Problem 9.20. Assume that the narrowband noise $n(t)$ is Gaussian and its power spectral density $S_N(f)$ is symmetric about the midband frequency f_c. Show that the in-phase and quadrature components of $n(t)$ are statistically independent.

Solution

The narrowband noise $n(t)$ can be expressed as:

$$\begin{aligned} n(t) &= n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t) \\ &= \mathrm{Re}\left[z(t) e^{j2\pi f_c t} \right] \end{aligned},$$

where $n_I(t)$ and $n_Q(t)$ are in-phase and quadrature components of $n(t)$, respectively. The term $z(t)$ is called the complex envelope of $n(t)$. The noise $n(t)$ has the power spectral density $S_N(f)$ that may be represented as shown below

We shall denote $R_{nn}(\tau)$, $R_{n_I n_I}(\tau)$ and $R_{n_Q n_Q}(\tau)$ as autocorrelation functions of $n(t)$, $n_I(t)$ and $n_Q(t)$, respectively. Then

$$\begin{aligned} R_{nn}(\tau) &= \mathbf{E}\left[n(t)n(t+\tau) \right] \\ &= \mathbf{E}\left\{ \left[n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t) \right] \cdot \left[n_I(t+\tau)\cos(2\pi f_c (t+\tau)) - n_Q(t+\tau)\sin(2\pi f_c (t+\tau)) \right] \right\} \\ &= \frac{1}{2}\left[R_{n_I n_I}(\tau) + R_{n_Q n_Q}(\tau) \right]\cos(2\pi f_c \tau) + \frac{1}{2}\left[R_{n_I n_I}(\tau) - R_{n_Q n_Q}(\tau) \right]\cos(2\pi f_c (2t+\tau)) \\ &\quad - \frac{1}{2}\left[R_{n_Q n_I}(\tau) - R_{n_I n_Q}(\tau) \right]\sin(2\pi f_c \tau) - \frac{1}{2}\left[R_{n_Q n_I}(\tau) + R_{n_I n_Q}(\tau) \right]\sin(2\pi f_c (2t+\tau)) \end{aligned}$$

Since $n(t)$ is stationary, the right-hand side of the above equation must be independent of t, this implies

$$R_{n_I n_I}(\tau) = R_{n_Q n_Q}(\tau) \qquad (1)$$

$$R_{n_I n_Q}(\tau) = -R_{n_Q n_I}(\tau) \qquad (2)$$

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Problem 9.20 continued

Substituting the above two equations into the expression for $R_{nn}(\tau)$, we have

$$R_{nn}(\tau) = R_{n_I n_I}(\tau)\cos(2\pi f_c \tau) - R_{n_Q n_I}(\tau)\sin(2\pi f_c \tau) \tag{3}$$

The autocorrelation function of the complex envelope $z(t) = n_I(t) + jn_Q(t)$ is

$$\begin{aligned} R_{zz}(\tau) &= E\left[z^*(t)z(t+\tau)\right] \\ &= 2R_{n_I n_I}(\tau) + j2R_{n_Q n_I}(\tau) \end{aligned} \tag{4}$$

From the bandpass to low-pass transformation of Section 3.8, the spectrum of the complex envelope z is given bye

$$S_Z(f) = \begin{cases} S_N\left(f + f_c\right) & f > -f_c \\ 0 & \text{otherwise} \end{cases}$$

Since $S_N(f)$ is symmetric about f_c, $S_Z(f)$ is symmetric about $f = 0$. Consequently, the inverse Fourier transform of $S_Z(f) = R_{zz}(\tau)$ must be real. Since $R_{zz}(\tau)$ is real valued, based on Eq. (4), we have

$$R_{n_Q n_I}(\tau) = 0,$$

which means the in-phase and quadrature components of $n(t)$ are uncorrelated. Since the in-phase and quadrature components are also Gaussian, this implies that they are also statistically independent.