{K-1} P(s_{i_1}) \log_2 \left(\frac{1}{p_{i_1}} \right) \sum_{i=0}^{K-1} P(s_{i_2}) \dots \sum_{i=0}^{K-1} P(s_{i_n}) \\ &= \sum_{i=0}^{K-1} p_{i_1} \log_2 \left(\frac{1}{p_{i_1}} \right) \\ &= H(S) \end{aligned}$$

Clearly, this result holds not only for $k=1$ but also $k=2, \dots, n$.

(c)

$$\begin{aligned} H(S^n) &= \sum_{i=0}^{M-1} P(\sigma_i) \log_2 \frac{1}{P(\sigma_i)} \\ &= \sum_{i=0}^{M-1} P(\sigma_i) \log_2 \frac{1}{P(s_{i_1}) P(s_{i_2}) \dots P(s_{i_n})} \\ &= \sum_{i=0}^{M-1} P(\sigma_i) \log_2 \frac{1}{P(s_{i_1})} + \sum_{i=0}^{M-1} P(\sigma_i) \log_2 \frac{1}{P(s_{i_2})} \\ &\quad + \dots + \sum_{i=0}^{M-1} P(\sigma_i) \log_2 \frac{1}{P(s_{i_n})} \end{aligned}$$

Using the result of part (b), we thus get

$$\begin{aligned} H(S^n) &= H(S) + H(S) + \dots + H(S) \\ &= n H(S) \end{aligned}$$

Problem 9.7

(a) The entropy of the source is

$$\begin{aligned} H(S) &= 0.7 \log_2 \frac{1}{0.7} + 0.15 \log_2 \frac{1}{0.15} + 0.15 \log_2 \frac{1}{0.15} \\ &= 0.258 + 0.4105 + 0.4105 \\ &= 1.079 \text{ bits} \end{aligned}$$

(b) The entropy of second-order extension of the source is

$$H(S^2) = 2 \times 1.079 = 2.158 \text{ bits}$$

Problem 9.8

The entropy of text is defined by the smallest number of bits that are required, on the average, to represent each letter.

According to Lucky[†], English text may be represented by less than 3 bits per character, because of the redundancy built into the English language. However, the spoken equivalent of English text has much less redundancy; its entropy is correspondingly much higher than 3 bits. It follows therefore from the source coding theorem that the number of bits required to represent (store) text is much smaller than the number of bits required to represent (store) its spoken equivalent.

Problem 9.9

(a) With K equiprobable symbols, the probability of symbol s_k is

$$p_k = P(s_k) = \frac{1}{K}$$

The average code-word length is

$$\begin{aligned} \bar{L} &= \sum_{k=0}^{K-1} p_k l_k \\ &= \frac{1}{K} \sum_{k=0}^{K-1} l_k \end{aligned}$$

The choice of a fixed code-word length $l_k=l_0$ for all k yields the value $\bar{L}=l_0$. With K symbols in the code, any other choice for l_k yields a value for \bar{L} no less than l_0 .

(b) Entropy of the source is

[†]R.W. Lucky, "Silicon Dreams", p.111 (St. Martin's Press, 1989).

$$\begin{aligned}
 H(S) &= \sum_{k=0}^{K-1} p_k \log_2 \left(\frac{1}{p_k} \right) \\
 &= \sum_{k=0}^{K-1} \frac{1}{K} \log_2 K = \log_2 K
 \end{aligned}$$

The coding efficiency is therefore

$$\eta = \frac{H(S)}{L} = \frac{\log_2 K}{l_0}$$

For $\eta=1$, we have

$$l_0 = \log_2 K$$

To satisfy this requirement, we choose

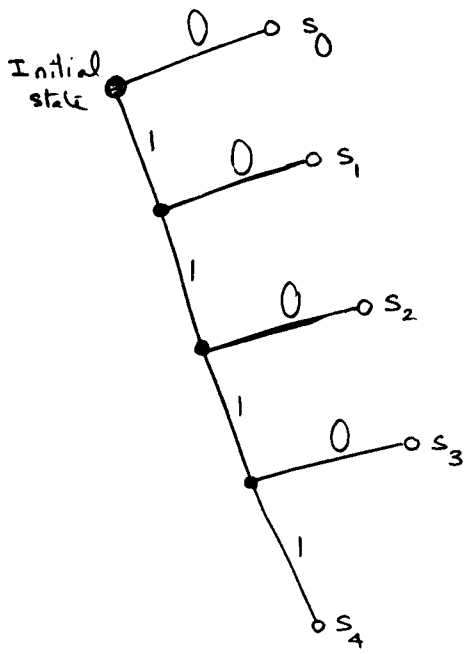
$$K = 2^{l_0}$$

where l_0 is an integer.

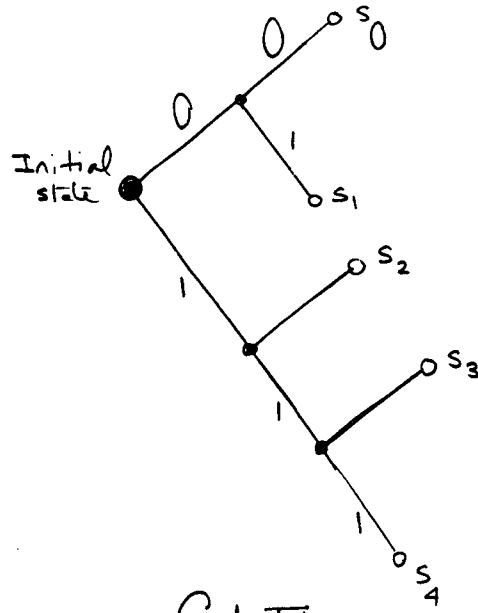
Problem 9.10

A prefix code is defined as a code in which no code word is the prefix of any other code word. By inspection, we see therefore that codes I and IV are prefix codes, whereas codes II and III are not.

To draw the decision tree for a prefix code, we simply begin from some starting node, and extend branches forward until each symbol of the code is represented. We thus have:



Code I

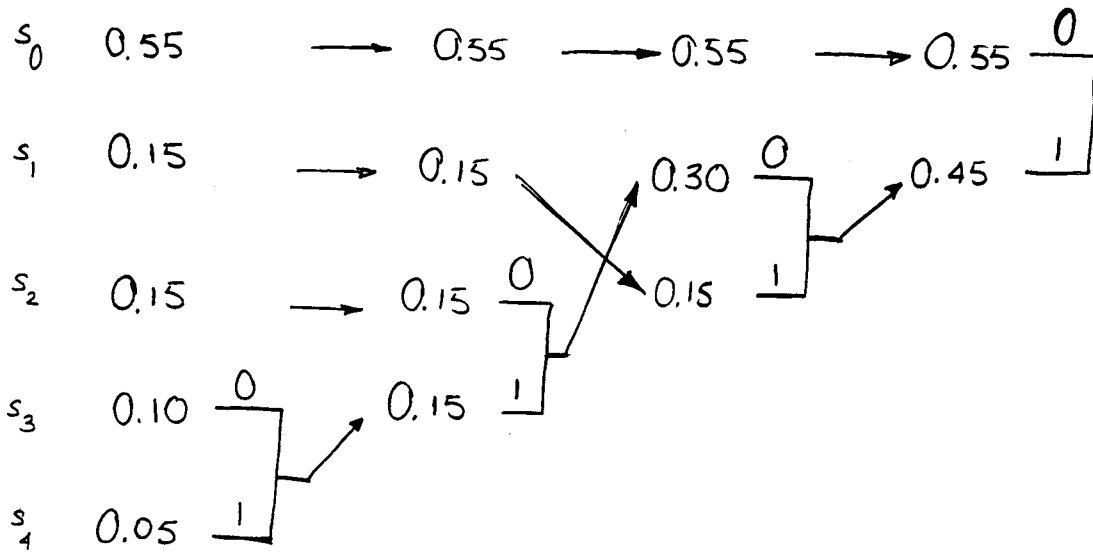


Code IV

Problem 9.11

We may construct two different Huffman codes by choosing to place a combined symbol as low or as high as possible when its probability is equal to that of another symbol.

We begin with the Huffman code generated by placing a combined symbol as low as possible:



The source code is therefore

s_0 0
 s_1 1 1
 s_2 1 0 0
 s_3 1 0 1 0
 s_4 1 0 1 1

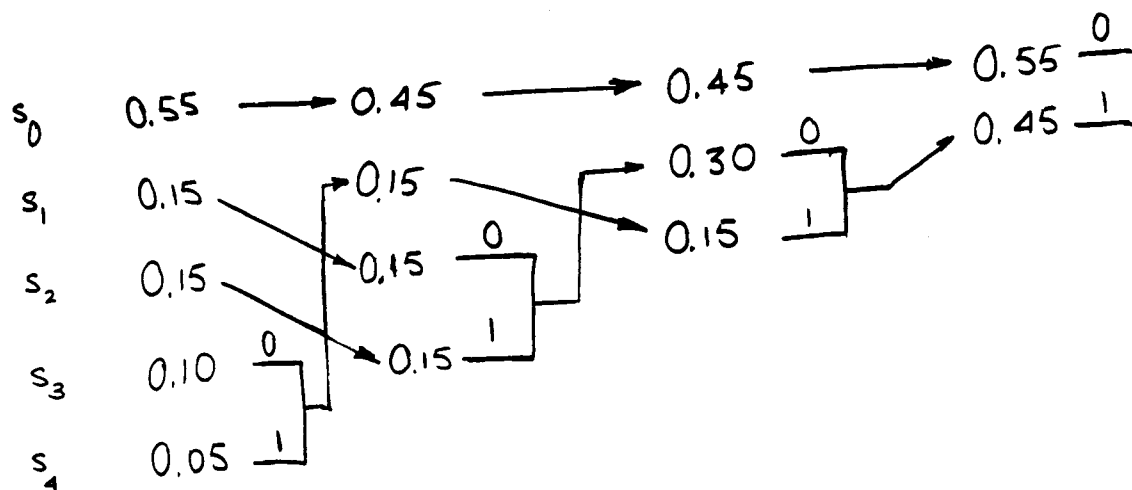
The average code-word length is therefore

$$\begin{aligned} L &= \sum_{k=0}^4 p_k l_k \\ &= 0.55(1) + 0.15(2) + 0.15(3) + 0.1(4) + 0.05(4) \\ &= 1.9 \end{aligned}$$

The variance of \bar{L} is

$$\begin{aligned} \sigma^2 &= \sum_{k=0}^4 p_k (l_k - L)^2 \\ &= 0.55(-0.9)^2 + 0.15(0.1)^2 + 0.15(1.1)^2 + 0.1(2.1)^2 + 0.05(2.1)^2 \\ &= 1.29 \end{aligned}$$

Next placing a combined symbol as high as possible, we obtain the second Huffman code:



Correspondingly, the Huffman code is

s_0	0
s_1	1 0 0
s_2	1 0 1
s_3	1 1 0
s_4	1 1 1

The average code-word length is

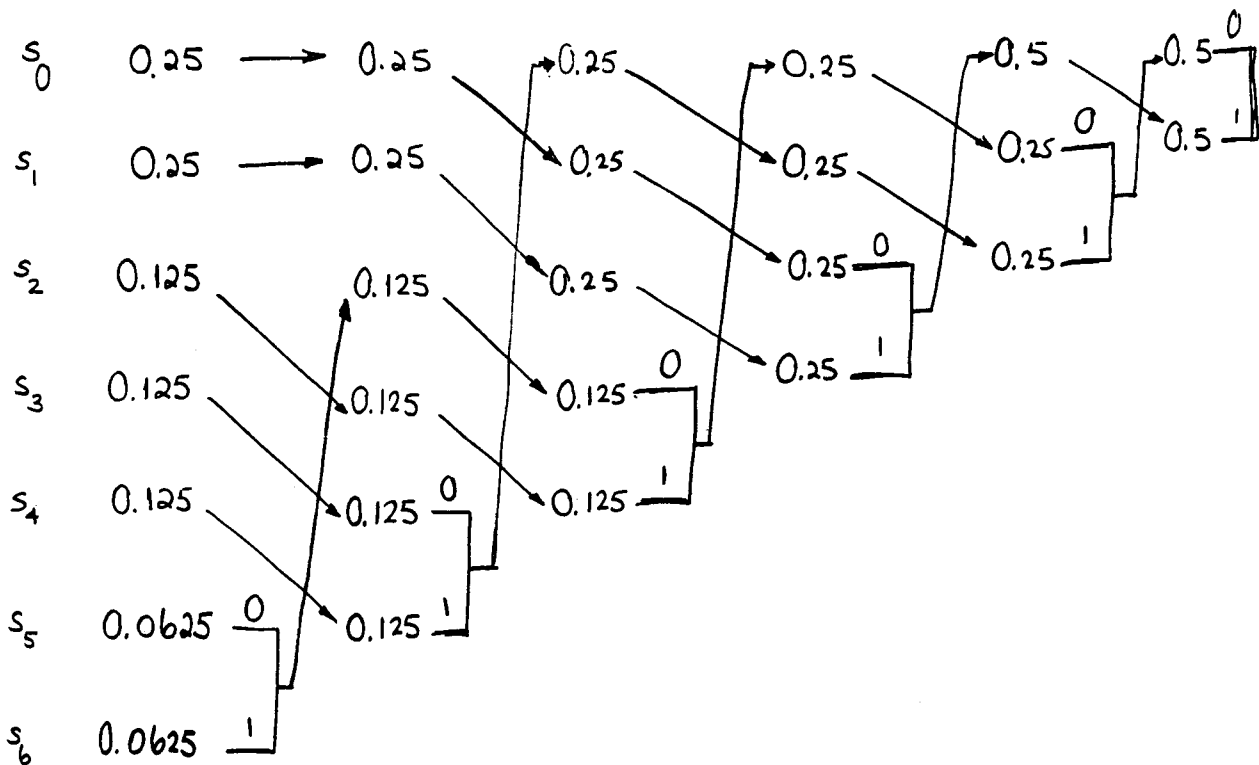
$$\begin{aligned} \bar{L} &= 0.55(1) + (0.15 + 0.15 + 0.1 + 0.05)(3) \\ &= 1.9 \end{aligned}$$

The variance of \bar{L} is

$$\begin{aligned} \sigma^2 &= 0.55(-0.9)^2 + (0.15 + 0.15 + 0.1 + 0.05)(1.1)^2 \\ &= 0.99 \end{aligned}$$

The two Huffman codes described herein have the same average code-word length but different variances.

Problem 9.12



The Huffman code is therefore

s_0	1 0
s_1	1 1
s_2	0 0 1
s_3	0 1 0
s_4	0 1 1
s_5	0 0 0 0
s_6	0 0 0 1

The average code-word length is

$$\begin{aligned} \bar{L} &= \sum_{k=0}^6 p_k l_k \\ &= 0.25(2)(2) + 0.125(3)(3) + 0.0625(4)(2) \\ &= 2.625 \end{aligned}$$

The entropy of the source is

$$\begin{aligned} H(S) &= \sum_{k=0}^6 p_k \log_2 \left(\frac{1}{p_k} \right) \\ &= 0.25(2) \log_2 \left(\frac{1}{0.25} \right) + 0.125(3) \log_2 \left(\frac{1}{0.125} \right) \\ &\quad + 0.0625(2) \log_2 \left(\frac{1}{0.0625} \right) \\ &= 2.625 \end{aligned}$$

The efficiency of the code is therefore

$$\eta = \frac{H(S)}{\bar{L}} = \frac{2.625}{2.625} = 1$$

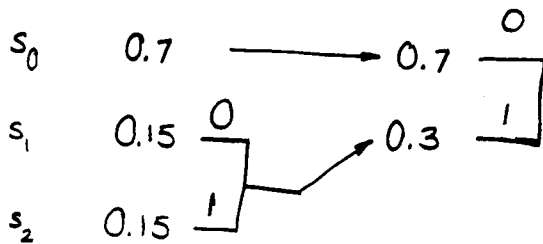
We could have shown that the efficiency of the code is 100% by inspection since

$$\eta = \frac{\sum_{k=0}^6 p_k \log_2(1/p_k)}{\sum_{k=0}^6 p_k l_k}$$

where $l_k = \log_2(1/p_k)$.

Problem 9.13

(a)



The Huffman code is therefore

s_0 0
 s_1 1 0
 s_2 1 1

The average code-word length is

$$\begin{aligned} \bar{L} &= 0.7(1) + 0.15(2) + 0.15(2) \\ &= 1.3 \end{aligned}$$

(b) For the extended source we have

Symbol	s_0s_0	s_0s_1	s_0s_2	s_1s_0	s_2s_0	s_1s_1	s_1s_2	s_2s_1	s_2s_2
Probability	0.49	0.105	0.105	0.105	0.105	0.0225	0.0225	0.0225	0.0225

Applying the Huffman algorithm to the extended source, we obtain the following source code:

s_0s_0	1
s_0s_1	0 0 1
s_0s_2	0 1 0
s_1s_0	0 1 1
s_2s_0	0 0 0 0
s_1s_1	0 0 0 1 0 0
s_1s_2	0 0 0 1 0 1
s_2s_1	0 0 0 1 1 0
s_2s_2	0 0 0 1 1 1

The corresponding value of the average code-word length is

$$\begin{aligned}\bar{L}_2 &= 0.49(1) + 0.105(3)(3) + 0.105(4) + 0.0225(4)(4) \\ &= 2.395 \text{ bits/extended symbol}\end{aligned}$$

$$\frac{\bar{L}_2}{2} = 1.1975 \text{ bits/symbol}$$

(c) The original source has entropy

$$\begin{aligned}H(S) &= 0.7 \log_2\left(\frac{1}{0.7}\right) + 0.15(2) \log_2\left(\frac{1}{0.15}\right) \\ &= 1.18\end{aligned}$$

According to Eq. (10-28),

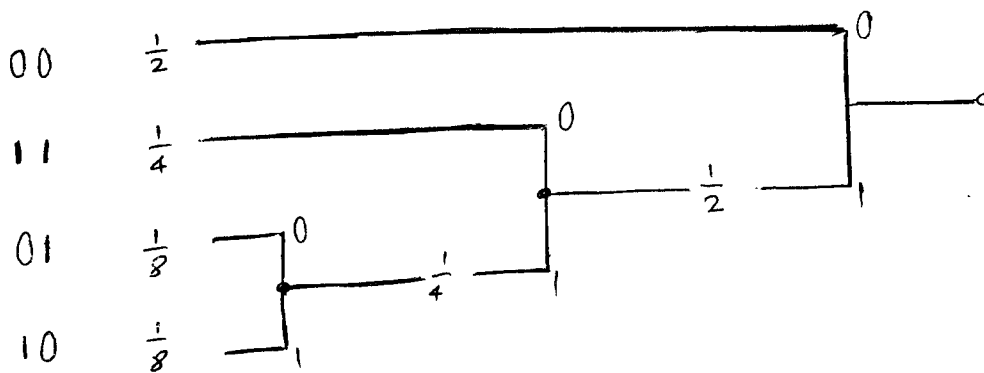
$$H(S) \leq \frac{L_n}{n} \leq H(S) + \frac{1}{n}$$

This is a condition which the extended code satisfies.

Problem 9.14

<u>Symbol</u>	<u>Huffman Code</u>	<u>Code-word length</u>
A	1	1
B	0 1 1	3
C	0 1 0	3
D	0 0 1	3
E	0 0 1 1	4
F	0 0 0 0 1	5
G	0 0 0 0 0	5

Problem 9.15



<u>Computer code</u>	<u>Probability</u>	<u>Huffman Code</u>
0 0	$\frac{1}{2}$	0
1 1	$\frac{1}{4}$	1 0
0 1	$\frac{1}{8}$	1 1 0
1 0	$\frac{1}{8}$	1 1 1

The number of bits used for the instructions based on the computer code, in a probabilistic sense, is equal to

$$2\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8}\right) = 2 \text{ bits}$$

On the other hand, the number of bits used for instructions based on the Huffman code, is equal to

$$1 \times \frac{1}{2} + 2 \times \frac{1}{4} + 3 \times \frac{1}{8} + 3 \times \frac{1}{8} = \frac{7}{4}$$

The percentage reduction in the number of bits used for instruction, realized by adopting the Huffman code, is therefore

$$100 \times \frac{1/4}{2} = 12.5\%$$

Problem 9.16

Initial step

Subsequences stored: 0
Data to be parsed: 1 1 1 0 1 0 0 1 1 0 0 0 1 0 1 1 0 1 0 0 ...

Step 1

Subsequences stored: 0, 1, 11
Data to be parsed: 1 0 1 0 0 1 1 0 0 0 1 0 1 1 0 1 0 0 ..

Step 2

Subsequences stored: 0, 1, 11, 10
Data to be parsed: 1 0 0 1 1 0 0 0 1 0 1 1 0 1 0 0

Step 3

Subsequences stored: 0, 1, 11, 10, 100
Data to be parsed: 1 1 0 0 0 1 0 1 1 0 1 0 0 ...

Step 4

Subsequences stored: 0, 1, 11, 10, 100, 110
Data to be parsed: 0 0 1 0 1 1 0 1 0 0 ...

Step 5

Subsequences stored: 0, 1, 11, 10, 100, 110, 00

Data to be parsed: 1 0 1 1 0 1 0 0

Step 6

Subsequences stored: 0, 1, 11, 10, 100, 110, 00, 101

Data to be parsed: 1 0 1 0 0 ...

Step 7

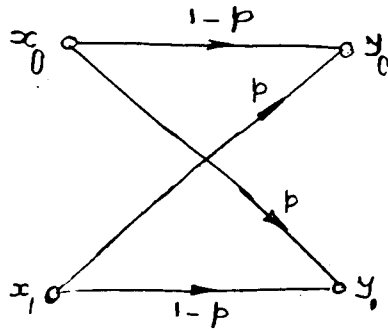
Subsequences stored: 0, 1, 11, 10, 100, 110, 00, 101, 1010

Data to be parsed: 0

Now that we have gone as far as we can go with data parsing for the given sequence, we write

Numerical positions	1	2	3	4	5	6	7	8	9
Subsequences	0,	1,	11,	10,	100,	110,	00,	101,	1010
Numerical representations			22,	21,	41,	31,	11,	42,	81
Binary encoded blocks			0101,	0100,	0100,	0110,	0010,	1001,	10000

Problem 9.17



$$P(x_0) = P(x_1) = \frac{1}{2}$$

$$\begin{aligned} P(y_0) &= (1 - p)P(x_0) + p P(x_1) \\ &= (1 - p) \left(\frac{1}{2}\right) + p \left(\frac{1}{2}\right) \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} P(y_1) &= p P(x_0) + (1 - p) P(x_1) \\ &= p \left(\frac{1}{2}\right) + (1 - p) \left(\frac{1}{2}\right) \\ &= \frac{1}{2} \end{aligned}$$

Problem 9.18

$$p(x_0) = \frac{1}{4}$$

$$p(x_1) = \frac{3}{4}$$

$$\begin{aligned} p(y_0) &= (1 - p) \left(\frac{1}{4}\right) + p\left(\frac{3}{4}\right) \\ &= \frac{1}{4} + \frac{p}{2} \end{aligned}$$

$$\begin{aligned} p(y_1) &= p\left(\frac{1}{4}\right) + (1 - p) \left(\frac{3}{4}\right) \\ &= \frac{3}{4} - \frac{p}{2} \end{aligned}$$

Problem 9.19

From Eq.(9-52) we may express the mutual information as

$$I(X, Y) = \sum_{j=0}^1 \sum_{k=0}^1 p(x_j, y_k) \log_2 \left(\frac{p(x_j, y_k)}{p(x_j) p(y_k)} \right)$$

The joint probability $p(x_j, y_k)$ has the following four possible values:

$$j = k = 0: \quad p(x_0, y_0) = p_0(1-p) = (1 - p_1) (1 - p)$$

$$j = 0, k = 1: \quad p(x_0, y_1) = p_0 p = (1 - p_1) p$$

$$j = 1, k = 0: \quad p(x_1, y_0) = p_1 p$$

$$j = k = 1: \quad p(x_1, y_1) = p(1 - p_1)$$

where $p_0 = p(x_0)$ and $p_1 = p(x_1)$

The mutual information is therefore

$$\begin{aligned}
I(X;Y) &= (1-p_1) (1-p) \log_2 \left(\frac{(1-p_1) (1-p)}{(1-p_1) ((1-p_1) (1-p) + p_1 p)} \right) \\
&+ (1-p_1) p \log_2 \left(\frac{(1-p_1) p}{(1-p_1) ((1-p_1) p + p_1 (1-p))} \right) \\
&+ p_1 p \log_2 \left(\frac{p_1 p}{p_1 ((1-p_1) (1-p) + p_1 p)} \right) \\
&+ p_1 (1-p) \log_2 \left(\frac{p_1 (1-p)}{p_1 ((1-p_1) p + p_1 (1-p))} \right)
\end{aligned}$$

Rearranging terms and expanding algorithms:

$$\begin{aligned}
I(X;Y) &= p \log_2 p + (1-p) \log_2(1-p) \\
&- [p_1(1-p) + (1-p_1)p] \log_2[p_1(1-p) + (1-p_1)p] \\
&- [p_1 p + (1-p_1) (1-p)] \log_2[p_1 p + (1-p_1) (1-p)]
\end{aligned}$$

Treating the transition probability p of the channel as a running parameter, we may carry out the following numerical evaluations:

$p=0$:

$$I(X;Y) = - p_1 \log_2 p_1 - (1 - p_1) \log_2 (1 - p_1)$$

$$p_1 = 0.5, \quad I(X;Y) = 1.0$$

$p=0.1$:

$$\begin{aligned}
I(X;Y) &= - 0.469 - (0.1 + 0.8 p_1) \log_2 (0.1 + 0.8 p_1) \\
&- (0.9 - 0.8 p_1) \log_2 (0.9 - 0.9 p_1)
\end{aligned}$$

$$p_1 = 0.5, \quad I(X;Y) = 0.531$$

$p=0.2$:

$$\begin{aligned}
I(X;Y) &= - 0.7219 - (0.2 + 0.6 p_1) \log_2 (0.2 + 0.6 p_1) \\
&- (0.8 - 0.6 p_1) \log_2 (0.8 - 0.6 p_1)
\end{aligned}$$

$$p_1 = 0.5, \quad I(X;Y) = 0.278$$

p=0.3:

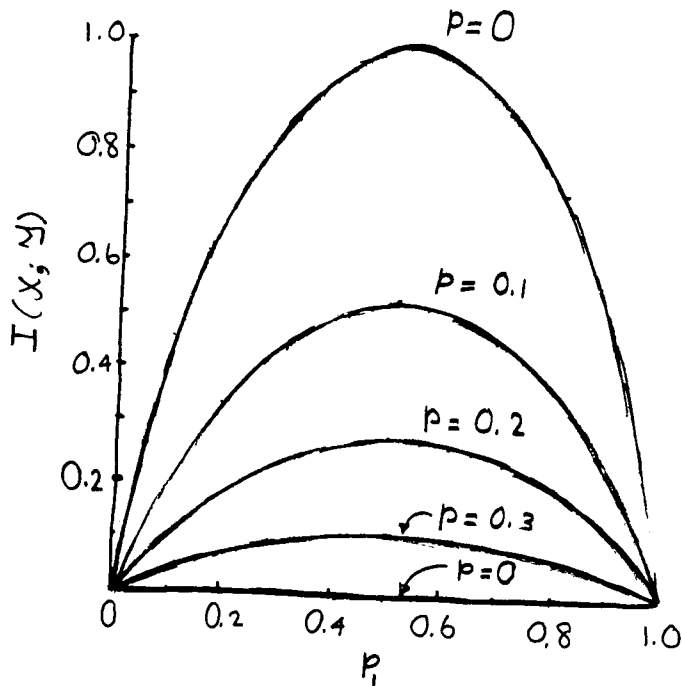
$$I(X;Y) = -0.88129 - (0.3 + 0.4 p_1) \log_2 (0.3 + 0.4 p_1) \\ - (0.7 - 0.4 p_1) \log_2 (0.7 - 0.4 p_1)$$

$$p_1 = 0.5, \quad I(X;Y) = 0.1187$$

p=0.5:

$$I(X;Y) = 0$$

Thus, treating the a priori probability p_1 as a variable and the transition probability p as a running parameter, we get the following plots:



Problem 9.20

From the plots of $I(X;Y)$ versus p_1 for p as a running parameter, that were presented in the solution to Problem 10.19 we observe that $I(X;Y)$ attains its maximum value at $p_1=0.5$ for any p . Hence, evaluating the mutual information $I(X;Y)$ at $p_1=0.5$, we get the channel capacity:

$$C = 1 + p \log_2 p + (1 - p) \log_2 (1 - p) \\ = 1 - H(p)$$

where $H(p)$ is the entropy function of p .

Problem 9.21

(a) Let

$$z = p_1(1 - p) + (1 - p_1)p = (1 - p_0)(1 - p) + p_0p$$

Hence,

$$\begin{aligned} p_1p + (1 - p_1)p &= 1 - p_1(1 - p) - (1 - p_1)p \\ &= 1 - z \end{aligned}$$

Correspondingly, we may rewrite the expression for the mutual information $I(X;Y)$ as

$$I(X;Y) = H(z) - H(p)$$

where

$$H(z) = -z \log_2 z - (1 - z) \log_2 (1 - z)$$

$$H(p) = -p \log_2 p - (1 - p) \log_2 (1 - p)$$

(b) Differentiating $I(X;Y)$ with respect to p_0 (or p_1) and setting the result equal to zero, we find that $I(X;Y)$ attains its maximum value at $p_0 = p_1 = 1/2$.

(c) Setting $p_0 = p_1 = 1/2$ in the expression for the variable z , we get:

$$z = 1 - z = 1/2$$

Correspondingly, we have

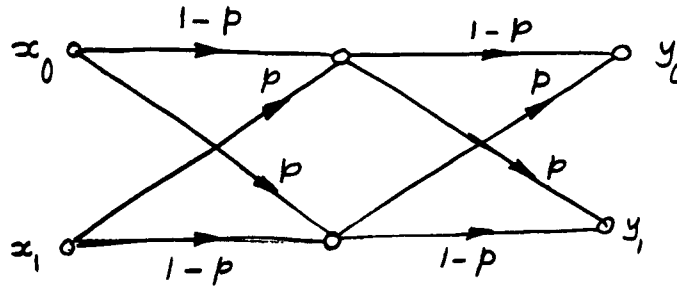
$$H(z) = 1$$

We thus get the following value for the channel capacity:

$$\begin{aligned} C &= I(X;Y) \Big|_{p_0 = p_1 = 1/2} \\ &= 1 - H(p) \end{aligned}$$

where $H(p)$ is the entropy function of the channel transition probability p .

Problem 9.22

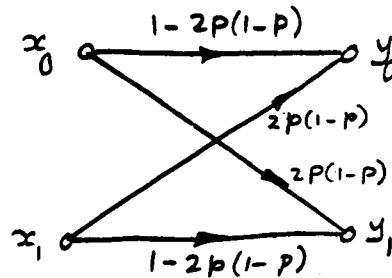


From this diagram, we obtain (by inspection)

$$P(y_0 | x_0) = (1 - p)^2 + p^2 = 1 - 2p(1 - p)$$

$$P(y_0 | x_1) = p(1 - p) + (1 - p)p = 2p(1 - p)$$

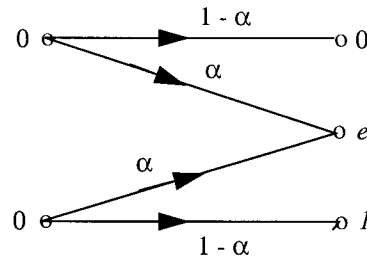
Hence, the cascade of two binary symmetric channels with a transition probability p is equivalent to a single binary symmetric channel with a transition probability equal to $2p(1 - p)$, as shown below:



Correspondingly, the channel capacity of the cascade is

$$\begin{aligned} C &= 1 - H(2p(1 - p)) \\ &= 1 - 2p(1 - p) \log_2 [2p(1 - p)] - (1 - 2p + 2p^2) \log_2 (1 - 2p + 2p^2) \end{aligned}$$

Problem 9.23



The mutual information for the erasure channel is

$$\mathbf{I}(\mathbf{X};\mathbf{Y}) = \sum_{j=0}^1 \sum_{k=0}^2 p(x_j, y_k) \log_2 \left(\frac{p(x_j, y_k)}{p(x_j)p(y_k)} \right) \quad (1)$$

The joint probabilities for the channel are

$$\begin{aligned} p(x_0, y_0) &= (1 - \alpha)p_0 & p(x_1, y_0) &= 0 & p(x_0, y_2) &= p_0\alpha \\ p(x_0, y_1) &= 0 & p(x_1, y_1) &= (1 - \alpha)p_1 & p(x_1, y_2) &= p_1\alpha \end{aligned}$$

where $p_0 + p_1 = 1$. Substituting these values in (1), we get

$$\mathbf{I}(\mathbf{X};\mathbf{Y}) = (1 - \alpha) \left[p_0 \log_2 \left(\frac{1}{p_0} \right) + (1 - p_0) \log_2 \left(\frac{1}{1 - p_0} \right) \right]$$

Since the transition probability $p = (1 - \alpha)$ is fixed, the mutual information $\mathbf{I}(\mathbf{X};\mathbf{Y})$ is maximized by choosing the a priori probability p_0 to maximize $H(p_0)$. This maximization occurs at $p_0 = 1/2$, for which $H(p_0) = 1$. Hence, the channel capacity C of the erasure channel is $1 - \alpha$.

Problem 9.24

(a) When each symbol is repeated three times, we have

<u>Messages</u>	<u>Unused signals</u>	<u>Channel outputs</u>
000	001	000
	010	001
	011	010
	100	100
	101	101
	110	110
111		111

We note the following:

1. The probability that no errors occur in the transmission of three 0s or three 1s is $(1 - p)^3$.
2. The probability of just one error occurring is $3p(1 - p)^2$.
3. The probability of two errors occurring is $3p^2(1 - p)$.
4. The probability of receiving all three bits in error is p^3 .

With the decision-making based on a majority vote, it is clear that contributions 3 and 4 lead to the probability of error

$$P_3 = 3p^2(1 - p) + p^3$$

(b) When each symbol is transmitted five times, we have

<u>Messages</u>	<u>Unused signals</u>	<u>Channel outputs</u>
00000		00000
	00001	00001
	00010	00010
	00011	00011
	⋮	⋮
	⋮	⋮
	11110	11110
11111		11111

The probability of zero, one, two, three, four, or five bit errors in transmission is as follows, respectively:

$$\begin{aligned}
 &(1-p)^5 \\
 &5p(1-p)^4 \\
 &10p^2(1-p)^3 \\
 &10p^3(1-p)^2 \\
 &5p^4(1-p) \\
 &p^5
 \end{aligned}$$

The last three contributions constitute the probability of error

$$P_e = p^5 + 5p^4(1-p) + 10p^3(1-p)^2$$

(a) For the general case of $n=2m + 1$, we note that the decision-making process (based on a majority vote) makes an error when $m+1$ bits or more out of the n bits of a message are received in error. The probability of i message bits being received in error is $\binom{n}{i} p^i (1-p)^{n-i}$. Hence, the probability of error is (in general)

$$P_e = \sum_{i=m+1}^n \binom{n}{i} p^i (1-p)^{n-i}$$

The results derived in parts (a) and (b) for $m=1$ and $m=2$ are special cases of this general formula.

Problem 9.25

The differential entropy of a random variable is independent of its mean. To evaluate the differential entropy of a Gaussian vector \mathbf{X} , consisting of the components X_1, X_2, \dots, X_n , we may simplify our task by setting the mean of \mathbf{X} equal to zero. Under this condition, we may express the joint probability density function of the Gaussian vector \mathbf{X} as

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(\sqrt{2\pi})^n \sigma_1 \sigma_2 \dots \sigma_n} \exp\left(-\frac{x_1^2}{2\sigma_1^2} - \frac{x_2^2}{2\sigma_2^2} - \dots - \frac{x_n^2}{2\sigma_n^2}\right)$$

The logarithm of $f_{\mathbf{X}}(\mathbf{x})$ is

$$\log_2 f_{\mathbf{X}}(\mathbf{x}) = -\log_2\left((2\pi)^{n/2} \sigma_1 \sigma_2 \dots \sigma_n\right) - \left(\frac{x_1^2}{2\sigma_1^2} + \frac{x_2^2}{2\sigma_2^2} + \dots + \frac{x_n^2}{2\sigma_n^2}\right) \log_2 e$$

Hence, the differential entropy of \mathbf{X} is

$$\begin{aligned} h(\mathbf{X}) &= - \int \int \dots \int f_{\mathbf{X}}(\mathbf{x}) \log_2(f_{\mathbf{X}}(\mathbf{x})) \, d\mathbf{x} \\ &= \log_2\left((2\pi)^{n/2} \sigma_1 \sigma_2 \dots \sigma_n\right) \int \int \dots \int f_{\mathbf{X}}(\mathbf{x}) \, d\mathbf{x} \\ &\quad + \log_2 e \int \int \dots \int \left(\frac{x_1^2}{2\sigma_1^2} + \frac{x_2^2}{2\sigma_2^2} + \dots + \frac{x_n^2}{2\sigma_n^2}\right) f_{\mathbf{X}}(\mathbf{x}) \, d\mathbf{x} \end{aligned}$$

We next note that

$$\begin{aligned} \int \int \dots \int f_{\mathbf{X}}(\mathbf{x}) \, d\mathbf{x} &= 1 \\ \int \int \dots \int x_i^2 f_{\mathbf{X}}(\mathbf{x}) \, d\mathbf{x} &= \sigma_i^2 \quad i = 1, 2, \dots, n \end{aligned}$$

Hence, we may simplify (1) as

$$\begin{aligned}
h(\mathbf{X}) &= \log_2 \left[(2\pi)^{n/2} \sigma_1 \sigma_2 \dots \sigma_n \right] + \frac{n}{2} \log_2 e \\
&= \log_2 \left[2\pi (\sigma_1^2 \sigma_2^2 \dots \sigma_n^2)^{1/n} \right]^{n/2} + \frac{n}{2} \log_2 e \\
&= \frac{n}{2} \left[2\pi (\sigma_1^2 \sigma_2^2 \dots \sigma_n^2)^{1/n} \right] + \frac{n}{2} \log_2 e \\
&= \frac{n}{2} \log_2 \left[2\pi e (\sigma_1^2 \sigma_2^2 \dots \sigma_n^2)^{1/n} \right]
\end{aligned}$$

When the individual variances are equal:

$$\begin{aligned}
\sigma_1^2 &= \sigma_2^2 = \dots = \sigma_n^2 = \sigma^2 \\
(\sigma_1^2 \sigma_2^2 \dots \sigma_n^2)^{1/n} &= \sigma^2
\end{aligned}$$

Correspondingly, the differential entropy of \mathbf{X} is

$$h(\mathbf{X}) = \frac{n}{2} \log_2(2\pi e \sigma^2)$$

Problem 9.26

(a) The differential entropy of a random variable X is

$$h(X) = - \int_{-\infty}^{\infty} f_X(x) \log_2 f_X(x) dx$$

The constraint on the value x of the random variable X is

$$|x| \leq M$$

Using the method ^{of} Lagrange multipliers, we find that $h(X)$, subject to this constraint, is maximized when

$$\int_{-M}^M [-f_X(x) \log_2 f_X(x) + \lambda f_X(x)] dx$$

is stationary. Differentiating $-f_X(x) \log_2 f_X(x) + \lambda f_X(x)$ with respect to $f_X(x)$, and then setting the result equal to zero, we get

$$-\log_2 e + \lambda = \log_2 f_X(x)$$

This shows that $f_X(x)$ is independent of x . Hence, for the differential entropy $h(X)$ under the constraints $|x| \leq M$ and $\int_{-\infty}^{\infty} f_X(x) dx = 1$ to be maximum, the random variable X must be uniformly distributed:

$$f_X(x) = \begin{cases} 1/2M, & -M \leq x < M \\ 0, & \text{otherwise} \end{cases}$$

(b) The maximum differential entropy of the uniformly distributed random variable X is

$$\begin{aligned} h(X) &= \int_{-M}^M \frac{1}{2M} \log_2(2M) dx \\ &= \frac{1}{2M} \log_2(2M) \int_{-M}^M dx \\ &= \log_2(2M) \end{aligned}$$

Problem 9.27

(a)

$$I(X;Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) \log_2 \left[\frac{f_X(x|y)}{f_X(x)} \right] dx dy \quad (1)$$

$$I(Y;X) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) \log_2 \left[\frac{f_Y(y|x)}{f_Y(y)} \right] dx dy \quad (2)$$

From Bayes' rule, we have

$$\frac{f_X(x|y)}{f_X(x)} = \frac{f_Y(y|x)}{f_Y(y)}$$

Hence, $I(X;Y) = I(Y;X)$.

(b) We note that

$$f_{X,Y}(x,y) = f_X(x|y)f_Y(y)$$

Therefore, we may rewrite (1) as

$$I(X;Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) \log_2 \left[\frac{f_{X,Y}(x,y)}{f_X(x)f_Y(y)} \right] dx dy$$

From the continuous version of the fundamental inequality, we have

$$\int_{-\infty}^{\infty} f_{X,Y}(x,y) \log_2 \left(\frac{f_{X,Y}(x,y)}{f_X(x)f_Y(y)} \right) dx dy \geq 0$$

which is satisfied with equality if and only if

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

Hence,

$$I(X;Y) \geq 0$$

(c) By definition, we have

$$h(X | Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) \log_2 \left[\frac{1}{f_X(x|y)} \right] dx dy \quad (3)$$

By definition, we also have

$$h(X) = \int_{-\infty}^{\infty} f_X(x) \log_2 \left[\frac{1}{f_X(x)} \right] dx$$

Since

$$\int_{-\infty}^{\infty} f_Y(y|x) dy = 1,$$

we may rewrite the expression for $h(X)$ as

$$\begin{aligned} h(X) &= \int_{-\infty}^{\infty} f_X \log_2 \left[\frac{1}{f_X(x)} \right] dx \int_{-\infty}^{\infty} f_Y(y|x) dx \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_Y(y|x) f_X(x) \log_2 \left[\frac{1}{f_X(x)} \right] dx dy \end{aligned}$$

But

$$f_Y(y|x) f_X(x) = f_{X,Y}(x,y)$$

Hence,

$$h(X) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) \log_2 \left[\frac{1}{f_X(x)} \right] dx dy \quad (4)$$

Next, we subtract (3) from (4), obtaining

$$\begin{aligned} h(X) - h(X|Y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) \left[\log_2 \frac{1}{f_X(x)} - \log_2 \frac{1}{f_X(x|y)} \right] dx dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) \log_2 \left[\frac{f_X(x|y)}{f_X(x)} \right] dx dy \\ &= I(X;Y) \end{aligned}$$

(d) Using the symmetric property derived in part (a), we may also write

$$I(Y;X) = h(Y) - h(Y|X)$$

Problem 9.28

By definition, we have

$$h(Y|X) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) \log_2 \left[\frac{1}{f_Y(y|x)} \right] dx dy$$

Since

$$f_{X,Y}(x,y) = f_Y(y|x)f_X(x)$$

we may also write

$$h(Y | X) = \int_{-\infty}^{\infty} f_X(x) dx \int_{-\infty}^{\infty} f_Y(y | x) \log_2 \left[\frac{1}{f_Y(y | x)} \right] dy \quad (1)$$

The variable Y is related to X as

$$Y = X + N$$

Hence, the conditional probability density function $f_Y(y | x)$ is identical to that of N except for a translation of x, the given value of X. Let $f_N(n)$ denote the probability density function of N. Then

$$f_Y(y | x) = f_N(y - x)$$

Correspondingly, we may write

$$\begin{aligned} \int_{-\infty}^{\infty} f_Y(y | x) \log_2 \left[\frac{1}{f_Y(y | x)} \right] dy &= \int_{-\infty}^{\infty} f_N(y - x) \log_2 \left[\frac{1}{f_N(y - x)} \right] dy \\ &= \int_{-\infty}^{\infty} f_N(n) \log_2 \left[\frac{1}{f_N(n)} \right] dn \\ &= h(N) \end{aligned} \quad (2)$$

where, in the second line, we have used $n = y - x$ and $dn = dy$ for a given x. Substituting Eq. (2) in (1):

$$\begin{aligned} h(Y | X) &= h(N) \int_{-\infty}^{\infty} f_X(x) dx \\ &= h(N) \end{aligned}$$

Problem 9.29

- (a) Channel bandwidth $B = 3.4 \text{ kHz}$
Received signal-to-noise ratio $\text{SNR} = 10^3 \equiv 30 \text{ dB}$

Hence, the channel capacity is

$$\begin{aligned} C &= B \log_2(1 + \text{SNR}) \\ &= 3.4 \times 10^3 \log_2(1 + 10^3) \\ &= 33.9 \times 10^3 \text{ bits/second} \end{aligned}$$

- (b) $4800 = 3.4 \times 10^3 \log_2(1 + \text{SNR})$

Solving for the unknown SNR, we get

$$\text{SNR} = 1.66 \equiv 2.2 \text{ dB}$$

Problem 9.30

With 10 distinct brightness levels with equal probability, the information in each level is $\log_2 10$ bits. With each picture frame containing 3×10^5 elements, the information content of each picture frame is $3 \times 10^5 \log_2 10$ bits. Thus, a rate of information transmission of 30 frames per second corresponds to

$$30 \times 3 \times 10^5 \log_2 10 = 9 \times 10^6 \log_2 10 \text{ bits/second}$$

That is, the channel capacity is

$$C = 9 \times 10^6 \log_2 10 \text{ bits/second}$$

$$C = B \log_2 (1 + \text{SNR})$$

With a signal-to-noise ratio $\text{SNR} = 10^3 = 30 \text{ dB}$, the channel bandwidth is therefore

$$\begin{aligned} B &= \frac{C}{\log_2(1 + \text{SNR})} \\ &= \frac{9 \times 10^6 \log_2 10}{\log_2 1001} \\ &= 3 \times 10^3 \text{ Hz} \end{aligned}$$

Problem 9.31

The information in each element is $\log_2 10$ bits.

The information in each picture is $[3 \log_2 (10)] \times 10^5$ bits.

The transmitted information rate is $[9 \log_2 (10)] \times 10^6$ bits/second.

The channel must have this capacity. From the information capacity theorem,

$$c = B \log_2(1 + \text{SNR}).$$

Therefore,

$$[9 \log_2(10)] \times 10^6 \text{ bits/second} = B \log_2(1 + 1000).$$

Solving for B, we get

$$B = 9 \times 10^6 \text{ Hz} \cdot \left(\frac{\log_2(10)}{\log_2(1001)} \right) = 3 \times 10^3 \text{ Hz}$$

Problem 9.32

Figure 1 shows the 64-QAM constellation. Under the condition that the transmitted signal energy per symbol is maintained the same as that in Fig. 1, we get the tightly packed constellation of Fig. 2. We now find that the Euclidean distance between adjacent signal points in the tightly packed constellation of Fig. 2 is larger than that of the 64-QAM constellation in Fig. 1. From Eq. (5.95) of the textbook, we recall that an increase in the minimum Euclidean distance d_{\min} results in a corresponding reduction in the average probability of symbol error. It follows therefore that, with the signal energy per symbol being maintained the same in the two constellations of Figs. 1 and 2, a digital communication systems using the tightly packed constellation of Fig. 2 produces a smaller probability of error than the corresponding 64-QAM system represented by Fig. 1.

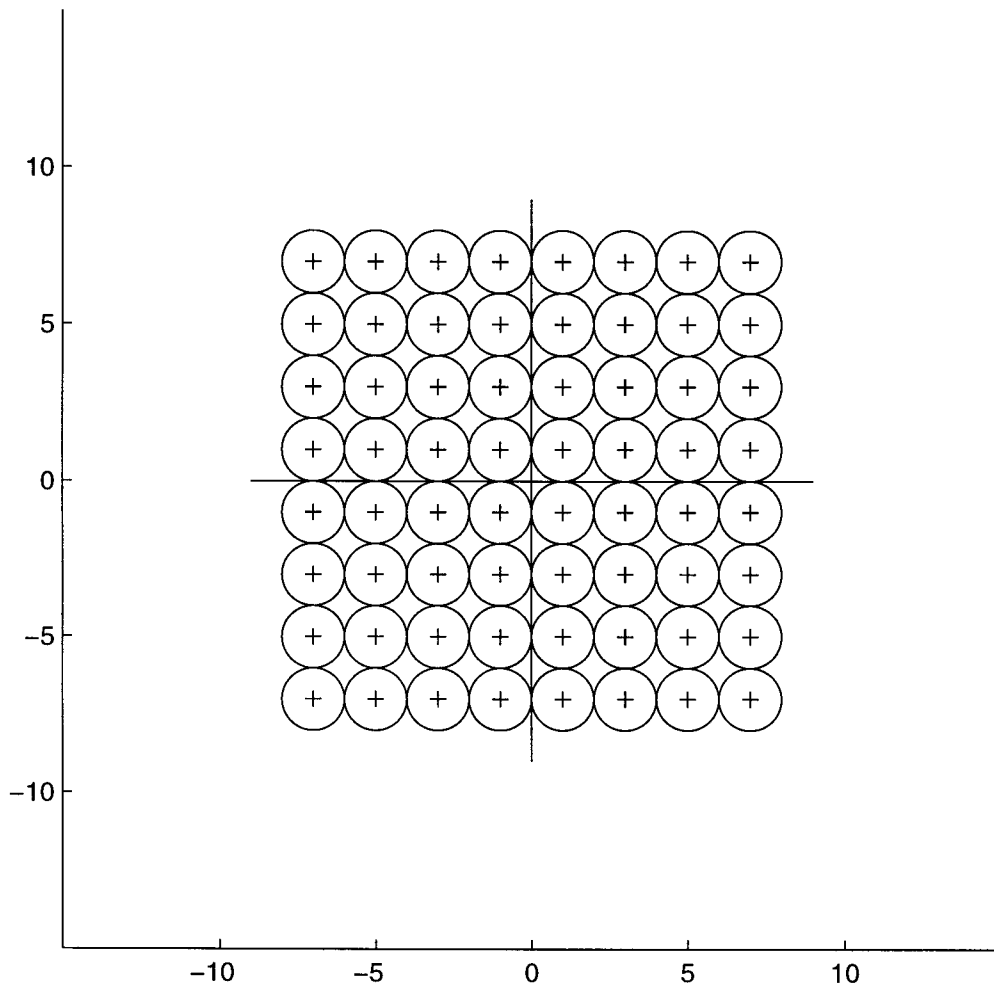


Figure 1

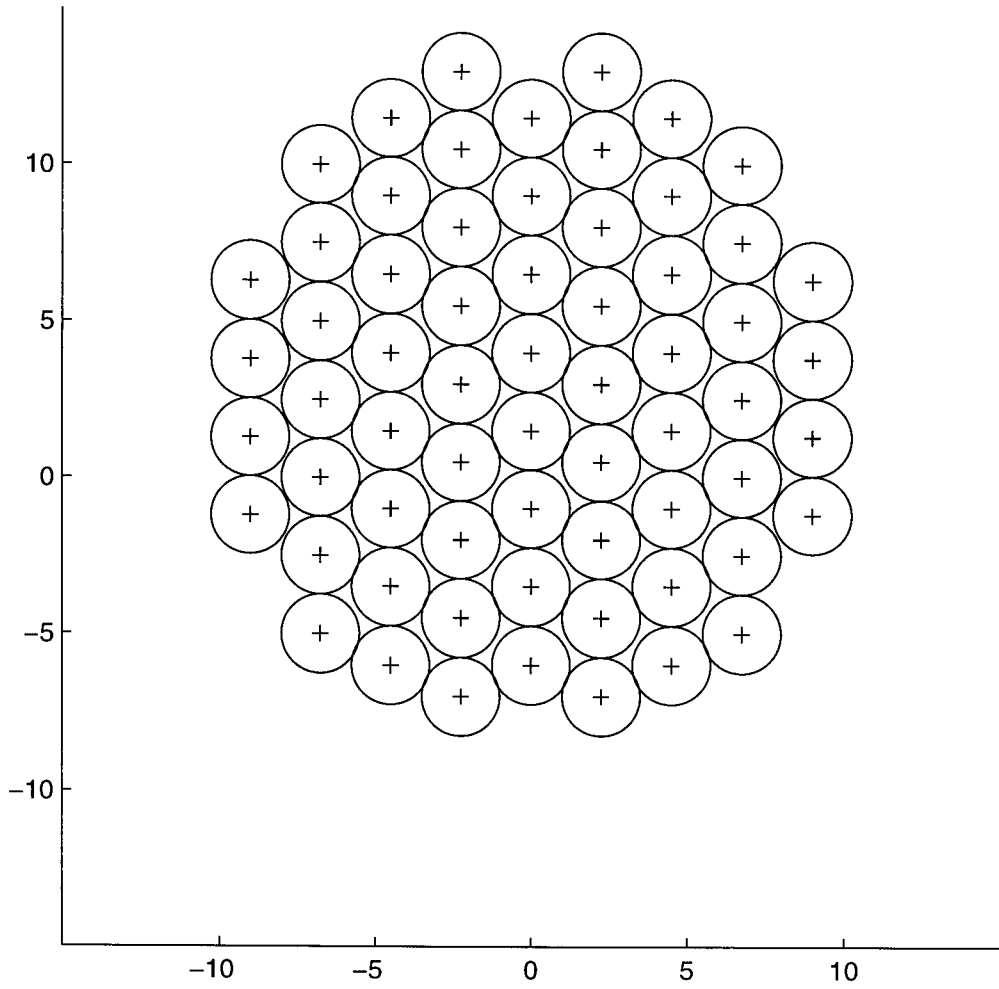


Figure 2

Problem 9.33

In the textbook, the capacity of the NEXT-dominated channel is derived as

$$C = \frac{1}{2} \int_{F_a} \log_2 \left(1 + \frac{|H_{\text{NEXT}}(f)|^2}{|H(f)|^2} \right) df$$

where F_a is the set of positive and negative frequencies for which $S_x(f) > 0$, where $S_x(f)$ is the power spectral density of the transmitted signal.

For the NEXT-dominated channel described in the question, the capacity is

$$\begin{aligned} C &= \frac{1}{2} \int_{F_a} \log_2 \left(1 + \frac{\beta f^{3/2}}{\exp(-\alpha \sqrt{f})} \right) df \\ &= \frac{1}{2} \int_{F_a} \log_2 \left(1 + \frac{\beta f^{3/2}}{\exp\left(-\frac{klf^{1/2}}{l_o}\right)} \right) df \end{aligned}$$

where β , k , l and f_o are all constants pertaining to the transmission medium. This formula for capacity can only be evaluated numerically for prescribed values of these constants.

Problem 9.34

For $k=1$, Eq. (9.38) reduces to

$$10 \log_2(\text{SNR}) = 6 \log_2 N + C_1 \text{ dB} \quad (1)$$

Expressing Eq. (3.33) in decibels, we have

$$10 \log_2(\text{SNR}) = 6R + 10 \log_{10} \left(\frac{3P}{m_{\max}^2} \right) \quad (2)$$

The number of bits per sample R , is defined by

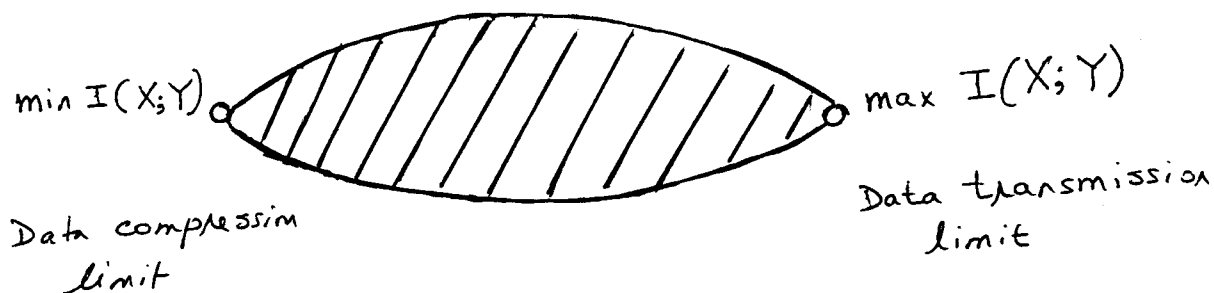
$$R = \log_2 N$$

We thus see that Eqs. (1) and (2) are equivalent, with

$$C_1 = 10 \log_{10} \left(\frac{3P}{m_{\max}^2} \right)$$

Problem 9.35

The rate distortion function and channel capacity theorem may be summed up diagrammatically as follows:



According to the rate distortion theory, the data compression limit set by minimizing the mutual information $I(X;Y)$ lies at the extreme left of this representation; here, the symbol Y represents the data compressed form of the source symbol X . On the other hand, according to the channel capacity theorem the data transmission limit is defined by maximizing the mutual information $I(X;Y)$ between the channel input X and channel output Y . This latter limit lies on the extreme right of the representation shown above.

1 Problem 9. 36

Matlab codes

```
% Computer Problem in Chapter 9
% Figure: The minimum achievable BER as a function of
% Eb/NO for several different code rates using binary signaling.
% This program calculates the Minimum required Eb/NO
% for BPSK signalling at unit power over AWGN channel
% given a rate and an allowed BER.
% Code is based on Brandon's C code.
% Ref: Brendan J. Frey, Graphical models for machine
% learning and digital communications, The MIT Press.
% Mathini Sellathurai

EbNo=double( [7.85168, 7.42122, 6.99319, 6.56785, 6.14714, 5.7329, 5.32711, ...
4.92926, 4.54106, 4.16568, 3.80312, 3.45317, 3.11902, 2.7981, 2.49337, 2.20617, ...
1.93251, 1.67587, 1.43313, 1.20671, 0.994633, 0.794801, 0.608808, 0.434862, ...
0.273476, 0.123322, -0.0148204, -0.144486, -0.266247, -0.374365, -0.474747, -0.5708, ...
-0.659038, -0.736526, -0.812523, -0.878333, -0.944802, -0.996262, -1.04468, ...
-1.10054, -1.14925, -1.18536, -1.22298, -1.24746, -1.27394, -1.31061, -1.34588, ...
-1.37178, -1.37904, -1.40388, -1.42553, -1.45221, -1.43447, -1.44392, -1.46129, ...
-1.45001, -1.50485, -1.50654, -1.50192, -1.45507, -1.60577, -1.52716, -1.54448, ...
-1.51713, -1.54378, -1.5684]);

rate= double([9.989372e-01, 9.980567e-01, 9.966180e-01, 9.945634e-01, ...
9.914587e-01, 9.868898e-01, 9.804353e-01, 9.722413e-01, 9.619767e-01, 9.490156e-01, ...
9.334680e-01, 9.155144e-01, 8.946454e-01, 8.715918e-01, 8.459731e-01, 8.178003e-01, ...
7.881055e-01, 7.565174e-01, 7.238745e-01, 6.900430e-01, 6.556226e-01, ...
6.211661e-01, 5.866480e-01, 5.525132e-01, 5.188620e-01, 4.860017e-01, 4.539652e-01, ...
4.232136e-01, 3.938277e-01, 3.653328e-01, 3.382965e-01, 3.129488e-01, 2.889799e-01, ...
2.661871e-01, 2.451079e-01, 2.251691e-01, 2.068837e-01, 1.894274e-01, ...
1.733225e-01, 1.588591e-01, 1.453627e-01, 1.326278e-01, 1.210507e-01, 1.101504e-01, ...
1.002778e-01, 9.150450e-02, 8.347174e-02, 7.598009e-02, 6.886473e-02, 6.266875e-02, ...
5.698847e-02, 5.188306e-02, 4.675437e-02, 4.239723e-02, 3.851637e-02, 3.476062e-02, ...
3.185243e-02, 2.883246e-02, 2.606097e-02, 2.332790e-02, 2.185325e-02, ...
1.941896e-02, 1.764122e-02, 1.586221e-02, 1.444108e-02, 1.314112e-02]);
N=66;

b=double([1e-5]); % Allowed BER
% Rate R (bits per channel usage)
r=double([1/32, 1/16,0.1,0.2,0.3,0.4,0.5, 0.6, 0.7, 0.8,0.85,0.95]);
```

```

le=zeros(1,length(r)); % initialize buffer for Eb/NO
for p=1:length(r)
    c = double(r(p)*(1.0+b*log(b)+(1.0-b)*log(1.0-b)/log(2.0)));
    i=N-1;
    % Minimum Eb/NO calculations
        while ( (i>=0) & (c>rate(i)) )
            i=i-1;
        end
    i=i+1;

if ( (i>0) | (i<N) )
    e =double( EbNo(i)+(EbNo(i-1)-EbNo(i))*(c-rate(i))/(rate(i-1)-rate(i)));
    le(p)=10*log10((10^(e/10))*c/r(p));
    display(le)
else
    display('values out of range')
end
end
plot(10*log10(r),le,'-')
xlabel('Rate (dB)')
ylabel('Minimum E_b/N_0 (dB)')
axis([10*log10(1/32), 0, -2 4])

```

```

% Computer Experiment in Chapter 9
% Program to create the figure for the minimum
% Eb/NO needed for error-free communication
% with a rate R code, over an AWGN channel
% using binary signaling
% This program calculates the Minimum required Eb/NO
% for BPSK signalling at unit power over AWGN channel
% given a rate and an allowed BER.
% Code is based on Brandon's C code.
% Ref: Brendan J. Frey, Graphical models for machine
% learning and digital communications, The MIT Press.
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EbNo= double([7.85168, 7.42122, 6.99319, 6.56785, 6.14714, 5.7329, 5.32711, ...
4.92926, 4.54106, 4.16568, 3.80312, 3.45317, 3.11902, 2.7981, 2.49337, 2.20617, ...
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-0.659038, -0.736526, -0.812523, -0.878333, -0.944802, -0.996262, -1.04468, ...
-1.10054, -1.14925, -1.18536, -1.22298, -1.24746, -1.27394, -1.31061, -1.34588, ...
-1.37178, -1.37904, -1.40388, -1.42553, -1.45221, -1.43447, -1.44392, -1.46129, ...
-1.45001, -1.50485, -1.50654, -1.50192, -1.45507, -1.60577, -1.52716, -1.54448, ...
-1.51713, -1.54378, -1.5684]);

rate=double( [9.989372e-01, 9.980567e-01, 9.966180e-01, 9.945634e-01, ...
9.914587e-01, 9.868898e-01, 9.804353e-01, 9.722413e-01, 9.619767e-01, 9.490156e-01, ...
9.334680e-01, 9.155144e-01, 8.946454e-01, 8.715918e-01, 8.459731e-01, 8.178003e-01, ...
7.881055e-01, 7.565174e-01, 7.238745e-01, 6.900430e-01, 6.556226e-01, ...
6.211661e-01, 5.866480e-01, 5.525132e-01, 5.188620e-01, 4.860017e-01, 4.539652e-01, ...
4.232136e-01, 3.938277e-01, 3.653328e-01, 3.382965e-01, 3.129488e-01, 2.889799e-01, ...
2.661871e-01, 2.451079e-01, 2.251691e-01, 2.068837e-01, 1.894274e-01, ...
1.733225e-01, 1.588591e-01, 1.453627e-01, 1.326278e-01, 1.210507e-01, 1.101504e-01, ...
1.002778e-01, 9.150450e-02, 8.347174e-02, 7.598009e-02, 6.886473e-02, 6.266875e-02, ...
5.698847e-02, 5.188306e-02, 4.675437e-02, 4.239723e-02, 3.851637e-02, 3.476062e-02, ...
3.185243e-02, 2.883246e-02, 2.606097e-02, 2.332790e-02, 2.185325e-02, ...
1.941896e-02, 1.764122e-02, 1.586221e-02, 1.444108e-02, 1.314112e-02]);
N=66;

b=double(0.5:-1e-5:1e-5); % Allowed BER
rrr=double([0.99,1/2,1/3,1/4,1/5,1/8]); % Rate R(bits/channel usage)
le=zeros(1,length(b));

for rr=1:length(rrr)
    r=rrr(rr);
    for p=1:length(b)
        c = double(r*(1.0+b(p))*log(b(p))+(1.0-b(p))*log(1.0-b(p))/log(2.0));
    end
end

```

```

        i=N-1;

        while ( (i>=0) & (c>rate(i)) )
            i=i-1;
        end
        i=i+1;

    if ( (i>0) | (i<N) )
        e = double(EbNo(i)+(EbNo(i-1)-EbNo(i))*(c-rate(i))/(rate(i-1)-rate(i)));
        le(p)=10*log10((10^(e/10))*c/r);

        else
            display('values out of range')
        end
    end
    plot(le,10*log10(b),'-')
end
xlabel('E_b/N_0 (dB)')
ylabel('Minimum BER')
axis([-2 1 -50 -10])

```

Answer to Problem 9.36

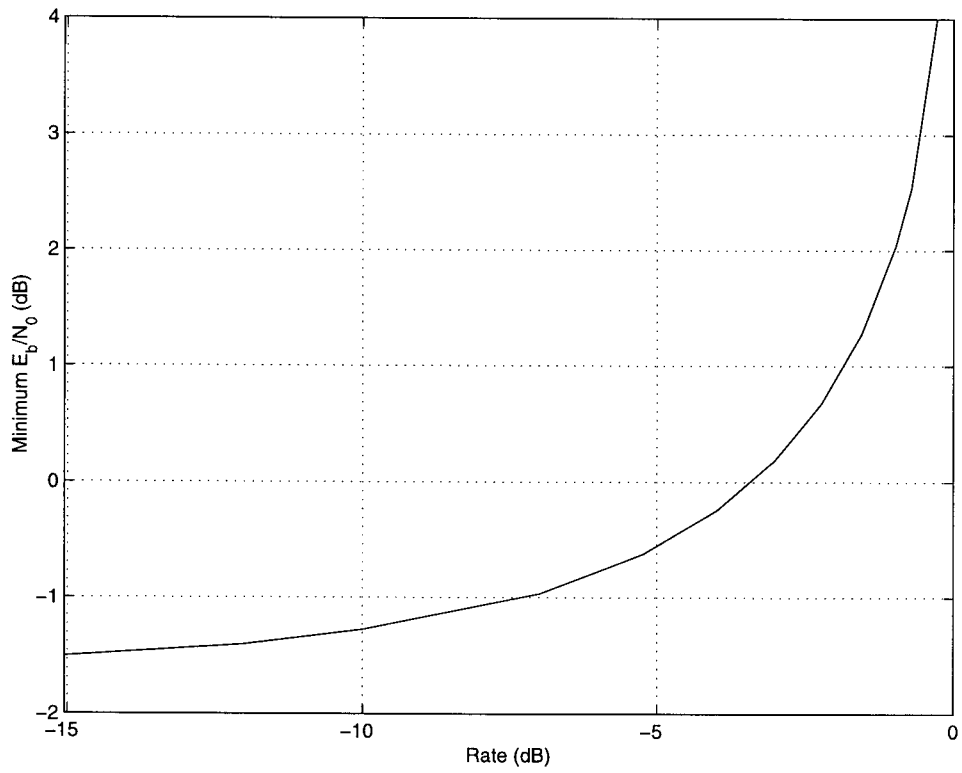


Figure 1: The minimum E_b/N_0 needed for error-free communication with a rate R code, over an AWGN channel using binary signaling

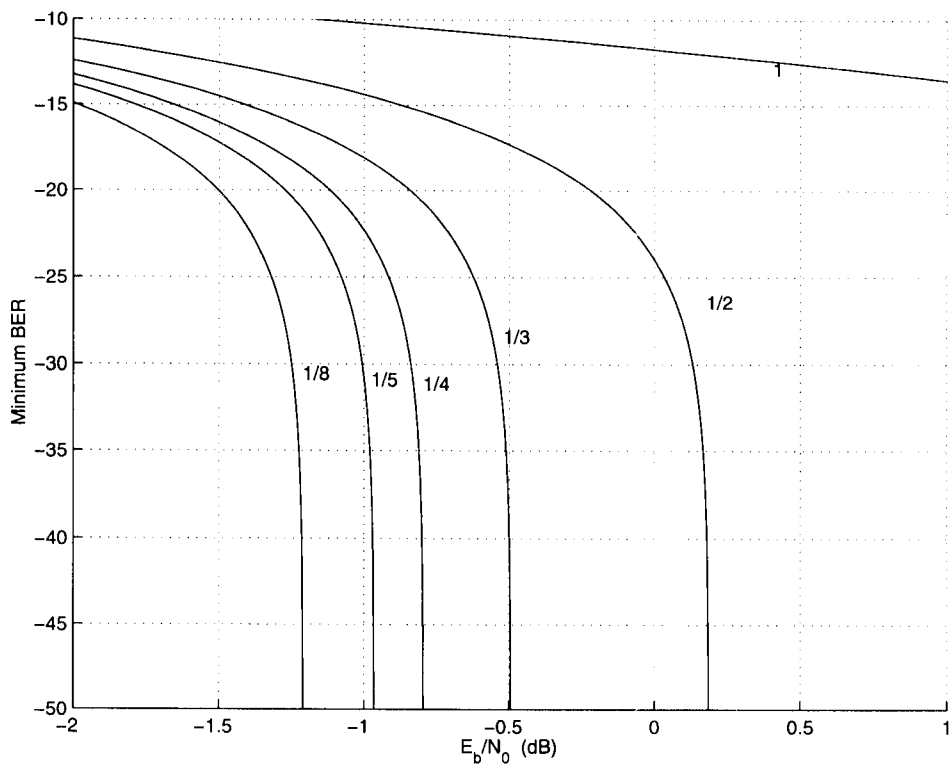


Figure 2: The minimum achievable BER as a function of E_b/N_0 for several different code rates using binary signaling