Problem 8.25 Consider a uniformly distributed random variable Z, defined by

$$f_{Z}(z) = \begin{cases} \frac{1}{2\pi}, & 0 \le z < 2\pi \\ 0, & \text{otherwise} \end{cases}$$

The two random variables *X* and *Y* are related to *Z* by X = sin(Z) and Y = cos(Z).

(a) Determine the probability density functions of X and Y.

(b) Show that X and Y are uncorrelated random variables.

(c) Are X and Y statistically independent? Why?

Solution

(a) The distribution function of *X* is formally given by

$$F_{X}(x) = \begin{cases} 0 & x \le -1 \\ \mathbf{P}[-1 \le X \le x] & -1 < x < 1 \\ 1 & x \ge 1 \end{cases}$$

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Analogous to Example 8.8, we have

$$\mathbf{P}(-1 \le X \le x) = \begin{cases} \mathbf{P}[\pi - \sin^{-1}(x) \le Z \le 2\pi + \sin^{-1}(x)] & -1 \le x \le 0\\ \frac{1}{2} + \mathbf{P}[0 \le Z \le \sin^{-1}(x)] + \mathbf{P}[\pi - \sin^{-1}(x) \le Z \le \pi] & 0 \le x \le 1 \end{cases}$$

$$=\begin{cases} \frac{\pi + 2\sin^{-1}(x)}{2\pi} & -1 \le x \le 0\\ \frac{1}{2} + \frac{2\sin^{-1}(x)}{2\pi} & 0 \le x \le 1 \end{cases}$$

$$=\frac{1}{2} + \frac{\sin^{-1}(x)}{\pi} \qquad -1 \le x \le 1$$

where the second line follows from the fact that the probability for a uniform random variable is proportional to the length of the interval. The distribution of Y follows from a similar argument (see Example 8.8).

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Problem 8.25 continued

(b) To show X and Y are uncorrelated, consider

$$\mathbf{E}[XY] = \mathbf{E}[\sin(Z)\cos(Z)]$$
$$= \mathbf{E}\left[\frac{\sin(2Z)}{2}\right]$$
$$= \frac{1}{4\pi} \int_{0}^{2\pi} \sin(2z) dz$$
$$= -\frac{1}{8\pi} \cos(2z) \Big|_{0}^{2\pi} = 0$$

Thus *X* and *Y* are uncorrelated.

(c) The random variables X and Y are not statistically independent since

$$\Pr[X|Y] \neq \Pr[X]$$

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