Problem 8.33. A random pulse has amplitude A and duration T but starts at an arbitrary time t_0. That is, the random process is defined as

$$X(t) = A\text{rect}(t + t_0)$$

where rect(t) is defined in Section 2.9. The random variable t_0 is assumed to be uniformly distributed over [0,T] with density

$$f_{t_0}(s) = \begin{cases} \dfrac{1}{T} & 0 \le s \le T \\ 0 & \text{otherwise} \end{cases}$$

(a) What is the autocorrelation function of the random process $X(t)$?
(b) What is the spectrum of the random process $X(t)$?

Solution

First note that the process $X(t)$ is not stationary. This may be demonstrated by computing the mean of $X(t)$ for which we use the fact that

$$f_X(x) = \int_{-\infty}^{\infty} f_X(x \mid s) f_{t_0}(s) ds$$

combined with the fact that

$$\begin{aligned} \mathbf{E}\left[X(t) \mid t_0\right] &= \int_{-\infty}^{\infty} x f_X(x \mid t_0) dx \\ &= \begin{cases} A & t_0 \le t \le t_0 + T \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Consequently, we have

$$\begin{aligned} \mathbf{E}\left[X(t)\right] &= \int_{-\infty}^{\infty} \mathbf{E}[X(t) \mid s] f_{t_0}(s) ds \\ &= \begin{cases} 0 & t < 0 \\ At/T & 0 \le t \le T \\ A(2 - t/T) & T < t \le 2T \\ 0 & t > 2T \end{cases} \end{aligned}$$

Thus the mean of the process is dependent on t, and the process is nonstationary.

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Problem 8.33 continued

We take a similar approach to compute the autocorrelation function. First we break the situation into a number of cases:

i) For any $t < 0$, $s < 0$, $t > 2T$, or $s > 2T$, we have that

$$\mathbf{E}[X(t)X(s)] = 0$$

ii) For $0 \le t < s \le 2T$, we first assume t_0 is known

$$\mathbf{E}[X(t)X(s) \mid t_0] = \begin{cases} A^2 & t > t_0,\ \ s < t_0 + T,\ \ 0 < t_0 < T \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} A^2 & \max(s - T, 0) < t_0 < \min(t, T) \\ 0 & \text{otherwise} \end{cases}$$

Evaluating the unconditional expectation, we have

$$\mathbf{E}[X(t)X(s)] = \int_{-\infty}^{\infty} \mathbf{E}[X(t)X(s) \mid w] f_{t_0}(w)dw$$
$$= \int_{\max(0,s-T)}^{\min(t,T)} A^2 \left(\frac{1}{T}\right) dw$$
$$= \frac{A^2}{T} \max\{\{\min(t,T) - \max(0, s-T)\}, 0\}$$

where the second maximum takes care of the case where the lower limit on the integral is greater than the upper limit.

iii) For $0 \le s < t \le 2T$, we use a similar argument to obtain

$$\mathbf{E}[X(t)X(s) \mid t_0] = \begin{cases} A^2 & \max(t - T, 0) < t_0 < \min(s, T) \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathbf{E}[X(t)X(s)] = \frac{A^2}{T} \max\{\{\min(s,T) - \max(0, t-T)\}, 0\}$$

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Problem 8.33 continued

Combining all of these results we have the autocorrelation is given by

$$\mathbf{E}\left[X(t)X(s)\right] = \begin{cases} \dfrac{A^2}{T}\max\{\{\min(t,T) - \max(0, s-T)\}, 0\} & 0 \le t < s \le 2T \\ \dfrac{A^2}{T}\max\{\{\min(s,T) - \max(0, t-T)\}, 0\} & 0 \le s < t \le 2T \\ 0 & \text{otherwise} \end{cases}$$

This result depends upon both *t* and *s*, not just t-s, as one would expect for a non-stationary process.