Problem 8.8 Show that the mean and variance of a Gaussian random variable X with the density function given by Eq. (8.48) are μ_X and σ_X^2.

Solution

Consider the difference $\mathbf{E}[X]-\mu_X$:

$$\mathbf{E}[X]-\mu_X = \int_{-\infty}^{\infty} \frac{(x-\mu_X)}{\sqrt{2\pi}\sigma_X} \exp\left\{-\frac{(x-\mu_X)^2}{2\sigma_X^{\ 2}}\right\} dx$$

Let $y = x-\mu_X$ and substitute

$$\begin{aligned}\mathbf{E}[X]-\mu_X &= \int_{-\infty}^{\infty} \frac{y}{\sqrt{2\pi}\sigma_X} \exp\left(-y^2 \Big/ 2\sigma_X^{\ 2}\right) dy \\ &= 0\end{aligned}$$

since integrand has odd symmetry. This implies $E[X]=\mu_X$. With this result

$$\begin{aligned}\mathrm{Var}(X) &= \mathbf{E}(x-\mu_X)^2 \\ &= \int_{-\infty}^{\infty} \frac{(x-\mu_X)^2}{\sqrt{2\pi}\sigma_X} \exp\left\{\frac{-(x-\mu_X)^2}{2\sigma_X^{\ 2}}\right\} dx\end{aligned}$$

In this case let

$$y = \frac{x-\mu_X}{\sigma_X}$$

and making the substitution, we obtain

$$\mathrm{Var}(X) = \sigma_X^{\ 2} \int_{-\infty}^{\infty} \frac{y^2}{\sqrt{2\pi}} \exp\left\{\frac{-y^2}{2}\right\} dy$$

Recalling the integration-by-parts, i.e., $\int u dv = uv - \int v du$, let $u = y$ and $dv = y\exp\left(\frac{-y^2}{2}\right)dy$. Then

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Problem 8.8 continued

$$\begin{aligned}
\mathrm{Var}(X) &= \sigma_X{}^2 \frac{(-)y}{\sqrt{2\pi}} \exp\left(-{y^2}\big/{2}\right)\Big|_{-\infty}^{\infty} + \sigma_X{}^2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy \\
&= 0 + \sigma_X{}^2 \bullet 1 \\
&= \sigma_X{}^2
\end{aligned}$$

where the second integral is one since it is integral of the normalized Gaussian probability density.