Problem 8.47 In a computer-communication network, the arrival time τ between messages is modeled with an exponential distribution function, having the density

$$f_T(\tau) = \begin{cases} \frac{1}{\lambda} e^{-\lambda\tau} & \tau \geq 0 \\ 0 & otherwise \end{cases}$$

a) What is the mean time between messages with this distribution?
b) What is the variance in this time between messages?

Solution (Typo in problem statement, should read $f_T(\tau)=(1/\lambda)\exp(-\tau/\lambda)$ for $\tau>0$)
(a) The mean time between messages is

$$\begin{aligned} \mathbf{E}[T] &= \int_0^\infty \tau f_T(\tau) d\tau \\ &= \int_0^\infty \frac{\tau}{\lambda} \exp(-\tau/\lambda) d\tau \\ &= -\tau \exp(-\tau/\lambda)\Big|_0^\infty + \int_0^\infty \exp(-\tau/\lambda) d\tau \\ &= 0 - \lambda \exp(-\tau/\lambda)\Big|_0^\infty \\ &= \lambda \end{aligned}$$

where the third line follows by integration by parts.

(b) To compute the variance, we first determine the second moment of T

$$\begin{aligned} \mathbf{E}[T^2] &= \int_0^\infty \tau^2 f_T(\tau) d\tau \\ &= \int_0^\infty \frac{\tau^2}{\lambda} \exp(-\tau/\lambda) d\tau \\ &= -\tau^2 \exp(-\tau/\lambda)\Big|_0^\infty + 2\int_0^\infty \tau \exp(-\tau/\lambda) d\tau \\ &= 0 + 2\lambda \mathbf{E}[T] \\ &= 2\lambda^2 \end{aligned}$$

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Problem 8.47 continued

The variance is then given by the difference of the second moment and the first moment squared (see Problem 8.23)

$$
\begin{aligned}
\mathrm{Var}(T) &= \mathbf{E}\left[T^2\right] - \left(\mathbf{E}\left[T\right]\right)^2 \\
&= 2\lambda^2 - \lambda^2 \\
&= \lambda^2
\end{aligned}
$$