**Problem 8.34** Given that a stationary random process X(t) has an autocorrelation function  $R_X(\tau)$  and a power spectral density  $S_X(f)$ , show that:

- (a) The autocorrelation function of dX(t)/dt, the first derivative of X(t) is equal to the negative of the second derivative of  $R_X(\tau)$ .
- (b) The power spectral density of dX(t)/dt is equal to  $4\pi^2 f^2 S_X(f)$ .

*Hint*: Use the results of Problem 2.24.

## **Solution**

(a) Let  $Y(t) = \frac{dX}{dt}(t)$ , and from the Wiener-Khintchine relations, we know the autocorrelation of Y(t) is the inverse Fourier transform of the power spectral density of Y. Using the results of part (b),

$$R_{Y}(f) = \mathbf{F}^{-1} [S_{Y}(f)]$$
  
=  $\mathbf{F}^{-1} [4\pi^{2} f^{2} S_{X}(f)]$   
=  $-\mathbf{F}^{-1} [(j2\pi f)^{2} S_{X}(f)]$ 

from the differential properties of the Fourier transform, we know that differentiation in the time domain corresponds to multiplication by  $j2\pi f$  in the frequency domain. Consequently, we conclude that

$$R_{Y}(f) = -\mathbf{F}^{-1} \Big[ (j2\pi f)^{2} S_{X}(f) \Big]$$
$$= -\frac{d^{2}}{d\tau^{2}} R_{X}(\tau)$$

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## **Problem 8.34 continued**

(b) Let  $Y(t) = \frac{dX}{dt}(t)$ , then the spectrum of Y(t) is given by (see Section 8.8)

$$S_{Y}(f) = \lim_{T \to \infty} \frac{1}{2T} \mathbf{E} \left[ \left| H_{T}^{Y}(f) \right|^{2} \right]$$

where  $H_T^Y(f)$  is the Fourier transform of Y(t) from -T to +T. By the properties of Fourier transforms  $H_T^Y(f) = (j2\pi f)H_T^X(f)$  so we have

$$S_{Y}(f) = \lim_{T \to \infty} \frac{1}{2T} \mathbf{E} \left[ \left| H_{T}^{Y}(f) \right|^{2} \right]$$
$$= \lim_{T \to \infty} \frac{1}{2T} \mathbf{E} \left[ \left| (j2\pi f) H_{T}^{X}(f) \right|^{2} \right]$$
$$= \lim_{T \to \infty} \frac{1}{2T} \left( 4\pi^{2} f^{2} \right) \mathbf{E} \left[ \left| H_{T}^{X}(f) \right|^{2} \right]$$
$$= 4\pi^{2} f^{2} S_{X}(f)$$

Note that the expectation occurs at a particular value of *f*; frequency plays the role of an index into a family of random variables.

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