Problem 8.53 A baseband signal is disturbed by a noise process $N(t)$ as shown by

$$X(t) = A\sin(0.3\pi t) + N(t)$$

where $N(t)$ is a stationary Gaussian process of zero mean and variance σ^2.

(a) What are the density functions of the random variables X_1 and X_2 where

$$X_1 = X(t)\big|_{t=1}$$

$$X_2 = X(t)\big|_{t=2}$$

(b) The noise process $N(t)$ has an autocorrelation function given by

$$R_N(\tau) = \sigma^2 \exp(-|\tau|)$$

What is the joint density function of X_1 and X_2, that is, $f_{X_1,X_2}(x_1,x_2)$?

Solution

(a) The random variable X_1 has a mean

$$\begin{aligned}\mathbf{E}[X(t_1)] &= \mathbf{E}[A\sin(0.3\pi) + N(t_1)] \\ &= A\sin(0.3\pi) + \mathbf{E}[N(t_1)] \\ &= A\sin(0.3\pi)\end{aligned}$$

Since X_1 is equal to $N(t_1)$ plus a constant, the variance of X_1 is the same as that of $N(t_1)$. In addition, since $N(t_1)$ is a Gaussian random variable, X_1 is also Gaussian with a density given by

$$f_{X_1}(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-(x-\mu_1)/2\sigma^2\right\}$$

where $\mu_1 = \mathbf{E}[X(t_1)]$. By a similar argument, the density function of X_2 is

$$f_{X_2}(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-(x-\mu_2)/2\sigma^2\right\}$$

where $\mu_2 = A\sin(0.6\pi)$.

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Problem 8-53 continued

(b) First note that since the mean of $X(t)$ is not constant, $X(t)$ is not a stationary random process. However, $X(t)$ is still a Gaussian random process, so the joint distribution of N Gaussian random variables may be written as Eq. (8.90). For the case of $N = 2$, this equation reduces to

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{2\pi|\Lambda|^{1/2}} \exp\left\{-(\mathbf{x}-\boldsymbol{\mu})\Lambda^{-1}(\mathbf{x}-\boldsymbol{\mu})^T / 2\right\}$$

where Λ is the 2x2 covariance matrix. Recall that $\text{cov}(X_1,X_2) = \mathbf{E}[(X_1-\mu_1)(X_2-\mu_2)]$, so that

$$\Lambda = \begin{bmatrix} \text{cov}(X_1,X_1) & \text{cov}(X_1,X_2) \\ \text{cov}(X_2,X_1) & \text{cov}(X_2,X_2) \end{bmatrix}$$
$$= \begin{bmatrix} R_N(0) & R_N(1) \\ R_N(1) & R_N(0) \end{bmatrix}$$
$$= \begin{bmatrix} \sigma^2 & \sigma^2\exp(-1) \\ \sigma^2\exp(-1) & \sigma^2 \end{bmatrix}$$

If we let $\rho = \exp(-1)$ then

$$|\Lambda| = \sigma^4(1-\rho^2)$$

and

$$\Lambda^{-1} = \frac{1}{\sigma^2(1-\rho^2)} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}$$

Making these substitutions into the above expression, we obtain upon simplification

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp\left\{-\frac{(x_1-\mu_1)^2 + (x_2-\mu_2)^2 - 2\rho(x_1-\mu_1)(x_2-\mu_2)}{2\sigma^2(1-\rho^2)}\right\}$$