Problem 8.22 Consider a random variable X defined by the double-exponential density where a and b are constants.

$$f_X(x) = a\exp(-b|x|) \qquad -\infty < x < \infty$$

(a) Determine the relationship between a and b so that $f_X(x)$ is a probability density function.
(b) Determine the corresponding distribution function $F_X(x)$.
(c) Find the probability that the random variable X lies between 1 and 2.

Solution

(a)

$$\int_{-\infty}^{\infty} f_X(x)dx = 1 \quad \Rightarrow 2\int_0^{\infty} a\exp(-bx)dx = 1$$

$$-\frac{2a}{b}\exp(-bx)\Big|_0^{\infty} = 1$$

$$\Rightarrow \frac{2a}{b} = 1 \;\; or \quad b = 2a$$

(b)

$$F_X(x) = \int_{-\infty}^{x} a\exp(-b|s|)ds$$

$$= \begin{cases} -\dfrac{a}{b}\exp(-b(-s))\Big|_{-\infty}^{x} & -\infty < x < 0 \\ \dfrac{1}{2} + -\dfrac{a}{b}\exp(-bs)\Big|_0^{x} & 0 < x < \infty \end{cases}$$

$$= \begin{cases} \dfrac{a}{b}\exp(bs) & -\infty < x < 0 \\ \dfrac{1}{2} + \dfrac{a}{b} - \dfrac{a}{b}\exp(-bs) & 0 \le x < \infty \end{cases}$$

$$= \begin{cases} \dfrac{1}{2}\exp(bx) & -\infty < x < 0 \\ 1 - \dfrac{1}{2}\exp(-bx) & 0 \le x < \infty \end{cases}$$

(c) The probability that $1 \le X \le 2$ is

$$F_X(2) - F_X(1) = \frac{1}{2}\left[\exp(-b) - \exp(-2b)\right]$$