Problem 9.20. Assume that the narrowband noise n(t) is Gaussian and its power spectral density $S_N(f)$ is symmetric about the midband frequency f_c . Show that the in-phase and quadrature components of n(t) are statistically independent.

Solution

The narrowband noise n(t) can be expressed as:

$$n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)$$
$$= \operatorname{Re}\left[z(t)e^{j2\pi f_c t}\right],$$

where $n_l(t)$ and $n_Q(t)$ are in-phase and quadrature components of n(t), respectively. The term z(t) is called the complex envelope of n(t). The noise n(t) has the power spectral density $S_N(f)$ that may be represented as shown below

We shall denote $R_{nn}(\tau)$, $R_{n_In_I}(\tau)$ and $R_{n_Qn_Q}(\tau)$ as autocorrelation functions of n(t), $n_I(t)$ and $n_Q(t)$, respectively. Then

$$\begin{split} R_{nn}(\tau) &= \mathbf{E} \Big[n(t)n(t+\tau) \Big] \\ &= \mathbf{E} \Big\{ \Big[n_{I}(t)\cos(2\pi f_{c}t) - n_{Q}(t)\sin(2\pi f_{c}t) \Big] \cdot \Big[n_{I}(t+\tau)\cos(2\pi f_{c}(t+\tau)) - n_{Q}(t+\tau)\sin(2\pi f_{c}(t+\tau)) \Big] \Big\} \\ &= \frac{1}{2} \Big[R_{n_{I}n_{I}}(\tau) + R_{n_{Q}n_{Q}}(\tau) \Big] \cos(2\pi f_{c}\tau) + \frac{1}{2} \Big[R_{n_{I}n_{I}}(\tau) - R_{n_{Q}n_{Q}}(\tau) \Big] \cos(2\pi f_{c}(2t+\tau)) \\ &- \frac{1}{2} \Big[R_{n_{Q}n_{I}}(\tau) - R_{n_{I}n_{Q}}(\tau) \Big] \sin(2\pi f_{c}\tau) - \frac{1}{2} \Big[R_{n_{Q}n_{I}}(\tau) + R_{n_{I}n_{Q}}(\tau) \Big] \sin(2\pi f_{c}(\tau) + r) \end{split}$$

Since n(t) is stationary, the right-hand side of the above equation must be independent of t, this implies

$$R_{n_i n_i}(\tau) = R_{n_Q n_Q}(\tau) \tag{1}$$

$$R_{n_{l}n_{Q}}(\tau) = -R_{n_{Q}n_{l}}(\tau)$$
⁽²⁾

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Problem 9.20 continued

Substituting the above two equations into the expression for $R_{nn}(\tau)$, we have

$$R_{nn}(\tau) = R_{n_{l}n_{l}}(\tau)\cos(2\pi f_{c}\tau) - R_{n_{Q}n_{l}}(\tau)\sin(2\pi f_{c}\tau)$$
(3)

The autocorrelation function of the complex envelope $z(t) = n_I(t) + jn_O(t)$ is

$$R_{zz}(\tau) = E\left[z^*(t)z(t+\tau)\right]$$

= $2R_{n_in_i}(\tau) + j2R_{n_on_i}(\tau)$ (4)

From the bandpass to low-pass transformation of Section 3.8, the spectrum of the complex envelope z is given bye

$$S_{Z}(f) = \begin{cases} S_{N}(f+f_{c}) & f > -f_{c} \\ 0 & \text{otherwise} \end{cases}$$

Since $S_N(f)$ is symmetric about f_c , $S_Z(f)$ is symmetric about f = 0. Consequently, the inverse Fourier transform of $S_Z(f) = R_{zz}(\tau)$ must be real. Since $R_{zz}(\tau)$ is real valued, based on Eq. (4), we have

$$R_{n_Q n_I}(\tau)=0,$$

which means the in-phase and quadrature components of n(t) are uncorrelated. Since the in-phase and quadrature components are also Gaussian, this implies that they are also statistically independent.