Problem 8.8 Show that the mean and variance of a Gaussian random variable *X* with the density function given by Eq. (8.48) are μ_X and σ_X^2 .

Solution

Consider the difference $\mathbf{E}[X]$ - μ_X :

$$\mathbf{E}[X] - \mu_X = \int_{-\infty}^{\infty} \frac{(x - \mu_X)}{\sqrt{2\pi\sigma_X}} \exp\left\{-\frac{(x - \mu_X)^2}{2{\sigma_X}^2}\right\} dx$$

Let $y = x - \mu_x$ and substitute

$$\mathbf{E}[X] - \mu_X = \int_{-\infty}^{\infty} \frac{y}{\sqrt{2\pi\sigma_X}} \exp\left(\frac{-y^2}{2\sigma_X^2}\right) dy$$
$$= 0$$

since integrand has odd symmetry. This implies $E[X] = \mu_X$. With this result

$$\operatorname{Var}(X) = \mathbf{E}(x - \mu_X)^2$$
$$= \int_{-\infty}^{\infty} \frac{(x - \mu_X)^2}{\sqrt{2\pi}\sigma_X} \exp\left\{\frac{-(x - \mu_X)^2}{2{\sigma_X}^2}\right\} dx$$

In this case let

$$y = \frac{x - \mu_X}{\sigma_X}$$

and making the substitution, we obtain

$$\operatorname{Var}(X) = \sigma_X^2 \int_{-\infty}^{\infty} \frac{y^2}{\sqrt{2\pi}} \exp\left\{\frac{-y^2}{2}\right\} dy$$

Recalling the integration-by-parts, i.e., $\int u dv = uv - \int v du$, let u = y and

$$dv = y \exp\left(\frac{-y^2}{2}\right) dy$$
. Then

Continued on next slide

Excerpts from this work may be reproduced by instructors for distribution on a not-for-profit basis for testing or instructional purposes only to students enrolled in courses for which the textbook has been adopted. Any other reproduction or translation of this work beyond that permitted by Sections 107 or 108 of the 1976 United States Copyright Act without the permission of the copyright owner is unlawful. page...8-8 Problem 8.8 continued

$$\operatorname{Var}(X) = \sigma_X^2 \frac{(-)y}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) \Big|_{-\infty}^{\infty} + \sigma_X^2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy$$
$$= 0 + \sigma_X^2 \cdot 1$$
$$= \sigma_X^2$$

where the second integral is one since it is integral of the normalized Gaussian probability density.

Excerpts from this work may be reproduced by instructors for distribution on a not-for-profit basis for testing or instructional purposes only to students enrolled in courses for which the textbook has been adopted. Any other reproduction or translation of this work beyond that permitted by Sections 107 or 108 of the 1976 United States Copyright Act without the permission of the copyright owner is unlawful. page...8-9