Problem 8.26 A Gaussian random variable has zero mean and a standard deviation of 10 V. A constant voltage of 5 V is added to this random variable.
(a) Determine the probability that a measurement of this composite signal yields a positive value.
(b) Determine the probability that the arithmetic mean of two independent measurements of this signal is positive.

Solution

(a) Let Z represent the initial Gaussian random variable and Y the composite random variable. Then

$$Y = 5 + Z$$

and the density function of Y is given by

$$f_y(y) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-(y-\mu)^2/2\sigma^2\right\}$$

where μ corresponds to a mean of 5V and σ corresponds to a standard deviation of 10V. The probability that Y is positive is

$$\begin{aligned}
P[Y>0] &= \frac{1}{\sqrt{2\pi}\sigma}\int_0^\infty \exp\left\{-(y-\mu)/2\sigma^2\right\}dy \\
&= \frac{1}{\sqrt{2\pi}}\int_{\frac{-\mu}{\sigma}}^\infty \exp\left(-s^2/2\right)ds \\
&= Q\left(\frac{-\mu}{\sigma}\right)
\end{aligned}$$

where, in the second line, we have made the substitution

$$s = \frac{y-\mu}{\sigma}$$

Making the substitutions for μ and σ, we have **P**[Y>0] = Q(- ½). We note that in Fig. 8.11, the values of $Q(x)$ are not shown for negative x; to obtain a numerical result, we use the fact that $Q(-x) = 1 - Q(x)$. Consequently, $Q(-½) = 1 - 0.3 = 0.7$.

Continued on next slide

Problem 8.26 continued

(b) Let W represent the arithmetic mean of two measurements Y_1 and Y_2, that is

$$W = \frac{Y_1 + Y_2}{2}$$

It follows that W is a Gaussian random variable with $\mathbf{E}[W] = \mathbf{E}[Y] = 5$. The variance of W is given by

$$\begin{aligned}
\mathrm{Var}(W) &= \mathbf{E}\left[(W - \mathbf{E}(W))^2\right] \\
&= \mathbf{E}\left[\left(\frac{Y_1 + Y_2}{2} - \frac{\mathbf{E}(Y_1) + \mathbf{E}(Y_2)}{2}\right)^2\right] \\
&= \frac{1}{4}\left(\mathbf{E}\left[(Y_1 - \mathbf{E}(Y_1))^2 + (Y_2 - \mathbf{E}(Y_2))^2 + 2(Y_1 - \mathbf{E}(Y_1))(Y_2 - \mathbf{E}(Y_2))\right]\right)
\end{aligned}$$

The first two terms correspond to the variance of Y. The third term is zero because the measurements are independent. Making these substitutions, the variance of W reduces to

$$\mathrm{Var}[W] = \sigma^2 / 2$$

Using the result of part (a), we then have

$$\mathbf{P}[W > 0] = Q\left(\frac{-\mu}{\left(\sigma/\sqrt{2}\right)}\right) = Q\left(-\frac{1}{\sqrt{2}}\right)$$