Problem 8.39 Assume the narrow-band process $X(t)$ described in Problem 8.38 is Gaussian with zero mean and variance σ_X^2.

(a) Calculate σ_X^2.

(b) Determine the joint probability density function of the random variables Y and Z obtained by observing the in-phase and quadrature components of $X(t)$ at some fixed time.

Solution

(a) The variance is given by

$$\begin{aligned}
\sigma_X^2 &= R(0) = \int_{-\infty}^{\infty} S(f)\,df \\
&= 2\left(\frac{1}{2}b_1h_1 + \frac{1}{2}b_2h_2\right) \\
&= 2\left(\frac{1}{2}\cdot 2\cdot 1 + \frac{1}{2}\cdot 1\cdot 1\right) \\
&= 3\ \textit{watts}
\end{aligned}$$

(b) The random variables Y and Z have zero mean, are Gaussian and have variance σ_X^2. If Y and Z are independent, the joint density is given by

$$\begin{aligned}
f_{Y,Z}(Y,Z) &= \frac{1}{\sqrt{2\pi}\sigma_X}\exp\left(-y^2 \Big/ 2\sigma_X^2\right)\cdot\frac{1}{\sqrt{2\pi}\sigma_X}\exp\left(-z^2 \Big/ 2\sigma_X^2\right) \\
&= \frac{1}{2\pi\sigma_X^2}\exp\left(-\frac{y^2+z^2}{2\sigma_X^2}\right)
\end{aligned}$$