Problem 9.23. In a receiver using coherent detection, the sinusoidal wave generated by the local oscillator suffers from a phase error $\theta(t)$ with respect to the carrier wave $\cos(2\pi f_c t)$. Assuming that $\theta(t)$ is a zero-mean Gaussian process of variance σ_{θ}^2 and that most of the time the maximum value of $\theta(t)$ is small compared to unity, find the mean-square error of the receiver output for DSB-SC modulation. The mean-square error is defined as the expected value of the squared difference between the receiver output and message signal component of a synchronous receiver output.

Solution

Based on the solution of Problem 9.22, we have the DSB-SC demodulator output is

$$y(t) = \frac{1}{2} A_c m(t) \cos[\theta(t)] + \frac{1}{2} n_I(t) \cos[\theta(t)] + \frac{1}{2} n_Q(t) \sin[\theta(t)]$$

Recall from Section 9. that the output of a synchronous receiver is

$$\frac{1}{2}A_cm(t) + \frac{1}{2}n_I(t)$$

The mean-square error (MSE) is defined by

$$MSE = \mathbf{E}\left[\left(y(t) - \frac{1}{2}A_c m(t)\right)^2\right]$$

Substituting the above expression for y(t), the mean-square error is

$$MSE = \mathbf{E}\left[\left[\frac{1}{2}A_{c}m(t)\left[\cos\left(\theta(t)\right)-1\right]+\frac{1}{2}n_{I}(t)\cos\left(\theta(t)\right)+\frac{1}{2}n_{Q}(t)\sin\left(\theta(t)\right)\right]^{2}\right]$$
$$= \frac{A_{c}^{2}}{4}\mathbf{E}\left[m^{2}(t)\left[\cos\left(\theta(t)\right)-1\right]^{2}\right]+\frac{1}{4}\mathbf{E}\left[n_{I}^{2}(t)\cos^{2}\left(\theta(t)\right)\right]+\frac{1}{4}\mathbf{E}\left[n_{Q}^{2}(t)\sin^{2}\left(\theta(t)\right)\right]$$

where we have used the independence of m(t), $n_I(t)$, $n_Q(t)$, and $\theta(t)$ and the fact that $\mathbf{E}[n_I(t)] = \mathbf{E}[n_Q(t)] = 0$ to eliminate the cross terms.

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Problem 9.23 continued

$$MSE = \frac{A_c^2}{4} \mathbf{E} \Big[m^2(t) \Big] \mathbf{E} \Big[\Big(1 - \cos(\theta(t)) \Big)^2 \Big] + \frac{1}{4} \mathbf{E} \Big[n_I^2(t) \Big] \mathbf{E} \Big[\cos^2(\theta(t)) \Big] + \frac{1}{4} \mathbf{E} \Big[n_Q^2(t) \Big] \mathbf{E} \Big[\sin^2(\theta(t)) \Big]$$
$$= \frac{A_c^2 P}{4} \mathbf{E} \Big[\Big(1 - \cos(\theta(t)) \Big)^2 \Big] + \frac{1}{4} N_0 W \mathbf{E} \Big[\cos^2(\theta(t)) \Big] + \frac{1}{4} N_0 W \mathbf{E} \Big[\sin^2(\theta(t)) \Big]$$
$$= \frac{A_c^2 P}{4} \mathbf{E} \Big[\Big(1 - \cos(\theta(t)) \Big)^2 \Big] + \frac{N_0 W}{2}$$

where we have used the equivalences of $\mathbf{E}[m^2(t)] = P$, and $\mathbf{E}[n_I^2(t)] = \mathbf{E}[n_Q^2(t)] = 2N_0W$. The last line uses the fact that $\cos^2(\theta(t)) + \sin^2(\theta(t)) = 1$. If we now use the relation that 1-cos $A = 2\sin^2(A/2)$, this expression becomes

$$MSE = A_c^2 P \mathbf{E} \left[\sin^4 \left(\frac{\theta(t)}{2} \right) \right] + \frac{N_0 W}{2}$$

Since the maximum value of $\theta(t) \le 1$, $\sin(\theta(t)) \approx \theta(t)$ and we have

$$MSE \approx A_c^2 P \mathbf{E} \left[\left(\frac{\theta(t)}{2} \right)^4 \right] + \frac{N_0 W}{2}$$
$$= \frac{3}{16} A_c^2 P \sigma_{\theta}^4 + \frac{N_0 W}{2}$$

where we have used the fact that if θ is a zero-mean Gaussian random variable then

$$\mathbf{E}\left[\theta^{4}\right] = 3\left(\mathbf{E}\left[\theta^{2}\right]\right)^{2} = 3\sigma_{\theta}^{4}$$

The mean square error is therefore $\frac{3}{16}A_c^2P\sigma_{\theta}^4 + \frac{1}{2}N_0W$.

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