Problem 8.49 Let X and Y be independent random variables with densities $f_X(x)$ and $f_Y(y)$, respectively. Show that the random variable $Z = X+Y$ has a density given by

$$f_Z(z) = \int_{-\infty}^{z} f_Y(z-s)\,f_X(s)ds$$

Hint: $\mathbf{P}[Z \le z] = \mathbf{P}[X \le z, Y \le z - X]$

<u>Solution</u> (*Typo in problem statement - should be "positive" independent random variables*)

Using the hint, we have that $F_Z(z) = \mathbf{P}[Z \le z]$ and

$$F_Z(z) = \int_{-\infty}^{z}\int_{-\infty}^{z-x} f_X(x)\,f_Y(y)dydx$$

To differentiate this result with respect to z, we use the fact that if

$$g(z) = \int_a^b h(x,z)dx$$

then

$$\frac{\partial g(z)}{dz} = \int_a^b \frac{\partial}{\partial z} h(x,z)dx + h(b,z)\frac{db}{dz} - h(a,z)\frac{da}{dz} \qquad (1)$$

Inspecting $F_Z(z)$, we identify $h(x,z)$

$$h(x,z) = \int_{-\infty}^{z-x} f_X(x)\,f_Y(y)dy$$

and $a = -\infty$ and $b = z$. We then obtain

$$\begin{aligned} f_Z(z) &= \frac{d}{dz}F_Z(z) \\ &= \int_{-\infty}^{z}\left[\frac{d}{dz}\int_{-\infty}^{z-x} f_X(x)\,f_Y(y)dy\right]dx + \int_{-\infty}^{z-z} f_X(z)\,f_Y(y)dy\frac{dz}{dz} - \int_{-\infty}^{z-(-\infty)} f_X(-\infty)\,f_Y(y)dy \cdot 0 \\ &= \int_{-\infty}^{z}\left[\frac{d}{dz}\int_{-\infty}^{z-x} f_X(x)\,f_Y(y)dy\right]dx \end{aligned}$$

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Problem 8-49 continued

where the second term of the second line is zero since the random variables are positive, and the third term is zero due to the factor zero. Applying the differentiation rule a second time, we obtain

$$f_Z(z) = \int_{-\infty}^{z}\left[0 + f_X(x) f_Y(z-x)\frac{d(z-x)}{dz} - f_X(x) f_Y(-\infty)\frac{d(-\infty)}{dz}\right]dx$$
$$= \int_{-\infty}^{z} f_X(x) f_Y(z-x)dx$$

which is the desired result.

An alternative solution is the following: we note that

$$\begin{aligned}\mathbf{P}[Z \le z \mid X = x] &= \mathbf{P}[X + Y \le z \mid X = x] \\ &= \mathbf{P}[x + Y \le z \mid X = x] \\ &= \mathbf{P}[x + Y \le z] \\ &= \mathbf{P}[Y \le z - x]\end{aligned}$$

where the third equality follows from the independence of *X* and *Y*. By differentiating both sides with respect to *z*, we see that

$$f_{Z|X}(z \mid x) = f_Y(z-x)$$

By the properties of conditional densities

$$f_{Z,X}(z,x) = f_X(x) f_{Z|X}(z \mid x) = f_X(x) f_Y(z-x)$$

Integrating to form the marginal distribution, we have

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)dx$$

If *Y* is a positive random variable then $f_Y(z\text{-}x)$ is zero for $x > z$ and the desired result follows.