Problem 8.47 In a computer-communication network, the arrival time τ between messages is modeled with an exponential distribution function, having the density

$$f_T(\tau) = \begin{cases} \frac{1}{\lambda} e^{-\lambda \tau} & \tau \ge 0\\ 0 & otherwise \end{cases}$$

a) What is the mean time between messages with this distribution?b) What is the variance in this time between messages?

Solution (Typo in problem statement, should read $f_T(\tau)=(1/\lambda)\exp(-\tau/\lambda)$ for $\tau>0$) (a) The mean time between messages is

$$\mathbf{E}[T] = \int_{0}^{\infty} \tau f_{T}(\tau) d\tau$$

= $\int_{0}^{\infty} \frac{\tau}{\lambda} \exp(-\tau/\lambda) d\tau$
= $-\tau \exp(-\tau/\lambda) \Big|_{0}^{\infty} + \int_{0}^{\infty} \exp(-\tau/\lambda) d\tau$
= $0 - \lambda \exp(-\tau/\lambda) \Big|_{0}^{\infty}$
= λ

where the third line follows by integration by parts.

(b) To compute the variance, we first determine the second moment of T

$$\mathbf{E}[T^{2}] = \int_{0}^{\infty} \tau^{2} f_{T}(\tau) d\tau$$

$$= \int_{0}^{\infty} \frac{\tau^{2}}{\lambda} \exp(-\tau/\lambda) d\tau$$

$$= -\tau^{2} \exp(-\tau/\lambda) \Big|_{0}^{\infty} + 2 \int_{0}^{\infty} \tau \exp(-\tau/\lambda) d\tau$$

$$= 0 + 2\lambda \mathbf{E}[T]$$

$$= 2\lambda^{2}$$

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Problem 8.47 continued

The variance is then given by the difference of the second moment and the first moment squared (see Problem 8.23)

$$\operatorname{Var}(T) = \mathbf{E}[T^{2}] - (\mathbf{E}[T])^{2}$$
$$= 2\lambda^{2} - \lambda^{2}$$
$$= \lambda^{2}$$

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