Problem 8.52 The oscillators used in communication systems are not ideal but often suffer from a distortion known as phase noise. Such an oscillator may be modeled by the random process

$$Y(t) = A\cos\left(2\pi f_c t + \phi(t)\right)$$

where $\phi(t)$ is a slowly varying random process. Describe and justify the conditions on the random process $\phi(t)$ such that $Y(t)$ is wide-sense stationary.

Solution

The first condition for wide-sense stationary process is a constant mean. Consider $t = t_0$, then

$$\mathbf{E}[Y(t_0)] = \mathbf{E}\left[A\cos\left(2\pi f_c t_0 + \phi(t_0)\right)\right]$$

In general, the function $\cos\theta$ takes from values -1 to +1 when θ varies from 0 to 2π. In this case θ corresponds to $2\pi f_c t_0 + \phi(t_0)$. If $\phi(t_0)$ varies only by a small amount then θ will be biased toward the point $2\pi f_c t_0 + \mathbf{E}[\phi(t_0)]$, and the mean value of $\mathbf{E}[Y(t_0)]$ will depend upon the choice of t_0. However, if $\phi(t_0)$ is uniformly distributed over $[0, 2\pi]$ then $2\pi f_c t_0 + \phi(t_0)$ will be uniformly distributed over $[0, 2\pi]$ when considered modulo 2π, and the mean $\mathbf{E}[Y(t_0)]$ will be zero and will not depend upon t_0.

Thus the first requirement is that $\phi(t)$ must be uniformly distributed over $[0,2\pi]$ for all t.

The second condition for a wide-sense stationary $Y(t)$ is that the autocorrelation depends only upon the time difference

$$\begin{aligned}\mathbf{E}[Y(t_1)Y(t_2)] &= \mathbf{E}\left[A\cos\left(2\pi f_c t_1 + \phi(t_1)\right)A\cos\left(2\pi f_c t_2 + \phi(t_2)\right)\right] \\ &= \frac{A^2}{2}\mathbf{E}\left[\cos\left(2\pi f_c(t_1+t_2) + \phi(t_1) + \phi(t_2)\right) + \cos\left(2\pi f_c(t_1-t_2) + \phi(t_1) - \phi(t_2)\right)\right]\end{aligned}$$

where we have used the relation $\cos A\cos B = \tfrac{1}{2}\left(\cos(A+B) + \cos(A-B)\right)$. In general, this correlation does not depend solely on the time difference t_2-t_1. However, if we assume:

We first note that if $\phi(t_1)$ and $\phi(t_2)$ are both uniformly distributed over $[0,2\pi]$ then so is $\psi = \phi(t_1) + \phi(t_2)$ (modulo 2π), and

$$\begin{aligned}\mathbf{E}\left[\cos\left(2\pi f_c(t_1+t_2) + \psi\right)\right] &= \frac{1}{2\pi}\int_0^{2\pi}\cos(2\pi f_c(t_1+t_2) + \psi)\,d\psi \\ &= 0\end{aligned} \qquad (1)$$

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Problem 8.52 continued

We consider next the term $R_Y(t_1,t_2)=\mathbf{E}\left[\cos\left(2\pi f_c(t_1-t_2)+\phi(t_1)-\phi(t_2)\right)\right]$ and three special cases:

(a) if $\Delta t = t_1 - t_2$ is small then $\phi(t_1)\approx\phi(t_2)$ since $\phi(t)$ is a slowly varying process, and

$$R_Y(t_1,t_2)=\frac{A^2}{2}\cos\left(2\pi f_c(t_1-t_2)\right)$$

(b) if Δt is large then $\phi(t_1)$ and $\phi(t_2)$ should be approximately independent and $\phi(t_1)-\phi(t_2)$ would be approximately uniformly distributed over $[0,2\pi]$. In this case

$$R_Y(t_1,t_2)\approx 0$$

using the argument of Eq. (1).

(c) for intermediate values of Δt, we require that

$$\phi(t_1)-\phi(t_2)\approx g(t_1-t_2)$$

for some arbitrary function $g(t)$.

Under these conditions the random process $Y(t)$ will be wide-sense stationary.