Problem 8.14 A discrete-time random process $\{Y_n: n = \ldots,-1,0,1,2, \ldots\}$ is defined by

$$Y_n = \alpha_0 Z_n + \alpha_1 Z_{n-1}$$

where $\{Z_n\}$ is a random process with autocorrelation function $R_Z(n) = \sigma^2 \delta(n)$. What is the autocorrelation function $R_Y(n,m) = \mathbf{E}[Y_n Y_m]$? Is the process $\{Y_n\}$ wide-sense stationary?

Solution

We implicitly assume that Z_n is stationary and has a constant mean μ_Z. Then the mean of Y_n is given by

$$\begin{aligned}\mathbf{E}[Y_n] &= \alpha_0 \mathbf{E}[Z_n] + \alpha_1 \mathbf{E}[Z_{n-1}] \\ &= (\alpha_0 + \alpha_1)\mu_Z\end{aligned}$$

The autocorrelation of Y is given by

$$\begin{aligned}\mathbf{E}[Y_n Y_m] &= \mathbf{E}[(\alpha_0 Z_n + \alpha_0 Z_{n-1})(\alpha_0 Z_m + \alpha_1 Z_{m-1})] \\ &= \alpha_0^2 \mathbf{E}[Z_n Z_m] + \alpha_1 \alpha_0 \mathbf{E}[Z_n Z_{m-1}] + \alpha_0 \alpha_1 \mathbf{E}[Z_{n-1} Z_m] + \alpha_1^2 \mathbf{E}[Z_{m-1} Z_{n-1}] \\ &= \alpha_0^2 \sigma^2 \delta(n-m) + \alpha_1 \alpha_0 \sigma^2 \delta(m-1-n) + \alpha_0 \alpha_1 \sigma^2 \delta(n-1-m) + \alpha_1^2 \delta(m-1-(n-1)) \\ &= (\alpha_0^2 + \alpha_1^2)\sigma^2 \delta(n-m) + \alpha_0 \alpha_1 \sigma^2 [\delta(n-m-1) + \delta(m-n-1)]\end{aligned}$$

Since the autocorrelation only depends on the time difference *n-m*, the process is wide-sense stationary with

$$R_Y(n) = (\alpha_0^2 + \alpha_1^2)\sigma^2 \delta(n) + \alpha_0 \alpha_1 \sigma^2 (\delta(n-1) + \delta(n+1))$$