Problem 8.9 Show that for a Gaussian random variable X with mean μ_X and variance σ_X^2 the transformation $Y = (X - \mu_X)/\sigma_X$, converts X to a normalized Gaussian random variable.

Solution

Let $y = \dfrac{x - \mu_X}{\sigma_X}$. Then

$$\begin{aligned}\mathbf{E}[Y] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y \exp\left(-y^2/2\right) dy \\ &= 0\end{aligned}$$

by the odd symmetry of the integrand. If $\mathbf{E}[Y] = 0$, then from the definition of Y, $\mathbf{E}[X] = \mu_X$. In a similar fashion

$$\begin{aligned}\mathbf{E}\left[Y^2\right] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^2 \exp\left(-\frac{y^2}{2}\right) dy \\ &= \frac{(-)y}{\sqrt{2\pi}} \cdot \exp\left\{-\frac{y^2}{2}\right\}\Bigg|_{-\infty}^{\infty} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-y^2/2\right) dy \\ &= 1\end{aligned}$$

where we use integration by parts as in Problem 8.8. This result implies

$$E\left(\frac{x - \mu_X}{\sigma_X}\right)^2 = 1$$

and hence $\mathbf{E}(x - \mu_X)^2 = \sigma_X^2$