Problem 9.23. In a receiver using coherent detection, the sinusoidal wave generated by the local oscillator suffers from a phase error $\theta(t)$ with respect to the carrier wave $cos(2\pi f_c t)$. Assuming that $\theta(t)$ is a zero-mean Gaussian process of variance σ_θ^2 and that most of the time the maximum value of $\theta(t)$ is small compared to unity, find the meansquare error of the receiver output for DSB-SC modulation. The mean-square error is defined as the expected value of the squared difference between the receiver output and message signal component of a synchronous receiver output.

Solution

Based on the solution of Problem 9.22, we have the DSB-SC demodulator output is

$$
y(t) = \frac{1}{2}A_c m(t) \cos[\theta(t)] + \frac{1}{2}n_t(t) \cos[\theta(t)] + \frac{1}{2}n_\varrho(t) \sin[\theta(t)]
$$

Recall from Section 9. that the output of a synchronous receiver is

$$
\frac{1}{2}A_c m(t) + \frac{1}{2}n_I(t)
$$

The mean-square error (MSE) is defined by

$$
MSE = E\left[\left(y(t) - \frac{1}{2}A_c m(t)\right)^2\right]
$$

Substituting the above expression for $y(t)$, the mean-square error is

$$
\begin{split} \text{MSE} &= \mathbf{E} \Bigg[\Bigg[\frac{1}{2} A_c m(t) \Big[\cos\big(\theta(t)\big) - 1 \Big] + \frac{1}{2} n_I(t) \cos\big(\theta(t)\big) + \frac{1}{2} n_Q(t) \sin\big(\theta(t)\big) \Bigg]^2 \Bigg] \\ &= \frac{A_c^2}{4} \mathbf{E} \Big[m^2(t) \Big[\cos\big(\theta(t)\big) - 1 \Big]^2 \Big] + \frac{1}{4} \mathbf{E} \Big[n_I^2(t) \cos^2\big(\theta(t)\big) \Big] + \frac{1}{4} \mathbf{E} \Big[n_Q^2(t) \sin^2\big(\theta(t)\big) \Big] \end{split}
$$

where we have used the independence of $m(t)$, $n_I(t)$, $n_O(t)$, and $\theta(t)$ and the fact that $\mathbf{E}[n_t(t)] = \mathbf{E}[\overline{n}_2(t)] = 0$ to eliminate the cross terms.

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Problem 9.23 continued

$$
\begin{split} \text{MSE} &= \frac{A_c^2}{4} \mathbf{E} \Big[m^2(t) \Big] \mathbf{E} \Big[\big(1 - \cos\big(\theta(t)\big) \big)^2 \Big] + \frac{1}{4} \mathbf{E} \Big[n_i^2(t) \Big] \mathbf{E} \Big[\cos^2\big(\theta(t)\big) \Big] + \frac{1}{4} \mathbf{E} \Big[n_0^2(t) \Big] \mathbf{E} \Big[\sin^2\big(\theta(t)\big) \Big] \\ &= \frac{A_c^2 P}{4} \mathbf{E} \Big[\big(1 - \cos\big(\theta(t)\big) \big)^2 \Big] + \frac{1}{4} N_0 W \mathbf{E} \Big[\cos^2\big(\theta(t)\big) \Big] + \frac{1}{4} N_0 W \mathbf{E} \Big[\sin^2\big(\theta(t)\big) \Big] \\ &= \frac{A_c^2 P}{4} \mathbf{E} \Big[\big(1 - \cos\big(\theta(t)\big) \big)^2 \Big] + \frac{N_0 W}{2} \end{split}
$$

where we have used the equivalences of $\mathbf{E}[m^2(t)] = P$, and $\mathbf{E}[n^2(t)] = \mathbf{E}[n^2(t)] = 2N_0W$. The last line uses the fact that $\cos^2(\theta(t)) + \sin^2(\theta(t)) = 1$. If we now use the relation that $1-\cos A = 2\sin^2(A/2)$, this expression becomes

$$
\text{MSE} = A_c^2 P \mathbf{E} \left[\sin^4 \left(\frac{\theta(t)}{2} \right) \right] + \frac{N_0 W}{2}
$$

Since the maximum value of $\theta(t) \ll 1$, $\sin(\theta(t)) \approx \theta(t)$ and we have

$$
\text{MSE} \approx A_c^2 P \mathbf{E} \left[\left(\frac{\theta(t)}{2} \right)^4 \right] + \frac{N_0 W}{2}
$$

$$
= \frac{3}{16} A_c^2 P \sigma_\theta^4 + \frac{N_0 W}{2}
$$

where we have used the fact that if θ is a zero-mean Gaussian random variable then

$$
\mathbf{E}[\theta^4] = 3(\mathbf{E}[\theta^2])^2 = 3\sigma_\theta^4
$$

The mean square error is therefore $\frac{3}{16}A_c^2P\sigma_\theta^4 + \frac{1}{2}N_0$ 3_{12} 4_{1} $\frac{3}{16}A_c^2P\sigma_{\theta}^4+\frac{1}{2}N_0W$.

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