Problem 8.31 A discrete-time random process is defined by

$$Y_n = \alpha Y_{n-1} + W_n \qquad n = \ldots, -1, 0, +1, \ldots$$

where the zero-mean random process W_n is stationary with autocorrelation function $R_W(k) = \sigma^2\delta(k)$. What is the autocorrelation function $R_y(k)$ of Y_n? Is Y_n a wide-sense stationary process? Justify your answer.

Solution

We partially address the question of whether Y_n is wide-sense stationary (WSS) first by noting that

$$\begin{aligned}\mathbf{E}[Y_n] &= \mathbf{E}[\alpha Y_{n-1} + W_n] \\ &= \alpha\mathbf{E}[Y_{n-1}] + \mathbf{E}[W_n] \\ &= \alpha\mathbf{E}[Y_{n-1}]\end{aligned}$$

since $\mathbf{E}[W_n] = 0$. To be WSS, the mean of the process must be constant and consequently, we must have that $\mathbf{E}[Y_n] = 0$ for all n, to satisfy the above relationship.

We consider the autocorrelation of Y_n in steps. First note that $R_Y(0)$ is given by

$$R_Y(0) = \mathbf{E}[Y_n Y_n] = \mathbf{E}[Y_n^2]$$

and that $R_Y(1)$ is

$$\begin{aligned}R_Y(1) &= \mathbf{E}[Y_n Y_{n+1}] \\ &= \mathbf{E}[Y_n(\alpha Y_n + W_n)] \\ &= \alpha\mathbf{E}[Y_n^2] + \mathbf{E}[Y_n W_n]\end{aligned}$$

Although not explicitly stated in the problem, we assume that W_n is independent of Y_n, thus $\mathbf{E}[Y_n W_n] = \mathbf{E}[Y_n]\mathbf{E}[W_n] = 0$, and so

$$R_Y(1) = \alpha R_Y(0)$$

We prove the result for general positive k by assuming $R_Y(k) = \alpha^k R_Y(0)$ and then noting that

$$\begin{aligned}R_Y(k+1) &= \mathbf{E}[Y_n Y_{n+k+1}] \\ &= \mathbf{E}[Y_n(\alpha Y_{n+k} + W_{n+k})] \\ &= \alpha\mathbf{E}[Y_n Y_{n+k}] + \mathbf{E}[Y_n W_{n+k}]\end{aligned}$$

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Problem 8.31 continued

To evaluate this last expression, we note that, since

$$\begin{aligned}Y_n &= \alpha Y_{n-1} + W_n \\ &= \alpha^2 Y_{n-2} + \alpha W_{n-1} + W_n \\ &= \alpha^3 Y_{n-3} + \alpha^2 W_{n-2} + \alpha W_{n-1} + W_n \\ &= \cdots\end{aligned}$$

we see that Y_n only depends on W_k for $k \le n$. Thus $\mathbf{E}[Y_n W_{n+k}] = 0$. Thus, for positive k, we have

$$\begin{aligned}R_Y(k+1) &= \alpha R_Y(k) \\ &= \alpha^{k+1} R_Y(0)\end{aligned}$$

Using a similar argument, a corresponding result can be shown for negative k. Combining the results, we have

$$R_Y(k) = \alpha^{|k|} R_Y(0)$$

Since the autocorrelation only depends on the time difference k, and the process is wide-sense stationary.