Problem 8.35 Consider a wide-sense stationary process $X(t)$ having the power spectral density $S_X(f)$ shown in Fig. 8.26. Find the autocorrelation function $R_X(\tau)$ of the process $X(t)$.

Solution

The Wiener-Khintchine relations imply the autocorrelation is given by the inverse Fourier transform of the power spectral density, thus

$$\begin{aligned} R(\tau) &= \int_{-\infty}^{\infty} S_X(f)\exp(j2\pi ft)df \\ &= \int_0^1 (1-f)\cos(2\pi ft)df \end{aligned}$$

where we have used the symmetry properties of the spectrum to obtain the second line. Integrating by parts, we obtain

$$\begin{aligned} R_X(\tau) &= (1-f)\frac{\sin(2\pi f\tau)}{2\pi\tau}\bigg|_0^1 + \int_0^1 \frac{\sin(2\pi f\tau)}{2\pi\tau}df \\ &= 0 + \frac{-\cos(2\pi f\tau)}{(2\pi\tau)^2}\bigg|_0^1 \\ &= \frac{1-\cos(2\pi\tau)}{(2\pi\tau)^2} \end{aligned}$$

Using the half-angle formula $\sin^2(\theta) = ½(1-\cos(2\theta)$, this result simplifies to

$$\begin{aligned} R_X(\tau) &= \frac{2\sin^2(\pi\tau)}{(2\pi\tau)^2} \\ &= \tfrac{1}{2}\operatorname{sinc}^2(\tau) \end{aligned}$$

where $\operatorname{sinc}(x) = \sin(\pi x)/\pi x$.