Problem 8.24 A random variable *R* is Rayleigh distributed with its probability density function given by

$$f_R(r) = \begin{cases} \dfrac{r}{b}\exp\left(-r^2/2b\right) & 0 \le r < \infty \\ 0 & otherwise \end{cases}$$

(a) Determine the corresponding distribution function

(b) Show that the mean of *R* is equal to $\sqrt{b\pi/2}$

(c) What is the mean-square value of *R*?

(d) What is the variance of *R*?

Solution

(a) The distribution of *R* is

$$\begin{aligned} F_R(r) &= \int_0^r f_R(s)ds \\ &= \int_0^r \frac{s}{b}\exp\left(-\frac{s^2}{2b}\right)ds \\ &= -\exp\left(-\frac{s^2}{2b}\right)\Bigg|_0^r \\ &= 1-\exp\left(-r^2/2b\right) \end{aligned}$$

(b) The mean value of *R* is

$$\begin{aligned} \mathbf{E}[R] &= \int_0^\infty s f_R(s)ds \\ &= \int_0^\infty \frac{s^2}{b}\exp\left(\frac{-s^2}{2b}\right)ds \\ &= \frac{1}{b}\sqrt{2\pi b}\left[\frac{1}{\sqrt{2\pi b}}\int_0^\infty s^2\exp\left(\frac{-s^2}{2b}\right)ds\right] \end{aligned}$$

The bracketed expression is equivalent to the evaluation of the half of the variance of a zero-mean Gaussian random variable which we know is *b* in this case, so

$$\mathbf{E}[R] = \frac{\sqrt{2\pi b}}{b}\frac{1}{2}(b) = \sqrt{\frac{\pi b}{2}}$$

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Problem 8.24 continued

(c) The second moment of R is

$$
\begin{aligned}
\mathbf{E}\left[R^2\right] &= \int_0^\infty s^2 f_R(s)ds \\
&= \int_0^\infty \frac{s^3}{b}\exp\left(\frac{-s^2}{2b}\right)ds \\
&= s^2 F_R(s)\Big|_0^\infty - \int_0^\infty 2s \;\; F_R(s)\, ds \\
&= s^2 F_R(s)\Big|_0^\infty - \int_0^\infty 2s\left(1-\exp\left(\frac{s^2}{2b}\right)\right)ds \\
&= s^2\left(F_R(s)-1\right)\Big|_0^\infty + 2b\int_0^\infty f_R(s)ds \\
&= 2b
\end{aligned}
$$

(d) The variance of R is

$$
\begin{aligned}
\mathrm{Var}(R) &= \mathbf{E}\left[R^2\right] - \left(\mathbf{E}[R]\right)^2 \\
&= 2b - \left(\sqrt{b\pi}\big/2\right)^2 \\
&= b\left(2 - \pi\big/2\right)
\end{aligned}
$$