Problem 8.44 The arrival times of two signals at a receiver are uniformly distributed over the interval [0,T]. The receiver will be jammed if the time difference in the arrivals is less than τ. Find the probability that the receiver will be jammed.

Solution

Let X and Y be random variables representing the arrival times of the two signals. The probability density functions of the random variables are

$$f_X(x) = \begin{cases} \dfrac{1}{T} & 0 < x < T \\ 0, & \text{otherwise} \end{cases}$$

and $f_Y(y)$ is similarly defined. Then the probability that the time difference between arrivals is less than τ is given by

$$\begin{aligned} \mathbf{P}\left[|X - Y| < \tau\right] &= \mathbf{P}\left[|X - Y| < \tau \mid X > Y\right]\mathbf{P}\left[X > Y\right] + \mathbf{P}\left[|X - Y| < \tau \mid Y > X\right]\mathbf{P}\left[Y > X\right] \\ &= \mathbf{P}\left[|X - Y| < \tau \mid X > Y\right] \end{aligned}$$

where the second line follows from the symmetry between the random variables X and Y, namely, $\mathbf{P}[X > Y] = \mathbf{P}[Y > X]$. If we only consider the case $X > Y$, then we have the conditions: $0 < X < T$ and $0 < Y < X < \tau + Y$. Combining these conditions we have $Y < X < \min(T, \tau + Y)$. Consequently,

$$\mathbf{P}\left[|X - Y| < \tau\right] = \int_0^T \int_y^{\min(T, \tau + y)} f_X(x) f_Y(y) dx\, dy$$

$$= \int_0^T \int_y^{\min(T, \tau + y)} \left(\frac{1}{T}\right)^2 dx\, dy$$

$$= \frac{1}{T^2} \int_0^T \left\{\min(T, \tau + y) - y\right\} dy$$

Combining the two terms of the integrand,

$$\begin{aligned} P\left[|X - Y| < \tau\right] &= \frac{1}{T^2} \int_0^T \min(T - y, \tau) dy \\ &= \frac{1}{T^2} \min\left(Ty - \frac{y^2}{2}, \tau y\right)\Bigg|_0^T \\ &= \min\left(\frac{1}{2}, \frac{\tau}{T}\right) \end{aligned}$$