Problem 8.53 A baseband signal is disturbed by a noise process $N(t)$ as shown by

$$
X(t) = A\sin(0.3\pi t) + N(t)
$$

where $N(t)$ is a stationary Gaussian process of zero mean and variance σ^2 .

(a) What are the density functions of the random variables X_1 and X_2 where

$$
X_1 = X(t)|_{t=1}
$$

$$
X_2 = X(t)|_{t=2}
$$

(b) The noise process *N*(*t*) has an autocorrelation function given by

$$
R_N(\tau) = \sigma^2 \exp(-|\tau|)
$$

What is the joint density function of X_1 and X_2 , that is, $f_{X_1, X_2}(x_1, x_2)$?

Solution

(a) The random variable X_1 has a mean

$$
\mathbf{E}[X(t_1)] = \mathbf{E}[A\sin(0.3\pi) + N(t_1)]
$$

= $A\sin(0.3\pi) + \mathbf{E}[N(t_1)]$
= $A\sin(0.3\pi)$

Since X_1 is equal to $N(t_1)$ plus a constant, the variance of X_1 is the same as that of $N(t_1)$. In addition, since $N(t_1)$ is a Gaussian random variable, X_1 is also Gaussian with a density given by

$$
f_{X_1}(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\left(x - \mu_1\right)/2\sigma^2\right\}
$$

where $\mu_1 = \mathbf{E}[X(t_1)]$. By a similar argument, the density function of X_2 is

$$
f_{X_2}(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\left(x - \mu_2\right)/2\sigma^2\right\}
$$

where $\mu_2 = A \sin(0.6\pi)$.

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Problem 8-53 continued

(b) First note that since the mean of $X(t)$ is not constant, $X(t)$ is not a stationary random process. However, *X*(*t*) is still a Gaussian random process, so the joint distribution of *N* Gaussian random variables may be written as Eq. (8.90). For the case of $N = 2$, this equation reduces to

$$
f_{\mathbf{x}}(\mathbf{x}) = \frac{1}{2\pi |\Lambda|^{1/2}} \exp\left\{ -(\mathbf{x} - \mathbf{\mu})\Lambda^{-1}(\mathbf{x} - \mathbf{\mu})^T / 2 \right\}
$$

where Λ is the 2x2 covariance matrix. Recall that $cov(X_1,X_2) = E[(X_1-\mu_1)(X_2-\mu_2)]$, so that

$$
\Lambda = \begin{bmatrix}\n\text{cov}(X_1, X_1) & \text{cov}(X_1, X_2) \\
\text{cov}(X_2, X_1) & \text{cov}(X_2, X_2)\n\end{bmatrix}
$$
\n
$$
= \begin{bmatrix}\nR_N(0) & R_N(1) \\
R_N(1) & R_N(0)\n\end{bmatrix}
$$
\n
$$
= \begin{bmatrix}\n\sigma^2 & \sigma^2 \exp(-1) \\
\sigma^2 \exp(-1) & \sigma^2\n\end{bmatrix}
$$

If we let $\rho = \exp(-1)$ then

$$
|\Lambda| = \sigma^4 (1 - \rho^2)
$$

and

$$
\Lambda^{-1} = \frac{1}{\sigma^2 (1 - \rho^2)} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}
$$

Making these substitutions into the above expression, we obtain upon simplification

$$
f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp\left\{-\frac{(x_1-\mu_1)^2 + (x_2-\mu_2)^2 - 2\rho(x_1-\mu_1)(x_2-\mu_2)}{2\sigma^2(1-\rho^2)}\right\}
$$

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