Problem 9.21. Suppose that the receiver bandpass-filter magnitude response $|H_{BP}(f)|$ has symmetry about $\pm f_c$ and noise bandwidth B_T. From the properties of narrowband noise described in Section 8.11, what is the spectral density $S_N(f)$ of the in-phase and quadrature components of the narrowband noise $n(t)$ at the output of the filter? Show that the autocorrelation of $n(t)$ is

$$R_N(\tau) = \rho(\tau)\cos(2\pi f_c \tau)$$

where $\rho(\tau) = \mathbf{F}^{-1}\left[S_N(f)\right]$; justify the approximation $\rho(\tau) \approx 1$ for $|\tau| < 1/B_T$.

Solution

Let the noise spectral density of the bandpass process be SH(f) then

$$S_H(f) = \frac{N_0}{2}\left|H_{BP}(f)\right|^2$$

From Section 8.11, the power spectral densities of the in-phase and quadrature components are given by

$$S_N(f) = \begin{cases} S_H(f-f_c) + S_H(f+f_c), & |f| \le B_T/2 \\ 0, & \text{otherwise} \end{cases}.$$

Since the spectrum $S_H(f)$ is symmetric about f_c, , the spectral density of the in-phase and quadrature components is

$$S_N(f) = \begin{cases} \left|H_{BP}(f-f_c)\right|^2 N_0 & |f| < B_T/2 \\ 0 & \text{otherwise} \end{cases} \qquad (1)$$

Note that if $|H_{BP}(f)|$ is symmetric about f_c then $|H_{BP}(f-f_c)|$ will be symmetric about 0. Consequently, the power spectral densities of the in-phase and quadrature components are symmetric about the origin. This implies that the corresponding autocorrelation functions are real valued (since they are related by the inverse Fourier transform). In Problem 9.20, we shown that if the autocorrelation function of the in-phase component is real valued then autocorrelation of $n(t)$ is $R_N(\tau) = R_{n_I n_I}(\tau)\cos(2\pi f_c \tau)$. If we denote

$$\rho(\tau) = R_{n_I n_I}(\tau) = \mathbf{F}^{-1}\left[S_N(f)\right] = N_O \mathbf{F}^{-1}\left[\left|H_{BP}(f-f_c)\right|^2\right]$$

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Problem 9.21 continued

then the autocorrelation of the bandpass noise is

$$R_N(\tau) = \rho(\tau)\cos\left(2\pi f_c t\right)$$

For $|\tau| \ll 1/B_T$ (*there is a typo in the text*), we have

$$\begin{aligned}\rho(\tau) &= \int_{-\infty}^{\infty} S_N(f)\exp\left(-j2\pi f\tau\right)df \\ &= \int_{0}^{\infty} S_N(f)\cos\left(2\pi f\tau\right)df\end{aligned}$$

due to the real even-symmetric nature of $S_N(f)$. If the signal has noise bandwidth B_T then

$$\begin{aligned}\rho(\tau) &\approx \int_{0}^{B_T} S_N(f)\cos\left(2\pi f\tau\right)df \\ &\approx \int_{0}^{B_T} S_N(f)\cos\left(0\right)df \\ &= \int_{0}^{B_T} S_N(f)df \\ &= \text{a constant}\end{aligned}$$

where the second line follows from the assumption that $|\tau| \ll 1/B_T$. With suitable scaling the constant can be set to one.