Problem 10.16. Show that if T is a multiple of the period of f_c, then the terms $\sin(2\pi f_c t)$ and $\cos(2\pi f_c t)$ are orthogonal over the interval $\left[t_0, T+t_0\right]$.

Solution

$$\begin{aligned}\int_{t_0}^{T+t_0} \sin(2\pi f_c t)\cos(2\pi f_c t)dt &= \int_{t_0}^{T+t_0} \frac{1}{2}\sin(4\pi f_c t)dt \\ &= \frac{1}{8\pi f_c}\left[-\cos(4\pi f_c t)\right]\Big|_{t_0}^{T+t_0} \\ &= -\frac{1}{8\pi f_c}\left[\cos(4\pi f_c (t_0+T)) - \cos(4\pi f_c t_0)\right] \\ &= \frac{-1}{4\pi f_c}\sin(4\pi f_c t_0 + 2\pi f_c T)\cdot\sin(2\pi f_c T)\end{aligned}$$

where we have used the equivalence $\cos A - \cos B = 2\sin[(A+B)/2]\sin[(B-A)/2)]$. If T is a multiple of the period of f_c, then $f_c T$ = integer, and $\sin(2\pi f_c T) = 0$.

Therefore, $\int_{t_0}^{t_0+T} \sin(2\pi f_c t)\cos(2\pi f_c t)dt = 0$. That is, $\sin(2\pi f_c t)$ and $\cos(2\pi f_c t)$ are orthogonal over the interval $[t_0, t_0+T]$.