Problem 8.25 Consider a uniformly distributed random variable Z, defined by

$$f_Z(z) = \begin{cases} \dfrac{1}{2\pi}, & 0 \le z < 2\pi \\ 0, & \text{otherwise} \end{cases}$$

The two random variables *X* and *Y* are related to *Z* by *X* = sin(*Z*) and *Y* = cos(*Z*).

(a) Determine the probability density functions of *X* and *Y*.
(b) Show that *X* and *Y* are uncorrelated random variables.
(c) Are *X* and *Y* statistically independent? Why?

Solution

(a) The distribution function of *X* is formally given by

$$F_X(x) = \begin{cases} 0 & x \le -1 \\ \mathbf{P}[-1 \le X \le x] & -1 < x < 1 \\ 1 & x \ge 1 \end{cases}$$

Analogous to Example 8.8, we have

$$\mathbf{P}(-1 \le X \le x) = \begin{cases} \mathbf{P}\left[\pi - \sin^{-1}(x) \le Z \le 2\pi + \sin^{-1}(x)\right] & -1 \le x \le 0 \\ \dfrac{1}{2} + \mathbf{P}\left[0 \le Z \le \sin^{-1}(x)\right] + \mathbf{P}\left[\pi - \sin^{-1}(x) \le Z \le \pi\right] & 0 \le x \le 1 \end{cases}$$

$$= \begin{cases} \dfrac{\pi + 2\sin^{-1}(x)}{2\pi} & -1 \le x \le 0 \\ \dfrac{1}{2} + \dfrac{2\sin^{-1}(x)}{2\pi} & 0 \le x \le 1 \end{cases}$$

$$= \frac{1}{2} + \frac{\sin^{-1}(x)}{\pi} \qquad -1 \le x \le 1$$

where the second line follows from the fact that the probability for a uniform random variable is proportional to the length of the interval. The distribution of *Y* follows from a similar argument (see Example 8.8).

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Problem 8.25 continued

(b) To show *X* and *Y* are uncorrelated, consider

$$\begin{aligned}\mathbf{E}[XY] &= \mathbf{E}[\sin(Z)\cos(Z)] \\ &= \mathbf{E}\left[\frac{\sin(2Z)}{2}\right] \\ &= \frac{1}{4\pi}\int_0^{2\pi}\sin(2z)dz \\ &= -\frac{1}{8\pi}\cos(2z)\Big|_0^{2\pi} = 0\end{aligned}$$

Thus *X* and *Y* are uncorrelated.

(c) The random variables *X* and *Y* are not statistically independent since

$$\Pr[X|Y] \neq \Pr[X]$$