Problem 8.50 Find the spectral density $S_Z(f)$ if

$$Z(t) = X(t)Y(t)$$

where $X(t)$ and $Y(t)$ are independent zero-mean random processes with

$$R_X(\tau) = a_1 e^{-\alpha_1|\tau|} \quad \text{and} \quad R_Y(\tau) = a_2 e^{-\alpha_2|\tau|}.$$

Solution

The autocorrelation of $Z(t)$ is given by

$$\begin{aligned} R_Z(\tau) &= \mathbf{E}[Z(t)Z(t+\tau)] \\ &= \mathbf{E}[X(t)X(t+\tau)Y(t)Y(t+\tau)] \\ &= \mathbf{E}[X(t)X(t+\tau)]\mathbf{E}[Y(t)Y(t+\tau)] \\ &= R_X(\tau)R_Y(\tau) \end{aligned}$$

By the Wiener-Khintchine relations, the spectrum of $Z(t)$ is given by

$$\begin{aligned} S_Z(f) &= \mathbf{F}^{-1}[R_X(\tau)R_Y(\tau)] \\ &= \mathbf{F}^{-1}[a_1 a_2 \exp(-(\alpha_1+\alpha_2)|\tau|)] \\ &= \frac{2a_1 a_2(\alpha_1+\alpha_2)}{(\alpha_1+\alpha_2)^2 + (2\pi f)^2} \end{aligned}$$

where the last line follows from the Fourier transform of the double-sided exponential (See Example 2.3).