Problem 8.34 Given that a stationary random process $X(t)$ has an autocorrelation function $R_X(\tau)$ and a power spectral density $S_X(f)$, show that:

(a) The autocorrelation function of $dX(t)/dt$, the first derivative of $X(t)$ is equal to the negative of the second derivative of $R_X(\tau)$.

(b) The power spectral density of $dX(t)/dt$ is equal to $4\pi^2 f^2 S_X(f)$.

Hint: Use the results of Problem 2.24.

Solution

(a) Let $Y(t) = \frac{dX}{dt}(t)$, and from the Wiener-Khintchine relations, we know the autocorrelation of $Y(t)$ is the inverse Fourier transform of the power spectral density of Y. Using the results of part (b),

$$\begin{aligned} R_Y(f) &= \mathbf{F}^{-1}\left[S_Y(f)\right] \\ &= \mathbf{F}^{-1}\left[4\pi^2 f^2 S_X(f)\right] \\ &= -\mathbf{F}^{-1}\left[(j2\pi f)^2 S_X(f)\right] \end{aligned}$$

from the differential properties of the Fourier transform, we know that differentiation in the time domain corresponds to multiplication by $j2\pi f$ in the frequency domain. Consequently, we conclude that

$$\begin{aligned} R_Y(f) &= -\mathbf{F}^{-1}\left[(j2\pi f)^2 S_X(f)\right] \\ &= -\frac{d^2}{d\tau^2} R_X(\tau) \end{aligned}$$

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Problem 8.34 continued

(b) Let $Y(t) = \frac{dX}{dt}(t)$, then the spectrum of *Y*(*t*) is given by (see Section 8.8)

$$S_Y(f) = \lim_{T\to\infty} \frac{1}{2T} \mathbf{E}\left[\left| H_T^Y(f) \right|^2 \right]$$

where $H_T^Y(f)$ is the Fourier transform of *Y*(*t*) from –*T* to +*T*. By the properties of Fourier transforms $H_T^Y(f) = (j2\pi f) H_T^X(f)$ so we have

$$\begin{aligned} S_Y(f) &= \lim_{T\to\infty} \frac{1}{2T} \mathbf{E}\left[\left| H_T^Y(f) \right|^2 \right] \\ &= \lim_{T\to\infty} \frac{1}{2T} \mathbf{E}\left[\left| (j2\pi f) H_T^X(f) \right|^2 \right] \\ &= \lim_{T\to\infty} \frac{1}{2T} \left(4\pi^2 f^2 \right) \mathbf{E}\left[\left| H_T^X(f) \right|^2 \right] \\ &= 4\pi^2 f^2 S_X(f) \end{aligned}$$

Note that the expectation occurs at a particular value of *f*; frequency plays the role of an index into a family of random variables.