Problem 8.30 Prove the following two properties of the autocorrelation function $R_X(\tau)$ of a random process $X(t)$:

(a) If $X(t)$ contains a dc component equal to A, then $R_X(\tau)$ contains a constant component equal to A^2.

(b) If $X(t)$ contains a sinusoidal component, then $R_X(\tau)$ also contains a sinusoidal component of the same frequency.

Solution

(a) Let $Y(t) = X(t) - A$ and $Y(t)$ is a random process with zero *dc* component. Then

$$\mathbf{E}[X(t)] = A$$

and

$$\begin{aligned} R_X(\tau) &= \mathbf{E}[X(t)X(t+\tau)] \\ &= \mathbf{E}[((X(t)-A)+A)((X(t+\tau)-A)+A)] \\ &= \mathbf{E}[(X(t)-A)(X(t+\tau)-A)] + \mathbf{E}(X(t+\tau)-A)A + \mathbf{E}(X(t)A) + A^2 \\ &= R_Y(\tau) + 0 + 0 + A^2 \end{aligned}$$

And thus $R_X(\tau)$ has a constant component A^2.

(b) Let $X(t) = Y(t) + A\sin(2\pi f_c t)$ where $Y(t)$ does not contain a sinusoidal component of frequency f_c.

$$\begin{aligned} R_X(\tau) &= \mathbf{E}[X(t)X(t+\tau)] \\ &= \mathbf{E}[(Y(t) + A\sin(2\pi f_c t))(Y(t+\tau) + A\sin(2\pi f_c t)) + \mathbf{E}[A^2 \sin(2\pi f_c t)\sin(2\pi f_c (t+\tau))]] \\ &= R_Y(\tau) + \ldots + \frac{A^2}{2}[\cos 2\pi f_c t + \cos 2\pi f_c (2t+\tau) + \theta] \\ &= R_Y(\tau) + \frac{A^2}{2}\cos(2\pi f_c \tau) \end{aligned}$$

And thus $R_X(\tau)$ has a sinusoidal component at f_c.