Problem 9.23. In a receiver using coherent detection, the sinusoidal wave generated by the local oscillator suffers from a phase error $\theta(t)$ with respect to the carrier wave $\cos(2\pi f_c t)$. Assuming that $\theta(t)$ is a zero-mean Gaussian process of variance σ_θ^2 and that most of the time the maximum value of $\theta(t)$ is small compared to unity, find the mean-square error of the receiver output for DSB-SC modulation. The mean-square error is defined as the expected value of the squared difference between the receiver output and message signal component of a synchronous receiver output.

Solution

Based on the solution of Problem 9.22, we have the DSB-SC demodulator output is

$$y(t) = \frac{1}{2}A_c m(t)\cos[\theta(t)] + \frac{1}{2}n_I(t)\cos[\theta(t)] + \frac{1}{2}n_Q(t)\sin[\theta(t)]$$

Recall from Section 9. that the output of a synchronous receiver is

$$\frac{1}{2}A_c m(t) + \frac{1}{2}n_I(t)$$

The mean-square error (MSE) is defined by

$$\text{MSE} = \mathbf{E}\left[\left(y(t) - \frac{1}{2}A_c m(t)\right)^2\right]$$

Substituting the above expression for $y(t)$, the mean-square error is

$$\begin{aligned}\text{MSE} &= \mathbf{E}\left[\left[\frac{1}{2}A_c m(t)\left[\cos(\theta(t)) - 1\right] + \frac{1}{2}n_I(t)\cos(\theta(t)) + \frac{1}{2}n_Q(t)\sin(\theta(t))\right]^2\right] \\ &= \frac{A_c^2}{4}\mathbf{E}\left[m^2(t)\left[\cos(\theta(t)) - 1\right]^2\right] + \frac{1}{4}\mathbf{E}\left[n_I^2(t)\cos^2(\theta(t))\right] + \frac{1}{4}\mathbf{E}\left[n_Q^2(t)\sin^2(\theta(t))\right]\end{aligned}$$

where we have used the independence of $m(t)$, $n_I(t)$, $n_Q(t)$, and $\theta(t)$ and the fact that $\mathbf{E}[n_I(t)] = \mathbf{E}[n_Q(t)] = 0$ to eliminate the cross terms.

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Problem 9.23 continued

$$
\begin{aligned}
\mathrm{MSE} &= \frac{A_c^2}{4}\mathbf{E}\left[m^2(t)\right]\mathbf{E}\left[\left(1-\cos\left(\theta(t)\right)\right)^2\right]+\frac{1}{4}\mathbf{E}\left[n_I^2(t)\right]\mathbf{E}\left[\cos^2\left(\theta(t)\right)\right]+\frac{1}{4}\mathbf{E}\left[n_Q^2(t)\right]\mathbf{E}\left[\sin^2(\theta(t)\right] \\
&= \frac{A_c^2 P}{4}\mathbf{E}\left[\left(1-\cos\left(\theta(t)\right)\right)^2\right]+\frac{1}{4}N_0W\mathbf{E}\left[\cos^2\left(\theta(t)\right)\right]+\frac{1}{4}N_0W\mathbf{E}\left[\sin^2(\theta(t)\right] \\
&= \frac{A_c^2 P}{4}\mathbf{E}\left[\left(1-\cos\left(\theta(t)\right)\right)^2\right]+\frac{N_0W}{2}
\end{aligned}
$$

where we have used the equivalences of $\mathbf{E}[m^2(t)] = P$, and $\mathbf{E}\left[n_I^2(t)\right]=\mathbf{E}\left[n_Q^2(t)\right]=2N_0W$. The last line uses the fact that $\cos^2(\theta(t))+\sin^2(\theta(t)) = 1$. If we now use the relation that $1-\cos A = 2\sin^2(A/2)$, this expression becomes

$$
\mathrm{MSE} = A_c^2 P\mathbf{E}\left[\sin^4\left(\frac{\theta(t)}{2}\right)\right]+\frac{N_0W}{2}
$$

Since the maximum value of $\theta(t) << 1$, $\sin(\theta(t)) \approx \theta(t)$ and we have

$$
\begin{aligned}
\mathrm{MSE} &\approx A_c^2 P\mathbf{E}\left[\left(\frac{\theta(t)}{2}\right)^4\right]+\frac{N_0W}{2} \\
&= \frac{3}{16}A_c^2 P\sigma_\theta^4+\frac{N_0W}{2}
\end{aligned}
$$

where we have used the fact that if θ is a zero-mean Gaussian random variable then

$$
\mathbf{E}\left[\theta^4\right] = 3\left(\mathbf{E}\left[\theta^2\right]\right)^2 = 3\sigma_\theta^4
$$

The mean square error is therefore $\frac{3}{16}A_c^2 P\sigma_\theta^4+\frac{1}{2}N_0W$.