Problem 8.53 A baseband signal is disturbed by a noise process N(t) as shown by

$$X(t) = A\sin(0.3\pi t) + N(t)$$

where N(t) is a stationary Gaussian process of zero mean and variance σ^2 .

(a) What are the density functions of the random variables X_1 and X_2 where

$$X_{1} = X(t)\big|_{t=1}$$
$$X_{2} = X(t)\big|_{t=2}$$

(b) The noise process N(t) has an autocorrelation function given by

$$R_N(\tau) = \sigma^2 \exp(-|\tau|)$$

What is the joint density function of X_1 and X_2 , that is, $f_{X_1,X_2}(x_1,x_2)$?

Solution

(a) The random variable X_1 has a mean

$$\mathbf{E}[X(t_1)] = \mathbf{E}[A\sin(0.3\pi) + N(t_1)]$$

= $A\sin(0.3\pi) + \mathbf{E}[N(t_1)]$
= $A\sin(0.3\pi)$

Since X_1 is equal to $N(t_1)$ plus a constant, the variance of X_1 is the same as that of $N(t_1)$. In addition, since $N(t_1)$ is a Gaussian random variable, X_1 is also Gaussian with a density given by

$$f_{X_1}(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\{-(x-\mu_1)/2\sigma^2\}$$

where $\mu_1 = \mathbf{E}[X(t_1)]$. By a similar argument, the density function of X_2 is

$$f_{X_2}(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\{-(x-\mu_2)/2\sigma^2\}$$

where $\mu_2 = A \sin(0.6\pi)$.

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Problem 8-53 continued

(b) First note that since the mean of X(t) is not constant, X(t) is not a stationary random process. However, X(t) is still a Gaussian random process, so the joint distribution of N Gaussian random variables may be written as Eq. (8.90). For the case of N = 2, this equation reduces to

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{2\pi |\Lambda|^{1/2}} \exp\{-(\mathbf{x} - \boldsymbol{\mu})\Lambda^{-1}(\mathbf{x} - \boldsymbol{\mu})^{T} / 2\}$$

where Λ is the 2x2 covariance matrix. Recall that $cov(X_1, X_2) = \mathbf{E}[(X_1 - \mu_1)(X_2 - \mu_2)]$, so that

$$\Lambda = \begin{bmatrix} \operatorname{cov}(X_1, X_1) & \operatorname{cov}(X_1, X_2) \\ \operatorname{cov}(X_2, X_1) & \operatorname{cov}(X_2, X_2) \end{bmatrix}$$
$$= \begin{bmatrix} R_N(0) & R_N(1) \\ R_N(1) & R_N(0) \end{bmatrix}$$
$$= \begin{bmatrix} \sigma^2 & \sigma^2 \exp(-1) \\ \sigma^2 \exp(-1) & \sigma^2 \end{bmatrix}$$

If we let $\rho = \exp(-1)$ then

$$\left|\Lambda\right| = \sigma^4 (1 - \rho^2)$$

and

$$\Lambda^{-1} = \frac{1}{\sigma^{2}(1-\rho^{2})} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}$$

Making these substitutions into the above expression, we obtain upon simplification

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp\left\{-\frac{(x_1-\mu_1)^2 + (x_2-\mu_2)^2 - 2\rho(x_1-\mu_1)(x_2-\mu_2)}{2\sigma^2(1-\rho^2)}\right\}$$

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