Problem 8.31 A discrete-time random process is defined by

$$Y_n = \alpha Y_{n-1} + W_n$$
  $n = ..., -1, 0, +1, ...$ 

where the zero-mean random process  $W_n$  is stationary with autocorrelation function  $R_w(k) = \sigma^2 \delta(k)$ . What is the autocorrelation function  $R_v(k)$  of  $Y_n$ ? Is  $Y_n$  a wide-sense stationary process? Justify your answer.

## **Solution**

We partially address the question of whether  $Y_n$  is wide-sense stationary (WSS) first by noting that

$$\begin{split} \mathbf{E}[Y_n] &= \mathbf{E}[\alpha Y_{n-1} + W_n] \\ &= \alpha \mathbf{E}[Y_{n-1}] + \mathbf{E}[W_n] \\ &= \alpha \mathbf{E}[Y_{n-1}] \end{split}$$

since  $\mathbf{E}[W_n] = 0$ . To be WSS, the mean of the process must be constant and consequently, we must have that  $\mathbf{E}[Y_n] = 0$  for all *n*, to satisfy the above relationship.

We consider the autocorrelation of  $Y_n$  in steps. First note that  $R_Y(0)$  is given by

$$R_{Y}(0) = \mathbf{E}[Y_{n}Y_{n}] = \mathbf{E}[Y_{n}^{2}]$$

and that  $R_{Y}(1)$  is

$$R_{Y}(1) = \mathbf{E}[Y_{n}Y_{n+1}]$$
$$= \mathbf{E}[Y_{n}(\alpha Y_{n} + W_{n})]$$
$$= \alpha \mathbf{E}[Y_{n}^{2}] + \mathbf{E}[Y_{n}W_{n}]$$

Although not explicitly stated in the problem, we assume that  $W_n$  is independent of  $Y_n$ , thus  $\mathbf{E}[Y_nW_n] = \mathbf{E}[Y_n]\mathbf{E}[W_n] = 0$ , and so

$$R_{\rm y}(1) = \alpha R_{\rm y}(0)$$

We prove the result for general positive k by assuming  $R_Y(k) = \alpha^k R_Y(0)$  and then noting that

$$R_{Y}(k+1) = \mathbf{E}[Y_{n}Y_{n+k+1}]$$
$$= \mathbf{E}[Y_{n}(\alpha Y_{n+k} + W_{n+k})]$$
$$= \alpha \mathbf{E}[Y_{n}Y_{n+k}] + \mathbf{E}[Y_{n}W_{n+k}]$$

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## **Problem 8.31 continued**

To evaluate this last expression, we note that, since

$$Y_{n} = \alpha Y_{n-1} + W_{n}$$
  
=  $\alpha^{2} Y_{n-2} + \alpha W_{n-1} + W_{n}$   
=  $\alpha^{3} Y_{n-3} + \alpha^{2} W_{n-2} + \alpha W_{n-1} + W_{n}$   
= ...

we see that  $Y_n$  only depends on  $W_k$  for  $k \le n$ . Thus  $\mathbf{E}[Y_n W_{n+k}] = 0$ . Thus, for positive k, we have

$$R_{Y}(k+1) = \alpha R_{Y}(k)$$
$$= \alpha^{k+1} R_{Y}(0)$$

Using a similar argument, a corresponding result can be shown for negative k. Combining the results, we have

$$R_{\rm Y}(k) = \alpha^{|k|} R_{\rm Y}(0)$$

Since the autocorrelation only depends on the time difference k, and the process is widesense stationary.

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