Problem 8.15 For the discrete-time process of Problem 8.14, use the discrete Fourier transform to approximate the corresponding spectrum. That is,

$$S_Y(k) = \sum_{n=0}^{N-1} R_Y(n) W^{kn}$$

If the sampling in the time domain is at n/T_s where $n = 0, 1, 2, \ldots, N$-1. What frequency does k correspond to?

Solution

Let $\beta_0 = \left(\alpha_0^2 + \alpha_1^2\right)\sigma^2$ and $\beta_1 = \alpha_0\alpha_1\sigma^2$. Then

$$\begin{aligned}
S_Y(k) &= \sum_{n=0}^{N-1} \left[\beta_0\delta(n) + \beta_1\left(\delta(n-1) + \delta(n+1)\right)\right] W^{kn} \\
&= \beta_0 W^0 + \beta_1\left(W^{-k} + W^{+k}\right) \\
&= \beta_0 + \beta_1\left(e^{\frac{-j2\pi k}{N}} + e^{\frac{+j2\pi k}{N}}\right) \\
&= \beta_0 + 2\beta_1 \cos\left(\frac{2\pi k}{N}\right)
\end{aligned}$$

The term $S_Y(k)$ corresponds to frequency $\dfrac{kf_s}{N}$ where $f_S = \dfrac{1}{T_S}$.