Problem 8.40 Find the probability that the last two digits of the cube of a natural number (1, 2, 3, …) will be 01.

Solution

Let a natural number be represented by concatenation xy where y represents last two digits and x represents the other digits. For example, the number 1345 has $y = 45$ and $x = 13$. Then

$$(xy)^3 = (x00 + y)^3 = (x^3 000000) + 3(x^2 0000)y + 3(x00)y^2 + y^3$$

where we have used the binomial expansion of $(a+b)^3$. The last digits of the first three terms on the right are clearly 00. Consequently, it is the last two digits of y^3 which determines the last two digits of $(xy)^3$. Checking the cube of all two digit numbers for 00 to 99, we find that: (a) y^3 ends in 1, only if y ends in 1; and (b) only the number $(01)^3$ gives 01 as the last two digits. From this counting argument, the probability is 0.01.